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Theory of residual stresses with application to an arterial geometry

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This paper presents a theory of residual stresses, with applications to biomechanics, especially to arteries. For a hyperelastic material, we use an initial local deformation tensor $\mathbf{K}$ as a descriptor of residual strain. This tensor, in general, is not the gradient of a global deformation, and a stress-free reference configuration, denoted $\overline{\mathscr{B}}$, therefore, becomes incompatible. Any compatible reference configuration $\mathscr{B}_0$ will, in general, be residually stressed. However, when a certain curvature tensor vanishes, there actually exists a compatible and stress-free configuration, and we show that the traditional treatment of residual stresses in arteries, using the opening–angle method, relates to such a situation.

Boundary value problems of nonlinear elasticity are preferably formulated on a fixed integration domain. For residually stressed bodies, three such formulations naturally appear: (i) a formulation relating to $\mathscr{B}_0$ with a non-Euclidean metric structure; (ii) a formulation relating to $\mathscr{B}_0$ with a Euclidean metric structure; and (iii) a formulation relating to the incompatible configuration $\overline{\mathscr{B}}$. We state these formulations, show that (i) and (ii) coincide in the incompressible case, and that an extra term appears in a formulation on $\overline{\mathscr{B}}$, due to the incompatibility.

1. Introduction

Essentially all blood vessels, and soft tissues in general, are subject to a residual stress when the applied load is removed. Although several studies had recognized the existence of residual stress in unloaded arteries, it was not until Vaishnav and Vossoughi [1] and Fung [2] reported their results that attention was drawn to residual stress in biomechanics. The residual stress in arteries is a compressive stress at the inner boundary and a tensile stress at the outer boundary. In addition, arteries may also experience residual stress in the axial direction. Residual stress is important from a physiological point of view since it redistributes the total stress and also gives a more uniform stress distribution; this is advantageous from an optimal operation point of view, since each part of the vessel wall carries a similar load.

Over the last decades, it has been suggested that growth and remodeling of the tissue may cause the residual stress, see the pioneering work in RODRIQUEZ *et al.* [3] and SKALAK *et al.* [4]. The term growth refers to a local process that increases the mass of the tissue, while remodeling is a change in the tissue structure that is achieved by reorganizing the existing constituents or by producing new constituents with a different organization. Both processes are natural processes in biological tissues and play an important role in the adaption to changes in their environment.

The general theory that we will use to model residual stress in this paper goes back to several sources, discussed by MAUGIN [5]. Firstly, it relates to the multiplicative decomposition of the deformation gradient, often used in plasticity theory. Secondly, there is the geometric line of development, initiated by Kondo with important contributions by TRUESDELL and NOLL [6] and NOLL [7]. Finally, there is the development of configurational or material forces of Eshelby, recently discussed by GURTIN [8]. This presentation will combine ideas of incompatible reference configurations, inherent in a multiplicative decomposition of the deformation gradient, with a geometrical viewpoint. Initial inspiration for this work was provided by the explicit construction of the incompatible reference configuration in JOHNSON and HOGER [9], and the use of the theory of Kondo in a biomechanics context by TAKAMIZAWA and MATSUDA [10].

When dealing with arteries, the prevailing method of describing residual stress is the opening–angle method, proposed by CHOUNG and FUNG [11], see also STÅLHAND *et al.* [12], and STÅLHAND and KLARBRING [13]. The method is based on the assumption that a radially cut artery opens up into a stress-free circular sector, the cut-open state. The residual strain is taken to be the strain produced by the deformation map shown in Fig. 1, i.e., the closure of a circular sector.

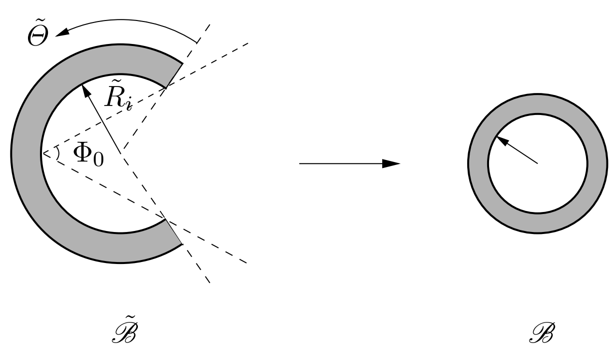

FIG. 1. A schematic picture of the opening–angle method. To the left is the cut–open configuration $\tilde{\mathscr{B}}$ with the opening angle Φ_0 and to the right is a residually stressed configuration. $\tilde{\Theta}$ is a circumferential coordinate, and $\tilde{R}_i$ is the inner radius in the cut–open state.

A more complete approach to residual stresses in arteries can be built on a general theory of residual stress as described above. One may then think that to obtain a stress-free configuration, the single cut of Fig. 1 is not enough (a thought supported by experiments). In fact, one can consider the idea that the body needs to be cut into infinitesimally small parts to become stress free. For a rotationally symmetric structure, as an artery, we may then think of concentric infinitesimal cylinders. A local tangent map **K**, as shown in Fig. 2, then represents the initial strain. Such a method was used for identification of residual stresses in OLSSON *et al.* [14].

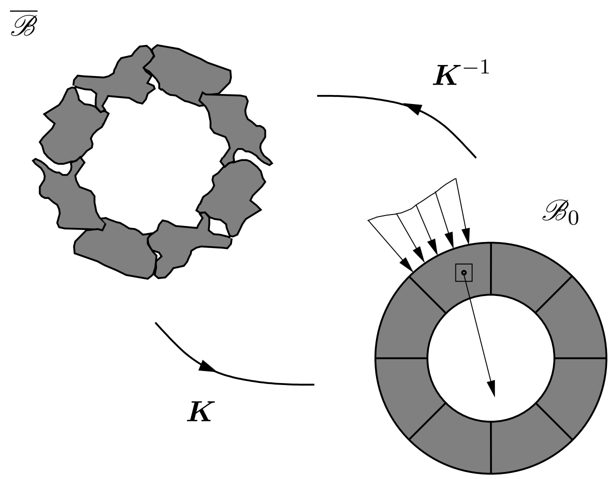

FIG. 2. A residually stressed reference configuration $\mathscr{B}_0$ is locally relaxed to form the stress free but incompatible configuration $\overline{\mathscr{B}}$.

In this paper we will show that the the opening–angle method is included in the general method. The connection is based on the result that when a certain curvature tensor vanishes, there exists a stress-free compatible reference configuration of the body, see BLUME [15] and KLARBRING and OLSSON [16], and related work in STEINMANN [17] and GANGHOFFER and HAUSSY [18].

A configuration of a body, denoted by $\mathscr{B}$ in this paper, is a set in the physical space occupied by material points. It is useful to take a such particular configuration, denoted by $\mathscr{B}_0$, as a reference, and any other configuration can then be mapped one-to-one onto this reference configuration. In the classical theory of elastic bodies it is usually assumed that $\mathscr{B}_0$ represents a stress-free state to which the body may relax when all external loads are removed. However, in the more general theory, which is necessary for modeling residual stresses, we may only assume that this relaxation takes place locally for each material point. This leads to an incompatible, but stress free, reference configuration, denoted $\overline{\mathscr{B}}$ in this paper. Thus, for a residually stressed body two essentially different reference configurations appear: the residually stressed, but compatible, configuration $\mathscr{B}_0$ and the stress-free but incompatible configuration $\overline{\mathscr{B}}$. When formulating the boundary value problems of static elasticity, it is necessary to decide which of these two reference configurations to use as a fixed integration domain. Takamizawa and Matsuda [10] used $\mathscr{B}_0$, but concluded that the initial stress induces a possibly non-Euclidean metric and the integration domain becomes a general Riemannian manifold. Noll [7], on the other hand, states the equilibrium equations in relation to $\overline{\mathscr{B}}$, but concludes that a non-classical term, due to incompatibility, appears. In the present paper, these results are given in a uniform setting and a second formulation on $\mathscr{B}_0$, that uses the Euclidean metric, is given. We do not make any definite conclusions as to which of the formulations is to be preferred in a particular application, but we do find the result that the non-Euclidean structure of the natural equilibrium equation on $\mathscr{B}_0$ disappears for an incompressible material. The derivation of the equilibrium equation of Noll [7] is based on a generalization of Piola's identity to the incompatible situation. A direct proof of this generalized Piola identity is given in the Appendix.

2. General theory of a residually stressed body

2.1. Geometry

Let an elastic body be represented by a subset $\mathscr{B}_0$ of the three–dimensional physical space with the Euclidean metric $\mathbf{G}$. A differentiable one–to–one map f deforms this body into another subset, say $\mathscr{B}$, with the Euclidean metric $\mathbf{g}$, i.e.,

$$f : \mathscr{B}_0 \to \mathscr{B}.$$

We call $\mathscr{B}_0$ a reference configuration (later we will construct other reference configurations) and $\mathscr{B}$ a spatial configuration. These and other configurations appearing in the theory are shown in Fig. 3.

Let the coordinates (X^1, X^2, X^3) represent a point in the reference configuration and (x^1, x^2, x^3) a point in the spatial configuration. We will use the abbreviation X^A and x^a, respectively, for these coordinates. The deformation can then be written in component form as

$$x^a = f^a(X^1, X^2, X^3), \quad a = 1, 2, 3.$$

Since f is differentiable, we can construct a tangent map (deformation gradient) $\mathbf{F}$ with components

$$F^a{}_A = \frac{\partial f^a}{\partial X^A}.$$

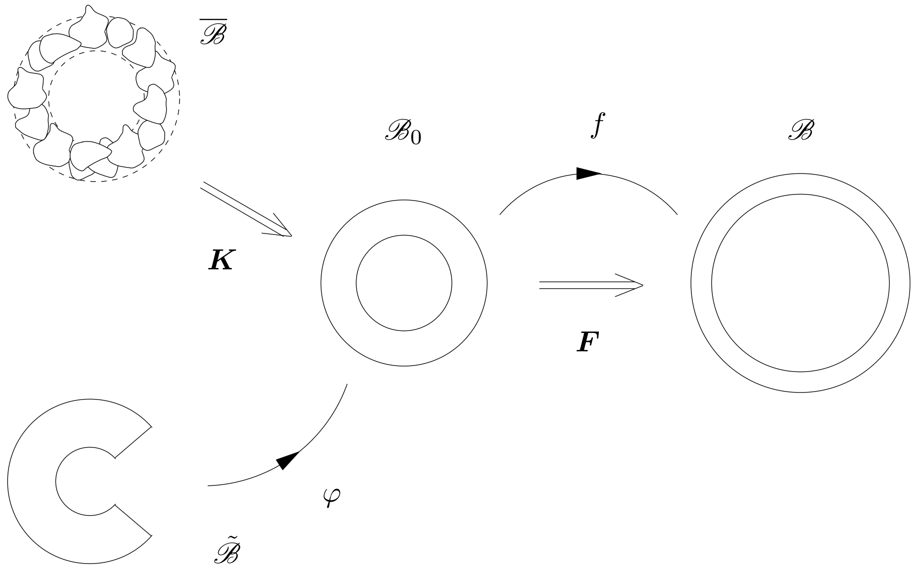

Fig. 3. Configurations, i.e., sets in the Euclidean space, used in the theory. f and φ are point mappings, while $\mathbf{F}$ and $\mathbf{K}$ are tangent maps. The drawing alludes to an arterial geometry, but is generally valid.

The deformation gradient is a two–point tensor that maps tangent vectors in $\mathscr{B}_0$ into tangent vectors in $\mathscr{B}$, i.e.,

$$\mathbf{F} : T_{\mathscr{B}_0} \to T_{\mathscr{B}},$$

where $T_{\mathscr{B}_0}$ and $T_{\mathscr{B}}$ denote the union of all tangent spaces (tangent bundle) in $\mathscr{B}_0$ and $\mathscr{B}$, respectively.

For later use we introduce the Einstein summation convention: for any term involving indices, a summation is enforced for an index that appears both as an upper and a lower index. Note also that we use lower case indices when referring to $\mathscr{B}$ and capital indices when referring to $\mathscr{B}_0$.

It is often assumed that the body in an unloaded reference configuration is stress free. This cannot be assumed to be generally true. As an example, take a thin rubber tube and turn it inside out. This inverted tube is unloaded but it is not stress free. A radial cut will relieve the stress and the tube will try to obtain its original shape. This example shows that an unloaded body is not necessarily stress free, and if our constitutive material law is such that it refers to a stress free state, as an hyperelastic law usually does, the choice of reference configuration must be made carefully.

Now, the reference configuration $\mathscr{B}_0$ introduced above cannot in general be assumed stress free. However, for an elastic body, it is true that each infinitesimal part of $\mathscr{B}_0$, or any other configuration, can be made stress-free by removing (cutting) it from the main body and letting it deform independently of its neighbors into a stress free state, see Johnson and Hoger [9] and Takamizawa and Matsuda [10]. This local deformation can be described by an invertible two–point tensor $\mathbf{K}^{-1}$ that takes vectors in a tangent space of $\mathscr{B}_0$ and places them into another tangent space. This later space is here taken as a tangent space of a subset $\overline{\mathscr{B}}$ of the physical space with Euclidean metric $\boldsymbol{\gamma}$, i.e., as $T_{\overline{\mathscr{B}}}$. Thus,

$$\mathbf{K}^{-1} : T_{\mathscr{B}_0} \rightarrow T_{\overline{\mathscr{B}}}. \tag{2.1}$$

Since rigid body rotation is not accommodated by stress, we may assume that $\mathbf{K}^{-1}$ is determined by unloading only up to such a rigid body rotation. The stress that may exist in the configuration $\mathscr{B}_0$ is termed *initial* stress and should be distinguished from the stress that exists in a particular spatial configuration $\mathscr{B}$ that is free of external loading, which is called *residual* stress. The inverse of $\mathbf{K}^{-1}$, denoted $\mathbf{K}$, may be called an *initial local deformation*.

Care should be taken in interpreting the subset $\overline{\mathscr{B}}$: since the mapping (2.1) is only between tangent spaces, any point mapping between $\mathscr{B}_0$ and $\overline{\mathscr{B}}$ is immaterial. For instance, it is plausible, and sometimes preferable, to let the sets $\mathscr{B}_0$ and $\overline{\mathscr{B}}$ coincide. Moreover, the tensor $\mathbf{K}^{-1}$ is certainly not, in general, the gradient of a deformation. An intuitive view of $\overline{\mathscr{B}}$ may be that it is a subset of Euclidean space where we have placed, at each point, an infinitesimal, locally deformed, material part, which does not fit together with its neighbors. In other words, it is an incompatible configuration of the body.

Since the material in $\overline{\mathscr{B}}$ is considered to be stress free, for a hyperelastic material, it is the local deformation out of this state that determines the stress in a general spatial configuration $\mathscr{B}$. This local deformation is a composition of $\boldsymbol{K}$ and the deformation gradient $\boldsymbol{F}$, and can be written

$$\mathbf{H} = \mathbf{F}\mathbf{K}, \qquad \mathbf{H} : T_{\overline{\mathscr{B}}} \rightarrow T_{\mathscr{B}}.$$

By introducing Greek indices when referring to $\overline{\mathscr{B}}$, we can write this total tangent map in components as

$$H^a{}_\alpha = F^a{}_A K^A{}_\alpha.$$

Volume elements (or forms) defined on the tangent spaces of the three configurations $\mathscr{B}_0$, $\mathscr{B}$ and $\overline{\mathscr{B}}$, are denoted $dv_{\mathscr{B}_0}$, $dv_{\mathscr{B}}$ and $dv_{\overline{\mathscr{B}}}$, respectively. These volume elements are related by the determinants of the tangent maps, i.e.,

$$dv_{\mathscr{B}} = \det \mathbf{F} dv_{\mathscr{B}_0}, \quad dv_{\mathscr{B}_0} = \det \mathbf{K} dv_{\overline{\mathscr{B}}}, \quad dv_{\mathscr{B}} = \det \mathbf{H} dv_{\mathscr{B}_0}, \tag{2.2}$$

where the last equation is a consequence of the first two. Note that the determinant of a two-point tensor depends on the metrics of the two involved manifolds. This fact is of some importance in Sec. 4.2, where further details are given.

Defining density functions ρ_0, ρ and ρ_{Ref} on the configurations $\mathscr{B}_0$, $\mathscr{B}$ and $\overline{\mathscr{B}}$, conservation of mass implies $\rho\, dv_{\mathscr{B}} = \rho_{\mathrm{Ref}}\, dv_{\overline{\mathscr{B}}} = \rho_0\, dv_{\mathscr{B}_0}$, and we then get from Eqs. (2.2) that

$$\rho \det \mathbf{F} = \rho_0, \quad \rho \det \mathbf{H} = \rho_{\mathrm{Ref}}, \quad \rho_0 \det \mathbf{K} = \rho_{\mathrm{Ref}}, \tag{2.3}$$

where the first two equations imply the last equation. The index Ref is used to indicate that the density of a stress-free material may act as a reference and as a known quantity in many natural problem formulations.

In the following we are particularly interested in the case when the material is incompressible, which, as discussed in OLSSON *et al.* [14], we take to imply that

$$\det \mathbf{H} = \det \mathbf{F} = \det \mathbf{K} = 1, \tag{2.4}$$

and which, with (2.3), gives $\rho = \rho_0 = \rho_{\mathrm{Ref}}$.

2.2. Balance and constitutive laws

The material is assumed to be hyperelastic with a strain–energy per unit mass Ψ. Since $\overline{\mathscr{B}}$ is regarded as stress free and the tangent map $\mathbf{H} = \mathbf{FK}$ is a local deformation out of this state, it is natural to take Ψ as a function of $\mathbf{H}$. Furthermore, objectivity requires that Ψ depends on $\mathbf{H}$ only through a dependence on the right Cauchy–Green deformation tensor $\mathbf{Z}$ with components

$$Z_{\alpha\beta} = H^a{}_\alpha\, g_{ab} H^b{}_\beta.$$

That is, $\Psi = \Psi(\mathbf{Z})$. This assumption is used together with the energy equation, with heat flux terms neglected:

$$\rho\dot{\Psi} = \sigma^{ab} d_{ab}, \tag{2.5}$$

where the superimposed dot means rate of change with respect to time, σ^{ab} are the components of the Cauchy stress tensor, and d_{ab} are the components of the rate–of–deformation tensor. Standard arguments then imply that

$$\sigma^{ab} = 2\rho H^a{}_\alpha \frac{\partial \Psi}{\partial Z_{\alpha\beta}} H^b{}_\beta. \tag{2.6}$$

For an incompressible material, the constraint (2.4) implies that the arguments that produced (2.6) as a consequence of (2.5) have to be modified. Instead of (2.6) we arrive at

$$\sigma^{ab} = -p g^{ab} + 2\rho H^a{}_\alpha \frac{\partial \Psi}{\partial Z_{\alpha\beta}} H^b{}_\beta, \tag{2.7}$$

where g^{ab} are the components of the inverse of the metric $\mathbf{g}$ and p is a multiplier sometimes called the hydrostatic pressure.

Finally, we need an equilibrium equation. On the domain $\mathscr{B}$ we classically have the following equation:

$$\nabla_b \sigma^{ab} + \rho b^a = 0, \tag{2.8}$$

where b^a are the components of an external force per unit mass vector and ∇_b is the covariant derivative associated with the metric in $\mathscr{B}$.

2.3. Existence of a stress-free compatible reference configuration

Now, it does happen in special cases that, in addition to the incompatible configuration $\overline{\mathscr{B}}$, there exists a compatible stress-free configuration of the body. In fact, as will be shown in Sec. 3.2, the opening angle method is based on such an assumption. However, in this section we will still be concerned with the general situation, which was discussed in TAKAMIZAWA and MATSUDA [10] and in KLARBRING and OLSSON [16]. It was concluded in [16] that an important theorem can be framed in terms of the curvature tensor $\mathbf{R}$ of the strain–like tensor $\mathbf{m}$ with components

$$m_{AB} = (\mathbf{K}^{-1})^{\alpha}{}_{A}\, \gamma_{\alpha\beta} (\mathbf{K}^{-1})^{\beta}{}_{B}, \tag{2.9}$$

where $\gamma_{\alpha\beta}$ are the components of a Euclidean metric tensor on $\overline{\mathscr{B}}$. The tensor $\mathbf{m}$ is a metric tensor, but not generally a Euclidean metric tensor. The components of the curvature tensor read

$$R_{ABCD} = m_{DK} \left(\frac{\partial}{\partial X^B} \Gamma^K_{AC} - \frac{\partial}{\partial X^A} \Gamma^K_{BC} + \Gamma^L_{AC} \Gamma^K_{LB} - \Gamma^L_{BC} \Gamma^K_{LA} \right),$$

where Γ^K_{AC} are the Christoffel symbols of the second kind of $\mathbf{m}$ viewed as a metric in $\mathscr{B}_0$. The theorem is a slight reinterpretation of a result of BLUME [15] and says that if $\mathscr{B}_0$ is simply connected and the curvature tensor vanishes, then there is a mapping φ^{-1} from $\mathscr{B}_0$ to a subset $\tilde{\mathscr{B}}$ of the Euclidean space with metric $\boldsymbol{\gamma}^*$ such that

$$\frac{\partial(\varphi^{-1})^{\tilde{\alpha}}}{\partial X^A} \gamma^*_{\tilde{\alpha}\tilde{\beta}} \frac{\partial(\varphi^{-1})^{\tilde{\beta}}}{\partial X^B} = (\mathbf{K}^{-1})^{\alpha}{}_{A}\, \gamma_{\alpha\beta} (\mathbf{K}^{-1})^{\beta}{}_{B}. \tag{2.10}$$

This means that $\mathbf{K}^{-1}$ and the gradient of φ^{-1} have the same stretch tensor and differ only by the product of an orthogonal tensor. Note that this correlation is local with the orthogonal tensor being possibly different at each point. The mapping φ^{-1} is locally invertible, but not necessarily globally invertible, e.g., its image could be overlapping. Assuming, however, that $\mathscr{B}_0$ is properly chosen so that φ^{-1} is globally invertible, we can define φ, the inverse of φ^{-1}, as a mapping from $\tilde{\mathscr{B}}$ to $\mathscr{B}_0$, see Fig. 3.

We now show that $\tilde{\mathscr{B}}$ can be used as a global compatible stress-free configuration on which stress calculations for the hyperelastic material can be based. Define, in the usual way, the deformation gradient of the composite mapping $f \circ \varphi$ (Fig. 3) as the tensor with components

$$\hat{H}^a{}_{\tilde{\alpha}} = \frac{\partial (f \circ \varphi)^a}{\partial \xi^{\tilde{\alpha}}}, \tag{2.11}$$

where (ξ^1, ξ^2, ξ^3) are coordinates in the set $\tilde{\mathscr{B}}$; such coordinates are labeled by Greek indices with a superposed tilde. Equation (2.10) then implies that

$$H^a{}_{\alpha} = \hat{H}^a{}_{\tilde{\alpha}}\, Q^{\tilde{\alpha}}{}_{\alpha}, \tag{2.12}$$

where $Q^{\tilde{\alpha}}_{\alpha}$ are the components of an orthogonal tensor $\mathbf{Q}$ mapping from a tangent space of $\overline{\mathscr{B}}$ to a tangent space of $\tilde{\mathscr{B}}$. Note that orthogonality of such a two-point tensor means that $\boldsymbol{\gamma}^* = \mathbf{Q}^T \boldsymbol{\gamma} \mathbf{Q}$. It can be shown, by use of Eq. (2.12), that (2.6) can be rewritten as

$$\sigma^{ab} = 2\rho \hat{H}^a{}_{\tilde{\alpha}} \frac{\partial \hat{\Psi}}{\partial \hat{Z}_{\tilde{\alpha}\tilde{\beta}}} \hat{H}^b{}_{\tilde{\beta}}\,, \tag{2.13}$$

where $\hat{\Psi} = \hat{\Psi}(\hat{\mathbf{Z}}) = \Psi(\mathbf{Z})$ for $\mathbf{Z}$ and $\hat{\mathbf{Z}}$ related as $Z_{\alpha\beta} = Q^{\tilde{\alpha}}_{\alpha} \hat{Z}_{\tilde{\alpha}\tilde{\beta}} Q^{\tilde{\beta}}_{\beta}$.

In analogy with (2.7), a constitutive law similar to (2.13) holds for the incompressible case.

Thus, in conclusion, when the curvature tensor $\mathbf{R}$ vanishes on $\mathscr{B}_0$, then there is a stress-free compatible configuration $\tilde{\mathscr{B}}$ of the body (possibly cut to make simply connected) and we can use this configuration as our reference configuration when calculating the stress.

3. Arterial geometry

In many applications an artery can be modeled as a rotationally symmetric body that also retains this symmetry when loaded. Therefore, when applying the theory of the previous section to arteries, we will use cylindrical coordinates which are denoted (R, Θ, Z) in $\mathscr{B}_0$ and (r, θ, z) in $\mathscr{B}$, respectively. These coordinates refer to radial, tangential and axial directions.

The mapping f is taken to be a radial expansion, or shrinking, of the cylinder, i.e.,

$$r = f^r(R), \qquad \theta = \Theta, \qquad z = Z,$$

with non-zero components of the deformation gradient given by

$$F^r{}_R = \frac{df^r(R)}{dR}, \qquad F^\theta{}_\Theta = 1, \qquad F^z{}_Z = 1.$$

The incompressibility constraint (2.4), $\det \mathbf{F} = 1$, implies that

$$r = f^r(R) = \sqrt{R^2 - R_i^2 + r_i^2}, \tag{3.1}$$

where R_i is the inner radius of the cylindrical tube $\mathscr{B}_0$ and r_i is the inner radius of the similar cylinder $\mathscr{B}$.

As indicated above, it is only the tangent spaces of $\overline{\mathscr{B}}$, and not the set itself, that play any direct role in the general theory. Therefore, without any loss of generality, we will let $\overline{\mathscr{B}}$ coincide with the set $\mathscr{B}_0$. The latter set now has a dual interpretation: it is a compatible configuration of the body, but it is also a set where we have placed individually unloaded parts that form an incompatible body structure. As a consequence of this identification, the tensor $\mathbf{K}$ will map both from and to $T_{\mathscr{B}_0}$ and its components will be indicated by indices R, Θ and Z, only. For simplicity, these components are assumed to form a diagonal matrix, i.e. only $K^R{}_R$, $K^\Theta{}_\Theta$ and $K^Z{}_Z$ are assumed to be non-zero. Furthermore, to comply with the assumption of rotational symmetry, they are also assumed to depend only on the radial coordinate. Moreover, the incompressibility constraint (2.4) indicates that $K^\Theta{}_\Theta = (K^R{}_R K^Z{}_Z)^{-1}$. The non-zero components of the tensor $\mathbf{H}$ then become

$$H^R{}_R = \frac{R}{\sqrt{R^2 - R_i^2 + r_i^2}} K^R{}_R, \qquad H^\Theta{}_\Theta = (K^R{}_R K^Z{}_Z)^{-1}, \qquad H^Z{}_Z = K^Z{}_Z. \tag{3.2}$$

We now conclude, through the constitutive equation (2.7), that given a configuration $\mathscr{B}$ and boundary conditions, the stresses are determined by $K^R{}_R$, $K^Z{}_Z$ and R_i. In particular, if $\mathscr{B}$ is an unloaded configuration the residual stresses are in this way parametrized by $K^R{}_R$, $K^Z{}_Z$ and R_i. In the next subsection we will see that the case of constant functions $K^R{}_R$ and $K^Z{}_Z$ has played a central historical role in the mechanical modeling of arteries.

The stress boundary conditions on the cylinder $\mathscr{B}$ are that a constant pressure λ acts on the inner surface, with radius r_i, and that the outer surface, with radius r_o, is stress free. Moreover, from (3.2) it follows that $\mathbf{H}$, and, thereby, from the specific form of $\mathbf{g}$ for cylindrical coordinates, also $\mathbf{Z}$, depend on the radial coordinate only. It was shown in Olsson *et al.* [14] that if the material behavior, i.e., the strain energy Ψ, varies only in the radial direction, then the equilibrium Eq. (2.8), with $b^a = 0$, implies that

- the hydrostatic pressure p depends on the radial coordinate only,
- everywhere in $\mathscr{B}$ it holds that

$$\sigma^{r\theta} = \sigma^{rz} = 0, \tag{3.3}$$

- and

$$\lambda = \int_{r_i}^{r_o} \left(r\sigma^{\theta\theta} - \frac{\sigma^{rr}}{r} \right) dr. \tag{3.4}$$

Note that the stress components in (3.4) are tensor components and, thus, do not have uniform physical dimensions.

Condition (3.3) is a constraint on the functional form of the strain energy.

3.1. An identification problem

When substituting (2.7) into (3.4), the hydrostatic pressure cancels, and (3.4) becomes a condition which relates constitutive constants and initial strain parameters to the pressure. Thus, a relation of the following form is obtained:

$$\lambda = \lambda(\boldsymbol{\kappa}, K^R{}_R, K^Z{}_Z, R_i, r_i, r_o), \tag{3.5}$$

where $\boldsymbol{\kappa}$ is a vector of constitutive parameters. This equation may be used to

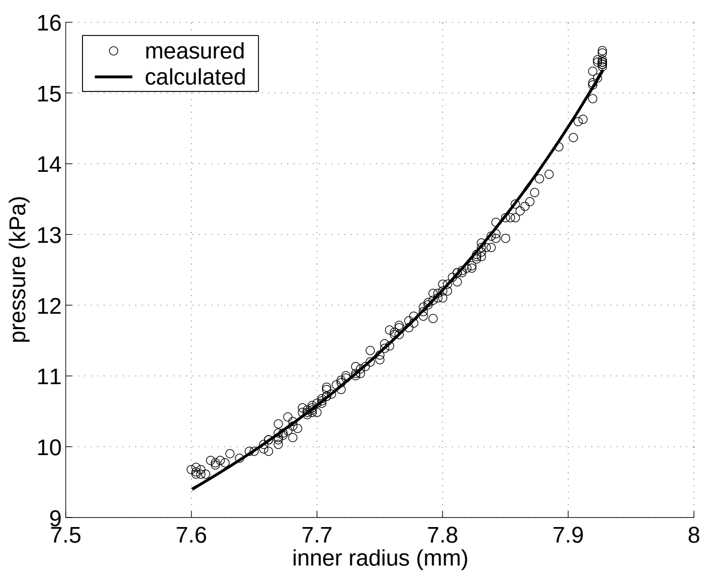

FIG. 4. The fitting of the model, in the least square sense, to measurements from SONNESON *et al.* [19]

state an inverse identification problem: if a series of measurements of pairs (λ, r_i) are at hand, we may attempt to find $(\boldsymbol{\kappa}, {K^R}_R, {K^Z}_Z, R_i, r_o)$ such that the measurements fit the predictions from (3.5) as closely as possible, e.g., in the least square sense. Such identification results were reported in Olsson *et al.* [14]. In vivo measurements of pairs (λ, r_i) on an abdominal aorta of a 47 year old female were taken from the study of Sonneson *et al.* [19]. The strain energy function was taken from Holzapfel *et al.* [20], meaning that $\boldsymbol{\kappa}$ contains three parameters. The fitting between model and measurements is shown in Fig. 4.

3.2. Compatible stress-free reference configuration

The prevailing method for describing residual strains and stresses in arteries is the opening–angle method, proposed by Choung and Fung [11], see Fig. 1. This method is based on the assumption that a radially cut arterial segment opens up into a stress-free circular sector (often referred to as the cut–open state). That is, we have a situation as the one described in Sec. 2.3 where a curvature tensor is zero and there exists a mapping φ^{-1}, which is the mapping that together with f describes the opening of the cut arterial segment. We may view the cutting as producing a simply connected domain. The coordinates of the circular sector $\tilde{\mathscr{B}}$ (see Fig. 3) are denoted $(\tilde{R}, \tilde{\Theta}, \tilde{Z})$ and the composite mapping $f \circ \varphi$, for an incompressible case, is defined by

$$r = \sqrt{\frac{\tilde{R}^2 - \tilde{R}_i^2}{\alpha\delta} + r_i^2}, \qquad \Theta = \alpha\tilde{\Theta}, \qquad z = \delta\tilde{Z}, \tag{3.6}$$

where $\tilde{R}_i$ is the inner radius of the circular sector $\tilde{\mathscr{B}}$, δ is its axial elongation and α describes the opening angle. From (3.6) we calculate the components $\hat{H}^a_{\tilde{\alpha}}$ as defined in (2.11):

$$\hat{H}^r_{\tilde{R}} = \frac{\tilde{R}}{\sqrt{\alpha\delta}\sqrt{\tilde{R}^2 - \tilde{R}_i^2 + \alpha\delta r_i^2}}, \qquad \hat{H}^\Theta_{\tilde{\Theta}} = \alpha, \qquad {\hat{H}^z}_{\tilde{Z}} = \delta.$$

The stresses can be now be calculated by use of a constitutive equation, and, thus, for a given configuration $\mathscr{B}$ they are represented by the three parameters α, δ and $\tilde{R}_i$. In the treatment of the arterial geometry by the more general method above, we concluded that stresses for a given $\mathscr{B}$ were described by two functions ${K^R}_R$ and ${K^Z}_Z$ and a constant R_i. If we assume that the two functions are constant through the thickness of the artery, the stresses are again described by three parameters. Therefore, we may expect that there is a particular choice of constants ${K^R}_R$, ${K^Z}_Z$ and parameter R_i that will make this special case of the general method identical to the opening–angle method. The key to such an identification is Eq. (2.12). This equation describes how local deformations, calculated by means of reference configurations $\overline{\mathscr{B}}$ and $\tilde{\mathscr{B}}$, respectively, need to be related in order to give the same stress. Equation (2.12) holds if

$$Q^{\tilde{R}}{}_{R} = 1, \qquad Q^{\tilde{\Theta}}{}_{\Theta} = \frac{R}{\tilde{R}}, \quad Q^{\tilde{Z}}{}_{Z} = 1, \tag{3.7}$$

and

$$K^{R}{}_{R} = \frac{\tilde{R}}{\alpha\delta R}, \qquad K^{Z}{}_{Z} = \delta. \tag{3.8}$$

The orthogonal tensor represented by (3.7) is such that material line elements form the same angle with coordinate surfaces in both $\tilde{\mathscr{B}}$ and $\mathscr{B}$.

From (3.8) we see that the functions $K^{R}{}_{R}$ and $K^{Z}{}_{Z}$ are constant when $\tilde{R}/R$ is so. This happens for a particular choice of R_i and $\tilde{R}_i$. By identifying (3.1) and (3.6) we can calculate the function φ. In particular, the radial coordinates are related as

$$R^2 - \frac{\tilde{R}^2}{\alpha\delta} = R_i^2 - \frac{\tilde{R}_i^2}{\alpha\delta}.$$

Thus, if parameters are set so that $R_i^2 = \tilde{R}_i^2/\alpha\delta$, then the factor $\tilde{R}/R = \sqrt{\alpha\delta}$ is constant and the opening-angle method can be mimicked by the general method with constant tensors $K^{R}{}_{R}$ and $K^{Z}{}_{Z}$ and this particular choice of R_i. Note that for other choices of R_i, i.e. of $\overline{\mathscr{B}}$, the general method can still contain the opening-angle method but not by a choice of constant functions $K^{R}{}_{R}$ and $K^{Z}{}_{Z}$.

4. Riemannian manifold

4.1. The tensor $\boldsymbol{m}$ as a metric on $\mathscr{B}_0$

The three configurations $\overline{\mathscr{B}}$, $\mathscr{B}_0$ and $\mathscr{B}$ and their coordinates have been assigned three metric tensors. The components of these metric tensors are denoted $\gamma_{\alpha\beta}$, G_{AB} and g_{ab}, respectively. Now, on the set $\mathscr{B}_0$ it is also interesting to use the tensor $\mathbf{m}$, the components of which are defined in (2.9), as a metric tensor. This is generally not an Euclidean metric since the tensor $\mathbf{K}^{-1}$ may not be the gradient of a deformation. The set $\mathscr{B}_0$ with the metric $\mathbf{m}$ constitutes a general Riemannian manifold, which we denote $(\mathscr{B}_0, \mathbf{m})$. Note that a change of metric does not affect the components $F^{a}{}_{A}$ of the deformation gradient since its definition is independent of the metric.

It may be intrinsically difficult to visualize $(\mathscr{B}_0, \mathbf{m})$ in the physical three-dimensional case, so for interpretation purposes we may think of a two-dimensional physical space, a situation that is shown in Fig. 5. The three configurations $\overline{\mathscr{B}}$, $\mathscr{B}_0$ and $\mathscr{B}$ are then flat two-dimensional surfaces, while the manifold $(\mathscr{B}_0, \mathbf{m})$

is a generally curved two-dimensional surface in an embedding three-dimensional space. This curved surface becomes flat in two ways: firstly, it may be non-uniformly stretched (flattened out) to become a configuration. Secondly, it may be teared into (infinitesimal) pieces, and these may be independently placed on the flat physical surface to form $\overline{\mathscr{B}}$. In some cases, a finite number of cuts may give a surface which can be unrolled to become flat. This is the case of zero curvature, when $\tilde{\mathscr{B}}$ exists, as in the opening–angle method.

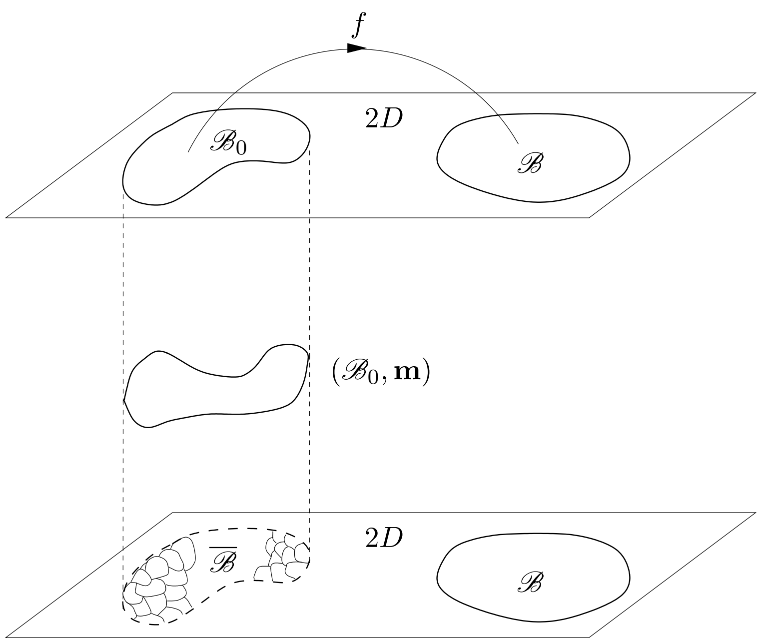

FIG. 5. If we view the Euclidean space as two-dimensional, the manifold $(\mathscr{B}_0, \mathbf{m})$ may be seen as a curved three-dimensional surface. This surface may then be made two-dimensional in two ways: On the top it is flattened, but stretched. On the bottom it is teared into infinitesimal pieces that generally do not fit together.

4.2. Determinants, volume elements and densities

The determinant of a two-point tensor, mapping between tangent spaces of two manifolds, depends on the metric tensors of the manifolds. This is clear from the following formula, which holds when the deformation gradient maps from a tangent space of $\mathscr{B}_0$, with metric $\mathbf{G}$, to a tangent space of $\mathscr{B}$, with metric $\mathbf{g}$:

$$\det \mathbf{F} = \frac{\sqrt{\det(g_{ab})}}{\sqrt{\det(G_{AB})}} \det(F^a{}_A), \tag{4.1}$$

where $\det(F^a{}_A)$ is the determinant of the matrix formed from the components of the deformation gradient $\mathbf{F}$, and similarly for $\det(g_{ab})$ and $\det(G_{AB})$.

If we regard $\mathbf{F}$ as a mapping from a tangent space of $\mathscr{B}_0$ with metric $\mathbf{m}$ to a tangent space of $\mathscr{B}$, its determinant will be different from that in (4.1). To indicate this, we write the corresponding operator with an index $\mathbf{m}$, and it holds that

$$\det_{\mathbf{m}}\mathbf{F} = \frac{\sqrt{\det(g_{ab})}}{\sqrt{\det(m_{AB})}}\det(F^a{}_A).$$

Formulas for the other determinants of two-point tensors are

$$\det\mathbf{K} = \frac{\sqrt{\det(G_{AB})}}{\sqrt{\det(\gamma_{\alpha\beta})}}\det(K^A{}_\alpha), \quad \det\mathbf{H} = \frac{\sqrt{\det(g_{ab})}}{\sqrt{\det(\gamma_{\alpha\beta})}}\det(H^a{}_\alpha).$$

A volume element of the manifold $(\mathscr{B}_0, \mathbf{m})$ is denoted $dv_{(\mathscr{B}_0,\mathbf{m})}$ and is defined by

$$dv_{\mathscr{B}} = (\det_{\mathbf{m}}\mathbf{F})\, dv_{(\mathscr{B}_0,\mathbf{m})}.$$

By introducing a density $\hat{\rho}_0$ on $(\mathscr{B}_0, \mathbf{m})$ by the mass conservation requirement $\hat{\rho}_0 dv_{(\mathscr{B}_0,\mathbf{m})} = \rho dv_{\mathscr{B}}$, we get

$$\hat{\rho}_0 = \rho \det_{\mathbf{m}}\mathbf{F}. \tag{4.2}$$

This equation together with $(2.3)^2$ implies that $\hat{\rho}_0 \det\mathbf{H} = \rho_{\mathrm{Ref}} \det_{\mathbf{m}}\mathbf{F}$. Now, from the definition of $\mathbf{m}$ and the above formulas for determinants, it follows that $\det_{\mathbf{m}}\mathbf{F} = \det\mathbf{H}$ and we conclude that

$$\hat{\rho}_0 = \rho_{\mathrm{Ref}}. \tag{4.3}$$

That is, the reference density ρ_{Ref} can be regarded both as the density of the incompatible configuration $\overline{\mathscr{B}}$ and as the density of the Riemannian manifold $(\mathscr{B}_0, \mathbf{m})$.

5. Boundary value problems

Boundary value problems of static elasticity are based on a constitutive law such as (2.6) and the equilibrium equation (2.8). As generally recognized, a possible inconvenience of such a formulation is that the equilibrium equation is a differential equation whose independent variable belongs to a domain that depends on the solution of the problem, i.e., $\mathscr{B}$ depends on the mapping f. When a stress-free compatible reference configuration is available, this difficulty is circumvented by making a Piola transform which defines the first Piola–Kirchhoff stress as a function on the stress-free configuration. Now, in the present situation, when there is no compatible stress-free configuration available, this idea has to be modified. We will discuss three different ways to choose a fixed set on which to write the equilibrium equations. Firstly, we formulate the boundary value problem on the Riemannian manifold $(\mathscr{B}_0, \mathbf{m})$. Secondly, we do the same for the initially stressed manifold $(\mathscr{B}_0, \mathbf{G})$ and, lastly, we formulate the problem in the local stress-free configuration $\overline{\mathscr{B}}$. Formulating the equilibrium equations on $\overline{\mathscr{B}}$ requires a modified Piola identity, which is derived in the Appendix (see also Noll [7]) and which is valid even if one configuration is incompatible.

The formulations are derived for the compressible case and the incompressible case is discussed in Sec. 5.4.

5.1. Boundary value problem on $\mathscr{B}_0$ with metric m

We use a Piola transformation to define the first Piola–Kirchhoff stress tensor on the manifold $(\mathscr{B}_0, \mathbf{m})$ as follows:

$$P_{\mathbf{m}}^{aA} = (\det_{\mathbf{m}}\mathbf{F})\,\sigma^{ab}(\mathbf{F}^{-1})^{A}{}_{b}. \tag{5.1}$$

Using this definition in (2.6) we obtain the constitutive relation

$$P_{\mathbf{m}}^{aA} = 2\,(\det_{\mathbf{m}}\mathbf{F})\,\rho H^{a}_{\alpha}\frac{\partial \Psi}{\partial Z_{\alpha\beta}}K^{A}{}_{\beta}. \tag{5.2}$$

From equations (4.2), (4.3) and the chain rule, we find that (5.2) can be further rewritten as

$$P_{\mathbf{m}}^{aA} = 2\rho_{\mathrm{Ref}}F^{a}{}_{B}\,\frac{\partial \tilde{\Psi}}{\partial C_{AB}}, \tag{5.3}$$

where $\tilde{\Psi}(\mathbf{C},\mathbf{K}) = \Psi(\mathbf{Z})$ and $\mathbf{C}$ has components $C_{AB} = g_{ab}F^{a}{}_{A}\,F^{b}{}_{B}$.

Since $\mathbf{F}$ is the gradient of a deformation f, $\det_{\mathbf{m}}\mathbf{F}$ is the Jacobian of f and the standard Piola identity, given in Marsden and Hughes [21], holds:

$$\hat{\nabla}_A P_{\mathbf{m}}^{aA} = \det{}_{\mathbf{m}}\mathbf{F}\nabla_b\sigma^{ab},$$

where $\hat{\nabla}_A$ is the covariant derivative on $(\mathscr{B}_0, \mathbf{m})$. We then obtain from (2.8) the equilibrium equation

$$\hat{\nabla}_A P_{\mathbf{m}}^{aA} + \rho_{\mathrm{Ref}}b^{a} = 0. \tag{5.4}$$

Equation (5.4), without body forces, is also given in Takamizawa and Matsuda [10]; it represents the equilibrium equations referring to $(\mathscr{B}_0, \mathbf{m})$.

To form a well defined boundary value problem, (5.4) needs to be combined with an appropriate boundary condition. One such condition is obtained by prescribing a traction vector $\mathbf{t}$ on the boundary of the domain. This vector is given through the Cauchy stress tensor by Cauchy's theorem as

$$t^{a} = \sigma^{ab}n_{b}, \tag{5.5}$$

where n_b are the components of an outward unit vector $\mathbf{n}$ at the boundary $\partial\mathscr{B}$. The boundary condition for the equilibrium equation (5.4) is obtained by using the Piola transformation (5.1) in (5.5). We then get

$$t^a = (\det_{\mathbf{m}}\mathbf{F})^{-1} P_{\mathbf{m}}^{aA} F^b{}_A\, n_b = P_{\mathbf{m}}^{aA} \hat{n}_A, \tag{5.6}$$

where we have used the notation

$$\hat{n}_A = (\det_{\mathbf{m}}\mathbf{F})^{-1} F^b{}_A\, n_b,$$

which is a covariant unit vector defined on $\partial\mathscr{B}_0$, the boundary of $\mathscr{B}_0$.

The complete boundary value problem, referring to the Riemannian manifold $(\mathscr{B}_0, \mathbf{m})$, can now be stated as:

B.V. Problem on $(\mathscr{B}_0, \mathbf{m})$

Given ρ_{Ref}, $\mathbf{K}$, $\mathbf{b}$ and $\mathbf{t}$, find f such that

$$\hat{\nabla}_A P_{\mathbf{m}}^{aA} + \rho_{\mathrm{Ref}} b^a = 0 \qquad \text{on} \quad \mathscr{B}_0$$

$$P_{\mathbf{m}}^{aA} = 2\rho_{\mathrm{Ref}} F^a{}_B \frac{\partial\tilde{\Psi}}{\partial C_{AB}} \qquad \text{on} \quad \mathscr{B}_0$$

$$t^a = P_{\mathbf{m}}^{aA} \hat{n}_A \qquad \text{on} \quad \partial\mathscr{B}_0.$$

5.2. Boundary value problem on $\mathscr{B}_0$ with metric G

To obtain a constitutive equation that fits the equilibrium equation written for the Euclidean manifold $(\mathscr{B}_0, \mathbf{G})$, we use the Piola transformation

$$P^{aA} = (\det\mathbf{F})\sigma^{ab}(\mathbf{F}^{-1})^A{}_b, \tag{5.7}$$

which gives

$$P^{aA} = 2\rho_0 F^a{}_B \frac{\partial\tilde{\Psi}}{\partial C_{AB}}. \tag{5.8}$$

By using $\rho_0 = \rho_{\mathrm{Ref}}(\det\mathbf{K})^{-1}$, the definition of $\det\mathbf{K}$ in Sec. 4.2, (2.9) and standard properties of determinants, we get

$$\rho_0 = \rho_{\mathrm{Ref}} \frac{\sqrt{\det(m_{AB})}}{\sqrt{\det(G_{CD})}}. \tag{5.9}$$

This means that the difference between constitutive equations for the manifold $(\mathscr{B}_0, \mathbf{G})$ and for the manifold $(\mathscr{B}_0, \mathbf{m})$ is that in the former case, when we consider ρ_{Ref} as fixed, the density (5.9), which appears in (5.8), must be dependent on the tangent map $\mathbf{K}$ through the metric $\mathbf{m}$.

Since det $\mathbf{F}$ is a Jacobian, the Piola transformation (5.7) gives a Piola identity which reads

$$\nabla_A P^{aA} = \det\mathbf{F}\nabla_b \sigma^{ab},$$

where ∇_A is the derivative operator of the manifold $(\mathscr{B}_0, \mathbf{G})$. The equilibrium equation (2.8) then becomes

$$\nabla_A P^{aA} + \rho_0 b^a = 0.$$

A boundary condition for this equilibrium equation is obtained by using the Piola transformation (5.7) in the same way as in (5.6). The result reads

$$t^a = P^{aA} n_A,$$

where

$$n_A = (\det\mathbf{F})^{-1} F^b{}_A\, n_b,$$

are the components of the outward unit vector at the boundary of $\mathscr{B}_0$. The following boundary value problem can now be formulated:

B.V. Problem on $(\mathscr{B}_0, \mathbf{G})$

Given ρ_{Ref}, $\mathbf{K}$, $\mathbf{b}$ and $\mathbf{t}$, find f such that

$$\nabla_A P^{aA} + \rho_0 b^a = 0 \qquad \text{on} \quad \mathscr{B}_0$$

$$P^{aA} = 2\rho_0 F^a{}_B \frac{\partial\tilde{\Psi}}{\partial C_{BA}} \qquad \text{on} \quad \mathscr{B}_0$$

$$t^a = P^{aA} n_A \qquad \text{on} \quad \partial\mathscr{B}_0$$

where

$$\rho_0 = \rho_{\text{Ref}} \frac{\sqrt{\det(m_{AB})}}{\sqrt{\det(G_{CD})}}.$$

5.3. Boundary value problem on $\overline{\mathscr{B}}$ with metric $\boldsymbol{\gamma}$

By using the Piola transformation

$$P^{a\alpha} = (\det\mathbf{H})\sigma^{ab}(\mathbf{H}^{-1})^\alpha{}_b, \tag{5.10}$$

it is shown in Theorem 1 of the Appendix (see also Noll [7]) that

$$\det\mathbf{H}\,\nabla_b \sigma^{ab} = \nabla_\alpha P^{a\alpha} + P^{a\alpha} s_\alpha,$$

where s_α vanishes if $\overline{\mathscr{B}}$ is compatible. That is, s_α is a measure of the incompatibility of the configuration. The equilibrium Eq. (2.8) can now be written as

$$\nabla_\alpha P^{a\alpha} + P^{a\alpha} s_\alpha + \rho_{\mathrm{Ref}} b^a = 0.$$

This equation is also derived in NOLL [7] and there referred to as Cauchy's modified equation of balance.

The constitutive Eq. (2.6) can be rewritten by using the transformation (5.10) and then it becomes

$$P^{a\alpha} = 2\rho_{\mathrm{Ref}} H^a{}_\beta \frac{\partial \Psi}{\partial Z_{\alpha\beta}}.$$

Also the boundary condition in $\overline{\mathscr{B}}$ is rewritten by using the Piola transformation (5.10) and becomes

$$t^a = P^{a\alpha} n_\alpha,$$

where

$$n_\alpha = (\det \mathbf{H})^{-1} H^b{}_\alpha n_b,$$

is a unit vector at the boundary $\partial\overline{\mathscr{B}}$. The following boundary value problem can now be formulated:

B.V. Problem on $\overline{\mathscr{B}}$

Given ρ_{Ref}, $\mathbf{K}$, $\mathbf{b}$ and $\mathbf{t}$, find f such that

$$\nabla_\alpha P^{a\alpha} + P^{a\alpha} s_\alpha + \rho_{\mathrm{Ref}} b^a = 0 \quad \text{on} \quad \overline{\mathscr{B}}$$

$$P^{a\alpha} = 2\rho_{\mathrm{Ref}} H^a{}_\beta \frac{\partial \Psi}{\partial Z_{\alpha\beta}} \quad \text{on} \quad \overline{\mathscr{B}}$$

$$t^a = P^{a\alpha} n_\alpha \quad \text{on} \quad \partial\overline{\mathscr{B}}.$$

5.4. Incompressibility

For an incompressible material, (2.4) and (5.9) give that

$$\sqrt{\det(G_{AB})} = \sqrt{\det(m_{AB})}. \tag{5.11}$$

Thus, for an incompressible material, the determinant of the Euclidean metric $\mathbf{G}$ on configuration $\mathscr{B}_0$ is equal to the determinant of the initial strain induced metric $\mathbf{m}$.

Now, an important conclusion, in relation to (5.11), is that the divergence of the first Piola–Kirchhoff stress tensor (that appears in the equilibrium equation) is dependent on the metric only trough its determinant. This can be seen from the following formula, which is a generalization of the formula for the divergence of a vector field given in Marsden and Hughes [21]:

$$(\hat{\nabla}_A P^{aA}_{\mathbf{m}})\mathbf{g}_a = \frac{1}{\sqrt{\det(m_{BC})}} \frac{\partial}{\partial X^A} \left(\sqrt{\det(m_{BC})} P^{aA}_{\mathbf{m}} \mathbf{g}_a \right), \tag{5.12}$$

where $\mathbf{g}_a$ is a natural basis in $\mathscr{B}$. A similar formula holds for the Piola–Kirchhoff stress P^{aA}, which, due to (5.11), turns out to coincide with $P^{aA}_{\mathbf{m}}$ in the incompressible case. Comparing this formula with (5.12), again taking account of (5.11), we find that

$$\hat{\nabla}_A P^{aA}_{\mathbf{m}} = \nabla_A P^{aA}. \tag{5.13}$$

This result implies that the boundary value problem in Sec. 5.1 coincide with that of Sec. 5.2. However, the metric tensors $\mathbf{G}$ and $\mathbf{m}$ can obviously be different even though their determinants coincide, so (5.13) does not imply that the non-Euclidean nature of the initially stretched configuration disappears.

5.5. Comparison of formulations

The present investigation does not give any conclusive answer as to when one formulation is to be preferred over another. Nevertheless, the equilibrium equation on $\overline{\mathscr{B}}$ obviously has a non-classical appearance due to the presence of s_α. The boundary value problem referring to $(\mathscr{B}_0, \mathbf{G})$ looks classical except for the density ρ_0, which depends on the initial deformation $\mathbf{K}$. For the boundary value problem referring to $(\mathscr{B}_0, \mathbf{m})$, this possible shortage is removed on the expense of having to deal with a non-Euclidean structure. The formulation based on $(\mathscr{B}_0, \mathbf{G})$ is probably the most intuitively appealing since this manifold can be thought of as a compatible configuration in physical space. Note that the opening-angle method is based on a cutting operation that generates a simply connected domain $\tilde{\mathscr{B}}$. Thus, in this special case, a fourth formulation, not covered in this section, is possible.

Appendix – Piola Identity

We will here derive a generalized Piola identity. This generalization is valid for configurations that are incompatible. Another proof of this theorem is given in Noll [7].

Theorem 1. ***A generalization of the Piola identity.*** *Given two configurations $\overline{\mathscr{B}}$ and $\mathscr{B}$ and a tangent map (not necessarily a deformation gradient) $\boldsymbol{H}$, the following formula, which is a generalization of the Piola identity, holds*

$$\nabla_b \sigma^{ab} = (\det \boldsymbol{H})^{-1} \left(\nabla_\alpha P^{a\alpha} + P^{a\alpha} s_\alpha \right),$$

where

$$P^{a\alpha} = (\det \boldsymbol{H})\sigma^{ab}(\boldsymbol{H}^{-1})^{\alpha}{}_{b} \quad \text{and} \quad s_{\alpha} = (\boldsymbol{H}^{-1})^{\gamma}{}_{b}\left(\nabla_{\gamma}H^{b}{}_{\alpha} - \nabla_{\alpha}H^{b}{}_{\gamma}\right),$$

and where the divergence operator on $\overline{\mathscr{B}}$ *is defined as*

$$\nabla_{\beta} = H^{b}{}_{\beta}\nabla_{b}.$$

To prove this theorem we need the following Lemma.

LEMMA 1. *The gradient of the inverted determinant* $(\det \boldsymbol{H})^{-1}$ *is given by*

$$\nabla_{b}(\det \boldsymbol{H})^{-1} = -(\det \boldsymbol{H})^{-1}\left((\boldsymbol{H}^{-1})^{\gamma}{}_{c}\nabla_{b}H^{c}{}_{\gamma}\right).$$

P r o o f of Lemma 1:

By using the chain rule we get

$$\begin{aligned}
\nabla_{b}(\det \mathbf{H})^{-1} &= \frac{\sqrt{\det(\gamma_{\eta\nu})}}{\sqrt{\det(g_{mn})}}\nabla_{b}(\det(H^{a}{}_{\alpha}))^{-1} \\
&= -\frac{\sqrt{\det(\gamma_{\eta\nu})}}{\sqrt{\det(g_{mn})}}(\det(H^{a}{}_{\alpha}))^{-2}\nabla_{b}(\det(H^{a}{}_{\alpha})) \\
&= -\frac{\sqrt{\det(\gamma_{\eta\nu})}}{\sqrt{\det(g_{mn})}}(\det(H^{a}{}_{\alpha}))^{-2}\frac{\partial \det(H^{a}{}_{\alpha})}{\partial H^{c}{}_{\gamma}}\nabla_{b}H^{c}{}_{\gamma},
\end{aligned}$$

and by noticing that

$$\frac{\partial \det(H^{a}{}_{\alpha})}{\partial H^{c}{}_{\gamma}} = (\mathbf{H}^{-1})^{\gamma}{}_{c}\det(H^{a}{}_{\alpha}),$$

we arrive at the final expression

$$\nabla_{b}(\det \mathbf{H})^{-1} = -(\det \mathbf{H})^{-1}\left((\mathbf{H}^{-1})^{\gamma}{}_{c}\nabla_{b}H^{c}{}_{\gamma}\right),$$

and Lemma 1 is proved. □

We are now ready to prove the theorem.

P r o o f of Theorem 1:

The first Piola–Kirchhoff stress tensor is defined as

$$P^{a\alpha} = (\det \mathbf{H})\sigma^{ab}(\mathbf{H}^{-1})^{\alpha}{}_{b}.$$

Solving for σ^{ab} yields

$$\sigma^{ab} = (\det \mathbf{H})^{-1}P^{a\alpha}H^{b}{}_{\alpha}. \tag{5.14}$$

We can now differentiate the left hand side of (5.14), and by using Liebnitz rule (differentiation of a product) we get

$$\nabla_b \sigma^{ab} = P^{a\alpha} H^b{}_\alpha \nabla_b (\det \mathbf{H})^{-1} + (\det \mathbf{H})^{-1} H^b{}_\alpha \nabla_b P^{a\alpha} + (\det \mathbf{H})^{-1} P^{a\alpha} \nabla_b H^b{}_\alpha. \tag{5.15}$$

Utilizing Lemma 1 for the first term on the right-hand side of (5.15) we obtain

$$P^{a\alpha} H^b{}_\alpha \nabla_b (\det \mathbf{H})^{-1} = -P^{a\alpha} H^b{}_\alpha (\det \mathbf{H})^{-1} \left((\mathbf{H}^{-1})^\gamma{}_c \nabla_b H^c{}_\gamma \right). \tag{5.16}$$

Now, using (5.16) and (5.15) we get

$$(\det \mathbf{H}) \nabla_b \sigma^{ab} = H^b{}_\alpha \nabla_b P^{a\alpha} + P^{a\alpha} \left(\nabla_b H^b{}_\alpha - H^b{}_\alpha (\nabla_b H^c{}_\gamma)(\mathbf{H}^{-1})^\gamma{}_c \right). \tag{5.17}$$

We now define the derivative operator on $\overline{\mathscr{B}}$ as the pull–back of the derivative operator on $\mathscr{B}$, that is,

$$\nabla_\beta = H^b{}_\beta \nabla_b.$$

Equation (5.17) can now be expressed in material setting as

$$(\det \mathbf{H}) \nabla_b \sigma^{ab} = \nabla_\alpha P^{a\alpha} + P^{a\alpha} s_\alpha,$$

where

$$s_\alpha = (\mathbf{H}^{-1})^\gamma{}_b \left(\nabla_\gamma H^b{}_\alpha - \nabla_\alpha H^b{}_\gamma \right)$$

and Theorem 1 is proved. □

Note that the covariant vector s_α is a measure of the incompatibility of the configuration. If s_α vanishes, as is the case when $\mathbf{H}$ is the derivative of a deformation, then we obtain the ordinary Piola identity.

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