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Effective yield strengths of random materials by an ε-self-consistent method.

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The problem of determining the effective yield strength domain of a material containing random distributed heterogeneities is dealt with. This material is represented by a set of microstructures, each occupying a volume of the order of the heterogeneities. A homogeneous comparison material is used, characterized by its own yield strength domain, in which these microstructures are placed. The equivalent homogeneous material is envisaged as the solution of a system of self-consistent equations. The problems of non-existence or non-uniqueness of the solutions of this system lead to modifying it, using an equality to "within ε". "Extremal" solutions are highlighted for each of the equations of the system transformed in this way, which bound the effective domain sought for. The proposed homogenization method is applied to a defect material and the result is compared with a structure calculation.

Key words: homogenization, yield design theory, self-consistent method.

1. Introduction

Determining the limit loads that can potentially be withstood by a mechanical system by yield design methods (Salençon [1]) presents almost inherent difficulties when the constituent materials are highly heterogeneous. A preliminary homogenization step proves indispensable.

In the favourable case where the materials are periodically heterogeneous, mechanically well-founded processes capable of integrating all the available microstructural information have been developed (Suquet [2], Bouchitte and Suquet [3], De Buhan and Salençon [4], De Buhan [5]) and applied in different situations (see De Buhan and Taliercio [6] for example). In the case where the periodicity assumption no longer holds, the complexity of the homogenization problem is quite different. This complexity on the one hand and the constant development of composites on the other hand, result in a very abundant bibliography. An exhaustive presentation can be found in Bornert *et al.* [7] of the methods relating to the different extensions of the self-consistent model, initially designed for linear behaviour, and followed by those of Hill [8]. These extensions (see for example Masson *et al.* [9], Ponte Castañeda [10] and Pastor and Ponte Castañeda [11]) for a comparison with numerical limit analysis results) find their justification in the insufficiency of the conventional extension (Gilormini [12]). The self-consistency concept has also been used with a non-linear comparison material in the case of porous media (Leblond and Perrin [13], Barthelemy and Dormieux [14]). The extremal heterogeneous model (Arminjon [15] and Arminjon *et al.* [16] for applications of this model) constitutes an alternative approach. It describes a continuous transition between the Reuss and Voigt bounds by variation of an experimentally fixed parameter. Other works specific to limit analysis or to yield design have been proposed for particular applications (Ponte Castañeda and De Botton [17], Sab [18]).

The object of the homogenization method presented in this paper is to predict the effective yield strength domain of a material containing random distributed heterogeneities. It is based on the concept of comparison material, the latter being homogeneous and directly characterized by its yield strength domain. It constitutes a generalization of the method developed in Turgeman and Guessab [19] necessary to make this method effectively predictive.

The basic ideas on which the proposed homogenization method is based are the following. The samples (denoted by E) under consideration are made of randomly distributed inclusions, which are placed within a homogeneous matrix of yield strength domain G. The inclusions are representative of the heterogeneous material (denoted by M_{ha}). For example, the inclusions are small volumes ω taken from distinct positions of M_{ha}. The smallest distance between these inclusions Fig. 1 in a given sample E is characterized by a number ρ, which ranges from 0 to 1 ($\rho = 0$ if two inclusions are tangential; $\rho = 1$ if the distance between both inclusions is infinite).

We submit the samples to two load processes that are: a load process of Reuss type (uniform stress boundary conditions) and a load process of Voigt type (uniform strain rate boundary conditions). We then determine, under some consistency conditions, the domain $K_R(E)$ (respectively $K_V(E)$) of admissible loads by any sample E for a loading of Reuss type (respectively of Voigt type).

We form the following self-consistent equations:

$$K_R(E) = G \quad \text{for every } E, \tag{1.1}$$

$$K_V(E) = G \quad \text{for every } E \tag{1.2}$$

which means that samples behave as if they were just composed of their own homogeneous matrix. In other words, the inclusions that are representative of

M_{ha} and that can be found in any sample E, do not alter or strengthen the homogeneous matrix of E. We therefore define the macroscopic yield strength domain of M_{ha} as the one of the matrix.

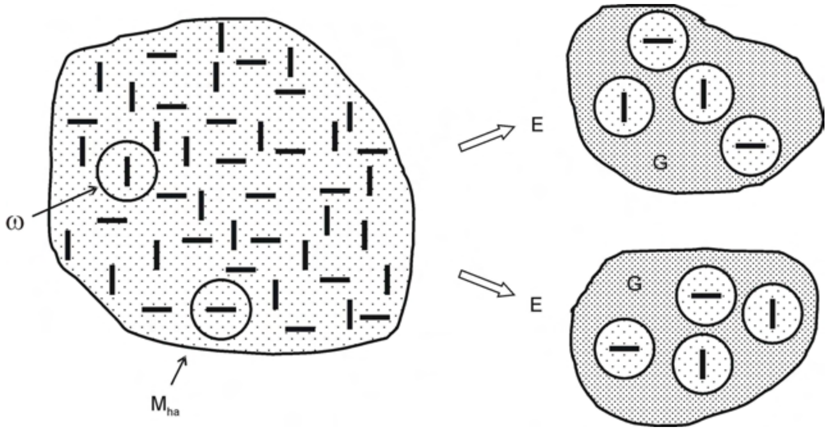

FIG. 1. Choice of inclusions representative of the heterogeneous material M_{ha} and examples of samples E.

The necessity of considering the two load processes is justified by the following results: Eq. (1.1) admits a greater solution $G_R^+(\rho)$ which increases according to ρ, Eq. (1.2) admits a smaller solution $G_V^-(\rho)$ which decreases according to ρ. Figure 2 shows the variations of the extremal solutions according to ρ in an ideal case in which the set of Eqs. (1.1) and (1.2) admits a unique solution G_h for $\rho = \rho_h$, $\rho_h \in [0.1[$. Even in this ideal case, the fact of considering only the

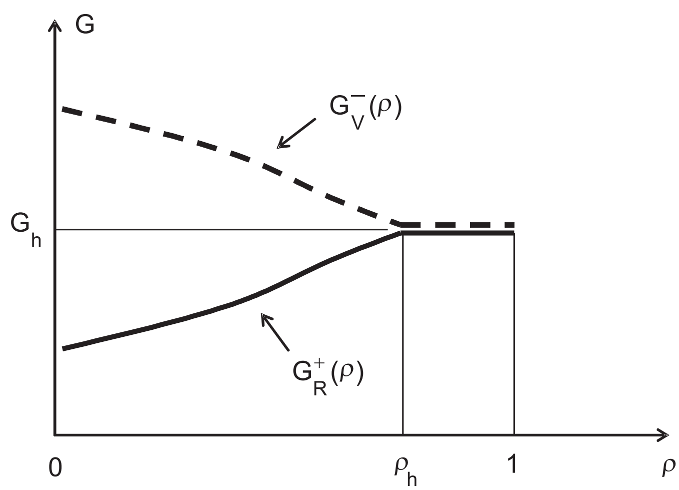

FIG. 2. Extremal solutions of self-coherent equations in an ideal case.

load process of Reuss type (respectively of Voigt) does not enable to distinguish G_h from the infinity of possible solutions that (1.1) presents (respectively (1.2)). On the other hand, the conjunction of these two equations enables to present the unique solution G_h.

Unfortunately, this system of equations does not in general admit any solution when the distance separating the inclusions is finite, and presents an infinity of solutions when this distance is infinite (Fig. 3). This means that the equation "response of the heterogeneous samples equals that of the same homogeneous samples constituted by the comparison material only" is not discriminant in the sense that it does not enable the comparison materials to be distinguished, either by absence of solutions, or on the contrary, by a multiplicity of solutions.

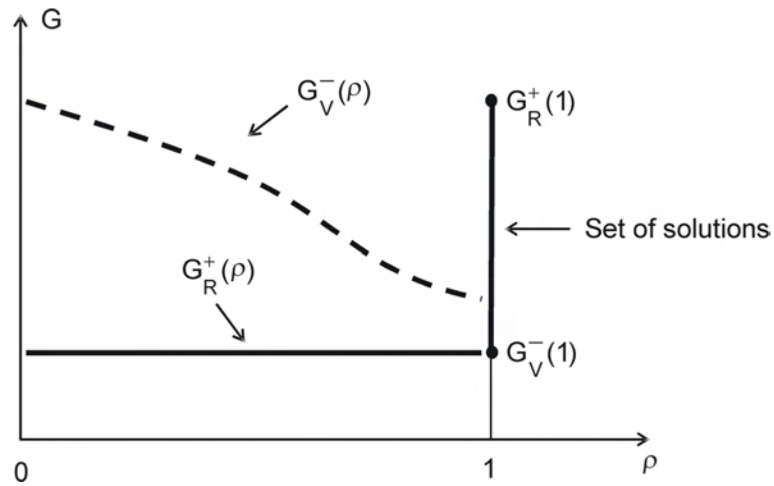

Fig. 3. Extremal solutions of self-coherent equations in a general case.

It follows that the only indications obtained on the effective yield strength domain sought for by this comparison process are the generalized Reuss or Voigt bounds, unless the parameter ρ is fixed arbitrarily.

In this work, we have sought to solve this problem linked to the non-existence or non-uniqueness of the solutions of the self-consistent system by remaining in the domain of finite distances between the inclusions and modifying the self-consistency equations using an approximated equality so that the set of solutions, characterized by its diameter, is non-void.

This process, based on a definition of the equality of convex sets to "within ε", leads to a system of ε-self-consistent equations, now parameterized by ρ and by ε. These parameters are fixed so that the diameter of the set of solutions is minimum. This amounts to making the choice of the pair (ρ, ε) that is the most discriminant.

Effective implementation of this characterization of the equivalent homogeneous material is complex since it assumes determination of the solutions of the ε-self-consistent system in order to deduce therefrom the diameter which is to be minimized. This difficulty is overcome by reducing the initial problem to separate resolution of each of the two equations which form the system. For this purpose we show that each of the two equations admits an extremal solution ($G_R^+(\rho,\varepsilon)$ for the load process of Reuss type, $G_V^-(\rho,\varepsilon)$ for the load process of Voigt type) that is a function of (ρ,ε), and that any solution of the ε-self-consistent system is bounded by these particular solutions (Fig. 4). The reciprocal of the latter property, necessary to achieve the required simplification, is not established in all cases. It is true under the assumptions that the extremal solutions are homothetic and that ε is not too large (but these conditions are certainly not necessary, as shown by the example dealt with in application). However, on account of its practical interest, it is used in a heuristic approach. The diameter of the set of solutions can then be replaced by the distance between the extremal solutions.

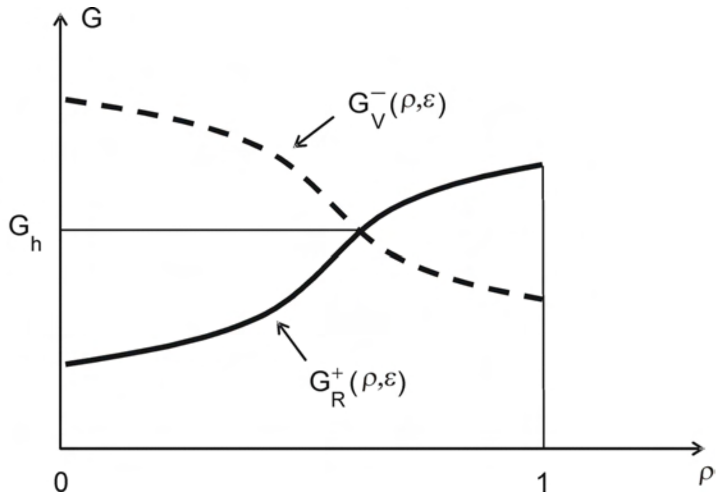

FIG. 4. Extremal solutions of ε-self-coherent equations in a general case.

The proposed method is applied to seeking the effective yield strength domain of a defect material. We deliberately chose to place ourselves in a simplified context (two-dimensional problem, representation of the heterogeneous material by a single microstructure) for the following two reasons: the first is that prediction of the effective yield strength domain can be obtained analytically with the consequence of being able to verify that the assumption on the reciprocal property referred to above is true in this case, without the extremal solutions however being homothetic; the second is that this simplified context enables the effective yield strength domain sought for to be approached by a structure calculation. The latter is performed on a series of volumes assumed to be representative of the heterogeneous material, each comprising a very large number of defects and being subjected to uniform strain rate boundary conditions or to uniform stress boundary conditions. Comparison of this direct calculation with the prediction provided by the proposed homogenization method constitutes a first step of the validation process.

2. Heterogeneous inclusions in a homogeneous comparison medium

2.1. Representation of the random material

A material M_{ha} containing random distributed heterogeneities is considered. This material occupies a domain of $\mathbb{R}^3$, of sufficiently large volume Ω_{ha} to be representative. The description of M_{ha} is made by means of a small volume ω of $\mathbb{R}^3$, for example that of a sphere of centre c and diameter δ_ω equal to one or more heterogeneities. We observe the microstructure which appears in ω, when the latter moves in Ω_{ha}. This microstructure is defined by the yield strength domain datum at each point of ω, including those of the possible contact conditions. We then represent M_{ha} by a finite set $\hat{\mu}s$ of microstructures μs_k $(k \in \hat{m} = \{1, ..., m\})$, each of which occupies a volume ω_k $(= \omega)$ and is assigned a weighting coefficient w_k $(k \in \hat{m})$ (ARMINJON *et al.* [16]). The yield strength domain at the point $\mathbf{x}$ of the microstructure $\mu s_k (x \in \omega_k, k \in \hat{m})$ is denoted $\varphi_k(\mathbf{x})$. We assume for the sake of simplification that the yield strength domains $\varphi_k(x)$ are in $\hat{C}$, the set of bounded closed convex sets of $\mathbb{R}_s^9$ which contain the zero stress tensor. The irreducible assumptions are those of convexity, which does not require the yield design (SALENÇON [1]), and the property of containing the zero tensor. Given $G \in \hat{C}$, its support function is denoted Π_G:

$$\forall \mathbf{D} \in \mathbb{R}_s^9 : \Pi_G (\mathbf{D}) = \max (\boldsymbol{\sigma} \cdot D, \quad \boldsymbol{\sigma} \in G) .$$

$\Pi_G (\mathbf{D})$ is equal to the dissipated power rate of the material of the yield strength domain G in the strain $\mathbf{D}$, for a unit volume.

The generalized REUSS G_R^g and VOIGT G_V^g bounds are associated with the set of microstructures $\hat{\mu}s$.

The REUSS bound G_R^g is formed by the stress tensors $S \in \mathbb{R}_s^9$, balanced by stress fields $\boldsymbol{\sigma}_k$ defined on the volume ω_k of $\mu s_k (k \in \hat{m})$ such that:

$$\mathrm{div} \boldsymbol{\sigma}_k (\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \omega_k; \quad \boldsymbol{\sigma}_k (\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = \mathbf{S} \cdot \mathbf{n}(\mathbf{x}), \quad \forall \mathbf{x} \in \partial \omega_k$$

$$\boldsymbol{\sigma}_k (\mathbf{x}) \in \varphi_k (\mathbf{x}) , \quad \forall \mathbf{x} \in \omega_k$$

where $\mathbf{n}(\mathbf{x})$ is the external normal at the point $\mathbf{x}$ of the bound $\partial \omega_k$ of ω_k.

The VOIGT bound G_V^g is formed by the stress tensors $\mathbf{S}$ such that:

$$\mathbf{S}\cdot\mathbf{D} \le \min\left(\sum_{k=1}^{n} w_k \langle \Pi_{\varphi_k}(\mathbf{x})(\mathbf{d}_k(\mathbf{x}))\rangle_{\omega_k}, \quad \mathbf{d}_k \in U_0(\mathbf{D})\right), \quad \forall\mathbf{D} \in \mathbb{R}_s^9$$

denoting:

- $U_0(\mathbf{D})$ the set of strain rate fields $\mathbf{d}$, defined on ω and derived from displacement velocity fields $\mathbf{v}$ such that: $\mathbf{v}(\mathbf{x}) = \mathbf{D}\,\mathbf{.}\,\mathbf{x}, \forall\mathbf{x} \in \partial\omega$;
- $\langle\ \rangle_\omega$ the mean value in ω.

The bounds G_R^g and G_V^g are in $\hat{C}$, on account of $\varphi_k(\mathbf{x}) \in \hat{C},\ \forall\mathbf{x} \in \omega_k, \forall k \in \hat{m}$.

A minimum condition for the heterogeneous material M_{ha} to be well represented by $\hat{\mu s}$ is that its yield strength domain G_h is such that:

$$G_R^g \subset G_h \subset G_V^g.$$

The chosen set $\hat{\mu s}$ is assumed to comply with this minimum condition.

2.2. Samples E

Samples E are made by placing N_E inclusions, each of them occupying a volume identical to ω, into a homogeneous matrix M(G). The volume Ω_E of a sample E, the number N_E of inclusions, and positions of the inclusions defined by that of their centre $c_i \quad (i \in \hat{N}_E = \{1,...,N_E\})$, are variable from one sample E to another. The matrix $M(G)$ is formed by a homogeneous material with a yield strength domain $G \in \hat{C}$. The inclusions are heterogeneous and have microstructures belonging to $\hat{\mu s}$. Each microstructure μs_k $(k \in \hat{m})$ is represented, in any sample E, proportionally to its weighting coefficient w_k. The yield strength domain at the point $\mathbf{x} \in \Omega_E$ is denoted $\Psi_E(\mathbf{x})$ and we therefore note $\Psi_E(\mathbf{x}) \in \{\mathrm{G}; \varphi_k(\mathbf{x}'), \forall\mathbf{x}' \in \omega, \forall k \in \hat{m}\ \}$.

Each inclusion is centred in a homothetic volume V (its zone of influence) of ω. The centre of the homothetic transformation associating V to ω is that of the inclusion. The homothetic transformation ratio is the maximum ratio such that two zones of influence are disjointed or tangent, and such that any zone of influence is contained in Ω_E. A sample E is generally characterized by the number:

$$\rho(E) = \frac{V-\omega}{V} \quad \Big(\in [0,1]\Big).$$

Each inclusion is surrounded by a minimum volume $V' = V\backslash\omega$ occupied by the homogeneous material $M(G)$.

The samples E form a set $\hat{E}$. The equivalence relation $\perp$ is considered in $\hat{E}$: two elements $E_1 \in \hat{E}$ and $E_2 \in \hat{E}$ are equivalent if and only if their matrix is formed by the same material $M(G)$ and if $\rho(E_1) = \rho(E_2)$. An element of $\hat{E}/\perp$ is denoted $(\dot{G,}\rho)$. A base $\mathcal{B}(G,\rho)$ formed by m mechanical systems B_k $(k \in \hat{m})$ is associated to each equivalence class $(\dot{G,}\rho)$. Each system B_k occupies a volume V_k equal to V, in which an inclusion of volume ω_k (equal to ω) and having the microstructure $\mu s_k (k \in \hat{m})$ is centered. The volume $V'_k = V_k \backslash \omega_k$ is formed by the homogeneous material $M(G)$. Thus any sample $E \in (\dot{G,}\rho)$ admits a breakdown into sub-sets that is either identical to an element of $\mathcal{B}(G,\rho)$ or homogeneous and formed by the material $M(G)$ Fig. 5.

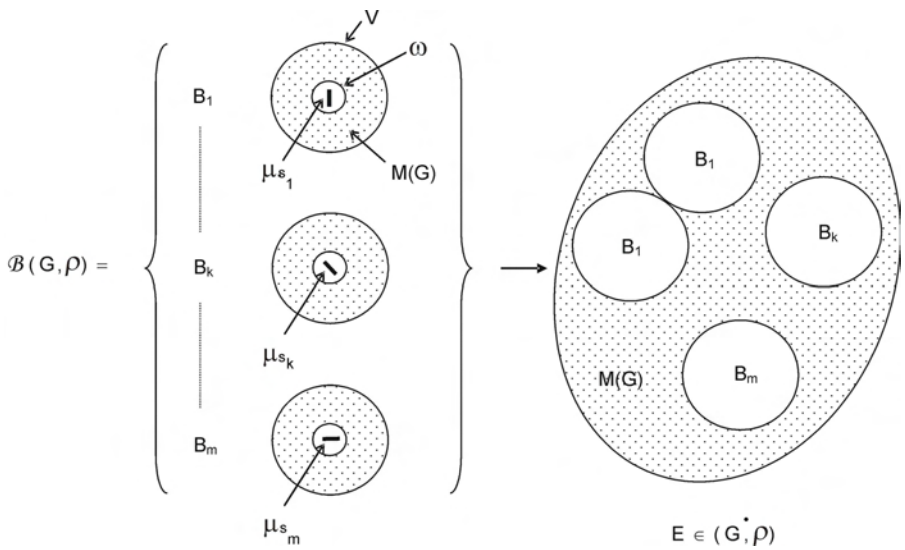

FIG. 5. Breakdown of sample E into the base elements and the homogeneous complementary region.

2.3. Consistent loads of a sample E

Let there be a sample $E \in ((\dot{G,}\rho))$ $(G \in \hat{C}, \rho \in [0,1])$, of volume Ω_E and bound $\partial\Omega_E$. It is subjected to a limit loading process corresponding to uniform stress (REUSS type) or uniform strain rate (VOIGT type) boundary conditions.

In the case of REUSS type conditions, the convex set $K_R(E) \in \hat{C}$ made up of loads $\mathbf{S}$, called R-consistent, is formed:

$$K_R(E) = \Big\{ \mathbf{S} \in \mathbb{R}^9_s / \exists \boldsymbol{\sigma} \text{ defined on } \Omega_E \text{ such that:}$$

$$\text{(2.1)} \qquad \operatorname{div} \boldsymbol{\sigma}(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \Omega_E; \quad \boldsymbol{\sigma}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = \mathbf{S} \cdot \mathbf{n}(\mathbf{x}), \quad \forall \mathbf{x} \in \partial\Omega_E$$

$$\text{(2.2)} \qquad \boldsymbol{\sigma}(\mathbf{x}) \in \Psi_E(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega_E$$

$$\text{(2.3)} \qquad \mathbf{S} \in G^g_V \Big\}$$

Any R-consistent load $\mathbf{S}$ is therefore balanced by a stress field $\boldsymbol{\sigma}$, called R-consistent, which is statically admissible (cf. (2.1)), meets the yield strength conditions (cf. (2.2)), and which complies with a consistency condition (cf. (2.3)).

In the case of VOIGT type conditions, the convex set $K_V(E) \in \hat{C}$ made up of loads $\mathbf{S}$, called V-consistent, is formed. For this purpose, the set $U_E(\mathbf{D})$ $(\mathbf{D} \in \mathbb{R}^9_s)$ is determined, the elements $\mathbf{d}(\mathbf{x})$ $(\mathbf{x} \in \Omega_E)$ of which are strain rate fields deriving from displacement velocity fields $\mathbf{v}(\mathbf{x})$, $\mathbf{x} \in \Omega_E$ (cf. (2.4)) that are kinematically admissible (cf. (2.5)) and comply with a consistency condition (2.6):

$$U_E(\mathbf{D}) = \Big\{ \mathbf{d}(\mathbf{x}), \mathbf{x} \in \Omega_E \ \text{ such that } \ \exists \mathbf{v}(\mathbf{x}), \mathbf{x} \in \Omega_E \ \text{ with:}$$

$$d_{ij} = (v_{i,j} + v_{j,i})/2 \ \ \big(i \in \{1,2,3\}; \ \ j \in \{1,2,3\}\big) \tag{2.4}$$

$$\mathbf{v}(\mathbf{x}) = \mathbf{D} \centerdot \mathbf{x}, \quad \forall \mathbf{x} \in \partial \Omega_E \tag{2.5}$$

$$\langle \mathbf{d}(\mathbf{x}) \rangle_{\omega'} = \mathbf{D} \Big\} \tag{2.6}$$

where $\omega' = N_E \centerdot \omega$ is the volume occupied by all the inclusions.

The elements $\mathbf{d}$ of $U_E(\mathbf{D})$ are called V-consistent for E in the direction $\mathbf{D}$.

We define:

$$K_V\,(E) = \Big\{ \mathbf{S} \in \mathbb{R}^9_s / \forall \mathbf{D} \in \mathbb{R}^9_s,\ \mathbf{S} \centerdot \mathbf{D} \leq \min(H_E\,(\mathbf{d})\,, \mathbf{d} \in U_E\,(\mathbf{D})\,) \Big\}$$

where $H_E(\mathbf{d}) = \langle \Pi_{\Psi_E(\mathbf{x})}(\mathbf{d}(\mathbf{x})) \rangle_{\Omega_E}$ is the mean dissipated power rate in Ω_E.

The convex sets $K_R(E)$ and $K_V(E)$ are such that $K_V(E) \supset K_R(E)$, $\forall E \in (\dot{G}, \rho)$

If the consistency condition (2.3) (respectively (2.6)) were omitted, the set $K_R(E)$ (respectively $K_V(E)$) would represent the set of loads potentially able to be withstood E for the REUSS type (respectively VOIGT type) loading process (SALENÇON [1]).

But then the mean fields in the matrix and in the inclusions would only be slightly correlative: the matrix would lose its potential status of representing the homogenized material. A consistency condition on the stresses symmetrical to (2.6) can on the contrary, be envisaged. The choice made, which favours a strong strain rate correlation and a low stress correlation, presents the advantage of generality as it only takes into account the conditions necessary for subsequent mathematical developments. From this point of view, it can be qualified as minimalistic.

The sets of R-consistent loads $K_R(G, \rho)$ and V-consistent loads $K_V(G, \rho)$ are likewise formed for the base $\mathcal{B}(G, \rho)$:

$$K_R(G,\rho) = \Big\{ \mathbf{S} \in G \cap G_V^g / \forall k \in \hat{m} : \exists \boldsymbol{\sigma}_k \text{ defined on } V_k \text{ such that:}$$

$$\operatorname{div} \boldsymbol{\sigma}_k = 0 \quad \text{on } V_k; \boldsymbol{\sigma}_k(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = \mathbf{S} \cdot \mathbf{n}(\mathbf{x}), \forall x \in \partial V_k$$

$$\boldsymbol{\sigma}_k(\mathbf{x}) \in \varphi_k(\mathbf{x}) \text{ if } \mathbf{x} \in \omega_k; \boldsymbol{\sigma}_k(\mathbf{x}) \in G \text{ if } \mathbf{x} \in V_k' = V_k \backslash \omega_k \Big\}; \tag{2.7}$$

$$K_V(G,\rho) = \Big\{ \mathbf{S} \in \mathbb{R}_s^9 / \forall D \in \mathbb{R}_s^9 :$$

$$\mathbf{S} \cdot \mathbf{D} \le \max\left(\Pi_G(\mathbf{D}), \min(H_\rho(\mathbf{d}_1,...,\mathbf{d}_m), (\mathbf{d}_1,...,\mathbf{d}_m) \in U_\rho(\mathbf{D}))\right) \Big\} \tag{2.8}$$

$$H_\rho(\mathbf{d}_1,...,\mathbf{d}_m) = \sum_{k=1}^{m} w_k(\langle \Pi_{\varphi_k}(\mathbf{x})(\mathbf{d}_k(\mathbf{x}))\rangle_{\omega_k}(1-\rho) + \langle \Pi_G(\mathbf{d}_k(\mathbf{x}))\rangle_{V'_k} . \rho)$$

$$U_\rho(\mathbf{D}) = \Big\{ (\mathbf{d}_1,..,\mathbf{d},..,\mathbf{d}_m) / \forall k \in \hat{m} : \mathbf{d}_k, \text{ defined on } V_k, \text{ derived from } v_k;$$

$$\mathbf{v}_k = (\mathbf{x})\mathbf{D} \,.\, \mathbf{x}, \forall \mathbf{x} \in \partial V_k; \sum_{k=1}^{m} w_k \langle \mathbf{d}_k(\mathbf{x})\rangle_{\omega_k} = \mathbf{D} \Big\}. \tag{2.9}$$

The family of stress fields $(\boldsymbol{\sigma}_k)_{k\in\hat{m}}$ associated with $\mathbf{S} \in K_R(G,\rho)$ enables an R-consistent stress field $\boldsymbol{\sigma}$ to be constructed for any $E \in (\dot{G},\rho)$, $\boldsymbol{\sigma}$ balancing $\mathbf{S}$. The sample E in fact admits a disintegration into elements of $\mathcal{B}(G,\rho)$ on which we set down: $\boldsymbol{\sigma} = \boldsymbol{\sigma}_k$ $(k \in \hat{m})$ and into complementary elements on which we set down $\boldsymbol{\sigma} = \mathbf{S}$. It follows that $\forall E \in (\dot{G},\rho)$ $K_R(E) \supset K_R(G,\rho)$.

We proceed in the same way to describe $U_E(\mathbf{D})$ $(\forall E \in (\dot{G},\rho), \forall \mathbf{D} \in \mathbb{R}_s^9)$ by means of the family of velocity fields $(\mathbf{v}_k)_{k\in\hat{m}}$ (cf. (2.9)). It follows that $\forall E \in (\dot{G},\rho)$ $K_V(E) \subset K_V(G,\rho)$.

The sets of R-consistent and V-consistent loads can then be bounded for any sample of a given equivalence class by those of the corresponding base:

$$\forall E \in (\dot{G},\rho) \;\; K_R(G,\rho) \subset K_R(E) \subset K_V(E) \subset K_V(G,\rho).$$

2.4. Problems posed by a self-consistent definition of G_h

It is sought to define the effective yield strength domain G_h of the heterogeneous material M_{ha} as being the solution of the system of self-consistency equations:

$$\exists \;\rho \in [0,1] \,/ K_R(G,\rho) = K_V(G,\rho) = G, \tag{2.10}$$

$$\Rightarrow \exists \;\rho \in [0,1] \,/ \forall E \in (\dot{G},\rho) \quad K_R(E) = K_V(E) = G. \tag{2.11}$$

This definition is intuitively satisfactory for it means that the homogeneous matrix of any sample E is not affected by the presence of any inclusions, for any REUSS type or VOIGT type condition, in compliance with the consistency conditions. In other words, the material $M(G_h)$ is a comparison material whose yield strength capacities are in agreement with those of the inclusions representing the material M_{ha}. It could be envisaged to define G_h as the solution of one or the other of the equations of the system (2.10), but with a mechanical meaning which would be weakened. However, each of these equations admits at least one solution for each value of ρ. And ρ cannot be fixed arbitrarily. It is in this respect that conjunction of the two equations appears opportune or even necessary. Unfortunately. the existence of solutions for $\rho < 1$ cannot in fact be proved, except for particular cases (TURGEMAN and GUESSAB [19]), (it can be shown that $\forall G \in \hat{C}$ such that $K_R(G,\rho) = G$ and $\forall G' \in \hat{C}$ such that $K_V(G',\rho) = G'$ we have $G' \supset G$, which is not a good omen for the existence of a solution in any case at all). And for $\rho = 1$, this system does not admit a single solution (indeed, $K_V(G,1) = G = K_R(G,1), \forall G \subset G_V^g$).

To make the concept of comparison material operational in the context of yield strength design, we propose considering approximate equalities in (2.10).

3. Characterization of the effective yield strength domain G_h

3.1. Distance, equality and inclusion to within ε in $\hat{C}$

A reference yield strength domain $I \in \hat{C}$ is chosen, with a support function Π_I. The domain I can for example be that of the matrix in the case where M_{ha} is a composite matrix-inclusions material. It can also be taken equal to G_R^g or to G_V^g. $\hat{C}$ is assigned a distance Δ associated to I:

$$\forall (G_1, G_2) \in \hat{C} \times \hat{C} : \Delta(G_1, G_2) = \max(|t(D)| / \Pi_I(\mathbf{D}), \mathbf{D} \in \mathbb{R}_s^9)$$

where:

$$t(\mathbf{D}) = \Pi_{G_1}(\mathbf{D}) - \Pi_{G_2}(\mathbf{D})$$

with the convention:

$$\begin{aligned} t(\mathbf{D})/\Pi_I(D) &= t(\mathbf{D}) \,.\, +\infty \qquad &&\text{if} \quad t(\mathbf{D}) \neq 0 \quad \text{and} \quad \Pi_I(\mathbf{D}) = 0,\\ t(\mathbf{D})/\Pi_I(\mathbf{D}) &= 0 &&\text{if} \quad t(\mathbf{D}) = 0 \quad \text{and} \quad \Pi_I(\mathbf{D}) = 0. \end{aligned}$$

The equality relation to within $\varepsilon (=_\varepsilon)$ and inclusion relation to within $\varepsilon (\subset_\varepsilon)$ are defined in $\hat{C}$ for $\varepsilon \in \mathbb{R}^+$:

$$\forall (G_1, G_2) \in \hat{C} \times \hat{C} : G_1 =_\varepsilon G_2 \Leftrightarrow \Delta(G_1, G_2) \leq \varepsilon$$

$$G_1 \subset_\varepsilon G_2 \Leftrightarrow \forall \mathbf{D} \in \mathbb{R}_s^9 : \Pi_{G_1}(\mathbf{D}) \leq \Pi_{G_2}(\mathbf{D}) + \varepsilon \Pi_I(\mathbf{D}).$$

They have the following properties ($G_i \in \hat{C}, i = 1, 2, 3$):

$$G_1 =_\varepsilon G_1; \quad G_1 =_\varepsilon G_2 \Rightarrow G_2 =_\varepsilon G_1;$$

$$G_1 =_\varepsilon G_2 \quad \text{and} \quad G_2 =_\varepsilon G_3 \Rightarrow G_1 =_{2\varepsilon} G_3;$$

$$G_1 =_\varepsilon G_2 \Rightarrow G_1 =_{\varepsilon'} G_2, \quad \forall \varepsilon' \geq \varepsilon$$

$$G_1 \subset_\varepsilon G_1; \quad G_1 \subset_\varepsilon G_2 \quad \text{and} \quad G_2 \subset_\varepsilon G_1 \Leftrightarrow G_1 =_\varepsilon G_2;$$

$$G_1 \subset_\varepsilon G_2 \quad \text{and} \quad G_2 \subset_\varepsilon G_3 \Rightarrow G_1 \subset_{2\varepsilon} G_3;$$

$$G_1 \subset_\varepsilon G_2 \Rightarrow G_1 \subset_{\varepsilon'} G_2, \quad \forall \varepsilon' \geq \varepsilon$$

We denote:

- $G_1 + G_2$ the convex of $\hat{C}$ whose support function is the sum of those of G_1 and G_2
- $\mu \,.\, G(\mu \in \mathbb{R}^+)$ the convex of $\hat{C}$ whose support function is the product of that of $G(\in \hat{C})$ by μ.

It follows that:

$$G_1 \subset_\varepsilon G_2 \Leftrightarrow G_1 \subset G_2 + \varepsilon.I$$

3.2. ε-self-consistent equations

The Eqs. (2.10) are replaced by the following ε-self-consistent equations for $\rho \in [0, 1[$ and $\varepsilon \geq 0$:

(3.1) $$K_R(G, \rho) =_\varepsilon G, \quad G \in \hat{C} \ (\Leftrightarrow K_R(G, \rho) \supset_\varepsilon G, \quad G \in \hat{C})$$

(3.2) $$K_V(G, \rho) =_{\varepsilon(1-\rho)} G, \quad G \in \hat{C} \ (\Leftrightarrow K_V(G, \rho) \subset_{\varepsilon(1-\rho)} G, \quad G \in \hat{C})$$

The set of solutions of (3.1) contained in G_V^g is denoted RES(ρ, ε) and the set of solutions of (3.2) containing G_R^g is denoted VES(ρ, ε). It will be shown (cf. propositions 1 and 2) that these sets are non-void.

Let VRES $(\rho, \varepsilon) =$ RES$(\rho, \varepsilon) \cap$ VES (ρ, ε). We than have, for any G belonging to VRES (ρ, ε):

(3.3) $$\forall E \in (G, \rho) \qquad K_R(E) =_\varepsilon G; \qquad K_V(E) =_\varepsilon G.$$

Mechanical interpretation of the solutions of the ε-self-consistent equations results from (3.3) and is analogous to that performed in the case where $\varepsilon = 0$.

The difference between the approximate equalities used (3.1) and (3.2) finds its justification in a property established further on (cf. (3.4)). This property shows that if an equality to within ε had been considered in (3.2), this equation would be all the easier to satisfy the more ρ tends to 1. The quality of representation of the heterogeneous material attributed to the comparison material $M(G)$, for the VOIGT type loading process would therefore not be intrinsic.

3.3. ε-self-consistent characterization of G_h

The diameter $\Phi\ (\rho, \varepsilon)(\in \mathbb{R}^+)$ of VRES(ρ, ε) is defined as follows:

$$\begin{aligned}\Phi(\rho, \varepsilon) = \max\big(\Delta(G_1, G_2),\ \ \forall G_i \in \mathrm{VRES}\,(\rho, \varepsilon),\ \ i = 1, 2\big) \quad &\text{if} \quad \mathrm{VRES}\,(\rho, \varepsilon) \neq \varnothing \\ = +\infty \quad &\text{if} \quad \mathrm{VRES}\,(\rho, \varepsilon) = \varnothing\end{aligned}$$

The mechanical meaning of the elements of VRES(ρ, ε) leads to looking for G_h in one of these sets. But in which one?

A pair (ρ, ε) such that $\Phi(\rho, \varepsilon)$ is large but finite is not discriminant in the following sense: very different elements G of VRES(ρ, ε) exist that present the same sensitivity to inclusions. If $\Phi(\rho, \varepsilon) = \infty$, the pair (ρ, ε) is not discriminant in $\hat{C}$ either, as then no convex G is distinguished. The yield strength domain G_h therefore must be located in the sets VRES(ρ, ε), the diameter of which is minimum (equal to Φ^*). Among these sets of diameter Φ^* (if there are several), those whose parameters (ρ, ε) are the most suitable, are those for which ε is minimum (according to (3.3)), and among the latter (if there are several) it will be established (cf. (3.17)) that ρ can be chosen in any manner. Whence the ε-self-consistent characterization of G_h:

G_h belongs to the set VRES(ρ^*, ε^*) the diameter Φ^* of which is minimum, with:

$$\varepsilon^* = \min(\varepsilon / \Phi(\rho, \varepsilon) = \Phi^*)$$

Note: This characterization of G_h must of course be tested by experimental validation. In any case, it does not enable a single element of $\hat{C}$ to be determined, but only a probable domain. The relevance of the latter depends on the quality of representation of the material which is not quantified by Φ^*.

3.4. ε-self-consistent equations study

The object of this study is effective resolution of G_h from its ε-self-consistent characterization. Firstly the variations of the sets of solutions RES(ρ, ε) and VES(ρ, ε) are established.

Let there be $G \in \hat{C}, \rho_0, \rho_1$ such that $0 \leq \rho_0 \leq \rho_1 < 1$ and $\varepsilon_0, \varepsilon_1$ such that $0 \leq \varepsilon_0 \leq \varepsilon_1$. We have (cf. Appendix 1):

$$K_R(G, \rho_0) \subset K_R(G, \rho_1) \tag{3.4}$$

$$K_V(G, \rho_1) \subset K_V(G, \rho_0).\frac{1 - \rho_1}{1 - \rho_0} + G\,.\,\frac{\rho_1 - \rho_0}{1 - \rho_0} \tag{3.5}$$

(3.6) $$\mathrm{RES}(\rho_0, \varepsilon) \subset \mathrm{RES}(\rho_1, \varepsilon)$$

(3.7) $$\mathrm{VES}(\rho_0, \varepsilon) \subset \mathrm{VES}(\rho_1, \varepsilon)$$

(3.8) $$\mathrm{RES}(\rho, \varepsilon_0) \subset \mathrm{RES}(\rho, \varepsilon_1)$$

(3.9) $$\mathrm{VES}(\rho, \varepsilon_0) \subset \mathrm{VES}(\rho, \varepsilon_1)$$

PROPOSITION 1. The set $\mathrm{RES}(\rho, \varepsilon)$ $(\varepsilon \geq 0, \rho \in [0, 1[)$ is non-void. It admits a larger element denoted $G_R^+(\rho, \varepsilon)$ (called extremal solution of (3.1)) which has the following properties:

(3.10) $$G_R^+(\rho_1, \varepsilon) \supset G_R^+(\rho_0, \varepsilon) \quad \text{if} \quad 1 > \rho_1 \geq \rho_0 \geq 0$$

(3.11) $$G_R^+(\rho, \varepsilon_1) \supset G_R^+(\rho, \varepsilon_0) \quad \text{if} \quad \varepsilon_1 \geq \varepsilon_0 \geq 0$$

(3.12) $$G_R^+(\rho, \varepsilon) \supset G_R^g$$

P r o o f. Let us assume that $\mathrm{RES}(\rho, \varepsilon)$ is non-void and that it contains two elements G_1 and G_2. It is shown that $G = G_1 \cup^C G_2$ is also contained in $\mathrm{RES}\,(\rho, \varepsilon)$ (where $\cup^C$ designates the convex shell of the union of two convex sets). For this purpose we compare $K_R(G, \rho)$ with $K = K_R(G_1, \rho) \cup^C K_R(G_2, \rho)$.

For all $\mathbf{S} \in K$, $\exists \mathbf{S}_i \in K_R(G_i, \rho)$ (i = 1, 2) and $\exists \lambda \in [0, 1]$ such that $\mathbf{S} = \lambda \mathbf{S}_1 + (1 - \lambda)\mathbf{S}_2$. For $i = 1, 2$, $\mathbf{S}_i$ is balanced by an R-consistent field $\boldsymbol{\sigma}_i$ for $\mathcal{B}(G_i, \rho)$. The field $\boldsymbol{\sigma} = \lambda \boldsymbol{\sigma}_1 + (1 - \lambda)\boldsymbol{\sigma}_2$ then balances $\mathbf{S}$ and it is R-consistent for $\mathcal{B}(G, \rho)$. Therefore:

(3.13) $$K \subset K_R(G, \rho) \ \ (\subset G)$$

(3.14) $$\Rightarrow \Delta(G, K_R(G, \rho)) \leq \Delta(G, K).$$

We have also:

(3.15) $$K + \varepsilon \centerdot I = (K_R(G_1, \rho) + \varepsilon \centerdot I) \cup^C (K_R(G_2, \rho) + \varepsilon \centerdot I) \supset G_1 \cup^C G_2$$

Therefore:

$$\Delta(G, K) \leq \varepsilon \ \ (\text{according to}, (3.13) \ \text{and} \ (3.15))$$

$$\Rightarrow \Delta(G, K_R(G, \rho)) \leq \varepsilon \ \ (\text{according to } (3.14)).$$

Consequently $G \in \mathrm{RES}\,(\rho, \varepsilon)$, which shows that $\mathrm{RES}\,(\rho, \varepsilon)$ admits a larger element noted $G_R^+(\rho, \varepsilon)$.

We have: $\forall G \in \hat{C} : K_R(G, \rho) \subset G_V^g$ (according to the consistency assumption (2.3)). Therefore if G satisfies (3.1) with $\varepsilon = 0$, we necessarily have $G \subset G_V^g$.

The series of convex sets (G_i) $(i \in \mathrm{IN})$ is considered with: $G_0 = G_V^g$; $G_{i+1} = K_R(G_i, \rho)$.

This is a decreasing series of nested convex sets. It admits a bound $G^* = K_R(G^*, \rho)$ which satisfies (3.1) with $\varepsilon = 0$ and which is in G_V^g. Therefore $\mathrm{RES}\,(\rho, 0)$ is non-void, and consequently $\mathrm{RES}\,(\rho, \varepsilon)$ is non-void.

Let i_0 be the smallest integer such that $\Delta(G_{i_0+1}, G_{i_0}) \le \varepsilon$. The convex G_{i_0} is the largest element of $\mathrm{RES}\,(\rho, \varepsilon)$ arising from the series $(G_i)(i \in \mathrm{IN})$. It is included in $G_R^+(\rho, \varepsilon)$ but we cannot assert that it is equal to $G_R^+(\rho, \varepsilon)$ except in the case $\varepsilon = 0$. This is the best approximation of $G_R^+(\rho, \varepsilon)$ which can be highlighted using this iterative process. Another possibility consists in solving (3.1) by fixing the form of its solutions, which is not in general obvious.

The property (3.10) results from (3.6) and the property (3.11) from (3.8); the property (3.12) is obtained by proving that $G_R^g \in \mathrm{RES}\,(\rho, 0)$, therefore to $\mathrm{RES}\,(\rho, \varepsilon)(\forall \varepsilon \ge 0)$.

PROPOSITION 2. The set $\mathrm{VES}\,(\rho, \varepsilon)$ $(\varepsilon \ge 0, \rho \in [0, 1[)$ is non-void. It admits a smaller element denoted $G_V^-(\rho, \varepsilon)$ (called extremal solution of (3.2)) which has the following properties:

$$G_V^-(\rho_0, \varepsilon) \supset G_V^-(\rho_1, \varepsilon) \quad \text{if} \;\; 1 > \rho_1 \ge \rho_0 \ge 0$$
$$G_V^-(\rho, \varepsilon_0) \supset G_V^-(\rho, \varepsilon_1) \quad \text{if} \;\; \varepsilon_1 \ge \varepsilon_0 \ge 0 \tag{3.16}$$
$$G_V^-(\rho, \varepsilon) \subset G_V^g$$

P r o o f. The proof is analogous to that of Proposition 1:

$\mathrm{VES}\,(\rho, \varepsilon)$ being assumed to be non-void, if it contains G_1 and G_2, then it contains $G_1 \cap G_2$ and consequently admits a smaller element.

$\forall G \in \hat{C}, K_V(G, \rho) \supset G \cup^C ((1-\rho)G_R^g + \rho G)$ due to the consistency assumption (2.6). It follows that any solution of (3.2) with $\varepsilon = 0$ contains G_R^g. This enables the series $(\tilde{G}_i)(i \in \mathrm{IN})$ to be considered with: $\tilde{G}_0 = G_R^g; \tilde{(}G_{i_0+1} = K_V(\tilde{G}_i, \rho)$, which is an increasing series of nested convex sets in $\hat{C}$. It admits a bound $\tilde{G}^* \in \mathrm{VES}\,(\rho, 0)$, and therefore to $\mathrm{VES}\,(\rho, \varepsilon)$ which is consequently non-void. Let i_0 be the smallest integer such that $\Delta(\tilde{(}G_{i_0+1}, \tilde{G}_{i_0}) \le \varepsilon(1-\rho)$. The convex $\tilde{G}_{i_0}$ is the smallest element of $\mathrm{VES}\,(\rho, \varepsilon)$ arising from the series $(\tilde{G}_i)(i \in \mathrm{IN})$. It contains $G_V^-(\rho, \varepsilon)$, but we cannot assert that it is equal to it, except in the case $\varepsilon = 0$.

PROPOSITION 3. The set $\mathrm{VRES}\,(\rho, \varepsilon)$ has the following properties:

$$\mathrm{VRES}\,(\rho_1, \varepsilon) \supset \mathrm{VRES}\,(\rho_0, \varepsilon) \qquad \text{if} \;\; 1 > \rho_1 \ge \rho_0 \ge 0 \;\; \text{and} \;\; \varepsilon \ge 0 \tag{3.17}$$

$$\mathrm{VRES}\,(\rho, \varepsilon_1) \supset VRES\,(\rho, \varepsilon_0) \quad \text{if} \;\; \varepsilon_1 \ge \varepsilon_0 \ge 0 \;\; \text{and} \;\; \rho \in [0, 1[\tag{3.18}$$

$$\forall \rho \in [0, 1[\,, \exists \varepsilon_\rho \ge 0 \quad \text{such that} \quad \forall \varepsilon \ge \varepsilon_\rho \mathrm{VRES}(\rho, \varepsilon) \ne \varnothing \tag{3.19}$$

P r o o f. We assume $\mathrm{VRES}\,(\rho_0, \varepsilon) \ne \varnothing$. Then the variations of $\mathrm{RES}\,(\rho, \varepsilon)$ and $\mathrm{VES}\,(\rho, \varepsilon)$ relatively to ρ and to ε imply (3.17) and (3.18).

We have:

$$G_R^g \in \mathrm{RES}\,(0,0)$$

$$K_V\left(G_R^g,\rho\right) \subset G_R^g \cup^C \left(G_V^g(1-\rho)+\rho G_R^g\right) \quad \text{(cf. (2.8) with}$$
$$\mathbf{v}_k(\mathbf{x}) = \mathbf{D}\,\boldsymbol{.}\,\mathbf{x} \;\text{ on } V_k\backslash\omega_k, \forall k \in \hat{m})$$

whence: $\left(K_V(G_R^g,0) \subset G_V^g \subset_{\varepsilon_0} G_R^g \quad \text{for} \quad \varepsilon_0 = \Delta(G_R^g,G_V^g)\right) \Rightarrow G_R^g \in \mathrm{VES}\,(0,\varepsilon_0)$

Therefore VRES $(0,\varepsilon_0)$ is non-void, whence (3.19).

3.5. Prediction of bounds of G_h

For any $(\rho,\varepsilon) \in [0,1[\times\mathbb{R}^+$, it results from Propositions 1 and 2 that:

$$G \in \mathrm{VRES}\,(\rho,\varepsilon) \Rightarrow G_V^-(\rho,\varepsilon) \subset G \subset G_R^+(\rho,\varepsilon) \tag{3.20}$$

Consequently, if VRES (ρ,ε) is non-void:

$$\Phi(\rho,\varepsilon) \leq \Delta(G_R^+(\rho,\varepsilon),G_V^-(\rho,\varepsilon))$$

We then set down:

$$\tilde{\Phi}(\rho,\varepsilon) = \Delta\left(G_R^+(\rho,\varepsilon),G_V-(\rho,\varepsilon)\right) \quad \text{if} \quad G_R^+(\rho,\varepsilon) \supset G_V^-(\rho,\varepsilon)$$
$$= +\infty \quad \text{if not}$$

On account of the properties (3.10) and (3.16), if the function $\tilde{\Phi}(\rho,\varepsilon)$ is finite in (ρ_0,ε) $(\rho_0 \in [0,1[)$ then it is finite for any (ρ,ε) with $\rho \in [\rho_0,1[$.

The function $\tilde{\Phi}(\rho,\varepsilon)$ constitutes an approximation of $\Phi(\rho,\varepsilon)$. We cannot assert that it is equal to it as the reciprocal of (3.20) is not established. We can show that this reciprocal is true if $G_V^-(\rho,\varepsilon)$ and $G_R^+(\rho,\varepsilon)$ are connected by a homothetic transformation of centre O and if ε is not too large $(\varepsilon\,\boldsymbol{.}\,I \subset G_R^g)$. This condition is sufficient but not necessary as proved by the case dealt with in application (cf. Sec. 4).

However the function $\tilde{\Phi}$ is obtained by solving the two equations of the ε-self-consistency system (3.1), (3.2) separately. This feature makes it much more accessible than the function Φ and incites the following heuristic property to be considered: $\tilde{\Phi}$ and Φ are minimal at the same point (ρ^*,ε^*). It follows that:

$$G_V^-(\rho^*,\varepsilon^*) \subset G_h \subset G_R^+(\rho^*,\varepsilon^*) \tag{3.21}$$

with:

$$\tilde{\Phi}^* = \min\left(\tilde{\Phi}(\rho,\varepsilon), \rho \in [0,1[, \varepsilon \geq 0\right)$$

$$\varepsilon^* = \min\left(\varepsilon/\tilde{\Phi}(\rho,\varepsilon) = \tilde{\Phi}^*\right) \tag{3.22}$$

If the problem (3.22) admits several solutions $(\varepsilon^*,\rho \in \hat{\rho}^*)$ with $\hat{\rho}^*$ a sub-set of [0,1[, the convex sets $G_V^-(\rho,\varepsilon^*)$ and $G_R^+(\rho,\varepsilon^*)$ are necessarily constant on $\hat{\rho}^*$. This justifies being able to choose ρ in any manner in $\hat{\rho}^*$.

4. Application: effective yield strength domain of a defect material

A medium provided with a privileged frame Oxy (of length unit u) is considered. This medium is composed of a heterogeneous material M_{ha} of a yield strength domain G_0, except on segments (called defect segments), located randomly, of identical length equal to 4u and parallel to Oy. The number of defect segments per area unit in the plane Oxy is 1/36. On these defect segments the yield strength domain is G_{def}. In the privileged frame Oxy, the expressions of the domains G_0 and G_{def} are as follows:

$$G_0 = g(\sigma_0, \sigma_0, \sigma_0)$$
$$G_{\text{def}} = g(0, 0, 0)$$

with $g(\sigma, \sigma', \sigma'') = \left\{ \boldsymbol{\sigma} \in \mathbb{R}_S^4 / |\sigma_x| \le \sigma; |\sigma_{xy}| \le \sigma'; |\sigma_y| \le \sigma'' \right\}$

For the yield strength domain g, the dissipated power rate in a strain **D** of components d_x, d_{xy}, d_y is equal to:

$$\Pi_g(\mathbf{D}) = \sigma|d_x| + 2\sigma'|d_{xy}| + \sigma''|d_y|$$

In a discontinuity of velocity $[\mathbf{v}] = \mathbf{v}_1 - \mathbf{v}_2$ (of components $[v_n], [v_t]$ in the frame $M\mathbf{nt}$) Fig. 6, the dissipated power rate is equal to:

$$\Pi_g([\mathbf{v}]) = \sigma \left| [v_n] \cos^2 \alpha - 0.5[v_t] \sin 2\alpha \right| + \sigma' \left| [v_n] \sin 2\alpha + [v_t] \cos 2\alpha \right| + \sigma'' \left| [v_n] \sin^2 \alpha + 0.5[v_t] \sin 2\alpha \right|$$

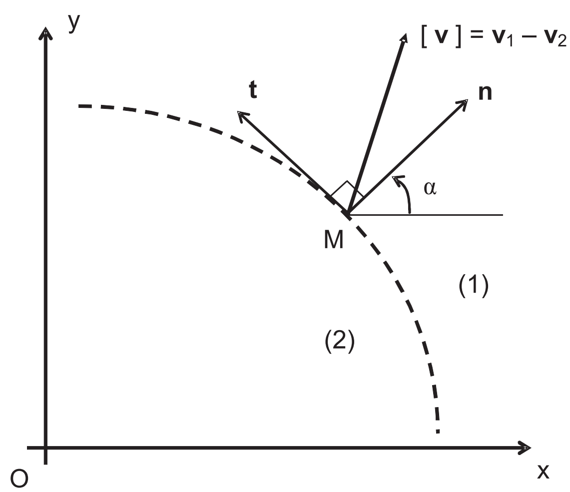

FIG. 6. Discontinuity of displacement velocity [**v**].

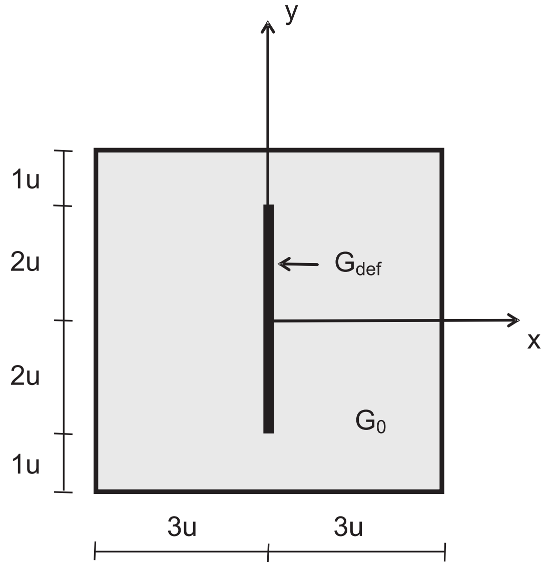

FIG. 7. Microstructure μs characterizing the defect material.

The heterogeneous material is characterized by a single microstructure μs occupying a volume ω, the plot of which in Oxy is a square with sides equal to 6u (see Fig. 7).

We obtain: $G_R^g = \sigma_0 \centerdot g\left(\frac{1}{3}, \frac{1}{3}, 1\right)$ and $G_V^g = \sigma_0 \centerdot g\left(\frac{29}{36}, \frac{29}{36}, 1\right)$.

The reference yield strength domain I is chosen equal to G_0.

The homogenization method is applied analytically by making the following simplifications: the comparison materials considered have yield strength domains of fixed form (that of g), as well as the convex sets $K_R(G,\rho)$ and $K_V(G,\rho)$. The convex sets $G_R^+(\rho,\varepsilon)$ and $G_V^-(\rho,\varepsilon)$ can then be obtained directly without using the iterative procedure (cf. Propositions 1 and 2).

The convex $K_R(G,\rho)$ is determined using the dual form of (2.7):

$$K_R(g,\rho) = \Big\{\mathbf{S} \in \mathbb{R}_s^4 / \mathbf{S} \centerdot \mathbf{D} \leq \min(\Pi_G(\mathbf{D}), \Pi_{G_V^g}(\mathbf{D}), \langle \Pi_{\mathbf{x}}(\mathbf{d})\rangle_V, \langle \mathbf{d}\rangle_V = \mathbf{D}), \quad \forall \mathbf{D} \in \mathcal{D}\Big\}$$

where:

- $\mathcal{D}$ is the set constituted by the effective strain rates D_i $(i = 1,2,3)$ necessary and sufficient to determine $K_R(G,\rho)$ (the non-zero components are d_{xx} for D_1; d_{xy} for D_2; d_{yy} for D_3);
- $\Pi_{\mathbf{x}}$ is the dissipated power rate at the point $\mathbf{x} \in V$.

When strain rate fields d are derived from discontinuous velocity fields v or when the microstructures include some voids, the average strain rate is written:

$$\langle \mathbf{d} \rangle_V = \frac{1}{2V} \int_{\partial V} (v_i \centerdot n_j + v_j \centerdot n_i)$$

Thus the convex sets $G_R^+(\rho, \varepsilon)$ and $G_V^-(\rho, \varepsilon)$ are obtained by applying the kinematic yield design method. We obtain:

$$G_R^+(\rho, \varepsilon) = \sigma_0 g(f_1(\rho, \varepsilon), f_1(\rho, \varepsilon), 1)$$

$$G_V^-(\rho, \varepsilon) = \sigma_0 g\left(\max\left(f_2(\rho, \varepsilon); \frac{1}{3} \right), \max\left(\frac{29}{36} - \varepsilon; \frac{1}{3} \right), 1 \right)$$

with

$$f_1\,(\rho, \varepsilon) = \min\left(\frac{1}{3} + \frac{\varepsilon}{\sqrt{1-\rho}}, \frac{29}{36} \right)$$

$$f_2(\rho, \varepsilon) = \min\left(0.73012 - 1.42122\,\varepsilon, \frac{1}{3\left(1 - \sqrt{1-\rho}\right)} - \varepsilon \right)$$

The velocity fields used are described in Appendix 2.

It is interesting to underline the fact that in this application the reciprocal of (3.20) is exact, under the assumptions concerning the form of the convex sets $K_R(G, \rho)$ and $K_V(G, \rho)$.

The function $\tilde{\Phi}(\rho, \varepsilon)$ reaches its minimum at a single point $(\rho^*, \varepsilon^*) = (0.735, 0.161)$, which leads to:

$$\sigma_0 g(0.502, 0.645, 1) \subset G_h \subset \sigma_0 g(0.645, 0.645, 1) \tag{4.1}$$

The sufficiently simple context of this application enables an approximation of the effective yield strength domain G_h to be determined by performing a structure calculation on a representative volume Ω of the heterogeneous material (see Fig. 8). This calculation also uses the kinematic yield design method. It is performed on 500 square samples comprising 25.600 defect segments distributed with the help of a digit programme that sets at random the coordinates of the medium point. The boundary conditions correspond to the load processes of VOIGT type (BC1) and REUSS type (BC2):

For the conditions BC1, Ω is cut into strips with widths of 6u parallel to Oy. In each of them, a discontinuity line is sought for dividing the strip into two rigid blocks and that is the least dissipative possible.

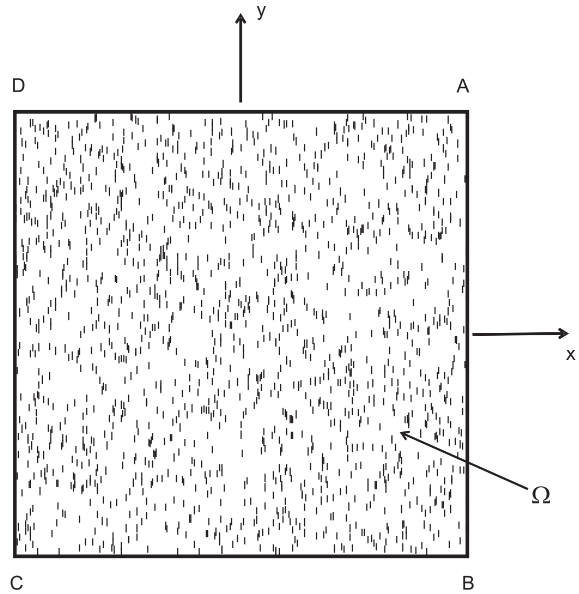

Fig. 8. A square sample composed of the defect material (comprising 1600 segments).

For the conditions BC2, the whole volume Ω is scanned to determine a single discontinuity line separating Ω into two rigid blocks. The mean results obtained on the 500 samples (with a maximum standard deviation of 0.0079) show that:

$$G_h \subset \sigma_0 g(0.751, 0.751, 1) \quad \text{for the conditions BC1} \tag{4.2}$$

$$G_h \subset \sigma_0 g(0.673, 0.673, 1) \quad \text{for the conditions BC2} \tag{4.3}$$

The so obtained upper bounds of G_h (cf. (4.2) and (4.3)) depend on the choice of the boundary conditions BC1 and BC2. This dependence doesn't mean that the Ω volume is not representative. More probably it results from the quality of the velocity fields involved in the structure calculation. If we agree with the fact that Ω volume is representative for the heterogeneous material, it is consistent to get the smaller of the two upper bounds as the best approach of G_h. Then we note a good agreement between the theoretical estimation (4.1) and the approach value of G_h (4.3).

5. Conclusion

Using a comparison material to determine the effective yield strength domain of a random material, gives rise to numerous problems when its behaviour is directly nonlinear. The first difficulty involves the adequate distance to be imposed between the inclusions representative of the heterogeneous material that are placed into the homogeneous comparison medium. This distance must enable the drastic REUSS or VOIGT conditions to be pertinently attenuated at the border of the inclusions which are, apart from the periodicity assumption, the only ones whose mechanical meaning can be established, provided that Ω_{ha} admits a breakdown into elements of volume ω. Combined use of the two types of boundary conditions referred to (REUSS and VOIGT), useful for mechanical interpretation of the comparison material (cf. (2.11) and (3.3)), appeared moreover to be able to fix this distance non-arbitrarily. However, except for particular cases, the self-consistency equations associated with each of these types of boundary conditions do not admit a common solution for a finite distance and admit an infinity of solutions for infinity.

Different methods, based on a disturbance of the equations, have been envisaged to overcome this difficulty. But the parameters that define these disturbances do not have any obvious physical sense and are consequently delicate to fix. Recourse to the notion of approximate equality proved from this point of view to be easier to master. It requires a characterization of the effective yield strength domain which fixes the value of the two parameters ρ and ε appearing in the self-consistency system. The one that is proposed in this work is based on the idea that the pair (ρ, ε) must be the one that is the most discriminant. However, this characterization would be complex to implement, and even out of reach, if it could not be reduced to separate resolution of the self-consistency equations. This simplification, necessary from a practical point of view, is not established in a general manner. It gives the developed method a heuristic nature, which can be cleared in certain cases. Another possibility of fixing the pair (ρ, ε) would be to use the property of the effective yield strength domain shown in TURGEMAN and GUESSAB [19] for $\varepsilon = 0$ and which can be generalized for $\varepsilon > 0$. But this property, which can be interpreted as being dual from (3.21), is only valid if ρ is smaller than a value, a function of ε, only an upper bound of which can be determined.

Another problem concerns the consistency conditions to be imposed on the stress and strain rate fields. We made the choice of minimum conditions basing ourselves on the requirements of mathematical developments. This choice presents the advantage of generality. Moreover, it enables a velocity formulation to be used for determining the set of R-consistent loads, which greatly facilitates analytical calculation.

The problem of validation of the proposed method is still outstanding. It was approached by dealing with the example of a defect material, studied directly on finite volumes which are representative on account of the large number of defect segments considered.

Appendix A.

Proofs of properties (3.4) to (3.9):

To simplify the notations, we place ourselves in the case where the material M_{ha} is represented by a single microstructure ($m = 1$). In this case the base $\mathcal{B}(G,\rho)$ comprises a single element: this is the mechanical system B, of volume V_i when $\rho = \rho_i$ ($i \in${0,1}), in which the microstructure representative of $M_{ha}$of volume ω is centred. The complementary volume $V_i' = V_i \backslash \omega$ is constituted by the homogeneous material $M(G)$.

(3.4): let there be $\mathbf{S} \in K_R(G,\rho_0)$. $\mathbf{S}$ is balanced by a field $\boldsymbol{\sigma}_0$, defined on V_0 and R-consistent. We consider in V_1 the field $\boldsymbol{\sigma}_1$ such that: $\boldsymbol{\sigma}_1(\mathbf{x}) = \boldsymbol{\sigma}_0(\mathbf{x}), \forall \mathbf{x} \in V_0$; $\boldsymbol{\sigma}_1(\mathbf{x}) = S, \forall \mathbf{x} \in V_1 \backslash V_0$. The field σ_1 is R-consistent for $B \in \mathcal{B}(G,\rho_1)$ and it balances $\mathbf{S}$. Consequently $\mathbf{S} \in K_R(G,\rho_1)$.

(3.5): the support function of $K_V(G,\rho_i)$ ($i \in${0,1}) is, for $\mathbf{D} \in \mathbb{R}_s^9$:

$$\mathit{\Pi}_{K_V(G,\rho_i)}(\mathbf{D}) = \max\Big(\mathit{\Pi}_G(\mathbf{D}), \min(H_{\rho_i}(\mathbf{D}),\ \ \mathbf{d} \in U_\rho(\mathbf{d}))\Big).$$

Let $U'_{\rho_1}(\mathbf{D}) = \big\{\mathbf{d}(\mathbf{x}), \mathbf{x} \in V_1$ such that on $V_0 : \mathbf{d}(\mathbf{x}) \in U_{\rho_0}(\mathbf{D})$ and on $V_1 \backslash V_0 : \mathbf{d}(\mathbf{x}) = \mathbf{D}\big\}$

We have $U_{\rho_1}(\mathbf{D}) \supset U'_{\rho_1}(\mathbf{D})$ and subsequently:

$$\min(H_{\rho_1}(\mathbf{d}), \mathbf{d} \in U_{\rho_1}(\mathbf{D})) \leq \min(H_{\rho_1}(\mathbf{d}), \mathbf{d} \in U'_{\rho_1}(\mathbf{D}))$$
$$= (H_{\rho_0}(\hat{\mathbf{d}}) \,.\, V_0 + \mathit{\Pi}_G(\mathbf{D}) \,.\, (V_1 \backslash V_0))\,/V_0))V_1\ V_1$$

with

$$(\hat{\mathbf{d}} \in U_{\rho_0}(\mathbf{D}) \quad \text{such that} \quad H_{\rho_0}(\hat{\mathbf{d}}) = \min(H_{\rho_0}(\mathbf{d}), \mathbf{d} \in U_{\rho_0}(\mathbf{D}).$$

Then:

$$\mathit{\Pi}_{K_V(G,\rho_1)}(\mathbf{D}) \leq \max\left(\mathit{\Pi}_G(\mathbf{D}), H_{\rho_0}(\hat{\mathbf{d}}) \,.\, \frac{V_0}{V_1} + \mathit{\Pi}_G(\mathbf{D}) \,.\, \left(1 - \frac{V_0}{V_1}\right)\right)$$

Whence (3.5), noting that $\omega = (1-\rho_1)V_1 = (1-\rho_0)V_0$.

$$\begin{aligned}(3.6) : G \in \text{RES}(\rho_0,\varepsilon) &\Rightarrow K_R(G,\rho_0) + \varepsilon \,.\, I \supset G \\ &\Rightarrow K_R(G,\rho_1) + \varepsilon \,.\, I \supset G \quad \text{(according to (3.4))} \\ &\Rightarrow G \in \text{RES}(\rho_1,\varepsilon)\end{aligned}$$

$$\begin{aligned}(3.7) : G \in \text{VES}(\rho_0,\varepsilon) &\Rightarrow K_V(G,\rho_0) \subset G + \varepsilon(1-\rho_0) \,.\, I \\ &\Rightarrow K_V(G,\rho_1) \subset G + \varepsilon(1-\rho_1) \,.\, I \quad \text{(according to (3.5))} \\ &\Rightarrow G \in \text{VES}(\rho_1,\varepsilon)\end{aligned}$$

The properties (3.8) and (3.9) are obvious.

Appendix B.

Determination of $K_R(g,\rho)$ is based on the velocity field Fig. 9, consisting of two rigid regions, separated by a velocity discontinuity line AD. For the effective strain $\mathbf{D}_1$ we have: $\mathbf{v}_0 = (1,0)$; for $\mathbf{D}_2$ we have: $\mathbf{v}_0 = (0,1)$. In the direction $\mathbf{D}_3$, the velocity field $\mathbf{v}(x,y) = (0,y)$ in V is used.

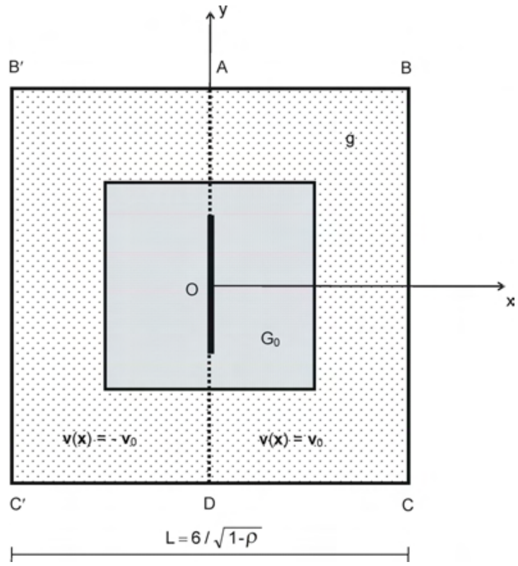

FIG. 9. Velocity field to determine $K_R(g,\rho)$.

We deduce therefrom:

$$K_R(g,\rho) = \Big\{\mathbf{S} \in \mathbb{R}^4_s / |S_x| \le \min(\sigma, a\sigma_0 + b\sigma, c\sigma_0);$$

$$|\mathbf{S}_{xy}| \le \min(\sigma', a\sigma_0 + b\sigma', c\sigma_0);$$

$$|\mathbf{S}_y| \le \min(\sigma'', (1-\rho)\sigma_0 + \rho\sigma'', \sigma_0)\Big\}$$

with $a = \sqrt{1-\rho}/3$; $\quad b = 1-\sqrt{1-\rho}$; $\quad c = 29/36$

Determination of $K_V(g,\rho)$ uses:

in the direction $\mathbf{D}_1$:

- the velocity field Fig. 6 with $\mathbf{v}_0 = (2.741379, 0)$; $\mathbf{v}(x,y) = (x,0)$ in V'.
- the velocity field Fig. 7 within the regions Z_i $(i = 1,2)$:

$$v_x(x,y) = x(|y|-3)/e + 3(-1)^i(e-|y|+3)/e;$$

$$v_y(x,y) = (-x^2/2e + 3(-1)^i x/e - 9/2e) \cdot |y|/y;$$

$$\mathbf{v}_0 = (3,0);\ \ \mathbf{v}(x,y) = (x,0) \text{ in } V'.$$

in the direction $\mathbf{D}_2$:

- the velocity field Fig. 10 with $\mathbf{v}_0 = (0, 2.5)$; $\mathbf{v}(x, y) = (0, x)$ in V'.

in the direction $\mathbf{D}_3$:$\mathbf{v}\,(x, y) = (0, y)$ in V.

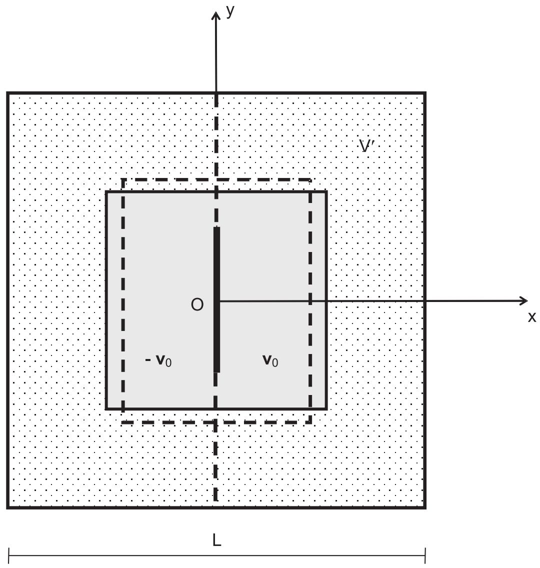

Fig. 10. Velocity field to determine $K_V(g, \rho)$ in the directions $\mathbf{D}_1$ and $\mathbf{D}_2$.

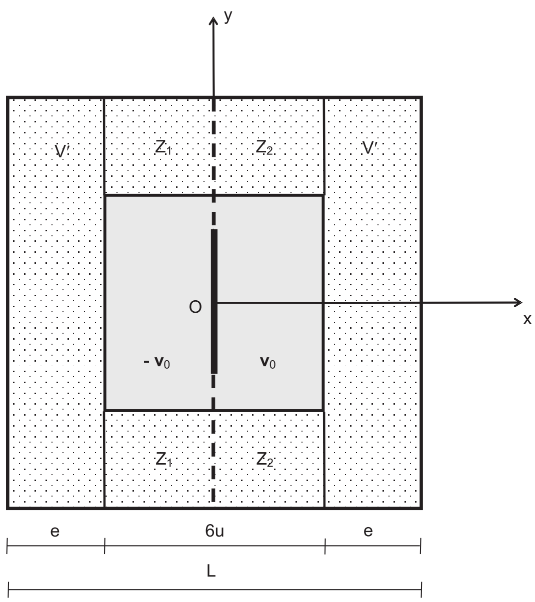

Fig. 11. Velocity field to determine $K_V(g, \rho)$ in the direction $\mathbf{D}_1$.

We deduce therefrom:

$$K_v(g,\rho) = \Big\{ \mathbf{S} \in \mathbb{R}^4_s / |\mathbf{S}_x| \leq \max(\sigma, \min(h_1(\rho), h_2(\rho)));$$
$$|\mathbf{S}_{xy}| \leq \max(\sigma', h_3(\rho));$$
$$|\mathbf{S}_y| \leq \max(\sigma'', (1-\rho)\sigma_0 + \rho\sigma'') \Big\}$$

with:

$$h_1(\rho) = 0.390804(1-\rho)\sigma_0 + \rho\sigma + 0.421222(1-\rho)\sigma';$$
$$h_2(\rho) = \frac{1}{3}(1-\rho)\sigma_0 + \left(\sqrt{1-\rho}/3(1-\sqrt{1-\rho})\right)(1-\rho)\sigma'';$$
$$h_3(\rho) = \frac{16}{36}(1-\rho)\sigma_0 + \sigma'\rho + \frac{13}{36}(1-\rho)\sigma''$$

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