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The interior Neumann problem for the Stokes resolvent system in a bounded domain in $\mathbb{R}^n$

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THE INTERIOR NEUMANN PROBLEM for the Stokes resolvent system is studied from the point of view of the potential theory. The existence and uniqueness results as well as boundary integral representations of the classical solution are given in the case of a bounded domain in $\mathbb{R}^n$, having a compact but not connected boundary of class $C^{1,\alpha}$ $(0 < \alpha \leq 1)$.

Key words: Stokes resolvent system, Neumann problem, fundamental solution, potential theory, boundary integral representations.

1. Introduction

LET $D' \subset \mathbb{R}^n$ and $D_1 \subset \mathbb{R}^n$ $(n \in \mathbb{N},\ n \geq 2)$ be two bounded domains with connected boundaries Γ' and Γ_1 of class $C^{1,\alpha}$ $(0 < \alpha \leq 1)$, such that $\overline{D}_1 \subset D'$. Also let $D \subset \mathbb{R}^n$ be the bounded domain given by $D = D' \setminus \overline{D}_1$, and let $\Gamma = \Gamma' \cup \Gamma_1$ be its boundary. We assume that the origin of $\mathbb{R}^n$ belongs to D_1, and denote by $\mathbf{n}$ the unit normal to Γ pointing outside the domain D (see Fig. 1).

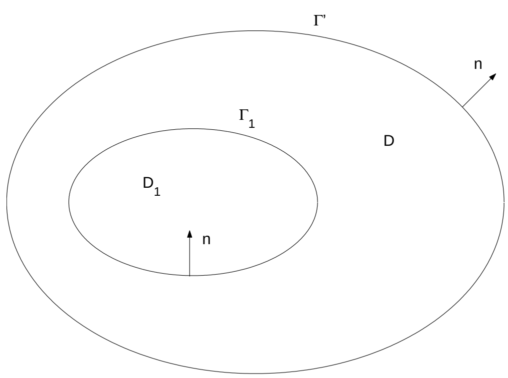

FIG. 1. Bounded domain in $\mathbb{R}^n$.

The following equations:

$$\nabla \cdot \mathbf{u} = 0, \quad -\nabla q + (\nabla^2 - \chi^2)\mathbf{u} + \mathbf{f} = \mathbf{0} \ \text{ in } D \tag{1.1}$$

determine the Stokes resolvent system in the bounded domain D. Note that χ^2 is a complex number such that $\chi^2 \in \mathbb{C} \setminus \{z \in \mathbb{C} : \ \mathrm{Re}z \leq 0, \ \mathrm{Im}z = 0\}$, $\mathbf{u} = (u_1, \ldots, u_n)$ and q are unknown functions, and $\mathbf{f} = (f_1, \ldots, f_n)$ is a given vector function. All functions occurring in this paper are complex-valued. In addition, ∇ is the n-dimensional gradient operator and ∇^2 denotes the Laplace operator.

The Stokes resolvent system (1.1) can be obtained by applying the Laplace transform to the system of the continuity and Navier–Stokes equations which, in the case $n = 2$ or $n = 3$, describes the low Reynolds number flow of a viscous incompressible fluid (for details see [12], Sec. 1.5). Thus, in this case, $\mathbf{u}$ and q are the Laplace transforms of the flow velocity and pressure fields, and $\mathbf{f}$ is the Laplace transform of a given body force.

The solution of the Stokes resolvent system can be used to obtain the existence, stability, and asymptotic properties of solutions to the Navier-Stokes equation, by applying some results of the functional analysis or pseudo-differential operator theory (for details see [2, 21]). On the other hand, the potential theory for the Stokes resolvent system in the general case $n \geq 2$ was developed by Varnhorn (see [25, 26]), and extension of this theory to the case of domains with connected boundaries of Lyapunov type (i.e., of class $C^{1,\alpha}$ $(0 < \alpha \leq 1)$) was recently obtained in [27]. In addition, the fundamental solution for the system of equations (1.1) in $\mathbb{R}^3_+$ was obtained by McCraken in [16]. Also the Dirichlet problems for the Stokes resolvent equations on bounded and exterior domains in $\mathbb{R}^n$ with compact but not connected boundaries of Lypaunov type have been studied recently in [11], and a mixed boundary value problem for the same equations has been treated in [10].

The aim of this paper is to use the potential theory in order to prove the existence and uniqueness result of the classical solution to the interior Neumann problem for the Stokes resolvent system (1.1), in the case of the bounded domain D with compact but not connected boundary of Lyapunov type.

2. The potential theory for the Stokes resolvent system

The first part of this paper is devoted to the presentation of the potential theory for the Stokes resolvent system.

2.1. Preliminary results

Let us assume that the fields $\mathbf{u}$ and q satisfy the system of equations (1.1). Then the corresponding Cauchy stress tensor $\mathbf{\Sigma}(\mathbf{u})$ is given by the relation

$$\mathbf{\Sigma}(\mathbf{u}) = -q\mathbf{I}_n + \nabla \mathbf{u} + (\nabla \mathbf{u})^T, \tag{2.1}$$

where $\mathbf{I}_n$ denotes the $n \times n$ identity matrix and $(\nabla \mathbf{u})^T$ is the transposed matrix to $\nabla \mathbf{u} = (\partial u_i / \partial x_j)_{i,j=1,\dots,n}$.

From the equations (1.1) we find that

$$\frac{\partial \Sigma_{kj}(\mathbf{u})}{\partial x_k} = \chi^2 u_j - f_j \text{ in } D, \quad j = 1, \dots, n, \tag{2.2}$$

where $\Sigma_{ij}(\mathbf{u})$ are the components of $\mathbf{\Sigma}(\mathbf{u})$, $i, j = 1, \dots, n$. Note that in (2.2) we have used the repeated-index summation convention. From now on, we take into account this rule.

Let us now denote by $\mathbf{T}$ a continuous vector field on Γ. Then the *interior Neumann problem* for the Stokes resolvent system (1.1) in the bounded domain D is the boundary value problem, due to the system of equations (1.1) and the boundary condition of the Neumann type

$$\mathbf{\Sigma}(\mathbf{u}) \cdot \mathbf{n} = \mathbf{T} \text{ on } \Gamma. \tag{2.3}$$

Let $(\cdot, \cdot) : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be the inner product given by relation

$$(z, \eta) = z_i \overline{\eta}_i, \tag{2.4}$$

for all $z = (z_1, \dots, z_n)$, $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{C}^n$, where $\overline{w}$ is the complex conjugate of $w \in \mathbb{C}$.

Using the equations (2.2) we get the following result (see e.g. [12] p. 24, for $\mathbf{f} = \mathbf{0}$):

LEMMA 1. *If the fields* $\mathbf{u} = (u_1, \dots, u_n)$ *and* q *satisfy the Stokes resolvent system* (1.1)*, then we have the identity*

$$\int_\Gamma \Sigma_{ij}(\mathbf{u}) n_j \overline{u}_i d\Gamma = \chi^2 \int_D (\mathbf{u}, \mathbf{u}) d\mathbf{x} + 2 \int_D E_{ij}(\mathbf{u}) \overline{E_{ij}(\mathbf{u})} d\mathbf{x} - \int_D (\mathbf{f}, \mathbf{u}) d\mathbf{x}, \tag{2.5}$$

where

$$E_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, \dots, n. \tag{2.6}$$

2.2. Uniqueness result of the classical solution to the interior Neumann problem (1.1), (2.3)

DEFINITION 1. *The pair* $(\mathbf{u}, q)$ *is a classical solution to the interior Neumann problem consisting of the system of equations (1.1) and the boundary condition (2.3) if* $(\mathbf{u}, q) \in (C^2(D) \cap C^1(\overline{D})) \times (C^1(D) \cap C^0(\overline{D}))$, $\mathbf{\Sigma}(\mathbf{u}) \cdot \mathbf{n} \in C^0(\Gamma)$, *and* $\mathbf{u}$ *and* q *satisfy the Equations (1.1) and the boundary condition (2.3) at each point of* D *and* Γ *respectively.*

Theorem 1. *The interior Neumann problem consisting of the system of equations* (1.1) *and the boundary condition of the Neumann type* (2.3) *has at most one classical solution* $(\mathbf{u}, q)$.

P r o o f. Let us assume that the pairs $(\mathbf{u}^{(1)}, q^{(1)})$ and $(\mathbf{u}^{(2)}, q^{(2)})$ are two classical solutions of the interior Neumann problem (1.1), (2.3), and let $(\mathbf{u}^{(0)}, q^{(0)})$ be their difference. Then applying the identity (2.5) to the fields $\mathbf{u}^{(0)}$ and $q^{(0)}$, one obtains the equality

$$\int_{\Gamma} \Sigma_{ij}(\mathbf{u}^{(0)}) n_j \overline{u_i^{(0)}} d\Gamma = \chi^2 \int_D |\mathbf{u}^{(0)}|^2 d\mathbf{x} + 2 \int_D E_{ij}(\mathbf{u}^{(0)}) \overline{E_{ij}(\mathbf{u}^{(0)})} d\mathbf{x}, \tag{2.7}$$

which, in view of the boundary condition $\mathbf{\Sigma}(\mathbf{u}^{(0)}) \cdot \mathbf{n} = \mathbf{0}$ on Γ, takes the form

$$\int_D \left[\chi^2 |\mathbf{u}^{(0)}|^2 + 2E_{ij}(\mathbf{u}^{(0)}) \overline{E_{ij}(\mathbf{u}^{(0)})} \right] d\mathbf{x} = 0. \tag{2.8}$$

In addition, since $|\arg \chi^2| < \pi$, we find that $\mathbf{u}^{(0)} = \mathbf{0}$ in $\overline{D}$, and in view of the homogeneous Stokes resolvent equation

$$\nabla q^{(0)} + (\chi^2 - \nabla^2)\mathbf{u}^{(0)} = \mathbf{0} \text{ in } D,$$

we deduce that $q^{(0)} = \alpha_0 \in \mathbb{C}$ in D. Finally, using the fact that $\mathbf{\Sigma}(\mathbf{u}^{(0)}) \cdot \mathbf{n} = \mathbf{0}$ on Γ, one obtains that $\alpha_0 = 0$. This completes the proof of Theorem 1. □

2.3. The fundamental solution of the Stokes resolvent system

Next, we refer to the system of the continuity and singularly Stokes resolvent equations

$$\nabla \cdot \mathbf{u} = 0, \quad -\nabla q + (\nabla^2 - \chi^2)\mathbf{u} + \mathbf{g}\delta(\mathbf{x}) = \mathbf{0}, \tag{2.9}$$

where $\mathbf{g} = (g_1, \ldots, g_n)$ is a constant vector and δ is the Dirac distribution or the delta function in $\mathbb{R}^n$. Also the fields $\mathbf{u} = (u_1, \cdots, u_n)$ and q are complex-valued, and $\chi^2 \in \mathbb{C} \setminus \{z \in \mathbb{C} : \ \mathrm{Re} z \le 0, \ \mathrm{Im} z = 0\}$.

2.3.1. The Green function and its associate pressure vector. The *unsteady Stokeslet* or the *free-space Green function* $\mathbf{G}^{\chi^2}(G_{ij}^{\chi^2})$ and the corresponding *pressure vector* $\mathbf{\Pi}^{\chi^2}(\Pi_i^{\chi^2})$ to the Stokes resolvent system are defined by the relations

$$u_i(\mathbf{x}) = \frac{1}{2\varpi_n} G_{ij}^{\chi^2}(\mathbf{x}) g_j, \quad q(\mathbf{x}) = \frac{1}{2\varpi_n} \Pi_j^{\chi^2}(\mathbf{x}) g_j, \tag{2.10}$$

where ϖ_n is the surface area of the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$.

Substituting the expressions (2.10) into the Eqs. (2.9), one obtains the equations

$$\frac{\partial G_{ij}^{\chi^2}(\mathbf{x})}{\partial x_i} = 0, \quad j = 1, \dots, n, \tag{2.11}$$

$$-\frac{\partial \Pi_j^{\chi^2}(\mathbf{x})}{\partial x_k} + (\nabla^2 - \chi^2) G_{kj}^{\chi^2}(\mathbf{x}) = -2\varpi_n \delta_{kj}\delta(\mathbf{x}), \quad j, k = 1, \dots, n. \tag{2.12}$$

Note that δ_{kj} is the Kronecker symbol, i.e., $\delta_{kj} = 1$ for $k = j$, and $\delta_{kj} = 0$ for $k \neq j$.

Let $\mathbf{\Sigma}^{\chi^2}(\mathbf{u})$ be the stress field corresponding to the fields $\mathbf{u}$ and q. Using the relations (2.1) and (2.10) we find that

$$\Sigma_{ik}^{\chi^2}(\mathbf{u})(\mathbf{x}) = \frac{1}{2\varpi_n} S_{ijk}^{\chi^2}(\mathbf{x}) g_j, \quad i, k = 1, \dots, n, \tag{2.13}$$

where $S_{ijk}^{\chi^2}$ are the components of the *stress tensor* $\mathbf{S}^{\chi^2}$, associated to the Green function and the pressure vector $\mathbf{G}^{\chi^2}$ and $\mathbf{\Pi}^{\chi^2}$, and having the form

$$S_{ijk}^{\chi^2}(\mathbf{x}) = -\Pi_j^{\chi^2}(\mathbf{x})\delta_{ik} + \frac{\partial G_{ij}^{\chi^2}(\mathbf{x})}{\partial x_k} + \frac{\partial G_{kj}^{\chi^2}(\mathbf{x})}{\partial x_i}, \quad i, j, k = 1, \dots, n. \tag{2.14}$$

The fundamental solution $(\mathbf{G}^{\chi^2}, \mathbf{\Pi}^{\chi^2})$ of the Stokes resolvent system (2.11), (2.12) can be obtained by the Fourier transform method in the form (see [2] and [28] for $n = 2$; [23] for $n = 3$; p. 81–82 [12], for $n = 2, 3$; p. 60 [25]; [26]; [27] in the general case $n \geq 2$):

$$\begin{aligned} G_{jk}^{\chi^2}(\mathbf{x}) &= \frac{\delta_{jk}}{|\mathbf{x}|^{n-2}} A_1(\chi|\mathbf{x}|) + \frac{x_j x_k}{|\mathbf{x}|^n} A_2(\chi|\mathbf{x}|), \\ \Pi_j^{\chi^2}(\mathbf{x}) &= 2\frac{x_j}{|\mathbf{x}|^n}, \quad j, k = 1, \dots, n, \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} A_1(z) &= 2\left(\frac{\left(\frac{z}{2}\right)^{m-1} K_{m-1}(z)}{\Gamma(m)} + 2\frac{\left(\frac{z}{2}\right)^{m} K_{m}(z)}{\Gamma(m) z^2} - \frac{1}{z^2} \right), \\ A_2(z) &= 2\left(\frac{n}{z^2} - 4\frac{\left(\frac{z}{2}\right)^{m+1} K_{m+1}(z)}{\Gamma(m) z^2} \right), \end{aligned} \tag{2.16}$$

$m = n/2$, $\Gamma(z)$ is the Gamma function, χ is the particular square root of $\chi^2 \in \mathbb{C} \setminus \{z \in \mathbb{C} : \ \mathrm{Re}z \le 0, \ \mathrm{Im}z = 0\}$, which has a positive real part, i.e., $\mathrm{Re}\chi > 0$, and K_ν is the modified Bessel function of the order $\nu \ge 0$. For details see e.g. [1].

2.3.2. The stress tensor associated with the Green function. Taking into account the relations (2.14), (2.15) and (2.16), one obtains the components of the stress tensor $\mathbf{S}^{\chi^2}$ in the form (see e.g. p. 61–62 [25]; [27]):

$$S_{ijk}^{\chi^2}(\mathbf{x}) = -2\left\{\delta_{ik}\frac{x_j}{|\mathbf{x}|^n}D_1(\chi|\mathbf{x}|) + \left(\delta_{kj}\frac{x_i}{|\mathbf{x}|^n} + \delta_{ij}\frac{x_k}{|\mathbf{x}|^n}\right)D_2(\chi|\mathbf{x}|)\right\} - 2\frac{x_i x_j x_k}{|\mathbf{x}|^{n+2}}D_3(\chi|\mathbf{x}|), \tag{2.17}$$

where

$$\begin{aligned} D_1(z) &= 8\frac{\left(\frac{z}{2}\right)^{m+1}K_{m+1}(z)}{\Gamma(m)z^2} - \frac{2n}{z^2} + 1, \\ D_2(z) &= 8\frac{\left(\frac{z}{2}\right)^{m+1}K_{m+1}(z)}{\Gamma(m)z^2} - \frac{2n}{z^2} + 2\frac{\left(\frac{z}{2}\right)^{m}K_{m}(z)}{\Gamma(m)}, \\ D_3(z) &= -16\frac{\left(\frac{z}{2}\right)^{m+2}K_{m+2}(z)}{\Gamma(m)z^2} + \frac{2n(n+2)}{z^2}. \end{aligned} \tag{2.18}$$

2.3.3. The pressure tensor associated with the stress tensor. Now, using the properties (2.11), (2.12) and (2.15) we deduce that

$$(\nabla_{\mathbf{x}}^2 - \chi^2)S_{ijk}^{\chi^2}(\mathbf{y}-\mathbf{x}) = \frac{\partial \Lambda_{ik}^{\chi^2}(\mathbf{x}-\mathbf{y})}{\partial x_j} \ \text{for } \mathbf{x} \ne \mathbf{y}, \ i,j,k = 1,\dots,n, \tag{2.19}$$

where (see [12] Chapter 2, for $n = 2, 3$; p. 61–62 [25], for $n \ge 2$)

$$\Lambda_{ik}^{\chi^2}(\mathbf{x}-\mathbf{y}) = \begin{cases} -2\delta_{ik}\chi^2 \ln r - 4\frac{\delta_{ik}}{r^2} + 8\frac{\widehat{x}_i\widehat{x}_k}{r^4} & \text{for } n = 2 \\ \frac{2\chi^2}{n-2}\delta_{ik}\frac{1}{r^{n-2}} - 4\delta_{ik}\frac{1}{r^n} + 4n\frac{\widehat{x}_i\widehat{x}_k}{r^{n+2}} & \text{for } n \ge 3, \end{cases} \tag{2.20}$$

denote the components of the *pressure tensor* $\mathbf{\Lambda}^{\chi^2}$ associated with the stress tensor $\mathbf{S}^{\chi^2}$, and $\widehat{\mathbf{x}} = \mathbf{x} - \mathbf{y} = (\widehat{x}_1, \dots, \widehat{x}_n)$, $r = |\widehat{\mathbf{x}}|$.

In addition, making use of the Eqs. (2.11) as well as the expression of the pressure vector $\mathbf{\Pi}^{\chi^2}$, we get the property

$$\frac{\partial S_{ijk}^{\chi^2}(\mathbf{y}-\mathbf{x})}{\partial x_j} = 0 \quad \text{for } \mathbf{x} \neq \mathbf{y}, \quad i,k = 1,\dots,n. \tag{2.21}$$

2.4. The potential theory for the Stokes resolvent system

Let us denote by $\tilde{\mathbf{f}} = (\tilde{f}_1,\dots,\tilde{f}_n)$ and $\tilde{\mathbf{h}} = (\tilde{h}_1,\dots,\tilde{h}_n)$ two complex vector-valued functions in the class $C^0(\varGamma)$.

2.4.1. The single- and double-layer potentials for the Stokes resolvent system. By the *single-layer potential* with density $\tilde{\mathbf{f}}$ we mean the complex vector-valued function $\mathbf{V}_{\chi^2,n}(\cdot,\tilde{\mathbf{f}})$ defined as follows:

$$\mathbf{V}_{\chi^2,n}(\mathbf{x},\mathbf{g}) = \int_{\varGamma} \mathbf{G}^{\chi^2}(\mathbf{x}-\mathbf{y}) \cdot \tilde{\mathbf{f}}(\mathbf{y}) d\varGamma(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^n \setminus \varGamma, \tag{2.22}$$

where $\mathbf{G}^{\chi^2}$ is the Green function of the Stokes resolvent system (see the relations (2.15) and (2.16)). Similarly, by the *double-layer potential* with density $\tilde{\mathbf{h}}$ we mean the complex vector-valued function $\mathbf{W}_{\chi^2,n}(\cdot,\tilde{\mathbf{h}})$ whose j^{th}-component has the form

$$(\mathbf{W}_{\chi^2,n})_j(\mathbf{x},\tilde{\mathbf{h}}) = \int_{\varGamma} S_{ijk}^{\chi^2}(\mathbf{y}-\mathbf{x}) n_k(\mathbf{y}) \tilde{h}_i(\mathbf{y}) d\varGamma(\mathbf{y}), \qquad \mathbf{x} \in \mathbb{R}^n \setminus \varGamma,\ j = 1,\dots,n, \tag{2.23}$$

where $\mathbf{S}^{\chi^2}$ is the stress tensor associated with the Green function $\mathbf{G}^{\chi^2}$ (see the relations (2.17) and (2.18)).

Now, let us denote by $P^s_{\chi^2,n}(\cdot,\tilde{\mathbf{f}})$ and $P^d_{\chi^2,n}(\cdot,\tilde{\mathbf{h}})$ the functions defined at each point $\mathbf{x} \in \mathbb{R}^n \setminus \varGamma$ by the relations

$$P^s_{\chi^2,n}(\mathbf{x},\tilde{\mathbf{f}}) = \int_{\varGamma} \varPi_i^{\chi^2}(\mathbf{x}-\mathbf{y}) \tilde{f}_i(\mathbf{y}) d\varGamma(\mathbf{y}), \tag{2.24}$$

$$P^d_{\chi^2,n}(\mathbf{x},\tilde{\mathbf{h}}) = \int_{\varGamma} \varLambda_{ik}^{\chi^2}(\mathbf{x}-\mathbf{y}) n_k(\mathbf{y}) \tilde{h}_i(\mathbf{y}) d\varGamma(\mathbf{y}), \tag{2.25}$$

where $\mathbf{\Pi}^{\chi^2}$ and $\mathbf{\Lambda}^{\chi^2}$ are the pressure vector and the pressure tensor respectively, associated with the Green function $\mathbf{G}^{\chi^2}$ and having the forms (2.15) and (2.20).

Taking into account the Eqs. (2.11), (2.12), (2.19) and (2.21), one obtains the result that the pairs $(\mathbf{V}_{\chi^2,n}(\cdot,\tilde{\mathbf{f}}), P^s_{\chi^2,n}(\cdot,\tilde{\mathbf{f}}))$ and $(\mathbf{W}_{\chi^2,n}(\cdot,\tilde{\mathbf{h}}), P^d_{\chi^2,n}(\cdot,\tilde{\mathbf{h}}))$ are classical solutions of the homogeneous Stokes resolvent system in $\mathbb{R}^n \setminus \Gamma$, i.e.,

$$\nabla \cdot \mathbf{V}_{\chi^2,n}(\cdot,\tilde{\mathbf{f}}) = 0, \quad -\nabla P^s_{\chi^2,n}(\cdot,\tilde{\mathbf{f}}) + (\nabla^2 - \chi^2)\mathbf{V}_{\chi^2,n}(\cdot,\tilde{\mathbf{f}}) = \mathbf{0} \quad \text{in } \mathbb{R}^n \setminus \Gamma, \tag{2.26}$$

$$\nabla \cdot \mathbf{W}_{\chi^2,n}(\cdot,\tilde{\mathbf{h}}) = 0, \quad -\nabla P^d_{\chi^2,n}(\cdot,\tilde{\mathbf{h}}) + (\nabla^2 - \chi^2)\mathbf{W}_{\chi^2,n}(\cdot,\tilde{\mathbf{h}}) = \mathbf{0} \quad \text{in } \mathbb{R}^n \setminus \Gamma. \tag{2.27}$$

The decay behavior of the single- and double-layer potentials $\mathbf{V}_{\chi^2,n}(\cdot,\tilde{\mathbf{f}})$ and $\mathbf{W}_{\chi^2,n}(\cdot,\tilde{\mathbf{h}})$ at infinity is given by the following relations (see e.g. pp. 78–79 [25]):

$$\mathbf{V}_{\chi^2,n}(\mathbf{x},\tilde{\mathbf{f}}) = O(|\mathbf{x}|^{-n}), \quad \mathbf{W}_{\chi^2,n}(\mathbf{x},\tilde{\mathbf{h}}) = O(|\mathbf{x}|^{1-n}) \ \text{ as } |\mathbf{x}| \to \infty. \tag{2.28}$$

Moreover, if the vector density $\tilde{\mathbf{h}}$ of the double-layer potential $\mathbf{W}_{\chi^2,n}(\cdot,\tilde{\mathbf{h}})$ satisfies the condition

$$\int_\Gamma \tilde{\mathbf{h}} \cdot \mathbf{n} d\Gamma = 0, \tag{2.29}$$

then we have

$$\mathbf{W}_{\chi^2,n}(\mathbf{x},\tilde{\mathbf{h}}) = O(|\mathbf{x}|^{-n}) \text{ as } |\mathbf{x}| \to \infty. \tag{2.30}$$

On the other hand, using the relations (2.14) it follows that the stress tensor $\mathbf{\Sigma}(\mathbf{V}_{\chi^2,n}(\cdot,\tilde{\mathbf{f}}))$ corresponding to the single-layer potential $\mathbf{V}_{\chi^2,n}(\cdot,\tilde{\mathbf{f}})$ has the following components:

$$\Sigma_{jk}(\mathbf{V}_{\chi^2,n}(\mathbf{x},\tilde{\mathbf{f}})) = -P^s_{\chi^2,n}(\mathbf{x},\tilde{\mathbf{f}})\delta_{jk} + \frac{\partial(\mathbf{V}_{\chi^2,n})_j(\mathbf{x},\tilde{\mathbf{f}})}{\partial x_k} + \frac{\partial(\mathbf{V}_{\chi^2,n})_k(\mathbf{x},\tilde{\mathbf{f}})}{\partial x_j} = \int_\Gamma S^{\chi^2}_{jik}(\mathbf{x}-\mathbf{y})\tilde{f}_i(\mathbf{y})d\Gamma(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^n \setminus \Gamma. \tag{2.31}$$

Now, let us denote by v a field defined in a domain U containing Γ. Then assuming that there exist the limiting values of this field at an arbitrary point $\mathbf{x}_0 \in \Gamma$, evaluated from D and $\mathbb{R}^n \setminus \overline{D}$ respectively, we denote these limiting values by $v^-(\mathbf{x}_0)$ and $v^+(\mathbf{x}_0)$. In particular, we use the notations $\mathbf{H}^+_{\chi^2,n}(\cdot,\tilde{\mathbf{f}})$ and

$\mathbf{H}^{-}_{\chi^2,n}(\cdot,\tilde{\mathbf{f}})$ for the limiting values of the normal stress due to the single layer potential $\mathbf{V}_{\chi^2,n}(\cdot,\tilde{\mathbf{f}})$ on both sides of Γ. Note that

$$(\mathbf{H}_{\chi^2,n})_j(\mathbf{x}_0,\tilde{\mathbf{f}}) = \int_{\Gamma} S_{jik}(\mathbf{x}-\mathbf{y})n_k(\mathbf{x})\tilde{f}_i(\mathbf{y})d\Gamma(\mathbf{y}), \quad \mathbf{x}\in U\setminus\Gamma, \tag{2.32}$$

where $\tilde{\mathbf{x}}$ is the unique projection of $\mathbf{x}\in U$ onto Γ.

The continuity behavior of the single- and double-layer potentials across the boundary Γ of the domain D is given by the following theorem (see p. 66 [25]; p. 199–201 [12]):

THEOREM 2. *Let* $\tilde{\mathbf{f}}$ *and* $\tilde{\mathbf{h}}$ *be two complex vector-valued densities in the class* $C^0(\Gamma)$, *and let* $\mathbf{V}_{\chi^2,n}(\cdot,\tilde{\mathbf{f}})$, $\mathbf{W}_{\chi^2,n}(\cdot,\tilde{\mathbf{h}})$ *and* $\mathbf{H}^{\pm}_{\chi^2,n}(\cdot,\tilde{\mathbf{f}})$ *be the complex vector-valued functions given by the relations* (2.22), (2.23) *and* (2.32). *Then for any point* $\mathbf{x}_0\in\Gamma$ *we have the relations*

$$\mathbf{V}^{+}_{\chi^2,n}(\mathbf{x}_0,\tilde{\mathbf{f}}) = \mathbf{V}^{-}_{\chi^2,n}(\mathbf{x}_0,\tilde{\mathbf{f}}) = \mathbf{V}_{\chi^2,n}(\mathbf{x}_0,\tilde{\mathbf{f}}), \tag{2.33}$$

$$\begin{aligned}\mathbf{W}^{+}_{\chi^2,n}(\mathbf{x}_0,\tilde{\mathbf{h}}) - \mathbf{W}^{*}_{\chi^2,n}(\mathbf{x}_0,\tilde{\mathbf{h}}) &= \varpi_n\tilde{\mathbf{h}}(\mathbf{x}_0)\\ &= \mathbf{W}^{*}_{\chi^2,n}(\mathbf{x}_0,\tilde{\mathbf{h}}) - \mathbf{W}^{-}_{\chi^2,n}(\mathbf{x}_0,\tilde{\mathbf{h}}),\end{aligned} \tag{2.34}$$

$$\begin{aligned}\mathbf{H}^{+}_{\chi^2,n}(\mathbf{x}_0,\tilde{\mathbf{f}}) - \mathbf{H}^{*}_{\chi^2,n}(\mathbf{x}_0,\tilde{\mathbf{f}}) &= -\varpi_n\tilde{\mathbf{f}}(\mathbf{x}_0)\\ &= \mathbf{H}^{*}_{\chi^2,n}(\mathbf{x}_0,\tilde{\mathbf{f}}) - \mathbf{H}^{-}_{\chi^2,n}(\mathbf{x}_0,\tilde{\mathbf{f}}),\end{aligned} \tag{2.35}$$

where

$$\begin{aligned}(\mathbf{W}^{*}_{\chi^2,n})_j(\mathbf{x}_0,\tilde{\mathbf{h}}) &= \overset{PV}{\int_{\Gamma}} \tilde{h}_i(\mathbf{y})S^{\chi^2}_{ijk}(\mathbf{y}-\mathbf{x}_0)n_k(\mathbf{y})d\Gamma(\mathbf{y}),\\ (\mathbf{H}^{*}_{\chi^2,n})_j(\mathbf{x}_0,\tilde{\mathbf{f}}) &= \overset{PV}{\int_{\Gamma}} \tilde{f}_i(\mathbf{y})S^{\chi^2}_{jik}(\mathbf{x}_0-\mathbf{y})n_k(\mathbf{x})d\Gamma(\mathbf{y})\end{aligned} \tag{2.36}$$

and PV *denotes the principal value.*

P r o o f. The proof of the properties (2.33)–(2.35) in the two- or three-dimensional steady case (i.e., for $n=2,3$ and the case $\chi=0$) is presented in [12], Chapter 3. The case $\chi^2\in\mathbb{C}\setminus\{z\in\mathbb{C}:\ \mathrm{Re}z\le 0,\ \mathrm{Im}z=0\}$, can be treated by using the relations

$$\begin{aligned}G^{\chi^2}_{kj}(\mathbf{x}-\mathbf{y}) &= G^{0}_{kj}(\mathbf{x}-\mathbf{y}) + G^{c}_{kj}(\mathbf{x}-\mathbf{y}),\\ S^{\chi^2}_{ijk}(\mathbf{y}-\mathbf{x})n_k(\mathbf{y}) &= S^{0}_{ijk}(\mathbf{y}-\mathbf{x})n_k(\mathbf{y}) + S^{c}_{ijk}(\mathbf{y}-\mathbf{x})n_k(\mathbf{y}),\end{aligned} \tag{2.37}$$

where $\mathbf{G}^0(G^0_{ij})$ is the steady Stokeslet and $\mathbf{S}^0(S^0_{ijk})$ is its associated stress tensor (which correspond to the case $\chi = 0$), given by (see p. 39 [12], for $n = 2, 3$; p. 16–17 [25])

$$G^0_{kj}(\mathbf{x}-\mathbf{y}) = \begin{cases} -\delta_{kj}\ln r + \dfrac{\widehat{x}_k\widehat{x}_j}{r^2} & \text{for } n = 2, \\ \dfrac{\delta_{kj}}{n-2}\dfrac{1}{r^{n-2}} + \dfrac{\widehat{x}_k\widehat{x}_j}{r^n} & \text{for } n \geq 3, \end{cases} \tag{2.38}$$

$$S^S_{ijk}(\mathbf{y}-\mathbf{x}) = 2n\frac{\widehat{x}_i\widehat{x}_j\widehat{x}_k}{r^{n+2}}, \quad n \geq 2, \tag{2.39}$$

and $\mathbf{G}^c(G^c_{ij})$ and $\mathbf{S}^c(S^c_{ijk})$ are continuous kernels. Note that $\widehat{\mathbf{x}} = \mathbf{x} - \mathbf{y} = (\widehat{x}_1, \ldots, \widehat{x}_n)$ and $r = |\widehat{\mathbf{x}}|$.

The decomposition formulas (2.37) yield that the kernel matrices $\mathbf{G}^0$ and $\mathbf{S}^0$ determine the continuity behavior of the potentials $\mathbf{V}_{\chi^2,n}(\cdot, \tilde{\mathbf{f}})$, $\mathbf{W}_{\chi^2,n}(\cdot, \tilde{\mathbf{h}})$ and $\mathbf{H}^{\pm}_{\chi^2,n}(\cdot, \tilde{\mathbf{f}})$. Therefore, the properties (2.33)–(2.35) are direct consequences of those corresponding to the case $\chi = 0$ (for details see e.g. [12], Sec. 3.4). □

2.4.2. Compactness of the single- and double-layer integral operators. For further considerations, we use the notations

$$\begin{aligned} \mathbf{W}^*_{\chi^2,n}(\mathbf{x}, \tilde{\mathbf{h}}) &= \overset{PV}{\int_{\Gamma}} \tilde{\mathbf{h}}(\mathbf{y}) \cdot \mathcal{K}_{\chi^2,n}(\mathbf{y}, \mathbf{x}) d\Gamma(\mathbf{y}), \\ \mathbf{H}^*_{\chi^2,n}(\mathbf{x}, \tilde{\mathbf{f}}) &= \overset{PV}{\int_{\Gamma}} \tilde{\mathbf{f}}(\mathbf{y}) \cdot \mathbf{D}_{\chi^2,n}(\mathbf{x}, \mathbf{y}) d\Gamma(\mathbf{y}), \end{aligned} \tag{2.40}$$

for any $\mathbf{x} \in \Gamma$, where $\mathcal{K}_{\chi^2,n}(\mathbf{y}, \mathbf{x})$ and $\mathbf{D}_{\chi^2,n}(\mathbf{x}, \mathbf{y})$ are the kernel matrices given by

$$\begin{aligned} (\mathcal{K}_{\chi^2,n})_{ij}(\mathbf{y}, \mathbf{x}) &= S^{\chi^2}_{ijk}(\mathbf{y}-\mathbf{x}) n_k(\mathbf{y}), \\ (\mathbf{D}_{\chi^2,n})_{ij}(\mathbf{x}, \mathbf{y}) &= S^{\chi^2}_{jik}(\mathbf{x}-\mathbf{y}) n_k(\mathbf{x}). \end{aligned} \tag{2.41}$$

Note that

$$\mathbf{D}_{\chi^2,n}(\mathbf{x}, \mathbf{y}) = (\mathcal{K}_{\chi^2,n}(\mathbf{x}, \mathbf{y}))^T, \quad \mathbf{x}, \mathbf{y} \in \Gamma, \mathbf{x} \neq \mathbf{y},$$

where the superscript T denotes the transpose of a matrix.

Let us now consider the single- and double-layer integral operators $\mathcal{V}_{\chi^2,n} : C^0(\Gamma) \to C^0(\Gamma)$ and $\mathbf{K}_{\chi^2,n} : C^0(\Gamma) \to C^0(\Gamma)$, given by the relations

$$(2.42)\qquad \begin{aligned} (\mathcal{V}_{\chi^2,n}\tilde{\mathbf{f}})(\mathbf{x}_0) &\equiv \mathbf{V}_{\chi^2,n}(\mathbf{x}_0, (2\varpi_n)^{-1}\tilde{\mathbf{f}}), \\ (\mathbf{K}_{\chi^2,n}\tilde{\mathbf{h}})(\mathbf{x}_0) &\equiv \mathbf{W}^*_{\chi^2,n}(\mathbf{x}_0, (2\varpi_n)^{-1}\tilde{\mathbf{h}}) \end{aligned}$$

for any $\mathbf{x}_0 \in \Gamma$ and all $\tilde{\mathbf{f}}, \tilde{\mathbf{h}} \in C^0(\Gamma)$. Using the formulas (2.37) and the assumption that $\Gamma \in C^{1,\alpha}$, it can be proved that both kernels $\mathcal{G}^{\chi^2}(\mathbf{x}-\mathbf{y})$ and $\mathcal{K}_{\chi^2,n}(\mathbf{y},\mathbf{x})$ of the integral operators $\mathcal{V}_{\chi^2,n}$ and $\mathbf{K}_{\chi^2,n}$ are weakly singular. Therefore, these operators are compact from $C^0(\Gamma)$ into $C^0(\Gamma)$.

Let us introduce the integral operator $\mathcal{H}_{\overline{\chi}^2,n} : C^0(\Gamma) \to C^0(\Gamma)$ given by the relation

$$(2.43)\qquad (\mathcal{H}_{\overline{\chi}^2,n}\tilde{\mathbf{f}})(\mathbf{x}_0) \equiv \mathbf{H}^*_{\overline{\chi}^2,n}(\mathbf{x}_0, (2\varpi_n)^{-1}\tilde{\mathbf{f}}), \quad \mathbf{x}_0 \in \Gamma,\ \tilde{\mathbf{f}} \in C^0(\Gamma),$$

where $\overline{\xi}$ means the complex conjugate of $\xi \in \mathbb{C}$. With respect to the inner product $\langle \cdot, \cdot \rangle : C^0(\Gamma) \times C^0(\Gamma) \to \mathbb{C}$ defined by

$$(2.44)\qquad \langle \mathbf{v}, \mathbf{w} \rangle \equiv \int_\Gamma \mathbf{v} \cdot \overline{\mathbf{w}} d\Gamma = \int_\Gamma v_j \overline{w}_j d\Gamma,$$

for all $\mathbf{v} = (v_1, \cdots, v_n), \mathbf{w} = (w_1, \cdots, w_n) \in C^0(\Gamma)$, the integral operators $\mathbf{K}_{\chi^2,n}$ and $\mathcal{H}_{\overline{\chi}^2,n}$ are adjoint, i.e., they satisfy the relation

$$(2.45)\qquad \langle \mathbf{K}_{\chi^2,n}\tilde{\mathbf{h}}, \tilde{\mathbf{f}} \rangle = \langle \tilde{\mathbf{h}}, \mathcal{H}_{\overline{\chi}^2,n}\tilde{\mathbf{f}} \rangle, \quad \tilde{\mathbf{f}}, \tilde{\mathbf{h}} \in C^0(\Gamma).$$

3. The interior Neumann problem

Using the potential theory for the Stokes resolvent system, we are now able to obtain the existence result of the classical solution to the interior Neumann problem (1.1), (2.3).

As above, let $D = D' \setminus \overline{D}_1 \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with boundary $\Gamma = \Gamma' \cup \Gamma_1$ of class $C^{1,\alpha}$ $(0 < \alpha \leq 1)$ and let $\chi^2 \in \mathbb{C} \setminus \{z \in \mathbb{C} :\ \mathrm{Re} z \leq 0, \mathrm{Im} z = 0\}$. Also, let $\mathbf{f} \in C^\lambda(D)$ be a Hölder continuous vector function in D with Hölder exponent $\lambda \in (0,1]$, and let $\mathbf{T} \in C^0(\Gamma)$ be given. First, we refer to the interior Neumann problem for the homogeneous Stokes resolvent system

$$(3.1)\qquad \nabla \cdot \mathbf{u} = 0, \quad -\nabla q + (\nabla^2 - \chi^2)\mathbf{u} = \mathbf{0} \text{ in } D$$

$$(3.2)\qquad \boldsymbol{\Sigma}(\mathbf{u}) \cdot \mathbf{n} = \mathbf{T} \text{ on } \Gamma.$$

3.1. Boundary integral representations of the solution

We have proved that this problem has at most one classical solution $(\mathbf{u}, q)$ (see Theorem 1). In order to show the existence of the solution to the Neumann problem (3.1)–(3.2), we consider the following boundary integral representations:

$$\mathbf{u}(\mathbf{x}) = \mathbf{V}_{\chi^2,n}(\mathbf{x}, (2\varpi_n)^{-1}\Psi),\ q(\mathbf{x}) = P^s_{\chi^2,n}\left(\mathbf{x}, (2\varpi_n)^{-1}\Psi\right),\ \mathbf{x} \in D, \tag{3.3}$$

where $\Psi \in C^0(\Gamma)$ is an unknown complex vector-valued density. Applying the boundary condition (3.2) to these boundary integral representations and using the jump formulas (2.35), we obtain the Fredholm integral equation of the second kind with unknown Ψ

$$\left(\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\chi^2,n}\right)\Psi = \mathbf{T} \text{ on } \Gamma. \tag{3.4}$$

Let us consider the homogeneous equation

$$\left(\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\chi^2,n}\right)\Psi_0 = \mathbf{0} \text{ on } \Gamma, \tag{3.5}$$

as well as its adjoint with respect to the inner product given by the formula (2.44)

$$\left(\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\overline{\chi}^2,n}\right)\mathbf{\Phi}_0 = \mathbf{0} \text{ on } \Gamma. \tag{3.6}$$

Then we have the following result:

Lemma 2. *Let $D = D'\setminus\overline{D}_1 \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with boundary $\Gamma = \Gamma' \cup \Gamma_1$ of class $C^{1,\alpha}$ $(0 < \alpha \leq 1)$ and let $\chi^2 \in \mathbb{C} \setminus \{z \in \mathbb{C} :\ \mathrm{Re} z \leq 0, \mathrm{Im} z = 0\}$. Then the null spaces of the operators*

$$\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\chi^2,n} : C^0(\Gamma) \to C^0(\Gamma), \quad \frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\overline{\chi}^2,n} : C^0(\Gamma) \to C^0(\Gamma) \tag{3.7}$$

are one-dimensional. Moreover, a basis of the space

$$\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\chi^2,n}\right) = \left\{\Psi_0 \in C^0(\Gamma) :\ \left(\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\chi^2,n}\right)\Psi_0 = \mathbf{0} \text{ on } \Gamma\right\},$$

is the set $\{\mathbf{N}_1\}$, where

$$\mathbf{N}_1(\mathbf{x}) = \begin{cases} \mathbf{n}(\mathbf{x}) & \text{if } \mathbf{x} \in \Gamma_1, \\ \mathbf{0} & \text{if } \mathbf{x} \in \Gamma', \end{cases} \tag{3.8}$$

and $\mathbf{n}$ is the unit normal to Γ pointing outside D.

P r o o f. Using the properties

$$\mathbf{V}_{\chi^2,n}(\mathbf{x},(2\varpi_n)^{-1}\mathbf{N}_1)=\mathbf{0},\quad \mathbf{x}\in\mathbb{R}^n \tag{3.9}$$

$$P^s_{\chi^2,n}\left(\mathbf{x},(2\varpi_n)^{-1}\mathbf{N}_1\right)=\begin{cases}1 & \text{if } \mathbf{x}\in D_1,\\ 0 & \text{if } \mathbf{x}\in D,\end{cases} \tag{3.10}$$

we find the relation

$$\mathbf{H}^-_{\chi^2,n}(\cdot,(2\varpi_n^{-1}\mathbf{N}_1)=\mathbf{0} \text{ on } \mathit{\Gamma}, \tag{3.11}$$

which shows that

$$\mathbf{N}_1\in\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n+\mathcal{H}_{\chi^2,n}\right). \tag{3.12}$$

Now, let $\mathit{\Psi}_0$ be an arbitrary function in the set $\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n+\mathcal{H}_{\chi^2,n}\right)$ and let $\mathbf{u}_0$ and q_0 be the fields defined by

$$\mathbf{u}_0=\mathbf{V}_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\mathit{\Psi}_0),\quad q_0=P^s_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\mathit{\Psi}_0) \text{ in } \mathbb{R}^n\setminus\mathit{\Gamma}. \tag{3.13}$$

Since $\mathit{\Psi}_0\in\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n+\mathcal{H}_{\chi^2,n}\right)$ it follows that

$$\mathbf{H}^-_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\Psi_0)=\mathbf{0} \text{ on } \mathit{\Gamma}, \tag{3.14}$$

i.e., $\mathbf{\Sigma}^-(\mathbf{u}_0)\cdot\mathbf{n}=\mathbf{0}$ on $\mathit{\Gamma}$. In addition, the fields $\mathbf{u}_0$ and q_0 satisfy the system of equations (3.1). Therefore, in view of uniqueness of the solution to the interior Neumann problem, we get

$$\mathbf{u}_0=\mathbf{0} \text{ in } \overline{D},\quad q_0=0 \text{ in } D. \tag{3.15}$$

In addition, using the uniqueness result of the classical solution of the following exterior Dirichlet problem (see p. 25 [12]):

$$\nabla\cdot\mathbf{u}_0=0,\quad -\nabla q_0+(\nabla^2-\chi^2)\mathbf{u}_0=\mathbf{0} \text{ in } \mathbb{R}^n\setminus\overline{D'},$$

$$\mathbf{u}_0=\mathbf{0} \text{ on } \mathit{\Gamma}',$$

$$(|\mathbf{u}_0||\nabla\mathbf{u}_0|)(\mathbf{x})=o(|\mathbf{x}|^{1-n}),\quad (|\mathbf{u}_0||q_0|)(\mathbf{x})=o(|\mathbf{x}|^{1-n}) \text{ as } |\mathbf{x}|\to\infty,$$

we deduce that

$$\mathbf{u}_0=\mathbf{0},\quad q_0=0 \text{ in } \mathbb{R}^n\setminus\overline{D'}. \tag{3.16}$$

Consequently, we have the relation

$$\mathbf{H}^{+}_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\Psi_0)=\mathbf{0} \text{ on } \Gamma'. \tag{3.17}$$

On the other hand, since the pair $(\mathbf{u}_0, q_0)$ is a classical solution to the interior Dirichlet problem

$$\nabla\cdot\mathbf{u}_0=0,\quad -\nabla q_0+(\nabla^2-\chi^2)\mathbf{u}_0=\mathbf{0} \text{ in } D_1,$$
$$\mathbf{u}_0=\mathbf{0} \text{ on } \Gamma_1,$$

it follows in view of the uniqueness result that (see Theorem 1)

$$\mathbf{u}_0=\mathbf{0},\quad q_0=c_1 \text{ in } D_1, \tag{3.18}$$

where $c_1\in\mathbb{C}$. Accordingly, we get the relations

$$\mathbf{H}^{+}_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\Psi_0)=-c_1\mathbf{n} \text{ on } \Gamma_1. \tag{3.19}$$

Now, taking into account the jump formulas (2.35), as well as the relations (3.14), (3.17) and (3.19), we deduce that

$$\Psi_0=\mathbf{0} \text{ on } \Gamma', \tag{3.20}$$

$$\Psi_0=c_1\mathbf{n} \text{ on } \Gamma_1 \tag{3.21}$$

or, equivalently,

$$\Psi_0=c_1\mathbf{N}_1 \text{ on } \Gamma, \tag{3.22}$$

where $\mathbf{N}_1$ is the function given by the relation (3.8). Consequently, the set $\{\mathbf{N}_1\}$ is a basis of the space $\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n+\mathcal{H}_{\chi^2,n}\right)$.

Finally, applying Fredholm's alternative (see e.g. [13]), we find that the null spaces of the operators

$$\frac{1}{2}\mathbf{I}_n+\mathcal{H}_{\chi^2,n}:C^0(\Gamma)\to C^0(\Gamma),\quad \frac{1}{2}\mathbf{I}_n+\mathbf{K}_{\overline{\chi}^2,n}:C^0(\Gamma)\to C^0(\Gamma)$$

(which are adjoint with respect to the inner product given by the formula (2.44)) have the same dimension, i.e.,

$$\dim\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n+\mathcal{H}_{\chi^2,n}\right)=\dim\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n+\mathbf{K}_{\overline{\chi}^2,n}\right)=1, \tag{3.23}$$

where

$$\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n+\mathbf{K}_{\overline{\chi}^2,n}\right)=\left\{\mathbf{\Phi}_0\in C^0(\Gamma):\ \left(\frac{1}{2}\mathbf{I}_n+\mathbf{K}_{\overline{\chi}^2,n}\right)\mathbf{\Phi}_0=\mathbf{0} \text{ on } \Gamma\right\}.$$

This completes the proof of Lemma 2. □

Using again Fredholm's alternative, we deduce that the Fredholm integral equation of the second kind (3.4) has a solution $\Psi \in C^0(\Gamma)$ if and only if the following orthogonality condition holds:

$$\int_{\Gamma} \mathbf{T} \cdot \overline{\mathbf{\Phi}}_0 d\Gamma = 0, \quad \forall\ \mathbf{\Phi}_0 \in \mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\overline{\chi}^2,n}\right). \tag{3.24}$$

The condition (3.24) is satisfied only in certain particular cases and is the consequence of the fact that we are looking for solutions to the interior Neumann problem (3.1)–(3.2) in the form of a single-layer potential without any completion. Note that this restriction does not appear in the case of a bounded domain with connected boundary, and the solution of the corresponding interior Neumann problem is expressed in terms of a single-layer potential (for details see e.g. p. 210 [12]; p. 70 [25]).

On the other hand, it is obvious that the result of Lemma 2 holds also for the operators

$$\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\chi^2,n} : C^0(\Gamma) \to C^0(\Gamma), \quad \frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\overline{\chi}^2,n} : C^0(\Gamma) \to C^0(\Gamma), \tag{3.25}$$

i.e.,

$$\dim \mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\chi^2,n}\right) = \dim \mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\overline{\chi}^2,n}\right) = 1, \tag{3.26}$$

and a basis of the null space $\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\overline{\chi}^2,n}\right)$ of the operator $\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\overline{\chi}^2,n}$ is the set $\{\mathbf{N}_1\}$, where the function $\mathbf{N}_1$ is given by the relation (3.8). Note that

$$\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\chi^2,n}\right) = \left\{\mathbf{\Phi} \in C^0(\Gamma) : \left(\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\chi^2,n}\right)\mathbf{\Phi} = \mathbf{0} \text{ on } \Gamma\right\}, \tag{3.27}$$

$$\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\overline{\chi}^2,n}\right) = \left\{\Psi_0 \in C^0(\Gamma) : \left(\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\overline{\chi}^2,n}\right)\Psi_0 = \mathbf{0} \text{ on } \Gamma\right\}. \tag{3.28}$$

Let $\{\mathbf{\Phi}_1\}$ be a basis of the null space $\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\chi^2,n}\right)$. Then $\{\overline{\mathbf{\Phi}}_1\}$ is a basis of the null space $\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\overline{\chi}^2,n}\right)$. Also let $\mathbf{u}_1$ and q_1 be the fields given by

$$\mathbf{u}_1(\mathbf{x}) = \mathbf{W}_{\chi^2,n}(\mathbf{x}, (2\varpi_n)^{-1}\mathbf{\Phi}_1), \quad q_1(\mathbf{x}) = P^s_{\chi^2,n}(\mathbf{x}, (2\varpi_n)^{-1}\mathbf{\Phi}_1), \quad \mathbf{x} \in \mathbb{R}^n \setminus \Gamma. \tag{3.29}$$

Applying the identity (2.5) to the fields $\mathbf{u}_1$ and q_1 in the bounded domain D, we obtain the formula

$$\int_D (\overline{\chi}^2|\mathbf{u}_1|^2 + 2E_{ij}(\mathbf{u}_1)\overline{E_{ij}(\mathbf{u}_1)})d\mathbf{x} = \int_\Gamma \{\overline{\mathbf{\Sigma}^-(\mathbf{u}_1)\cdot\mathbf{n}}\}\cdot\mathbf{u}_1^- d\Gamma. \tag{3.30}$$

Since $\mathbf{\Phi}_1 \in \mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\chi^2,n}\right)$, it follows that $\mathbf{u}_1^+ = \mathbf{0}$ on Γ and thus, in view of the jump formulas (2.34), we deduce that $\mathbf{u}_1^- = -\mathbf{\Phi}_1$ on Γ. Therefore, the formula (3.30) becomes

$$\int_D (\overline{\chi}^2|\mathbf{u}_1|^2 + 2E_{ij}(\mathbf{u}_1)\overline{E_{ij}(\mathbf{u}_1)})d\mathbf{x} = -\int_\Gamma \{\overline{\mathbf{\Sigma}^-(\mathbf{u}_1)\cdot\mathbf{n}}\}\cdot\mathbf{\Phi}_1 d\Gamma. \tag{3.31}$$

On the other hand, from the identity

$$-\frac{1}{2}\mathbf{\Phi}_1 = \mathbf{K}_{\chi^2,n}\mathbf{\Phi}_1 \text{ on } \Gamma$$

and the regularizing properties of the double-layer integral operator $\mathbf{K}_{\chi^2,n} : C^0(\Gamma) \to C^0(\Gamma)$, we find that (see e.g. [15] in the case $\chi = 0$; [24]; [27])

$$\mathbf{\Phi}_1 \in C^{1,\alpha}(\Gamma).$$

Hence the normal stress due to the double-layer potential $\mathbf{u}_1$ has equal limiting values on both sides of Γ (see p. 47, 103 [7]; [12] Theorem 3.4.11, in the case $n = 3$, $\chi = 0$), i.e.,

$$\mathbf{\Sigma}^-(\mathbf{W}_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\mathbf{\Phi}_1))\cdot\mathbf{n} = \mathbf{\Sigma}^+(\mathbf{W}_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\mathbf{\Phi}_1))\cdot\mathbf{n} \text{ on } \Gamma. \tag{3.32}$$

Further, integrating both sides of the equation

$$\nabla\cdot\mathbf{W}_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\mathbf{\Phi}_1) = 0 \text{ in } D$$

over the domain D, and using the divergence theorem, as well as the boundary condition

$$\mathbf{W}^-_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\mathbf{\Phi}_1) = -\mathbf{\Phi}_1 \text{ on } \Gamma,$$

we obtain the relation

$$\int_\Gamma \mathbf{\Phi}_1\cdot\mathbf{n}d\Gamma = 0, \tag{3.33}$$

which, in view of the properties (2.30), yields the result

$$\mathbf{W}_{\chi^2,n}(\mathbf{x},(2\varpi_n)^{-1}\mathbf{\Phi}_1) = O(|\mathbf{x}|^{-n}) \text{ as } |\mathbf{x}|\to\infty. \tag{3.34}$$

This result is sufficient to show that the fields

$$\mathbf{u}_1 = \mathbf{W}_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\mathbf{\Phi}_1), \quad q_1 = P^d_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\mathbf{\Phi}_1)$$

satisfy the far-field conditions

$$(|\mathbf{u}_1||\nabla\mathbf{u}_1|)(\mathbf{x}) = o(|\mathbf{x}|^{1-n}), \quad (|\mathbf{u}_1||q_1|)(\mathbf{x}) = o(|\mathbf{x}|^{1-n}) \text{ as } |\mathbf{x}| \to \infty. \tag{3.35}$$

In addition, the fields $\mathbf{u}_1$ and q_1 satisfy the system of equations

$$\nabla \cdot \mathbf{u}_1 = 0, \quad -\nabla q_1 + (\nabla^2 - \chi^2)\mathbf{u}_1 = \mathbf{0} \text{ in } C\overline{D'},$$

as well as the property

$$\mathbf{u}_1^+ = \mathbf{W}^+_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\mathbf{\Phi}_1) = \mathbf{0} \text{ on } \mathit{\Gamma}'.$$

Taking into account the uniqueness result of the solution to the exterior Dirichlet problem (see p. 25 [12]), we thus deduce that

$$\mathbf{u}_1 = \mathbf{0}, \quad q_1 = 0 \text{ in } \mathbb{R}^n \setminus \overline{D'} \tag{3.36}$$

and hence

$$\begin{aligned}\mathbf{\Sigma}^-(\mathbf{W}_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\mathbf{\Phi}_1)) \cdot \mathbf{n} \\ = \mathbf{\Sigma}^+(\mathbf{W}_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\mathbf{\Phi}_1)) \cdot \mathbf{n} = \mathbf{0} \qquad \text{on } \mathit{\Gamma}'.\end{aligned} \tag{3.37}$$

Also the relation $\mathbf{W}^+_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\mathbf{\Phi}_1) = \mathbf{0}$ on $\mathit{\Gamma}_1$ (note that the plus sign applies here for the internal side of $\mathit{\Gamma}_1$) together with the uniqueness result of the solution to the interior Dirichlet problem (see p. 25 [12]) lead to

$$\mathbf{W}_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\mathbf{\Phi}_1) = \mathbf{0}, \; P^d_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\mathbf{\Phi}_1) = c_1^0 \text{ in } D_1, \tag{3.38}$$

and thus

$$\begin{aligned}\mathbf{\Sigma}^-(\mathbf{W}_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\mathbf{\Phi}_1)) \cdot \mathbf{n} \\ = \mathbf{\Sigma}^+(\mathbf{W}_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\mathbf{\Phi}_1)) \cdot \mathbf{n} = -c_1^0\mathbf{n} \qquad \text{on } \mathit{\Gamma}_1,\end{aligned} \tag{3.39}$$

where $c_1^0 \in \mathbb{C}$.

Now, in view of the relations (3.29), (3.37) and (3.39), the formula (3.31) becomes

$$\begin{aligned}&\int_D (\overline{\chi}^2|\mathbf{u}_1|^2 + 2E_{ij}(\mathbf{u}_1)\overline{E_{ij}(\mathbf{u}_1)})d\mathbf{x} \\ &= -\int_{\mathit{\Gamma}} \overline{\left\{\mathbf{\Sigma}^-(\mathbf{W}_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\mathbf{\Phi}_1)) \cdot \mathbf{n}\right\}} \cdot \mathbf{\Phi}_1 d\mathit{\Gamma} = \overline{c_1^0}\int_{\mathit{\Gamma}_1} \mathbf{\Phi}_1 \cdot \mathbf{n} d\mathit{\Gamma}_1.\end{aligned} \tag{3.40}$$

If

$$\int_D (\overline{\chi}^2|\mathbf{u}_1|^2 + 2E_{ij}(\mathbf{u}_1)\overline{E_{ij}(\mathbf{u}_1)})d\mathbf{x} = 0,$$

then $\mathbf{u}_1 = \mathbf{0}$ in D, and hence $\mathbf{u}_1^- = \mathbf{0}$ on Γ. In addition, $\mathbf{u}_1^+ = \mathbf{0}$ on Γ, and thus, according to the jump formulas (2.34), we obtain $\mathbf{\Phi}_1 \equiv \mathbf{0}$. This result contradicts the property $\mathbf{\Phi}_1 \neq \mathbf{0}$ on Γ. (Note that the set $\{\mathbf{\Phi}_1\}$ is a basis of the null space $\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\chi^2,n}\right)$, and hence $\mathbf{\Phi}_1 \neq \mathbf{0}$ on Γ.) Therefore, we must have

$$\overline{c_1^0} \int_{\Gamma_1} \mathbf{\Phi}_1 \cdot \mathbf{n} d\Gamma_1 \neq 0, \tag{3.41}$$

i.e.,

$$\int_{\Gamma_1} \mathbf{\Phi}_1 \cdot \mathbf{n} d\Gamma_1 \neq 0, \quad c_1^0 \neq 0. \tag{3.42}$$

3.2. The completion of the boundary integral representations (3.3)

Recall that the boundary integral representation of the velocity field for the interior Neumann problem in terms of a single-layer potential without any completion leads to the boundary integral equation (3.4), which admits solutions in $C^0(\Gamma)$ only if the condition (3.24) holds.

Let us now consider the completed boundary integral representations

$$\mathbf{u}(\mathbf{x}) = \mathbf{V}_{\chi^2,n}(\mathbf{x}, (2\varpi_n)^{-1}\mathbf{\Psi}) + \beta_1 \mathbf{W}_{\chi^2,n}(\mathbf{x}, (2\varpi_n)^{-1}\mathbf{\Phi}_1), \quad \mathbf{x} \in D, \tag{3.43}$$

$$q(\mathbf{x}) = P^s_{\chi^2,n}(\mathbf{x}, (2\varpi_n)^{-1}\mathbf{\Psi}) + \beta_1 P^d_{\chi^2,n}(\mathbf{x}, (2\varpi_n)^{-1}\mathbf{\Phi}_1), \quad \mathbf{x} \in D, \tag{3.44}$$

where $\beta_1 \in \mathbb{C}$ is an unknown constant, $\mathbf{\Psi} \in C^0(\Gamma)$ is an unknown vector density, and the set $\{\mathbf{\Phi}_1\}$ is a basis of the space $\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\chi^2,n}\right)$.

Applying the boundary condition (3.2) to the boundary integral representations (3.43) and (3.44), and using the jump formulas (2.35), we obtain the following Fredholm integral equation of the second kind with unknown density $\mathbf{\Psi}$:

$$\left(\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\chi^2,n}\right)\mathbf{\Psi} = \mathbf{T} - \beta_1 \mathbf{\Sigma}^-(\mathbf{W}_{\chi^2,n}(\cdot, (2\varpi_n)^{-1}\mathbf{\Phi}_1) \cdot \mathbf{n} \text{ on } \Gamma. \tag{3.45}$$

Now, according to the properties (3.39) and (3.42), we can choose the number $\beta_1 \in \mathbb{C}$ such that

$$\beta_1 = \left[\int_\Gamma \left\{\mathbf{\Sigma}^-(\mathbf{W}_{\chi^2,n}(\cdot, (2\varpi_n)^{-1}\mathbf{\Phi}_1)) \cdot \mathbf{n}\right\} \cdot \mathbf{\Phi}_1 d\Gamma\right]^{-1} \int_\Gamma \mathbf{T} \cdot \mathbf{\Phi}_1 d\Gamma. \tag{3.46}$$

Therefore, we get the relation

$$\int_{\Gamma} \left\{ \mathbf{T} - \beta_1 \boldsymbol{\Sigma}^- \left(\mathbf{W}_{\chi^2,n}(\cdot, (2\varpi_n)^{-1}\boldsymbol{\Phi}_1) \right) \cdot \mathbf{n} \right\} \cdot \boldsymbol{\Phi}_1 d\Gamma = 0, \tag{3.47}$$

which is just the condition required by Fredholm's alternative in order to have a solution of the Eq. (3.45) in the space $C^0(\Gamma)$. Recall that $\{\overline{\boldsymbol{\Phi}}_1\}$ is a basis of the space $\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\overline{\chi}^2,n}\right)$.

Concluding the above arguments, we obtain the following property:

THEOREM 3. *Let $D = D' \setminus \overline{D}_1 \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with boundary $\Gamma = \Gamma' \cup \Gamma_1$ of class $C^{1,\alpha}$ $(0 < \alpha \leq 1)$ and let $\chi^2 \in \mathbb{C} \setminus \{z \in \mathbb{C} :$ $\mathrm{Re}z \leq 0, \mathrm{Im}z = 0\}$. Also, let $\mathbf{T} \in C^0(\Gamma)$ be given. Assume that the set $\{\boldsymbol{\Phi}_1\}$ is a basis of the space $\mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\chi^2,n}\right)$. Then there exists the uniquely determined constant $\beta_1 \in \mathbb{C}$ such that the Fredholm integral equation of the second kind* (3.45) *has a solution $\boldsymbol{\Psi} \in C^0(\Gamma)$. Moreover, the boundary integral representations* (3.43) *and* (3.44)*, obtained with the density $\boldsymbol{\Psi}$ and the constant β_1, determine the unique classical solution of the interior Neumann problem* (3.1)–(3.2).

Taking into account the previous property, we can obtain the existence and uniqueness result for the classical solution of the interior Neumann problem associated with the non-homogeneous Stokes resolvent system

$$\nabla \cdot \mathbf{u} = 0, \quad -\nabla q + (\nabla^2 - \chi^2)\mathbf{u} = -\mathbf{f} \text{ in } D \tag{3.48}$$

$$\boldsymbol{\Sigma}(\mathbf{u}) \cdot \mathbf{n} = \mathbf{T} \text{ on } \Gamma. \tag{3.49}$$

This result is given by the following theorem:

THEOREM 4. *Let $D = D' \setminus \overline{D}_1$ be a bounded domain with boundary $\Gamma = \Gamma' \cup \Gamma_1$ of class $C^{1,\alpha}$ $(0 < \alpha \leq 1)$ and let $\chi^2 \in \mathbb{C} \setminus \{z \in \mathbb{C} :$ $\mathrm{Re}z \leq 0, \mathrm{Im}z = 0\}$. Also, let $\mathbf{f} \in C^{\lambda}(D)$ be a Hölder continuous vector function in D $(0 < \lambda \leq 1)$, and let $\mathbf{T} \in C^0(\Gamma)$ be given. Then the boundary integral representations*

$$\mathbf{u}(\mathbf{x}) = \mathbf{V}_{\chi^2,n}(\mathbf{x}, (2\varpi_n)^{-1}\boldsymbol{\Psi}) + \mathbf{W}_{\chi^2,n}(\mathbf{x}, (2\varpi_n)^{-1}\boldsymbol{\Phi}) + \frac{1}{2\varpi_n} \int_D \mathcal{G}^{\chi^2}(\mathbf{x} - \mathbf{y}) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y}, \tag{3.50}$$

$$q(\mathbf{x}) = P^s_{\chi^2,n}(\mathbf{x}, (2\varpi_n)^{-1}\boldsymbol{\Psi}) + P^d_{\chi^2,n}(\mathbf{x}, (2\varpi_n)^{-1}\boldsymbol{\Phi}) + \frac{1}{2\varpi_n} \int_D \boldsymbol{\Pi}^{\chi^2}(\mathbf{x} - \mathbf{y}) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y}, \tag{3.51}$$

$\mathbf{x} \in D$, *determine the unique classical solution of the interior Neumann problem* (3.48)–(3.49), *where* $\mathbf{\Psi} \in C^0(\Gamma)$ *is a solution of the Fredholm integral equation of the second kind*

$$\left(\frac{1}{2}\mathbf{I}_n + \mathcal{H}_{\chi^2,n}\right)\mathbf{\Psi} = \mathbf{T}^0 - \mathbf{\Sigma}^-(\mathbf{W}_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\mathbf{\Phi}))\cdot\mathbf{n} \text{ on } \Gamma, \tag{3.52}$$

$\mathbf{T}^0 = (T_1^0, \cdots, T_n^0)$ *is the vector function with the components*

$$T_j^0(\mathbf{x}) = T_j(\mathbf{x}) - \frac{1}{2\varpi_n} n_k(\mathbf{x}) \int_D S_{jik}^{\chi^2}(\mathbf{x}-\mathbf{y}) f_i(\mathbf{y}) d\mathbf{y}, \quad \mathbf{x} \in \Gamma, \tag{3.53}$$

$j = 1, \ldots, n$, *and the function* $\mathbf{\Phi} \in \mathcal{N}\left(\frac{1}{2}\mathbf{I}_n + \mathbf{K}_{\chi^2,n}\right)$ *is uniquely determined in the form* $\mathbf{\Phi} = \beta_1\mathbf{\Phi}_1$, *with*

$$\beta_1 = \left[\int_\Gamma \left\{\mathbf{\Sigma}^-(\mathbf{W}_{\chi^2,n}(\cdot,(2\varpi_n)^{-1}\mathbf{\Phi}_1))\cdot\mathbf{n}\right\}\cdot\mathbf{\Phi}_1 d\Gamma\right]^{-1} \int_\Gamma \mathbf{T}^0\cdot\mathbf{\Phi}_1 d\Gamma. \tag{3.54}$$

4. Conclusions

In this paper we have used the results of the potential theory for the Stokes resolvent system in order to obtain the existence and uniqueness result of the classical solution to the interior Neumann problem, associated with the Stokes resolvent system in a bounded domain with compact but not connected boundary.

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