Arch. Mech., **59**, 2, pp. 173–197, Warszawa 2007

Cauchy problem for quasilinear hyperbolic systems with coefficients functionally dependent on solutions

M. ZDANOWICZ[1)], Z. PERADZYŃSKI[2)]

[1)] *Institute of Mathematics*
University of Białystok
Akademicka 2, 15-267 Białystok, Poland

[2)] *Institute of Applied Mathematics and Mechanics*
Warsaw University
Banacha 2, 02–097 Warsaw, Poland

The Cauchy problem for a quasilinear hyperbolic systems with coefficients functionally dependent on the solutions is studied.We assume that the coefficients are continuous nonlinear operators in the Banach space $C^1(\mathbb{R})$ satisfying some additional assumptions. Under these assumptions we prove the uniqueness and existence of local in time C^1 solutions, provided that the initial data are also of class C^1.

Key words: functional dependence on solutions, characteristic curves, Cauchy problem, hyperbolic system.

1. Introduction

In this paper we discuss the case of a quasilinear hyperbolic system with coefficients functionally dependent on the solution

$$u_t + A[u]u_x = S[u], \tag{1.1}$$

$$u(0,x) = u^0(x), \quad x \in \mathbb{R}. \tag{1.2}$$

The problem studied in this paper comes from modeling plasma in the stationary plasma thruster [6]. In this type of thrusters plasma moves across a radial magnetic field in a cylindrically symmetric channel. The magnetic field is strong enough to magnetize the electrons but not the ions ($m_e/m_i \approx 10^{-5}$). Thus the ions are moving practically along the axis of the device in agreement with the electric field, whereas the electrons in average have azimuthal as well as slow axial motion. The majority of electrons are thus stuck orbiting in the region of high radial magnetic field near the thruster exit plane, while the ions are accelerated and produce the thrust. In addition, the electrons ionize the neutral atoms which are injected through the anode.

One of the easiest ways of modeling the plasma is to treat it as a three-component fluid consisting of neutral atoms with density N_a and velocity V_a, electrons with density n_e, axial velocity V_e and azimuthal velocity V_θ and ions with density n_i and velocity V_i.

The ion component could be treated as a cold fluid with zero temperature, while electrons are relatively hot. Hence apart from the continuity and momentum equations the energy equation that defines the temperature of electrons, is also needed. The set of these three equations describing the motion of these three fluids must be supplemented by the Poisson equation for the electric field. It appears that this system is difficult to solve numerically because of various physical and numerical instabilities and the existence of different time scales. However several essential approximations are possible. First of all we can make the assumption of quasineutrality saying that the density of electrons is equal to density of ions, $n_e = n_i$. This assumption is often used in the description of plasma and it is well justified in our case. In this approximation the Poisson equation is redundant and must be dropped out. Then taking into account that in the case of xenon the electron mass m_e is much less than the ion mass $m_e \approx 10^{-5} m_i$, we can neglect the time and space derivatives in the electron momentum equations to obtain the Ohm's law type of equation, expressing the electric field through the other variables [1]:

$$\frac{e}{m_i}E = -\nu V_e - \frac{k}{nm_i}\frac{\partial(T_e n)}{\partial x},$$

where ν – effective collision frequency, T_e – temperature of electrons, n – ion density (we assume that ion density and electron density are equal), k – the Boltzmann constant.

Using the expression for the total current density $I = n(V_i - V_e)$ we have

$$\frac{e}{m_i}E = \nu\left(\frac{I}{n} - V_i\right) - \frac{k}{nm_i}\frac{\partial(T_e n)}{\partial x}. \tag{1.3}$$

After these simplifications one obtains the following system of four equations:

- neutral continuity equation

$$\frac{\partial N_a}{\partial t} + V_a\frac{\partial N_a}{\partial x} = -\beta N_a n, \tag{1.4}$$

- ion continuity and axial momentum equations

$$\frac{\partial n}{\partial t} + \frac{\partial (V_i n)}{\partial x} = \beta N_a n, \tag{1.5}$$

$$\frac{\partial V_i}{\partial t} + V_i\frac{\partial V_i}{\partial x} + \frac{k}{nm_i}\frac{\partial(T_e n)}{\partial x} = \nu\left(\frac{I}{n} - V_i\right) - \beta N_a(V_i - V_a), \tag{1.6}$$

- temperature equation for T_e

$$\frac{1}{\sqrt{T_e}}\left(\frac{\partial T_e^{3/2}}{\partial t}+\frac{\partial(T_e^{3/2}V_e)}{\partial x}\right)=Q(T_e,n,V_i,x). \tag{1.7}$$

Here β is the ionization coefficient, Q – the source term in the energy equation containing gains (Joule heating) and losses due to collisions with walls and atoms.

The boundary condition for the electric field $\int_0^L E dx = U_0$, saying that the applied voltage is equal to U_0, allows to obtain (from "Ohm's law") the total current, which depends on time but not on x

$$I(t)=\left(\int_0^L \frac{\nu}{n}dx\right)^{-1}\left(\frac{m_i}{e}U_0+\int_0^L\left[\nu V_i+\frac{k}{nm_i}\frac{\partial(T_e n)}{\partial x}\right]dx\right). \tag{1.8}$$

System (1.4)–(1.8) is hyperbolic. The right-hand side of this system depends functionally on the solution, because I in Eq. (1.6) is expressed by (1.8).

Near the anode the electron velocity V_a is rather large. This introduces a small time scale that is inconvenient for numerical computation. In such a case it is reasonable to neglect the time derivative in Eq. (1.7).

When we omit the time derivative in the equation for temperature, then temperature is expressed functionally by n, N_a and V_i. Finally we have a hyperbolic system consisting of three equations (for n, V_i, N_a), whose coefficients depend functionally on the solution.

After such a reduction our system can serve as an example which characterizes the situation often encountered in computational physics. Suppose that we have the following hyperbolic system of $n+1$ equations:

$$\begin{aligned} u_t + A(t,x,u,v)u_x &= b(t,x,u,v),\\ v_t + f(t,x,u,v)v_x &= g(t,x,u,v). \end{aligned} \tag{1.9}$$

The characteristic velocities in this system are defined by the eigenvalues of the matrix of this system. So the characteristic times are proportional to the inverse of these eigenvalues. Therefore if for example $|f|$ is much larger than the absolute value of remaining eigenvalues (i.e. the eigenvalues of A), then for numerical computations it is reasonable to neglect the time derivative in the last equation. In this way we come to a system with functional dependence not only on the right-hand side but also on the characteristics.

In the following we will be concerned with the existence and uniqueness theorem of a local in time solution of the initial problem for quasilinear hyperbolic system (1.1) with two independent variables (t,x) and coefficients functionally dependent on the solution. We show also that the solution depends continuously (in C^1 topology) on the initial data. In [5] the existence of Lipschitz continuous solutions is proved, but the continuous dependence on the initial data is shown in the supremum norm, not in the Lipschitz norm.

The proof of existence is based on the method of successive approximations and basically follows the reasoning used in [4]. Hence we need the results for the linear systems. In the linear case

$$u_t + A(t,x)u_x = b(t,x) + B(t,x)u, \tag{1.10}$$

$$u(0,x) = u^0(x), \quad x \in \mathbb{R}, \tag{1.11}$$

due to the definition of hyperbolicity, the $n \times n$ matrix A has real eigenvalues and n independent eigenvectors at each (t,x). Thus A can be represented as

$$A = L^{-1}DL,$$

where L is a nonsingular $n \times n$ matrix. The rows $L_1, \ldots, L_n$ are left eigenvectors of A, whereas the columns of L^{-1} are the right eigenvectors of A. $D = \operatorname{diag}[\xi_1, \ldots, \xi_n]$ and functions ξ_i, $i = 1, \ldots, n$, are eigenvalues of A corresponding to eigenvectors L_i. We do not assume here and in the following that ξ_i are different. $B(t,x)$ is $n \times n$ matrix and $b(t,x)$, $u(t,x)$ are column vectors.

For problem (1.10)–(1.11) we recall the following theorem [4]:

Theorem 1. *Assume that $D(t,x)$, $L(t,x)$, $b(t,x)$, $B(t,x)$ are bounded matrix functions of class $C^k([0,\infty) \times \mathbb{R})$ with bounded derivatives up to order k on $[0,\infty) \times \mathbb{R}$. If the initial condition $u^0(x)$ is of class $C^k(\mathbb{R})$ with bounded derivatives up to order k on $\mathbb{R}$, then there exists a unique global solution $u \in C^k([0,\infty) \times \mathbb{R})$. The solution depends continuously on the initial condition (1.2) in the C^k topology on every finite strip $[0,T] \times \mathbb{R}$.*

Let

$$X_0 = \left\{ u \in C(\mathbb{R}); \|u\|_0 := \sup_{x \in \mathbb{R}} \sqrt{\sum_{i=1}^{n} u_i^2} < \infty \right\}.$$

Similarly

$$X_1 = \{ u \in C^1(\mathbb{R}); \|u\|_1 := \|u\|_0 + \|u_x\|_0 < \infty \}.$$

In contrast to the results shown in [3, 5], we confine ourself to the functional dependence with respect to the variable x only. Thus in $A[u]$, $S[u]$, etc., u is treated as a function of x, parametrically dependent on t. Similarly we admit that the operators A, S, etc., are parametrically depending on t. To simplify the notation, dependence on t will not be marked explicitly. In this way the linear case (1.10) is a special case of (1.1). We assume also that for a given u from a ball $B_r^1(u^0)$ with radius r, centered at u^0 and open in X_1, the matrix $A[u]$ ($t \in [0,T]$) has real eigenvalues $\xi_1[u], \dots, \xi_n[u]$ and can be diagonalized

$$A[u] = L^{-1}[u]D[u]L[u],$$

$$D[u] = \operatorname{diag}[\xi_1[u], \dots, \xi_n[u]], \quad L[u] = \begin{bmatrix} L_1[u] \\ \vdots \\ L_n[u] \end{bmatrix}.$$

The rows of the nonsingular matrix $L[u]$ are the left eigenvectors of $A[u]$.

Multiplying (1.1) on the left by $L[u]$, we obtain the characteristic form of equations

$$L[u]\, u_t + D[u]\, L[u]\, u_x = Z[u], \tag{1.12}$$

where $Z[u] = L[u]\, S[u]$.

Let $u^0(x)$ be an initial condition (1.2) for the system (1.1). We will assume that there exists a ball $B_r^1(u^0)$ in X_1 such that for all $t \in [0,T]$, the following conditions hold:

(A_1) $K : B_r^1(u^0) \to X_1$ and for some constant $C < \infty$: $\|K[v]\|_1 \le C$ for all $v \in B_r^1(u^0)$, where K denotes L, L^{-1}, D, Z.

(A_2) L is a continuous nonlinear operator, $L : B_r^1(u^0) \to X_1$. In addition, we assume that L is Fréchet differentiable as an operator acting from X_0 to X_0, i.e. $L : B_r^0(u^0) \to X_0$, where $B_r^0(u^0)$ is the closure of $B_r^1(u^0)$ in the metric of the space X_0 and

$$\exists_{C>0} \forall_{v \in B_r^0(u^0) \subset X_0} \forall_{h \in X_0} \|L'(v)h\|_0 \le C\|h\|_0.$$

(A_3) $L[v]$ is of C^1 class with respect to the parameter t and there is a constant C such that $\left\| \dfrac{\partial}{\partial t} L[v] \right\|_0 \le C$, $v \in B_r^0(u^0)$.

(A_4) For $|x - \bar{x}| \le \delta$ and all $v \in B_r^1(u^0)$, there is a constant C and a function $N(\delta)$, $N(\delta) \to 0$ as $\delta \to 0$ such that

$$\left| \frac{\partial}{\partial x} K[v](t,x) - \frac{\partial}{\partial x} K[v](t,\bar{x}) \right| \le C|v_x(x) - v_x(\bar{x})| + C|v(x) - v(\bar{x})| + N(\delta),$$

where K stands for L, D, Z. $|\cdot|$ denotes the Euclidean metric.

(A_5) There exists a constant C that $\|K[v] - K[\bar{v}]\|_0 \le C\|v - \bar{v}\|_0$ for $v \in B_r^1(u^0)$, where K stands for L, L^{-1}, D, Z.

Although the constants appearing in all these assumptions can be different, we can take C as the greatest one.

As an example of the operator which satisfies assumptions (A_1)–(A_5) can serve the operator

$$K[v](t,x) = k\left(t, x, v, \int\limits_{\mathbb{R}} g_1(x,y)v(y)dy, \int\limits_0^x g_2(x,y)v(y)dy\right), \tag{1.13}$$

where $v \in X_1$, k is a continuously differentiable function with respect to all variables. Moreover we assume, that the integrals $\int_{\mathbb{R}} g_1(x,y)v(t,y)dy$, $\int_0^x g_2(x,y)$ $v(y)dy$ exist and are continuously differentiable with respect to x and derivatives are uniformly bounded as a function of x.

The particular case of system of partial differential-functional equations (1.1) with operators of the type (1.13) is a quasilinear hyperbolic system.

Let us now formulate the theorem which is our main result.

Theorem 2. *Under the conditions (A_1)–(A_5), there exists a local in time unique solution of class C^1 of the problem (1.1)–(1.2).*

2. Prolonged system

Let us define the prolongation of system (1.1) which will help us to estimate the growth of solution of system (1.12) as well as its derivatives.
We introduce the new unknown vector function p by

$$p(t,x) = L[u(t,\cdot)]\, u_x, \tag{2.1}$$

and useful denotation for function $v \in X_1$ independent of t

$$L_t[v] = \frac{\partial}{\partial t} L[v]. \tag{2.2}$$

Thus if $u = u(t,x)$ then

$$L_t[u] = \left.\frac{\partial}{\partial t} L[v]\right|_{v=u}. \tag{2.3}$$

The Frechét derivative of $L[u]$ acting on ω will be denoted by

$$L'[u]\omega := L'(u;\omega), \qquad u \in X_1,\ \omega \in X_0. \tag{2.4}$$

Now we formally differentiate all equations of (1.12) with respect to x and we obtain

$$\left(\frac{\partial}{\partial x} L[u]\right) u_t + L[u]u_{tx} + \left(\frac{\partial}{\partial x} D[u]\right) L[u]u_x + D[u]\frac{\partial}{\partial x}(L[u]u_x) = \frac{\partial}{\partial x} Z[u]. \tag{2.5}$$

For the derivative u_{tx} we have

$$L[u]\, u_{tx} = \frac{\partial}{\partial t}\,(L[u]\, u_x) - \left(\frac{\partial}{\partial t} L[u]\right) u_x,$$

where by assumption (A_2) and by (2.3) we can develop $\dfrac{\partial}{\partial t} L[u]$ as follows:

$$\frac{\partial}{\partial t} L[u] = L'(u; u_t) + L_t[u]. \tag{2.6}$$

Finally, expressing u_t and u_x from (1.12) and (2.1) in terms of p we obtain the prolonged system:

$$u_t = L^{-1}[u] Z[u] - L^{-1}[u] D[u] p, \tag{2.7}$$

$$\begin{aligned} \frac{\partial p}{\partial t} + D[u]\,\frac{\partial p}{\partial x} &= \frac{\partial}{\partial x} Z[u] - \left(\frac{\partial}{\partial x} L[u]\right) L^{-1}[u] Z[u] \\ &+ \left(\left(\frac{\partial}{\partial x} L[u]\right) L^{-1}[u] D[u] + L'\Big(u; L^{-1}[u] Z[u] - L^{-1}[u] D[u] p\Big) L^{-1}[u] \right. \\ &\left. + L_t[u] L^{-1}[u] - \frac{\partial}{\partial x} D[u]\right) p, \end{aligned} \tag{2.8}$$

$$u(0,x) = u^0(x), \tag{2.9}$$

$$p(0,x) = p^0(x) = L[u^0]\, u^0_x. \tag{2.10}$$

Definition 1. We define a linear mapping

$$\mathcal{P} : (C([0,T] \times \mathbb{R}))^n \longmapsto ((C([0,T] \times [0,T] \times \mathbb{R}))^n,$$

acting on vector functions $f(t,x) = [f_1(t,x), \ldots, f_n(t,x)]^T$ by

$$(\mathcal{P} f)_k(t, \bar{t}, \bar{x}) = f_k(t, x_k(t; \bar{t}, \bar{x})), \qquad k = 1, \ldots, n. \tag{2.11}$$

Thus $\mathcal{P}$ acts in this way that in the k-th component f_k of vector f it substitutes for x the expression of the k-th family of characteristics $x_k(t; \bar{t}, \bar{x})$.

Since

$$\sup_{(t,\bar{t},\bar{x}) \in [0,T] \times [0,T] \times \mathbb{R}} |f_k(t, x_k(t; \bar{t}, \bar{x}))| = \sup_{(t,x) \in [0,T] \times \mathbb{R}} |f_k(t,x)|\,,$$

then $\mathcal{P}$ is bounded and hence continuous. For convenience we will use the following denotation:

$$\mathcal{P}_t f = (\mathcal{P} f)(t, \cdot, \cdot).$$

Let us notice that the left-hand side of (2.7) is the directional derivative along the characteristic curves:

$$\frac{d}{dt}(\mathcal{P}_t p) = \mathcal{P}_t f, \tag{2.12}$$

where

$$f = \frac{\partial}{\partial x}Z[u] - \left(\frac{\partial}{\partial x}L[u]\right)L^{-1}[u]Z[u] + \left(\left(\frac{\partial}{\partial x}L[u]\right)L^{-1}[u]D[u]\right.$$
$$\left. + L'\Big(u; L^{-1}[u]Z[u] - L^{-1}[u]D[u]p\Big)L^{-1}[u] + L_t[u]L^{-1}[u] - \frac{\partial}{\partial x}D[u]\right)p.$$

Integrating (2.12) along characteristics with respect to t from 0 to $\bar{t}$ we obtain

$$p(\bar{t},\bar{x}) = \mathcal{P}_0 p + \int\limits_0^{\bar{t}} (\mathcal{P}_t f)\,dt. \tag{2.13}$$

To derive Eqs. (2.7)–(2.8) we need, in principle, to assume that $u(t,x) \in C^2$. However, the integral form (2.13) permits us to look for weaker solutions which are only continuous, although p_k is differentiable along the k-th characteristics $(k = 1,\ldots,n)$.

It is worth pointing out that system (2.7)–(2.8) is expressed in Riemann invariants, i.e. it has a diagonal form, whereas (1.1), in general, is not.

Let (u,p) belong to the space $X_0 \times X_0$ with norm

$$\|(u,p)\|_* := (C^3+1)\|u\|_0 + C\|p\|_0.$$

If (u,p) is in a ball $B^*_r(u^0,p^0)$ centered at (u^0,p^0) and open in $X_0 \times X_0$, then the function u stays in the ball $B^1_r(u^0)$. From assumptions (A_2) and (A_5) we get

$$\|u_x - u^0_x\|_0 = \|L^{-1}[u]\,p - L^{-1}[u^0]\,p^0\|_0 \le C\|p-p^0\|_0 + C\|u-u^0\|\|p^0\|_0.$$

Since

$$\|p^0\|_0 = \|L[u^0]u^0_x\|_0 \le C^2,$$

we have

$$\|u_x - u^0_x\|_0 + \|u-u^0\|_0 \le (C^3+1)\|u-u^0\|_0 + C\|p-p^0\|_0 < r.$$

Now we will show that if there exists a solution $(u(t,x),p(t,x))$ of Eqs. (2.7)–(2.8) (p in the sense of Eq. (2.13)) then it must stay in $B^*_r(u^0,p^0)$ for some finite time $t \in [0,t_*)$, where t_* is defined by (2.18).

The following estimations hold in the ball $B_r^*(u^0, p^0)$:

$$(2.14)\qquad \begin{aligned} |u_t(t,x)| &\le \|L^{-1}[u]\|_0 \, \|Z[u]\|_0 + \|L^{-1}[u]\|_0 \, \|D[u]\|_0 \, \|p\|_0 \\ &\le C^2(1 + \|p\|_0). \end{aligned}$$

Since

$$\|p\|_0 \le \|p - p^0\|_0 + \|p^0\|_0 \le \frac{r}{C} + C^2,$$

we have

$$(2.15)\qquad |u_t(t,x)| \le C_u,$$

where $C_u = C^2 + C^4 + Cr$.

Since p_k is differentiable along the k-th family of characteristics, then by (2.8) we can write:

$$(2.16)\qquad \begin{aligned} \left|\frac{d}{dt}(\mathcal{P}_t p)\right| \le \left\|\frac{\partial}{\partial x} Z[u]\right\|_0 + \left\|\frac{\partial}{\partial x} L[u]\right\|_0 \left\|L^{-1}[u]\right\|_0 \, \|Z[u]\|_0 \\ + \left(\left\|\frac{\partial}{\partial x} L[u]\right\|_0 \|L^{-1}[u]\|_0 \|D[u]\|_0\right. \\ + C \left\|L^{-1}[u]Z[u] - L^{-1}[u]D[u]p\right\|_0 \|L^{-1}[u]\|_0 \\ \left. + \|L_t[u]\|_0 \|L^{-1}[u]\|_0 + \left\|\frac{\partial}{\partial x} D[u]\right\|_0 \right) \|p\|_0 \\ \le C + C^3 + \|p\|_0 (C + C^2 + C^3 + C^4) + C^4 (\|p\|_0)^2. \end{aligned}$$

Hence

$$(2.17)\qquad \left|\frac{d}{dt}(\mathcal{P}_t p)\right| \le C_p,$$

where $C_p = C + 2C^3 + C^4 + C^5 + C^6 + C^8 + r(1 + C + C^2 + C^3 + 2C^5) + C^2 r^2$.
As for any function $\varphi(t) \in C^1$:

$$\frac{d}{dt}|\varphi(t)| \le \left|\frac{d}{dt}\varphi(t)\right|,$$

then we obtain from (2.15), (2.17) the conditions:

$$\begin{aligned} \frac{\partial}{\partial t}|u(t,x) - u^0(x)| &\le C_p, \\ \frac{\partial}{\partial t}|\mathcal{P}_t p - \mathcal{P}_0 p| &\le C_u, \end{aligned}$$

which imply

$$|u(t,x) - u^0(x)| \leq \bar{t}\, C_p,$$
$$|\mathcal{P}_t p - \mathcal{P}_0 p| \leq \bar{t}\, C_u.$$

Because C_u, C_p are constants independent of x therefore we have

$$\|(u,p) - (u^0,p^0)\|_* = (C^3+1)\|u-u^0\|_0 + C\|p-p^0\|_0$$
$$\leq \bar{t}\left(C_u(C^3+1) + C_p\, C\right) =: \bar{t}\, C_*.$$

If

$$t_* = \min\left\{\frac{r}{C_*}, T\right\}, \tag{2.18}$$

then we see that the solution must indeed stay in $B_r^*(u^0,p^0)$ (hence it is bounded) for $t \in [0,t_*)$.

3. Characteristics

The characteristic curve $x = x_k(t;\bar{t},\bar{x})$ of the k-th family passing through the point $(\bar{t},\bar{x})$ is the solution of the equation

$$\frac{dx}{dt} = \xi_k[u](t,x), \qquad t \in [0,\bar{t}\,], \tag{3.1}$$

with the following initial condition:

$$x_k(t;\bar{t},\bar{x})|_{t=\bar{t}} = \bar{x}. \tag{3.2}$$

For $u \in B_r^1(u^0)$ the function $\xi_k[u](t,x)$ is bounded and has a bounded derivative with respect to x. Hence it satisfies the Lipschitz condition with respect to x and therefore initial problem (3.1)–(3.2) has a unique solution. Through each point $(\bar{t},\bar{x}) \in [0,t_*) \times \mathbb{R}$ there passes one and only one characteristic of the k-th family, which is defined for $t \in [0,t_*)$.

4. Uniqueness

We shall show the following

Lemma 1. *If there exists a solution of the Cauchy problem (1.1)–(1.2), then it is unique.*

P r o o f. Assume that $u(t,x)$ and $\bar{u}(t,x)$ are two different solutions of problem (1.1)–(1.2) and moreover

$$u(0,x) = \bar{u}(0,x) = u^0(x).$$

For abbreviation we will write

$$\bar{L} = L[\bar{u}], \qquad \bar{D} = D[\bar{u}], \qquad \bar{Z} = Z[\bar{u}].$$

We form the difference

$$v(t,x) = u(t,x) - \bar{u}(t,x), \qquad v(0,x) = [0,\dots,0]^T, \tag{4.1}$$

for which it holds

$$\bar{L}v_t + \bar{D}\bar{L}v_x = Z - \bar{Z} - (L-\bar{L})u_t - (DL - \bar{D}\bar{L})u_x. \tag{4.2}$$

The form (4.2) of the system suggests introducing a new unknown function

$$\bar{v} = \bar{L}\,v.$$

By observing that

$$\begin{aligned}
\frac{\partial \bar{v}}{\partial t} &= \left(\frac{\partial}{\partial t}\bar{L}\right) v + \bar{L}\,\frac{\partial v}{\partial t} = \Bigg(L_t[\bar{u}] + L'(\bar{u};\bar{u}_t)\Bigg) v + \bar{L}\,\frac{\partial v}{\partial t} \\
&= \Bigg(L_t[\bar{u}] + L'(\bar{u};\bar{L}^{-1}\,\bar{Z} - \bar{L}^{-1}\,\bar{D}\,\bar{p})\Bigg) v + \bar{L}\,\frac{\partial v}{\partial t}, \\
\frac{\partial \bar{v}}{\partial x} &= \left(\frac{\partial}{\partial x}\bar{L}\right) v + \bar{L}\,\frac{\partial v}{\partial x},
\end{aligned}$$

we may write the system (4.2) in the Riemann invariants

$$\begin{aligned}
\frac{\partial \bar{v}}{\partial t} + \bar{D}\,\frac{\partial \bar{v}}{\partial x} &= Z - \bar{Z} - (L-\bar{L})\,u_t - (D\,L - \bar{D}\,\bar{L})\,u_x \\
&\quad + \left(L_t[\bar{u}] + L'(\bar{u};\bar{L}^{-1}\,\bar{Z} - \bar{L}^{-1}\,\bar{D}\,\bar{p}) + \bar{D}\,\frac{\partial}{\partial x}\bar{L}\right)\,\bar{L}^{-1}\,\bar{v}.
\end{aligned} \tag{4.3}$$

Hence we have

$$\begin{aligned}
\frac{d}{d\,t}(\mathcal{P}_t\bar{v}) &= \mathcal{P}_t Z - \mathcal{P}_t\bar{Z} - \mathcal{P}_t\Big(\Big(L-\bar{L}\Big)u_t\Big) - \mathcal{P}_t\Bigg(\Big(DL - \bar{D}\bar{L}\Big)u_x\Bigg) \\
&\quad + \mathcal{P}_t\left(\left(L_t[\bar{u}] + L'(\bar{u};\bar{L}^{-1}\,\bar{Z} - \bar{L}^{-1}\,\bar{D}\,\bar{p}) + \bar{D}\frac{\partial}{\partial x}\bar{L}\right)\,\bar{L}^{-1}\bar{v}\right).
\end{aligned} \tag{4.4}$$

After integrating from 0 to $\bar{t}$ we obtain

$$
\begin{aligned}
|\bar{v}(\bar{t},\bar{x})| \;\le\; & \int_0^{\bar{t}} \|Z[u]-Z[\bar{u}]\|_0 \, dt + \int_0^{\bar{t}} \|L[u]-L[\bar{u}]\|_0 \, \|u_t\|_0 \, dt \\
& + \int_0^{\bar{t}} \|D[u]\,L[u]-D[\bar{u}]\,L[\bar{u}]\|_0 \, \|u_x\|_0 \, dt \\
& + \int_0^{\bar{t}} \|L_t[\bar{u}]\|_0 \, \left\|L^{-1}[\bar{u}]\right\|_0 \, \|\bar{v}\|_0 \, dt \\
& + \int_0^{\bar{t}} \left\|L'\left(\bar{u};\bar{L}^{-1}\,\bar{Z}-\bar{L}^{-1}\,\bar{D}\,\bar{p}\right)\right\|_0 \, \left\|L^{-1}[\bar{u}]\right\|_0 \, \|\bar{v}\|_0 \, dt \\
& + \int_0^{\bar{t}} \|D[\bar{u}]\|_0 \, \left\|\frac{\partial}{\partial x}L[\bar{u}]\right\|_0 \, \left\|L^{-1}[\bar{u}]\right\|_0 \, \|\bar{v}\|_0 \, dt.
\end{aligned}
$$

Now we easily arrive at the following estimations:

- $\|u_t(t,x)\|_0 \le \|L^{-1}[u]\|_0\|Z[u]\|_0 + \|L^{-1}[u]\|_0\|D[u]\|_0 \le 2C^2,$
- $\|D[u]\,L[u]-D[\bar{u}]\,L[\bar{u}]\|_0 \le C\|L[u]-L[\bar{u}]\|_0 + C\|D[u]-D[\bar{u}]\|_0 \le 2C^2\|u-\bar{u}\|_0,$
- $\|u_x(t,x)\|_0 = \left\|L^{-1}[u]\,p\right\|_0 \le C\,\|p\|_0 \le r + C^3,$
- $\|L_t[u]\|_0 \le C,$
- $\|L'(\bar{u};\bar{u}_t)\|_0 = \left\|L'\left(\bar{u};L^{-1}[u]\,Z[u]-L^{-1}[u]\,D[u]\,p\right)\right\|_0$
  $\le C\left\|L^{-1}[u]\,Z[u]-L^{-1}[u]\,D[u]\,p\right\|_0 \le C^3+C^5+C^2r,$
- $\|D[\bar{u}]\|_0 \left\|\dfrac{\partial}{\partial x}L[\bar{u}]\right\|_0 \le C^2,$
- $\|L^{-1}[u]\|_0 \le C.$

As a consequence of these inequalities we can write

(4.5)
$$\begin{aligned}|\bar v_k(\bar t,\bar x)| &\le C_1\int_0^{\bar t}\|u(t,x)-\bar u(t,x)\|_0\,dt+C_2\int_0^{\bar t}\|\bar v(t,x)\|_0\,dt\\ &\le C_1\int_0^{\bar t}\|L^{-1}[u](t,x)\|_0\,\|\bar v(t,x)\|_0\,dt\\ &\quad +C_2\int_0^{\bar t}\|\bar v(t,x)\|_0\,dt\\ &\le C_3\int_0^{\bar t}\|\bar v(t,x)\|_0\,dt,\end{aligned}$$

where $C_1 = C+2C^3+2C^5+2C^2r$, $C_2 = C^2+C^3+C^4+C^6+C^3r$, $C_3 = C_1C+C_2$.

Finally we obtain

(4.6)
$$\|\bar v(\bar t,x)\|_0 \le \sqrt{n}\,C_3\int_0^{\bar t}\|\bar v(t,x)\|_0 dt.$$

Applying the Gronwall's lemma we conclude that $\|\bar v(t,x)\|_0 \equiv 0$, i.e. $\bar u(t,x) \equiv u(t,x)$, which completes the proof.

5. Existence

LEMMA 2. *There exists a solution of (1.1)–(1.2) and it is of class* $C^1([0,t_*)\times\mathbb{R})$.

To prove the existence of the solution of (1.1)–(1.2) we use the method of successive approximations.

We start with an arbitrary admissible initial function $\overset{(0)}{u}(t,x)$, e.g.

(5.1)
$$\overset{(0)}{u}(t,x) = u^0(x).$$

To shorten the notation we will write $\overset{(s)}{L}$ instead of $L[\overset{(s)}{u}]$ and the same for the other operators.

Assume that the approximation $\overset{(s)}{u}(t,x) \in C^1$ has been constructed. The next approximation $\overset{(s+1)}{u}(t,x)$ we define as the solution of the linear Cauchy problem

(5.2)
$$\overset{(s)}{L}\,\overset{(s+1)}{u_t} + \overset{(s)}{D}\,\overset{(s)}{L}\,\overset{(s+1)}{u_x} = \overset{(s)}{Z},$$

(5.3)
$$\overset{(s+1)}{u}(0,x) = u^0(x).$$

The existence theorem for linear systems asserts the existence of a solution of class C^1 if the coefficients and the initial condition are of class C^1. Therefore for $t \in [0, t_*)$ there exists a solution $\overset{(s+1)}{u}(t,x) \in B^1_r(u^0)$.

5.1. Successive approximations for prolonged system

Denoting

$$\overset{(0)}{p} = L[u^0]\,\frac{du^0(x)}{dx}, \qquad \overset{(s+1)}{p} = \overset{(s)}{L}\,\overset{(s+1)}{u_x}, \tag{5.4}$$

we consider the linear system

$$\overset{(s+1)}{u_t} = \overset{(s)}{L^{-1}}\overset{(s)}{Z} - \overset{(s)}{L^{-1}}\overset{(s)}{D}\overset{(s)}{p}, \tag{5.5}$$

$$\begin{aligned} \overset{(s+1)}{p_t} + \overset{(s)}{D}\overset{(s+1)}{p_x} = \frac{\partial}{\partial x}\overset{(s)}{Z} - \left(\frac{\partial}{\partial x}\overset{(s)}{L}\right)\overset{(s)}{L^{-1}}\overset{(s)}{Z} + \left(\frac{\partial}{\partial x}\overset{(s)}{L}\right)\overset{(s)}{L^{-1}}\overset{(s)}{D}\overset{(s)}{p} \\ + L'(\overset{(s)}{u};\overset{(s)}{L^{-1}}\overset{(s)}{Z} - \overset{(s)}{L^{-1}}\overset{(s)}{D}\overset{(s)}{p})\,\overset{(s)}{L^{-1}}\overset{(s+1)}{p} \\ + \left(L_t[\overset{(s)}{u}]\,\overset{(s)}{L^{-1}} - \frac{\partial}{\partial x}\overset{(s)}{D}\right)\overset{(s+1)}{p}, \end{aligned} \tag{5.6}$$

$$\overset{(s+1)}{u}(0,x) = \overset{(0)}{u}, \tag{5.7}$$

$$\overset{(s+1)}{p}(0,x) = \overset{(0)}{p}. \tag{5.8}$$

Using induction we will demonstrate that for any $s = 0, 1, \ldots$ the solution of (5.5)–(5.8) exists and it is defined for each $t \in [0, t_*)$ and moreover, $(\overset{(s)}{u}, \overset{(s)}{p})$ stays in $B^*_r(u^0, p^0)$.

Assume that for $t \in [0, t_*)$

$$(\overset{(s)}{u}, \overset{(s)}{p}) \in B^*_r(u^0, p^0).$$

If so, then the same estimates as those in Sec. 2 are true for $(\overset{(s+1)}{u}, \overset{(s+1)}{p})$. Therefore $(\overset{(s+1)}{u}, \overset{(s+1)}{p})$ is defined for $t \in [0, t_*)$ and stays in $B^*_r(u^0, p^0)$.

Since $(\overset{(0)}{u}, \overset{(0)}{p}) \in B^*_r(u^0, p^0)$ for any time, then $(\overset{(s)}{u}, \overset{(s)}{p})_{s=0,1,\ldots} \in B^*_r(u^0, p^0)$ for $t \in [0, t_*)$.

5.2. Uniform convergence of $\{\overset{(s)}{u}\}$

We shall show the following:

Lemma 3. *The sequence $\{\overset{(s)}{u}\}$ is convergent in a Banach space $C([0,t_*)\times\mathbb{R})$.*

P r o o f. We define the new unknown vector function

$$\overset{(s+1)}{r}(t,x) = \overset{(s)}{L}\left(\overset{(s+1)}{u} - \overset{(s)}{u}\right), \qquad s = 0,1,\dots$$

with the initial condition

$$\overset{(s+1)}{r}(0,x) = [0,\dots,0]^T .$$

It is easy to verify that derivatives of $\overset{(s+1)}{r}$ are given by

$$\begin{aligned}\frac{\partial}{\partial t}\overset{(s+1)}{r} &= \left(L_t[\overset{(s)}{u}] + L'(\overset{(s)}{u};\overset{(s)}{u}_t)\right)\left(\overset{(s+1)}{u} - \overset{(s)}{u}\right) + \overset{(s)}{L}\left(\overset{(s+1)}{u_t} - \overset{(s)}{u_t}\right)\\ &= \left(L_t[\overset{(s)}{u}] + L'\left(\overset{(s)}{u};\overset{(s)}{L}{}^{-1}\overset{(s)}{Z} - \overset{(s)}{L}{}^{-1}\overset{(s)}{D}\overset{(s)}{p}\right)\right)\left(\overset{(s+1)}{u} - \overset{(s)}{u}\right)\\ &\qquad + \overset{(s)}{L}\left(\overset{(s+1)}{u_t} - \overset{(s)}{u_t}\right),\end{aligned}$$

$$\frac{\partial}{\partial x}\overset{(s+1)}{r} = \left(\frac{\partial}{\partial x}\overset{(s)}{L}\right)\left(\overset{(s+1)}{u} - \overset{(s)}{u}\right) + \overset{(s)}{L}\left(\overset{(s+1)}{u_x} - \overset{(s)}{u_x}\right).$$

From (5.2) we obtain the system involving $\overset{(s+1)}{r}$

$$\begin{aligned}\frac{\partial}{\partial t}\overset{(s+1)}{r} + \overset{(s)}{D}\frac{\partial}{\partial x}\overset{(s+1)}{r} &= \overset{(s)}{Z} - \overset{(s-1)}{Z}\\ &\quad + L_t[\overset{(s)}{u}]\,\overset{(s)}{L}{}^{-1}\overset{(s+1)}{r}\\ &\quad + L'\left(\overset{(s)}{u};\overset{(s)}{L}{}^{-1}\overset{(s)}{Z} - \overset{(s)}{L}{}^{-1}\overset{(s)}{D}\overset{(s)}{p}\right)\overset{(s)}{L}{}^{-1}\overset{(s+1)}{r}\\ &\quad + \overset{(s)}{D}\left(\frac{\partial}{\partial x}\overset{(s)}{L}\right)\overset{(s)}{L}{}^{-1}\overset{(s+1)}{r}\\ &\quad - \left(\overset{(s)}{L} - \overset{(s-1)}{L}\right)\overset{(s)}{u_t}\\ &\quad - \left(\overset{(s)}{D}\overset{(s)}{L} - \overset{(s-1)}{D}\overset{(s-1)}{L}\right)\overset{(s)}{u_x}.\end{aligned} \tag{5.9}$$

We now proceed to reduce the problem of solving the last system with the initial conditions $\overset{(s+1)}{r}(0,x) = [0,\dots,0]^T$ to that of solving an integral system. Along the characteristic curves we have

$$\begin{aligned}
\frac{d}{d\,t}\left(\mathcal{P}_t \overset{(s+1)}{r}\right) &= \mathcal{P}_t \overset{(s)}{Z} - \mathcal{P}_t \overset{(s-1)}{Z} \\
&+ \mathcal{P}_t \left(L_t[\overset{(s)}{u}]\, \overset{(s)}{L}{}^{-1} \overset{(s+1)}{r} \right) \\
&+ \mathcal{P}_t \left(L'\big(\overset{(s)}{u}; \overset{(s)}{L}{}^{-1} \overset{(s)}{Z} - \overset{(s)}{L}{}^{-1} \overset{(s)}{D} \overset{(s)}{p} \big)\, \overset{(s)}{L}{}^{-1} \overset{(s+1)}{r} \right) \\
&+ \mathcal{P}_t \left(\overset{(s)}{D} \left(\frac{\partial}{\partial x} \overset{(s)}{L} \right) \overset{(s)}{L}{}^{-1} \overset{(s+1)}{r} \right) \\
&- \mathcal{P}_t \left(\left(\overset{(s)}{L} - \overset{(s-1)}{L} \right) \overset{(s)}{u}_t \right) \\
&- \mathcal{P}_t \left(\left(\overset{(s)}{D} \overset{(s)}{L} - \overset{(s-1)}{D} \overset{(s-1)}{L} \right) \overset{(s)}{u}_x \right).
\end{aligned}$$

Integrating each of these equations along the corresponding characteristics with respect to t from 0 to $\bar{t}$, we obtain

$$\begin{aligned}
\overset{(s+1)}{r}(\bar{t},\bar{x}) &= \int_0^{\bar{t}} \left(\mathcal{P}_t \overset{(s)}{Z} - \mathcal{P}_t \overset{(s-1)}{Z} \right) dt \\
&+ \int_0^{\bar{t}} \mathcal{P}_t \left(L_t[\overset{(s)}{u}]\; \overset{(s)}{L}{}^{-1} \overset{(s+1)}{r} \right) dt \\
&+ \int_0^{\bar{t}} \mathcal{P}_t \left(L'\big(\overset{(s)}{u}; \overset{(s)}{L}{}^{-1} \overset{(s)}{Z} - \overset{(s)}{L}{}^{-1} \overset{(s)}{D} \overset{(s)}{p} \big)\, \overset{(s)}{L}{}^{-1} \overset{(s+1)}{r} \right) dt \\
&+ \int_0^{\bar{t}} \mathcal{P}_t \left(\overset{(s)}{D} \left(\frac{\partial}{\partial x} \overset{(s)}{L} \right) \overset{(s)}{L}{}^{-1} \overset{(s+1)}{r} \right) dt \\
&- \int_0^{\bar{t}} \mathcal{P}_t \left(\left(\overset{(s)}{L} - \overset{(s-1)}{L} \right) \overset{(s)}{u}_t \right) dt \\
&- \int_0^{\bar{t}} \mathcal{P}_t \left(\left(\overset{(s)}{D} \overset{(s)}{L} - \overset{(s-1)}{D} \overset{(s-1)}{L} \right) \overset{(s)}{u}_x \right) dt.
\end{aligned}$$

We are able to estimate $\overset{(s+1)}{r}$ following similarly to (4.6):

$$\overset{(s+1)}{r} \le C_r \int_0^{\bar t} \| \overset{(s)}{r} \|_0 dt + C_r \int_0^{\bar t} \| \overset{(s+1)}{r} \|_0 dt, \tag{5.10}$$

for positive constants C_r.

We define the next quantity

$$\overset{(i)}{Q}(\bar t) = \max_{t \in [0, \bar t]} \| \overset{(i)}{r}(t,x) t \|_0 .$$

Using it, we rewrite (5.10) in the form

$$\overset{(s+1)}{Q}(\bar t) \le C_r \int_0^{\bar t} \overset{(s)}{Q}(t) dt + C_r \int_0^{\bar t} \overset{(s+1)}{Q}(t) dt. \tag{5.11}$$

For every $t_1 \ge \bar t$ it is easily seen that

$$\overset{(s+1)}{Q}(\bar t) \le C_r \int_0^{t_1} \overset{(s)}{Q}(t) dt + C_r \int_0^{\bar t} \overset{(s+1)}{Q}(t) dt.$$

After applying the Gronwall's inequality we get

$$\overset{(s+1)}{Q}(\bar t) \le C_r \, e^{C_r \, \bar t} \int_0^{t_1} \overset{(s)}{Q}(t) dt \le C_r \, e^{C_r \, t_1} \int_0^{t_1} \overset{(s)}{Q}(t) \, dt = C_4 \int_0^{t_1} \overset{(s)}{Q}(t) dt,$$

where $C_4 = C_r \, e^{C_r \, t_*}$.

This result holds for every $t_1 \ge \bar t$, hence in particular for $t_1 = \bar t$:

$$\overset{(s+1)}{Q}(\bar t) \le C_4 \int_0^{\bar t} \overset{(s)}{Q}(t) dt. \tag{5.12}$$

Applying s times the formula (5.12)

$$\overset{(s+1)}{Q}(\bar t) \le C_4^s \int_0^{\bar t} dt \int_0^{t} d\tau_1 \ldots \int_0^{\tau_{s-1}} \overset{(1)}{Q}(\tau_s) d\tau_{s-1},$$

and observing the fact that

$$\overset{(1)}{Q}(\bar t) = \max_{t\in[0,\bar t]} \Big\| \overset{(0)}{r}(t,x)\Big\|_0 \le \max_{t\in[0,t_*)} \Big\| L[\overset{(0)}{u}](\overset{(1)}{u} - \overset{(0)}{u})\Big\|_0 =: C_Q, \tag{5.13}$$

we conclude that

$$\overset{(s+1)}{Q}(\bar t) \le \frac{(C_4\,\bar t)^s}{s!} C_Q, \qquad s = 0,1,\ldots. \tag{5.14}$$

We are now in a position to show that $\{\overset{(s)}{u}\}$ is a Cauchy sequence in the Banach space $C([0,t_*)\times\mathbb{R})$ with the supremum norm $|||\cdot|||_0 = \max\limits_{t\in[0,t_*)} \|\cdot\|_0$. Let $k > m$. Using (5.14) we obtain an upper bound for the difference between any two approximations of u:

$$\begin{aligned}
\Big\|\overset{(k)}{u} - \overset{(m)}{u}\Big\|_0 &\le \Big\|\overset{(k)}{u} - \overset{(k-1)}{u}\Big\|_0 + \cdots + \Big\|\overset{(m+1)}{u} - \overset{(m)}{u}\Big\|_0 \\
&= \Big\|\overset{(k-1)}{L^{-1}}\,\overset{(k)}{r}\Big\|_0 + \cdots + \Big\|\overset{(m)}{L^{-1}}\,\overset{(m+1)}{r}\Big\|_0 \\
&\le CC_Q\left(\frac{(C_4\,t)^{k-1}}{(k-1)!} + \cdots + \frac{(C_4\,t)^m}{m!}\right) \\
&\le CC_Q\frac{(C_4\,t)^m}{m!}\left(1 + \frac{C_4\,t}{m+1} + \frac{(C_4\,t)^2}{(m+1)(m+2)} + \ldots \right. \\
&\qquad \left. + \frac{(C_4\,t)^{k-1-m}}{(m+1)\ldots(k-1)}\right) \\
&\le CC_Q\frac{(C_4\,t)^m}{m!}\left(1 + \frac{C_4\,t}{1!} + \frac{(C_4\,t)^2}{2!} + \cdots + \frac{(C_4\,t)^{k-1-m}}{(k-1-m)!}\right) \\
&\le CC_Q\frac{(C_4\,t)^m}{m!}\,e^{C_4\,t}.
\end{aligned}$$

Hence we deduce that the sequence $\{\overset{(s)}{u}\}$ satisfies the Cauchy criterion in a Banach space $C([0,t_*)\times\mathbb{R})$:

$$\max_{t\in[0,t_*)} \Big\|\overset{(k)}{u} - \overset{(m)}{u}\Big\|_0 \le C\,C_Q\frac{(C_4\,t_*)^m}{m!}\,e^{C_4\,t_*} \longrightarrow 0, \qquad \text{as} \quad m\to+\infty. \tag{5.15}$$

5.3. Uniform convergence of $\{\overset{(s)}{p}\}$ on compact subsets of $\mathbb{R}$

LEMMA 4. *The sequence of functions $\{\overset{(s)}{p}\}$ is uniformly convergent on compact subsets of $\mathbb{R}$ for fixed $t\in[0,t_*)$.*

We begin by proving equi-continuity (with respect to x) of functions of the sequence $\{\overset{(s)}{p}\}$ for $t \in [0, t_*)$. We will show that there exists a function $\tilde{M}(\delta)$, $\tilde{M}(\delta) \to 0$ as $\delta \to 0$ such that it obeys the formula written below for $t \in [0, t_*)$ and for all s:

$$\left|\overset{(s+1)}{p}(t,x) - \overset{(s+1)}{p}(t,\bar{x})\right| \le \tilde{M}(\delta), \qquad \text{if} \quad |x - \bar{x}| \le \delta. \tag{5.16}$$

Considering (5.6) along the characteristic curves, we obtain

$$\begin{aligned} \frac{d}{dt}\left(\mathcal{P}_t \overset{(s+1)}{p}\right) &= \mathcal{P}_t\left(\frac{\partial}{\partial x}\overset{(s)}{Z} - \left(\frac{\partial}{\partial x}\overset{(s)}{L}\right)\overset{(s)}{L}{}^{-1}\overset{(s)}{Z}\right) \\ &+\mathcal{P}_t\left(\left(\frac{\partial}{\partial x}\overset{(s)}{L}\right)\overset{(s)}{L}{}^{-1}\overset{(s)}{D}\overset{(s)}{p}\right) \\ &+\mathcal{P}_t\left(L_t[\overset{(s)}{u}]\,\overset{(s)}{L}{}^{-1}\overset{(s+1)}{p}\right) \\ &+\mathcal{P}_t\left(+L'\big(\overset{(s)}{u};\overset{(s)}{L}{}^{-1}\overset{(s)}{Z} - \overset{(s)}{L}{}^{-1}\overset{(s)}{D}\overset{(s)}{p}\big)\,\overset{(s)}{L}{}^{-1}\overset{(s+1)}{p}\right) \\ &-\mathcal{P}_t\left(\left(\frac{\partial}{\partial x}\overset{(s)}{D}\right)\overset{(s+1)}{p}\right). \end{aligned} \tag{5.17}$$

Integrating from 0 to $\bar{t}$, we get:

$$\begin{aligned} \overset{(s+1)}{p}(\bar{t},\bar{x}) &= \mathcal{P}_0\overset{(s+1)}{p} \\ &+\int_0^{\bar{t}}\mathcal{P}_t\left(\frac{\partial}{\partial x}\overset{(s)}{Z} - \left(\frac{\partial}{\partial x}\overset{(s)}{L}\right)\overset{(s)}{L}{}^{-1}\overset{(s)}{Z}\right)dt \\ &+\int_0^{\bar{t}}\mathcal{P}_t\left(\left(\frac{\partial}{\partial x}\overset{(s)}{L}\right)\overset{(s)}{L}{}^{-1}\overset{(s)}{D}\overset{(s)}{p}\right)dt \\ &+\int_0^{\bar{t}}\mathcal{P}_t\left(L_t[\overset{(s)}{u}]\,\overset{(s)}{L}{}^{-1}\overset{(s+1)}{p}\right)dt \\ &+\int_0^{\bar{t}}\mathcal{P}_t\left(L'\big(\overset{(s)}{u};\overset{(s)}{L}{}^{-1}\overset{(s)}{Z} - \overset{(s)}{L}{}^{-1}\overset{(s)}{D}\overset{(s)}{p}\big)\,\overset{(s)}{L}{}^{-1}\overset{(s+1)}{p}\right)dt \\ &-\int_0^{\bar{t}}\mathcal{P}_t\left(\left(\frac{\partial}{\partial x}\overset{(s)}{D}\right)\overset{(s+1)}{p}\right)dt. \end{aligned} \tag{5.18}$$

Now we shall estimate the expressions under the integral signs in (5.19) evaluated at the points $(\bar{t},\bar{x})$, $(\bar{t},\bar{\bar{x}})$. To shorten the notation, we denote $\bar{\bar{\mathcal{P}}}_t f$ for any function f along the characteristic curves crossing the point $(\bar{t},\bar{\bar{x}})$.

Since $\mathcal{P}_0 \overset{(s+1)}{p}$ is a continuous vector function of $\bar{x}$, then it is uniformly continuous on any compact set. Therefore for $|\bar{x}-\bar{\bar{x}}| \le \delta$ there exists a function $N_0(\delta) \to 0$ as $\delta \to 0$, such that:

$$\left|\mathcal{P}_0 \overset{(s+1)}{p} - \bar{\bar{\mathcal{P}}}_0 \overset{(s+1)}{p}\right| \le N_0(\delta). \tag{5.19}$$

From condition (A_4), for $|\bar{x}-\bar{\bar{x}}| \le \delta$ we see that

$$\begin{aligned}\left|\frac{\partial}{\partial x}\overset{(s)}{Z}_k(t,x_k(t;\bar{t},\bar{x})) - \frac{\partial}{\partial x}\overset{(s)}{Z}_k(t,x_k(t;\bar{t},\bar{\bar{x}}))\right| &\le \\ C\Big(\Big|\overset{(s)}{u}_x(t,x_k(t;\bar{t},\bar{x})) - \overset{(s)}{u}_x(t,x_k(t;\bar{t},\bar{\bar{x}}))\Big| \\ + \Big|\overset{(s)}{u}(t,x_k(t;\bar{t},\bar{x})) - \overset{(s)}{u}(t,x_k(t;\bar{t},\bar{\bar{x}}))\Big|\Big) &+ N(\delta),\end{aligned}$$

where $N(\delta) \to 0$ as $\delta \to 0$.
Hence

$$\begin{aligned}\left|\mathcal{P}_t\left(\frac{\partial}{\partial x}\overset{(s)}{Z}\right) - \bar{\bar{\mathcal{P}}}_t\left(\frac{\partial}{\partial x}\overset{(s)}{Z}\right)\right| &\le n\,N(\delta) \\ + nC\max_{k=1,\dots,n}\Big(\Big|\overset{(s)}{u}_x(t,x_k(t;\bar{t},\bar{x})) &- \overset{(s)}{u}_x(t,x_k(t;\bar{t},\bar{\bar{x}}))\Big| \\ + \Big|(s)\ u(t,x_k(t;\bar{t},\bar{x})) &- \overset{(s)}{u}(t,x_k(t;\bar{t},\bar{\bar{x}}))\Big|\Big).\end{aligned} \tag{5.20}$$

LEMMA 5. *Assume that K stands for the operators L, D and V stands for $L^{-1}Z$, $L^{-1}D$. Then there is a constant $C_5 > 0$ such that for $|\bar{x}-\bar{\bar{x}}| \le \delta$ and $s = 0,1,2,\dots$ there holds*

$$\begin{aligned}&\left|\mathcal{P}_t\left(\left(\frac{\partial}{\partial x}K[\overset{(s)}{u}]\right)V[\overset{(s)}{u}]\,\overset{(s)}{p}\right) - \bar{\bar{\mathcal{P}}}_t\left(\left(\frac{\partial}{\partial x}K[\overset{(s)}{u}]\right)V[\overset{(s)}{u}]\,\overset{(s)}{p}\right)\right| \\ &\le C_5\Big\{\delta + N(\delta) + \max_{k=1,\dots,n}\Big(\Big|\overset{(s)}{u}(t,x_k(t;\bar{t},\bar{x})) - \overset{(s)}{u}(t,x_k(t;\bar{t},\bar{\bar{x}}))\Big| \\ &\quad + \Big|\overset{(s)}{u}_x(t,x_k(t;\bar{t},\bar{x})) - \overset{(s)}{u}_x(t,x_k(t;\bar{t},\bar{\bar{x}}))\Big| \\ &\quad + \Big|\overset{(s)}{p}(t,x_k(t;\bar{t},\bar{x})) - \overset{(s)}{p}(t,x_k(t;\bar{t},\bar{\bar{x}}))|\Big)\Big\}.\end{aligned}$$

P r o o f. As it follows from (A_1): $\Big\|\dfrac{\partial}{\partial x}V[\overset{(s)}{u}]\Big\|_0 \le C^2$. Hence we have

$$|V[\overset{(s)}{u}](t, x_k(t;\bar t,\bar x)) - V[\overset{(s)}{u}](t, x_k(t;\bar t,\bar{\bar x}))| \le C^2 |x_k(t;\bar t,\bar x) - x_k(t;\bar t,\bar{\bar x})|.$$

Similarly, since $\Big\|\dfrac{\partial}{\partial x}L[u]\Big\|_0 \le C$ then by the theorem on differentiability of solutions of ODE with respect to initial data [2]:

$$|x_k(t;\bar t,\bar x) - x_k(t;\bar t,\bar{\bar x})| \le C|\bar x - \bar{\bar x}|, \qquad \text{for any } \bar x, \bar{\bar x} \in \mathbb{R}.$$

By assumptions (A_1) and (A_4) we have $(C_6 > 0)$

$$\begin{aligned}
&\bigg|\bigg(\mathcal{P}_t\bigg(\bigg(\frac{\partial}{\partial x}K[\overset{(s)}{u}]\bigg)V[\overset{(s)}{u}]\ \overset{(s)}{p}\bigg)\bigg)_k - \bigg(\bar{\bar{\mathcal{P}}}_t\bigg(\bigg(\frac{\partial}{\partial x}K[\overset{(s)}{u}]\bigg)V[\overset{(s)}{u}]\ \overset{(s)}{p}\bigg)\bigg)_k\bigg| \\
&\le \bigg|\frac{\partial}{\partial x}K_k[\overset{(s)}{u}](t, x_k(t;\bar t,\bar x)) - \frac{\partial}{\partial x}K_k[\overset{(s)}{u}](t, x_k(t;\bar t,\bar{\bar x}))\bigg| \\
&\quad\times \Big|V[\overset{(s)}{u}](t, x_k(t;\bar t,\bar x))\Big|\ \Big|\overset{(s)}{p}\ (t, x_k(t;\bar t,\bar x))\Big| \\
&\quad+ \bigg|\frac{\partial}{\partial x}K_k[\overset{(s)}{u}](t, x_k(t;\bar t,\bar{\bar x}))\bigg|\ \Big|V[\overset{(s)}{u}](t, x_k(t;\bar t,\bar x))\Big|\ \Big|\overset{(s)}{p}\ (t, x_k(t;\bar t,\bar x)) \\
&\quad- \overset{(s)}{p}\ (t, x_k(t;\bar t,\bar{\bar x}))\Big| \\
&\quad+ \bigg|\frac{\partial}{\partial x}K[\overset{(s)}{u}](t, x_k(t;\bar t,\bar{\bar x}))\bigg|\ \Big|V[\overset{(s)}{u}](t, x_k(t;\bar t,\bar x)) - V[\overset{(s)}{u}](t, x_k(t;\bar t,\bar{\bar x}))\Big| \\
&\quad\times \Big|\overset{(s)}{p}\ (t, x_k(t;\bar t,\bar{\bar x}))\Big| \\
&\le C_6\bigg\{\delta + N(\delta) + \Big|\overset{(s)}{u}\ (t, x_k(t;\bar t,\bar x)) - \overset{(s)}{u}\ (t, x_k(t;\bar t,\bar{\bar x}))\Big| \\
&\quad+ \Big|\overset{(s)}{u}_x\ (t, x_k(t;\bar t,\bar x)) - \overset{(s)}{u}_x\ (t, x_k(t;\bar t,\bar{\bar x}))\Big| \\
&\quad+ \Big|\overset{(s)}{p}\ (t, x_k(t;\bar t,\bar x)) - \overset{(s)}{p}\ (t, x_k(t;\bar t,\bar{\bar x}))\Big|\bigg\},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\bigg|\mathcal{P}_t\bigg(\bigg(\frac{\partial}{\partial x}K[\overset{(s)}{u}]\bigg)V[\overset{(s)}{u}]\ \overset{(s)}{p}\bigg) - \bar{\bar{\mathcal{P}}}_t\bigg(\bigg(\frac{\partial}{\partial x}K[\overset{(s)}{u}]\bigg)V[\overset{(s)}{u}]\ \overset{(s)}{p}\bigg)\bigg| \\
&\le nC_6\bigg\{\delta + N(\delta) + \bigg(\max_{k=1,\dots,n}\Big|\overset{(s)}{u}\ (t, x_k(t;\bar t,\bar x)) - \overset{(s)}{u}\ (t, x_k(t;\bar t,\bar{\bar x}))\Big|
\end{aligned}$$

$$+\left|\overset{(s)}{u}_x(t,x_k(t;\bar t,\bar x))-\overset{(s)}{u}_x(t,x_k(t;\bar t,\bar{\bar x}))\right|+\left|\overset{(s)}{p}(t,x_k(t;\bar t,\bar x))-\overset{(s)}{p}(t,x_k(t;\bar t,\bar{\bar x}))\right|\Bigg)\Bigg\}.$$

□

By Lemma 5 we have

$$(5.21)\quad \left|\mathcal{P}_t\left(\left(\frac{\partial}{\partial x}\overset{(s)}{L}\right)\overset{(s)}{L^{-1}}\overset{(s)}{Z}\right)-\bar{\bar{\mathcal{P}}}_t\left(\left(\frac{\partial}{\partial x}\overset{(s)}{L}\right)\overset{(s)}{L^{-1}}\overset{(s)}{Z}\right)\right|$$
$$+\left|\mathcal{P}_t\left(\left(\frac{\partial}{\partial x}\overset{(s)}{L}\right)\overset{(s)}{L^{-1}}\overset{(s)}{D}\overset{(s)}{p}\right)-\bar{\bar{\mathcal{P}}}_t\left(\left(\frac{\partial}{\partial x}\overset{(s)}{L}\right)\overset{(s)}{L^{-1}}\overset{(s)}{D}\overset{(s)}{p}\right)\right|$$
$$+\left|\mathcal{P}_t\left(\left(\frac{\partial}{\partial x}\overset{(s)}{D}\right)\overset{(s+1)}{p}\right)-\bar{\bar{\mathcal{P}}}_t\left(\left(\frac{\partial}{\partial x}\overset{(s)}{D}\right)\overset{(s+1)}{p}\right)\right|$$
$$\le C_7\Bigg\{\delta+N(\delta)+\max_{k=1,\dots,n}\Bigg(\left|\overset{(s)}{u}(t,x_k(t;\bar t,\bar x))-\overset{(s)}{u}(t,x_k(t;\bar t,\bar{\bar x}))\right|$$
$$+\left|\overset{(s)}{u}_x(t,x_k(t;\bar t,\bar x))-\overset{(s)}{u}_x(t,x_k(t;\bar t,\bar{\bar x}))\right|$$
$$+\left|\overset{(s)}{p}(t,x_k(t;\bar t,\bar x))-\overset{(s)}{p}(t,x_k(t;\bar t,\bar{\bar x}))\right|$$
$$+\left|\overset{(s+1)}{p}(t,x_k(t;\bar t,\bar x))-\overset{(s+1)}{p}(t,x_k(t;\bar t,\bar{\bar x}))\right|\Bigg)\Bigg\},$$

where C_7 is a positive constant.

For fixed $t\in[0,t_*)$ and for x belonging to any compact set, the functions $L_t[\overset{(s)}{u}]$ and $L'[\overset{(s)}{u}]$ are uniformly continuous. Hence for all s there exists a function $N_L(\delta)\to 0$ as $\delta\to 0$, such that there holds ($C_8>0$):

$$(5.22)\quad \Bigg|\mathcal{P}_t\left(\left(L_{,t}[\overset{(s)}{u}]+L'(\overset{(s)}{u};\overset{(s)}{L^{-1}}\overset{(s)}{Z}-\overset{(s)}{L^{-1}}\overset{(s)}{D}\overset{(s)}{p})\right)\overset{(s)}{L^{-1}}\overset{(s+1)}{p}\right)$$
$$-\bar{\bar{\mathcal{P}}}_t\left(\left(L_t[\overset{(s)}{u}]+L'(\overset{(s)}{u};\overset{(s)}{L^{-1}}\overset{(s)}{Z}-\overset{(s)}{L^{-1}}\overset{(s)}{D}\overset{(s)}{p})\right)\overset{(s)}{L^{-1}}\overset{(s+1)}{p}\right)\Bigg|$$
$$\le C_8\Bigg\{N_L(\delta)+\max_{k=1,\dots,n}\Bigg(\left|\overset{(s)}{u}(t,x_k(t;\bar t,\bar x))-\overset{(s)}{u}(t,x_k(t;\bar t,\bar{\bar x}))\right|$$
$$+\left|\overset{(s)}{p}(t,x_k(t;\bar t,\bar x))-\overset{(s)}{p}(t,x_k(t;\bar t,\bar{\bar x}))\right|$$
$$+\left|\overset{(s+1)}{p}(t,x_k(t;\bar t,\bar x))-\overset{(s+1)}{p}(t,x_k(t;\bar t,\bar{\bar x}))\right|\Bigg)\Bigg\}.$$

Let us point out that it will be useful to replace $\overset{(s)}{u_x}$ by $\overset{(s)}{p}$ on the right-hand side of (5.20) and (5.21). The operator L^{-1} is bounded in the ball $B^1_r(u^0)$ and the function $\overset{(s-1)}{L^{-1}}$ satisfies the Lipschitz condition with respect to x. For this reason, for $t \in [0, t_*)$ we obtain

$$\begin{aligned}
&\Big| \overset{(s)}{u}_x (t, x_k(t; \bar{t}, \bar{x})) - \overset{(s)}{u}_x (t, x_k(t; \bar{t}, \bar{\bar{x}}))\Big| \\
&\qquad = \left| \overset{(s-1)}{L^{-1}} \overset{(s)}{p} \Big|_{(t, x_k(t;\bar{t},\bar{x}))} - \overset{(s-1)}{L^{-1}} \overset{(s)}{p} \Big|_{(t, x_k(t;\bar{t},\bar{\bar{x}}))} \right| \\
&\qquad \le \Big| \overset{(s-1)}{L^{-1}} (t, x_k(t; \bar{t}, \bar{x}))\Big| \Big| \overset{(s)}{p} (t, x_k(t; \bar{t}, \bar{x})) - \overset{(s)}{p} (t, x_k(t; \bar{t}, \bar{\bar{x}}))\Big| \\
&\qquad + \Big| \overset{(s)}{L^{-1}} (t, x_k(t; \bar{t}, \bar{x})) - \overset{(s)}{L^{-1}} (t, x_k(t; \bar{t}, \bar{\bar{x}}))\Big| \Big| \overset{(s)}{p} (t, x_k(t; \bar{t}, \bar{\bar{x}}))\Big| \\
&\le C\bigg(\Big| \overset{(s)}{p} (t, x_k(t; \bar{t}, \bar{x})) - \overset{(s)}{p} (t, x_k(t; \bar{t}, \bar{\bar{x}}))\Big| + Big|x_k(t; \bar{t}, \bar{x}) - x_k(t; \bar{t}, \bar{\bar{x}})\Big|\bigg) \\
&\qquad \le C\Big| \overset{(s)}{p} (t, x_k(t; \bar{t}, \bar{x})) - \overset{(s)}{p} (t, x_k(t; \bar{t}, \bar{\bar{x}}))\Big| + C^2\Big|\bar{x} - \bar{\bar{x}}\Big| \\
&\qquad \le C_9\bigg(\Big| \overset{(s)}{p} (t, x_k(t; \bar{t}, \bar{x})) - \overset{(s)}{p} (t, x_k(t; \bar{t}, \bar{\bar{x}}))\Big| + \delta\bigg), \qquad k = 1, \dots, n.
\end{aligned}$$

where $C_9 = \max\{C, C^2\}$.

For fixed t and for x belonging to any compact set, functions $\overset{(s)}{u}$, $(s = 0, 1, 2, \dots)$, are uniformly continuous. Hence, for all s, there exists a function $N_u(\delta) \to 0$ as $\delta \to 0$, such that

$$\max_{k=1,\dots,n} \left| \overset{(s)}{u} (t, x_k(t; \bar{t}, \bar{x})) - \overset{(s)}{u} (t, x_k(t; \bar{t}, \bar{\bar{x}}))\right| \le N_u(\delta) \qquad \text{if} \qquad |\bar{x} - \bar{\bar{x}}| \le \delta.$$

Summarizing, we see that the sequence $\{\overset{(s)}{p}\}$ satisfies the condition

$$\begin{aligned}
\text{(5.23)} \qquad &\left| \overset{(s+1)}{p} (\bar{t}, \bar{x}) - \overset{(s+1)}{p} (\bar{t}, \bar{\bar{x}})\right| < N_0(\delta) \\
&\quad + C_{10}\left(\int_0^{\bar{t}} (N_u(\delta) + N(\delta) + N_L(\delta) + \delta)\, dt \right.\\
&\quad + \int_0^{\bar{t}} \max_{k=1,\dots,n} \Big| \overset{(s)}{p} (t, x_k(t; \bar{t}, \bar{x})) - \overset{(s)}{p} (t, x_k(t; \bar{t}, \bar{\bar{x}}))\Big| dt \\
&\quad \left. + \int_0^{\bar{t}} \max_{k=1,\dots,n} \Big| \overset{(s+1)}{p} (t, x_k(t; \bar{t}, \bar{x})) - \overset{(s+1)}{p} (t, x_k(t; \bar{t}, \bar{\bar{x}}))\Big| dt \right),
\end{aligned}$$

where $C_{10} = \max\{C_9(nC + C_7) + C_7 + C_8; nC + C_7 + C_8\}$.

Now we define a new function

$$M_{s+1}(\bar{t},\delta) = \max_{\substack{k=1,\dots,n\\ i=0,\dots,s+1}} \sup_{\substack{|\bar{x}-\bar{\bar{x}}|\le\delta\\ t\le\bar{t}}} \left| \overset{(i)}{p}(t, x_k(t;\bar{t},\bar{x})) - \overset{(i)}{p}(t, x_k(t;\bar{t},\bar{\bar{x}})) \right|. \tag{5.24}$$

From (5.23) we obtain the following formula:

$$M_{s+1}(\bar{t},\delta) \le N_0(\delta) + \bar{t}\, C_{10}\Big(N_u(\delta) + N(\delta) + N_L(\delta) + \delta\Big) + 2C_{10}\int_0^{\bar{t}} M_{s+1}(t,\delta)dt. \tag{5.25}$$

The next step is to apply the Gronwall's inequality to the last expression

$$M_{s+1}(\bar{t},\delta) \le N_0(\delta)\, e^{2C_{10}\bar{t}} + \frac{\bar{t}}{2}\Big(N_u(\delta) + N(\delta) + N_L(\delta) + \delta\Big)\,(e^{2C_{10}\bar{t}} - 1). \tag{5.26}$$

Because $N_0(\delta), N_u(\delta), N(\delta), N_L(\delta) \to 0$ as $\delta \to 0$, we conclude that, for $\bar{t} \in [0,t_*)$, $M_{s+1}(\bar{t},\delta) \to 0$ as $\delta \to 0$. Consequently we take the definition of the function $\tilde{M}(\delta)$ in the form

$$\tilde{M}(\delta) = N_0(\delta)\, e^{2C_{10}t_*} + \frac{t_*}{2}\Big(N_u(\delta) + N(\delta) + N_L(\delta) + \delta\Big)\,(e^{2C_{10}t_*} - 1). \tag{5.27}$$

By the Arzela–Ascoli theorem, if functions of a sequence are equi-bounded and equi-continuous, then there exists a uniformly convergent subsequence. Therefore some subsequence $\{\overset{(s_k)}{p}\}$ converges uniformly on any compact subset of $\mathbb{R}$ to the continuous function $p(t,x)$ for fixed $t \in [0,t_*)$. We have proved that the sequence $\{\overset{(s)}{u}\}$ converges uniformly to the continuous function $u(t,x)$. Hence we are able to consider a subsequence $\{\overset{(s_k)}{u}\}$, which obviously is uniformly convergent to $u(t,x)$. Under the assumptions for the operator L^{-1} we have

$$\|L^{-1}[\overset{(s_k)}{u}] - L^{-1}[u]\|_0 \le C\,\|\overset{(s_k)}{u} - u\|_0.$$

Consequently, for fixed $t \in [0,t_*)$, $L^{-1}[\overset{(s_k)}{u}]$ converges uniformly to $L^{-1}[u]$ and $L^{-1}[\overset{(s_k)}{u}]\,\overset{(s_k)}{p}$ uniformly converges on any compact subset of $\mathbb{R}$ to the continuous function $L^{-1}[u]\,p$. Note that $\overset{(s_k)}{u_x} = L^{-1}[\overset{(s_k)}{u}]\,\overset{(s_k)}{p}$. We conclude that the function $u(t,x)$ is continuously differentiable for $t \in [0,t_*)$ and x belonging to any closed interval in $\mathbb{R}$ and $u_x = L^{-1}[u]\,p$. Therefore any uniformly converging subsequence of $\{\overset{(s)}{p}\}$ must converge to $p = L[u]\,u_x$ on compact subset of $\mathbb{R}$.

Passing in (5.2) to the limit when s tends to infinity, we see that the function $u(t,x)$ satisfies (1.12). Continuity of the derivative with respect to t is a consequence of continuity of the right-hand side of the system (1.1) and the derivative u_x. By the uniqueness lemma, the function $u(t,x)$ is a $C^1([0,t_*)\times\mathbb{R})$ solution of the Cauchy problem (1.1–1.2).

References

1. S. Barral, K. Makowski, Z. Peradzyński, N. Gascon, M. Dudeck, *Wall material effects in stationary plasma thruster II. Near-wall and in-wall conductivity*, Physics of Plasma, **10**, 10, 2003.
2. P. Hartman, *Ordinary differential equations*, John Wiley and Sons, Inc., New York–London–Sydney 1964.
3. Z. Kamont, *Hyperbolic functional differential inequalities and applications*, Mathematics and its Applications, 486. Kluwer Academic Publishers, Dordrecht 1999.
4. B.L. Rozhdestvenski, N.N. Yanenko, *Systems of quasilinear equations and their applications to gas dynamics*, "Nauka", Moscow 1968.
5. J. Turo, *Existence and uniqueness of solutions quasilinear hyperbolic systems of partial differential equations*, Math. Slovaca, **37**, 4, 375–389, 1987.
6. V.V. Zhurin, H.R. Kaufman, R.S. Robinson, *Physics of closed drifts thrusters*, Plasma Sources Sci. Technol., **8**, R1–R20, 1999.

Received November 7, 2006.