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Secant stress/strain relations of orthotropic elastic damage with dual properties*)

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A constitutive framework of orthotropic elastic damage in initially-isotropic materials is presented. The constitutive equations are developed within the phenomenological approach of Continuum Damage Mechanics. Focus is made on secant stress/strain relations that can be derived by the application of the so-called damage-effect tensors, namely the fourth-order operators that define the linear transformations between nominal and effective stress and strain quantities. In the attempt to provide selected forms of anisotropic damage approaching general orthotropy, several proposals of damage-effect tensors are formulated. Such fourth-order operators are obtained from the complete orthotropic representations as particular instances that satisfy a duality requirement between compliance- and stiffness-based derivations. A complete family of solutions based on a specific non-singular tensor generator is derived in full invariant form.

Key words: Continuum Damage Mechanics, orthotropic damage, fourth-order damage-effect tensor, second-order damage tensor, tensor inverses, dual secant relations.

1. Introduction

Starting from the pioneering contributions by Kachanov [11] and Rabotnov [18], Continuum Damage Mechanics (CDM) has reached nowadays a rather consolidated stage of development. This includes, among other features (for instance the ultimate coupling between plasticity and damage, see e.g. Maier and Hueckel [16]), the constitutive modeling of *anisotropic* elastic stiffness degradation in quasi-brittle materials such as e.g. concrete, rocks, composites. The CDM formulations are typically based on the introduction of damage variables

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of various tensor orders, i.e. scalars, vectors, second- and fourth-order tensors (see the comprehensive reference lists provided in both research articles, e.g. Zheng and Betten [24], Carol *et al.* [6, 7], Betten [3], and specific monographs dedicated to the subject that are now available in the literature, e.g. Lemaitre [14], Krajcinovic [12], Skrzypek and Ganczarski [21]).

The present authors have contributed to the topic with a proposal of a plasticity-type unified theoretical framework of elastic stiffness degradation and damage based on a loading surface [6], and with the formulation of constitutive models for anisotropic elastic stiffness degradation in initially-isotropic materials [7, 8, 19]. The latter models are characterized by second-order symmetric tensor damage variables with evolution laws expressed in terms of a (non-holonomic) pseudo-logarithmic rate of damage. The resulting secant elastic relations correspond to a restricted form of orthotropic material behavior that we refer to as Valanis-type damage, see Valanis [22] and Zysset and Curnier [25].

During these investigations, the request of deriving more general forms of orthotropic elastic stiffness degradation arose spontaneously, together with the desire of preserving at the same time a full *duality* between alternative compliance- and stiffness-based derivations of the constitutive relations. These dual properties can be read-off directly from the structure of the so-called *damage-effect tensors*, namely the fourth-order operators that, based on the underlying tensor damage variables, define the linear transformations between nominal and effective stress and strain quantities. The damage-effect tensors may be either postulated directly or derived on the basis of specific considerations (e.g. micromechanical, an approach that is not followed here) and may be adopted to prescribe in practice the secant relations of elastic damage (Murakami and Ohno [17], Cordebois and Sidoroff [9], Betten [1], Lu and Chow [15]). Summaries of the different proposals of the damage-effect tensors available in the literature are reported e.g. by Lam and Zhang [13], Zheng and Betten [24], Voyiadjis and Park [23], Betten [3].

Recently, the authors have attempted, in a companion paper [20], a generalization of these previous propositions by providing a set of *dual orthotropic damage-effect tensors* which are obtained from the general fourth-order orthotropic representations as specific instances that satisfy the duality requirement. The present note reconsiders some of these new proposals, provides additional instances of dual damage-effect tensors including those that complete a solution family based on a specific non-singular tensor generator and categorizes systematically all the solutions belonging to such a family.

This paper has a theoretical character. The approach followed in the paper, its target and content are mainly algebraic. We believe that, from this point of view, it attempts in contributing to the understanding of key issues like those of the definition of the damaged stiffness or increased compliance, possibly through the use of damage-effect tensors. Often in the CDM literature, to the damage-effect tensors are given forms in which some terms of the general representation are retained, some not, without apparent justifications, neither physical, nor algebraic. This paper makes an effort towards a systematic approach of treating damage-effect tensors (or directly elasticity tensors), at least from an algebraic point of view, by privileging the role of duality. Though it is not said that these convenient forms should be necessarily attached to damage cases of physical meaning, they highlight out as likely the first to be checked for a corresponding physical interpretation.

The paper is just dealing with secant relations, which is a single ingredient of a constitutive formulation of elastic damage (which further necessitates the characterization of other aspects, like for instance the definition of a damage domain and of appropriate damage evolution laws). However, secant relations are actually one of the key features of the formulation of a CDM model, often crucial also in the subsequent definition of these further aspects of the formulation, as we have experienced in the derivations of our previous CDM models presented in [7, 8, 19].

The secant CDM relations of the elastic-damage model and the definition of the damage-effect tensors are provided first in Sec. 2. There, the general orthotropic representations of fourth-order symmetric damage-effect tensors and secant compliance and stiffness tensors, are introduced in terms of three 'shear-like' and six 'non-shear' coefficients and the corresponding tensor addends, and the requirement of duality is precisely stated. To elucidate the type of representations that embed the sought dual structure, a few examples of both the symmetric and non-symmetric dual damage-effect tensors are reported in Sec. 3, including a new particular symmetric instance that lacks only two 'shear-like' coefficients and embeds all 'non-shear' coefficients. Then, Sec. 4 outlines a complete family of symmetric solution instances based on a specific 'shear-like' generator, starting from the more general one that includes all the 'non-shear' coefficients, going through new solution instances that involve just subsets of the 'non-shear' coefficients (and hold with or without constraints on the coefficients), to arrive finally at the sole 'shear-like' generator itself. All the solution instances of the family are expressed in complete invariant form and are resumed in the synoptic Table 1 at the end of the section. A few final comments are gathered in the closing section. Appendix A at the end of the paper collects the lengthy expressions of some of the terms entering the various solution instances.

Notation

Compact or index tensor notation is used throughout. Second-order tensors are identified by boldface characters (e.g. $\mathbf{w}, \boldsymbol{\phi}, \boldsymbol{\epsilon}, \boldsymbol{\sigma}$), whereas fourth-order tensors are denoted by blackboard-bold fonts (e.g. $\mathbb{A}, \mathbb{C}, \mathbb{E}$). Symbols '$\cdot$' and

':' between tensors of various orders denote the inner products with single and double contraction, respectively. Superscript $^{\mathrm{T}}$ indicates the transpose operation applied either to second-order tensors, or to fourth-order tensors; componentwise $(w^{\mathrm{T}})_{ij} = w_{ji}$ and $(A^{\mathrm{T}})_{ijkl} = A_{klij}$. The dyadic product of second-order tensors is indicated with '$\otimes$' and defined as $(\mathbf{A} \otimes \mathbf{B}) : \mathbf{C} = (\mathbf{B} : \mathbf{C})\,\mathbf{A}$, for any second-order tensors $\mathbf{A}, \mathbf{B}, \mathbf{C}$, whereas '$\overline{\underline{\otimes}}$' denotes the symmetrized dyadic product of second-order tensors defined as $(\mathbf{A} \overline{\underline{\otimes}} \mathbf{B}) : \mathbf{C} = \mathbf{A} \cdot \mathbf{C}^s \cdot \mathbf{B}^{\mathrm{T}}$, for any second-order tensors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, where $\mathbf{C}^s = (\mathbf{C}+\mathbf{C}^{\mathrm{T}})/2$ is the symmetric part of $\mathbf{C}$; componentwise $(\mathbf{A}\otimes\mathbf{B})_{ijkl} = A_{ij}\,B_{kl}$ and $(\mathbf{A} \overline{\underline{\otimes}} \mathbf{B})_{ijkl} = (A_{ik}\,B_{jl}+A_{il}\,B_{jk})/2$. $\mathbf{I}$ and $\mathbb{I}^s = \mathbf{I} \overline{\underline{\otimes}} \mathbf{I}$ are respectively the second-order and symmetric (major and minor symmetries) fourth-order identity tensors; componentwise $I_{ij} = \delta_{ij}$ and $I^s_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2$, where δ_{ij} is the Kronecker delta ($\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$). $\mathbb{I}^s$ maps any second-order tensor $\mathbf{A}$ onto its symmetric part $\mathbf{A}^s$, i.e. $\mathbb{I}^s : \mathbf{A} = \mathbf{A}^s$, and any symmetric second-order tensor $\mathbf{B} = \mathbf{B}^{\mathrm{T}}$ onto itself, i.e. $\mathbb{I}^s : \mathbf{B} = \mathbf{B}$. Symbol 'tr' denotes the trace operator applied to second-order tensors, i.e. $\operatorname{tr}\mathbf{A} = \mathbf{I} : \mathbf{A} = A_{ii}$. For more detailed definitions see e.g. Rizzi and Carol [19, Appendix A].

2. Secant relations of orthotropic elastic damage

At any damage state the *nominal* (small) strain tensor $\boldsymbol{\epsilon}$ and stress tensor $\boldsymbol{\sigma}$ are related by the following secant elastic constitutive laws:

$$\boldsymbol{\epsilon} = \mathbb{C}(\mathbb{C}_0, \boldsymbol{\mathcal{D}}) : \boldsymbol{\sigma} \,; \qquad \boldsymbol{\sigma} = \mathbb{E}(\mathbb{E}_0, \bar{\boldsymbol{\mathcal{D}}}) : \boldsymbol{\epsilon} \,, \tag{2.1}$$

where $\mathbb{C}$ and $\mathbb{E}$ are the current positive-definite fourth-order compliance and stiffness tensors, inverse of each other (i.e. $\mathbb{C} : \mathbb{E} = \mathbb{E} : \mathbb{C} = \mathbb{I}^s$) and endowed with both the major and minor symmetries. The current values of compliance $\mathbb{C}(\mathbb{C}_0, \boldsymbol{\mathcal{D}})$ and stiffness $\mathbb{E}(\mathbb{E}_0, \bar{\boldsymbol{\mathcal{D}}})$ start from their initial values $\mathbb{C}_0$, $\mathbb{E}_0$ in the undamaged state and evolve as functions of generally-defined tensor damage variables $\boldsymbol{\mathcal{D}}$, or of dual tensor damage variables $\bar{\boldsymbol{\mathcal{D}}}$ (overbars denote dual quantities). Assuming that the undamaged behavior is isotropic, the initial compliance and stiffness are expressed by the classical elastic relations

$$\mathbb{C}_0 = \frac{1+\nu_0}{E_0}\,\mathbf{I} \overline{\underline{\otimes}} \mathbf{I} - \frac{\nu_0}{E_0}\,\mathbf{I}\otimes\mathbf{I} \,; \mathbb{E}_0 = 2\,G_0\,\mathbf{I} \overline{\underline{\otimes}} \mathbf{I} + \Lambda_0\,\mathbf{I}\otimes\mathbf{I} \,, \tag{2.2}$$

in terms of undamaged Poisson's ratio ν_0 and Young's modulus E_0, or undamaged shear modulus G_0 and the first Lamé's constant Λ_0. Alternatively, the undamaged bulk modulus K_0 could be employed instead of Λ_0 through the standard relations $3K_0 = 3\Lambda_0 + 2G_0 = E_0/(1 - 2\nu_0)$. This would privilege a more convenient volumetric/deviatoric representation of Eq. (2.2) displaying directly the volumetric/deviatoric decoupling of the isotropic response.

Through a purely phenomenological approach, the damage-state relations $\mathbb{C} = \mathbb{C}(\mathbb{C}_0, \boldsymbol{\mathcal{D}})$ and $\mathbb{E} = \mathbb{E}(\mathbb{E}_0, \bar{\boldsymbol{\mathcal{D}}})$ are derived here by following steps that are typical of the CDM framework (see e.g. the references quoted in the Introduction and the schemes provided in [19, 8]): i) a constitutive law is introduced for the undamaged material behavior, which relates *effective* strain and stress quantities, $\boldsymbol{\epsilon}_{\text{eff}}$ and $\boldsymbol{\sigma}_{\text{eff}}$, acting in the intact material between microcracks: $\boldsymbol{\epsilon}_{\text{eff}} = \mathbb{C}_0 : \boldsymbol{\sigma}_{\text{eff}}$, $\boldsymbol{\sigma}_{\text{eff}} = \mathbb{E}_0 : \boldsymbol{\epsilon}_{\text{eff}}$, where $\mathbb{C}_0$ and $\mathbb{E}_0$ are given here by Eq. (2.2); ii) a relation between nominal and effective (stress or strain) quantities is assumed, in linear form, by introducing a non-singular fourth-order *damage-effect tensor* which is a function of the damage variables, e.g. $\mathbb{A}(\boldsymbol{\mathcal{D}})$ in the stress relation $\boldsymbol{\sigma}_{\text{eff}} = \mathbb{A}(\boldsymbol{\mathcal{D}}) : \boldsymbol{\sigma}$; iii) a second link between nominal and effective states is postulated through an 'equivalence' principle, specifically that of *'energy equivalence'* (Cordebois and Sidoroff [9]), $\boldsymbol{\sigma} : \boldsymbol{\epsilon}/2 = \boldsymbol{\sigma}_{\text{eff}} : \boldsymbol{\epsilon}_{\text{eff}}/2$, which automatically renders secant stiffness and compliance enjoying major symmetry. The following nominal/effective relations are then consistently assumed/obtained:

$$\begin{aligned} \boldsymbol{\sigma}_{\text{eff}} &= \mathbb{A}\,(\boldsymbol{\mathcal{D}}) : \boldsymbol{\sigma}, & \boldsymbol{\epsilon} &= \mathbb{A}^{\mathrm{T}}(\boldsymbol{\mathcal{D}}) : \boldsymbol{\epsilon}_{\text{eff}}, \\ \boldsymbol{\epsilon}_{\text{eff}} &= \bar{\mathbb{A}}^{\mathrm{T}}(\bar{\boldsymbol{\mathcal{D}}}) : \boldsymbol{\epsilon}, & \boldsymbol{\sigma} &= \bar{\mathbb{A}}\,(\bar{\boldsymbol{\mathcal{D}}}) : \boldsymbol{\sigma}_{\text{eff}}\ , \end{aligned} \tag{2.3}$$

and compliance and stiffness are expressed by the symmetric forms:

$$\mathbb{C}(\mathbb{C}_0, \boldsymbol{\mathcal{D}}) = \mathbb{A}^{\mathrm{T}}(\boldsymbol{\mathcal{D}}) : \mathbb{C}_0 : \mathbb{A}\,(\boldsymbol{\mathcal{D}})\ ; \qquad \mathbb{E}(\mathbb{C}_0, \bar{\boldsymbol{\mathcal{D}}}) = \bar{\mathbb{A}}\,(\bar{\boldsymbol{\mathcal{D}}}) : \mathbb{E}_0 : \bar{\mathbb{A}}^{\mathrm{T}}(\bar{\boldsymbol{\mathcal{D}}})\ , \tag{2.4}$$

where $\mathbb{A}\,(\boldsymbol{\mathcal{D}}) = \bar{\mathbb{A}}^{-1}(\bar{\boldsymbol{\mathcal{D}}})$ and $\bar{\mathbb{A}}\,(\bar{\boldsymbol{\mathcal{D}}}) = \mathbb{A}^{-1}(\boldsymbol{\mathcal{D}})$ are *dual* non-singular fourth-order damage-effect tensors, inverse of each other (i.e. $\mathbb{A} : \bar{\mathbb{A}} = \bar{\mathbb{A}} : \mathbb{A} = \mathbb{I}^s$) and endowed with minor symmetries (not necessarily major symmetries).

Concerning the dual underlying damage variables $\boldsymbol{\mathcal{D}}$ and $\bar{\boldsymbol{\mathcal{D}}}$ entering the dependence of the damage-effect tensors with the damage state, the model of focus in the present paper makes use of positive-definite symmetric second-order tensor variables: the so-called integrity tensor $\bar{\boldsymbol{\phi}}$ of Valanis [22], varying between $\mathbf{I}$ and $\mathbf{0}$, or its inverse $\boldsymbol{\phi} = \bar{\boldsymbol{\phi}}^{-1}$, with complementary variation between $\mathbf{I}$ and ∞. The square-root tensors $\mathbf{w} = \boldsymbol{\phi}^{1/2}$, $\bar{\mathbf{w}} = \bar{\boldsymbol{\phi}}^{1/2}$ are as well employed in notation to express explicitly the functional dependence of $\mathbb{A}$ and $\bar{\mathbb{A}}$ on the damage variables. Positive-definiteness of tensors $\boldsymbol{\phi}$, $\bar{\boldsymbol{\phi}}$, $\mathbf{w}$, $\bar{\mathbf{w}}$ is assumed in the context of the development of the complete CDM constitutive equation [7, 8, 19]. The way these tensors vary at increasing, irreversible damage, is not considered in this paper, which focusses just on current state secant relations, without addressing evolution equations.

Now, since either the damage-effect tensor $\mathbb{A}(\mathbf{w})$ or the damage-effect tensor $\bar{\mathbb{A}}(\bar{\mathbf{w}})$ could be postulated independently as the source ingredient of the constitutive formulation, we are interested in seeking particular instances of the general orthotropic representations of $\mathbb{A}(\mathbf{w})$ and $\bar{\mathbb{A}}(\bar{\mathbf{w}})$, with the property that their inverses do display the structure of the transposes of the tensors obtained by replacing $\mathbf{w}$ with its dual inverse $\bar{\mathbf{w}}$ (or *vice versa*). Indeed, notice that inverse tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ play the same role in Eqs. (2.3), (2.4), except for a transpose operation (which obviously matters only if the damage-effect tensors do not enjoy major symmetry). The resulting damage-effect tensors are said then to possess *dual structures*.

General representations of orthotropic fourth-order tensors can be obtained either by algebraic decomposition (e.g. Walpole [26]), or through representation theorems (e.g. Betten [2], Boehler [5]), and could be used for either the compliance and stiffness or for the damage-effect tensors themselves. The damage-effect tensors could be represented in both the symmetric and non-symmetric forms. In the present paper we focus mainly on symmetric representations. Non-symmetric expansions of the damage-effect tensors are considered in [20]. Then, the general representation of a symmetric damage-effect tensor $\mathbb{A}(\mathbf{w})$ representing orthotropic damage in initially-isotropic materials (Lam and Zhang [13]) can be given as follows, according to the ordering of tensor terms adopted by Zysset and Curnier [25]:

$$\begin{aligned}\mathbb{A} = a_1\,\mathbf{I}\otimes\mathbf{I} + a_2\,\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I} + a_3\,\mathbf{w}\otimes\mathbf{w} \\ + a_4\,(\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{w}) + a_5\,\mathbf{w}^2\otimes\mathbf{w}^2 + a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w} \\ + a_7\,(\mathbf{w}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}) + a_8\,(\mathbf{w}^2\otimes\mathbf{w} + \mathbf{w}\otimes\mathbf{w}^2) + a_9\,(\mathbf{w}^2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}^2),\end{aligned} \tag{2.5}$$

where the 9 scalar coefficients a_i, $i = 1-9$, are any arbitrary functions of the three principal invariants of $\mathbf{w}$ (which can be classically defined as follows, ${}^{\mathrm{w}}I_1 = \operatorname{tr}\mathbf{w}$, ${}^{\mathrm{w}}I_2 = (\operatorname{tr}^2\mathbf{w} - \operatorname{tr}\mathbf{w}^2)/2$, ${}^{\mathrm{w}}I_3 = \det\mathbf{w} = \operatorname{tr}\mathbf{w}^3/3 + \operatorname{tr}^3\mathbf{w}/6 - \operatorname{tr}\mathbf{w}\,\operatorname{tr}\mathbf{w}^2/2$, and enter the Cayley–Hamilton theorem applied to $\mathbf{w}$, i.e. $\mathbf{w}^3 - {}^{\mathrm{w}}I_1\,\mathbf{w}^2 + {}^{\mathrm{w}}I_2\,\mathbf{w} - {}^{\mathrm{w}}I_3\mathbf{I} = \mathbf{0}$). In the non-symmetric case each of the three coefficients a_7, a_8, a_9 would split into a pair of two coefficients, namely a_{71}, a_{72}, a_{81}, a_{82}, a_{91}, a_{92} (e.g. a_{71} and a_{72} attached respectively to $\mathbf{w}\otimes\mathbf{I}$ and $\mathbf{I}\otimes\mathbf{w}$, and so on), for a total of $6 + 6 = 12$ coefficients.

Notice that the three terms embedding the symmetrized dyadic products '$\overline{\underline{\otimes}}$' in representation (2.5) are attached to the three coefficients a_2, a_4, a_6, that we may label *'shear-like'*, and affect only the diagonal entries of a 6×6 matrix representation of the damage-effect tensor in the principal axes of damage. The six supplemental rank-one updates provided by the addends with standard dyadic products '$\otimes$' are attached to the remaining six coefficients a_1, a_3, a_5, a_7, a_8, a_9, that we may label *'non-shear'*, to distinguish them from the previous ones, and affect only the upper-left 3×3 submatrix representation of $\mathbb{A}$ (these issues are discussed in details in [20]). The subdivision of terms and coefficients in these two classes is just formal and concerns the algebraic structure of Eq. (2.5): it attaches names to objects for further reference; no implications in the mechanical sense are really implied at this level. For example, this subdivision does not necessarily correspond to the possibility of defining three independent shears, as discussed e.g. by Blinowski and Rychlewski [4]. Notice also that, during the process of irreversible damage, the type of material symmetry that is described by representation (2.5) may change, but within the algebraic structure therein assumed.

Representations similar to (2.5) hold as well for the dual damage-effect tensor $\bar{\mathbb{A}}$, in terms of the dual square root integrity variable $\bar{\mathbf{w}}$ and dual coefficients with bars, $\bar{a}_i$, $i = 1-9$ (generally functions of the three principal invariants of $\bar{\mathbf{w}}$, ${}^{\bar{\mathbf{w}}}I_1 = \mathrm{tr}\ \bar{\mathbf{w}}, {}^{\bar{\mathbf{w}}}I_2 = (\mathrm{tr}^2\ \bar{\mathbf{w}} - \mathrm{tr}\ \bar{\mathbf{w}}^2)/2$, ${}^{\bar{\mathbf{w}}}I_3 = \det \bar{\mathbf{w}}$), and also for the current compliance $\mathbb{C}$ and stiffness $\mathbb{E}$ in terms of the damage variables $\boldsymbol{\phi}$, $\bar{\boldsymbol{\phi}}$ and analogous scalar coefficients c_i, e_i, $i = 1-9$. The links between alternative representations of each fourth-order tensor in terms of either $\boldsymbol{\phi}$ or $\mathbf{w}$ (and of $\bar{\boldsymbol{\phi}}$ or $\bar{\mathbf{w}}$) can be obtained through the representation of isotropic functions $\boldsymbol{\phi} = \mathbf{w}^2$, $\mathbf{w} = \boldsymbol{\phi}^{1/2}$ (and $\bar{\boldsymbol{\phi}} = \bar{\mathbf{w}}^2$, $\bar{\mathbf{w}} = \bar{\boldsymbol{\phi}}^{1/2}$).

Notice that a natural constraint on representation (2.5) (and dual one for $\bar{\mathbb{A}}$) arises from Eq. (2.3) in the absence of damage: nominal and effective quantities have to coincide and the linear transformations must reduce to the (symmetric) identity. Then, for $\mathbf{w} = \mathbf{I}$ (undamaged state), $\mathbb{A}(\mathbf{I}) = \mathbb{I}^s = \mathbf{I} \underline{\overline{\otimes}} \mathbf{I}$, that is, when all the scalar coefficients are evaluated in $\mathbf{w} = \mathbf{I}$:

$$(2.6) \qquad [a_2 + 2a_4 + a_6](\mathbf{I}) = 1, \qquad [a_1 + a_3 + a_5 + a_7 + a_8 + a_9](\mathbf{I}) = 0\ .$$

3. Significative examples of dual damage-effect tensors

Considering representation (2.5) for $\mathbb{A}$ and a dual one for $\bar{\mathbb{A}}$, the point under consideration here is precisely that of seeking particular instances of such general representations (possibly with a reduced number of tensor terms) that correspond to each other through an inversion operation spanning the same set of terms. In other words, if a coefficient is taken out from the complete orthotropic representation of $\mathbb{A}$, say e.g. a_1, while the others are kept in, the dual coefficient $\bar{a}_1$, and none other, should disappear as well in the dual representation of $\bar{\mathbb{A}}$.

The task of seeking instances that solve the problem at hand has been tackled in [20]. A set of solution instances has been advanced, based on either a rigorous treatment, whenever possible (that takes advantage of matrix representations on the damage-effect tensors in the principal axes of damage), or on guessing procedures and guided searches, as well as on the use of tensor multiplication tables and on the repeated application of Sherman–Morrison's formula for the inversion of a rank-one update of a given tensor.

The rigorous analysis showed that solution sets including all the 'shear-like' coefficients a_2, a_4, a_6 are possible and that the unknown dual 'shear-like' coefficients $\bar{a}_2, \bar{a}_4, \bar{a}_6$ can be readily expressed in closed form as follows (their determination is indeed decoupled from that of the 'non-shear' coefficients):

$$\begin{aligned}
\bar{a}_2 &= \frac{a_2^2 + a_4(a_2\,{}^{\mathrm{w}}I_1 + a_4\,{}^{\mathrm{w}}I_2 + a_6\,{}^{\mathrm{w}}I_3)}{d_s}\,, \\
\bar{a}_4 &= \frac{a_2\,a_6 - a_4^2}{d_s}\ {}^{\mathrm{w}}I_3\,, \qquad (3.1)\\
\bar{a}_6 &= \frac{a_4(a_2 + a_4\,{}^{\mathrm{w}}I_1 + a_6\,{}^{\mathrm{w}}I_2) + a_6^2\ {}^{\mathrm{w}}I_3}{d_s}\ {}^{\mathrm{w}}I_3\,,
\end{aligned}$$

where

$$\begin{aligned}
d_s = {a_2}^3 + {a_2}^2\ (2\,a_4\,{}^{\mathrm{w}}I_1 + a_6\,{}^{\mathrm{w}}I_2) + {a_4}^3\ ({}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3) \qquad (3.2)\\
+ 2\,a_4\,{a_6}^2\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3 + {a_6}^3\,{{}^{\mathrm{w}}I_3}^2 + {a_4}^2\,a_6({{}^{\mathrm{w}}I_2}^2 + {}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)\\
+ a_2\left({a_4}^2\,({{}^{\mathrm{w}}I_1}^2 + {}^{\mathrm{w}}I_2) + {a_6}^2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3 + a_4\,a_6\ ({}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2 + 3\,{}^{\mathrm{w}}I_3)\right).
\end{aligned}$$

P r o o f. To prove quickly these relations in an independent way, consider representation (2.5) for $\mathbb{A}$ in terms of $\mathbf{w}$ and dual one for $\bar{\mathbb{A}} = \mathbb{A}^{-1}$ in terms of $\bar{\mathbf{w}} = \mathbf{w}^{-1}$. Since it is required that $\mathbb{A} : \bar{\mathbb{A}} = \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}$, one expands the product of $\mathbb{A}$ and $\bar{\mathbb{A}}$ and retains only 'shear-like' terms with symmetrized dyadic product $\underline{\overline{\otimes}}$ (since those are the only ones that can fully span the identity, see observations in the paragraph after Eq. (2.5)). By performing such operation, whereby one may make use of tensor multiplication tables as those provided in [19, Appendix A], one gets:

$$\begin{aligned}
\mathbb{A} : \bar{\mathbb{A}} = (a_2\bar{a}_2 + 2a_4\bar{a}_4 + a_6\bar{a}_6)\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} + a_6\bar{a}_2\,\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} \qquad (3.3)\\
+ (a_4\bar{a}_2 + a_6\bar{a}_4)\,(\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{w}) + a_2\bar{a}_6\,\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}}\\
+ (a_2\bar{a}_4 + a_4\bar{a}_6)\,(\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}}) + a_4\bar{a}_4\,(\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\mathbf{w} + \mathbf{w}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}})\\
+ \ldots\ (\text{'non-shear' terms with } \otimes)\,.
\end{aligned}$$

Next $\bar{\mathbf{w}}$ is expressed in terms of $\mathbf{w}$ through the Cayley-Hamilton theorem applied to $\mathbf{w}$, $\bar{\mathbf{w}}=1/{}^{\mathrm{w}}I_3\ (\mathbf{w}^2 - {}^{\mathrm{w}}I_1\,\mathbf{w} - {}^{\mathrm{w}}I_2\,\mathbf{I})$ and substituted in the last three 'shear-like' terms of Eq. (3.3). Also, Rivlin's tensorial identities [20, Appendix B], are employed to reduce the arising terms in $\mathbf{w}^2$:

$$\begin{aligned}
\mathbf{w}^2\,\underline{\overline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{w}^2 &= -{}^{\mathrm{w}}I_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} - \mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} + {}^{\mathrm{w}}I_1\,(\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{w}) + \ldots\ ;\\
\mathbf{w}^2\,\underline{\overline{\otimes}}\,\mathbf{w} + \mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w}^2 &= -{}^{\mathrm{w}}I_3\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} + {}^{\mathrm{w}}I_1\,\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} + \ldots\ ; \qquad (3.4)\\
\mathbf{w}^2\,\underline{\overline{\otimes}}\,\mathbf{w}^2 &= {}^{\mathrm{w}}I_2\,\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} - {}^{\mathrm{w}}I_3\,(\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{w}) + \ldots\ .
\end{aligned}$$

By doing so, one gets:

$$\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}} = \frac{1}{{}^{\mathrm{w}}I_3}\left({}^{\mathrm{w}}I_1\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} - (\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{w})\right) + \dots\ ;$$

$$\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}} = \frac{1}{{}^{\mathrm{w}}I_3}\left({}^{\mathrm{w}}I_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} - \mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w}\right) + \dots\ ; \tag{3.5}$$

$$\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\mathbf{w} + \mathbf{w}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}} = \frac{1}{{}^{\mathrm{w}}I_3}\left(-\,{}^{\mathrm{w}}I_3\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} - {}^{\mathrm{w}}I_1\,\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} + {}^{\mathrm{w}}I_2\,(\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{w})\right) + \dots .$$

Finally, one collects terms $\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}$, $\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w}$, $\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{w}$ and, to impose $\mathbb{A} : \bar{\mathbb{A}} = \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}$ at any damage state $\mathbf{w}$, sets to 1 the coefficient premultiplying $\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}$ and to 0 the coefficients premultiplying $\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w}$ and $\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{w}$, to gather the following 3×3 linear system of equations in the unknowns $\bar{a}_2, \bar{a}_4, \bar{a}_6$:

$$\begin{bmatrix} a_2 & a_2{}^{\mathrm{w}}I_2/{}^{\mathrm{w}}I_3 + a_4 & a_2{}^{\mathrm{w}}I_1/{}^{\mathrm{w}}I_3 + a_4{}^{\mathrm{w}}I_2/{}^{\mathrm{w}}I_3 + a_6 \\ -a_6{}^{\mathrm{w}}I_3 & a_2 + a_4{}^{\mathrm{w}}I_1 & a_4 \\ -a_4{}^{\mathrm{w}}I_3 & -a_4{}^{\mathrm{w}}I_2 - a_6{}^{\mathrm{w}}I_3 & a_2 \end{bmatrix} \cdot \begin{Bmatrix} \bar{a}_2 \\ \bar{a}_4 \\ \bar{a}_6 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \tag{3.6}$$

which renders Eqs. (3.1)–(3.2). Notice that d_s, Eq. (3.2), is the determinant of the coefficient matrix in Eq. (3.6).

On the other hand, locating specific subsets of the 'non-shear' coefficients a_1, a_3, a_5, a_7, a_8, a_9 and dual ones $\bar{a}_1, \bar{a}_3, \bar{a}_5, \bar{a}_7, \bar{a}_8, \bar{a}_9$ that correspond to each other in the dual structures, turns out to be more involved. Particular solution instances of this sort that miss one or two of the three 'shear-like' terms can be conveniently identified from Eq. (3.1), in particular those containing only either 'shear-like' coefficients a_2, $\bar{a}_2$ or a_6, $\bar{a}_6$ (in such cases the dual coefficients simply become reciprocal, i.e. $a_2 = 1/\bar{a}_2$, $\bar{a}_2 = 1/a_2$ or $a_6 = 1/\bar{a}_6$, $\bar{a}_6 = 1/a_6$). These two possibilities are further explored in the present paper, specifically the second one concerning 'shear-like' coefficients $a_6, \bar{a}_6$, which originates the complete solution family presented in Sec. 4. A new general solution based on 'shear-like' coefficients $a_2, \bar{a}_2$ and containing all six 'non-shear' coefficients is derived as well in the paper and given below in the present section.

Before presenting the solution instances of the family characterized by 'shear-like' coefficients $a_6, \bar{a}_6$ (Sec. 4), a few particular instances of damage-effect tensors endowed with dual structures and based on both 'shear-like' generators

$a_2\,\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I}$, $\bar{a}_2\,\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I}$ and $a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}$, $\bar{a}_6\,\bar{\mathbf{w}}\,\overline{\underline{\otimes}}\,\bar{\mathbf{w}}$ are given below as illustrative examples of the sought correspondence. All the listed cases hold without placing any constraint on the coefficients.

A first case is readily apparent:

SOLUTION (2.1). The isotropic case in which only the two-coefficients sets (a_1, a_2) and $(\bar{a}_1, \bar{a}_2)$ are kept in the expansions of $\mathbb{A}$ and $\bar{\mathbb{A}}$ (in the following notations the pivoting 'shear-like' terms are always presented first):

$$\mathbb{A} = a_2\,\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I} + a_1\,\mathbf{I}\otimes\mathbf{I}, \qquad \bar{\mathbb{A}} = \bar{a}_2\,\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I} + \bar{a}_1,\ \mathbf{I}\otimes\mathbf{I}\,, \tag{3.7}$$

with the classical 'isotropic' inversion relations

$$\bar{a}_2 = \frac{1}{a_2} \qquad \bar{a}_1 = -\frac{a_1}{a_2\,(3a_1 + a_2)}\,. \tag{3.8}$$

This assumption may lead to a general form of isotropic damage based on two independent scalar damage variables (if the two coefficient functions a_1, a_2 are taken as independent), or to a restricted form of isotropic damage based on a single scalar damage variable (if the two coefficient functions a_1, a_2 are linked to each other, as for instance is the case in the classical scalar damage models of the so-called $(1-D)$-type), see e.g. JU [10]. *Solution* instance *(2.1)* is based on 'shear-like' generators $a_2\,\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I}$, $\bar{a}_2\,\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I}$ (number 2 as the first label digit) and contains *a single* 'non-shear' coefficient (number 1 as the second label digit).

A new solution instance based on the same 'shear-like' generators $a_2\,\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I}$, $\bar{a}_2\,\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I}$ and representing a full generalization of *Solution (2.1)* that contains all *six* 'non-shear' coefficients $a_1, a_3, a_5, a_7, a_8, a_9$ (and thus without constraints on the coefficients) can be derived as follows in full invariant form:

SOLUTION (2.6). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the seven-coefficients sets $(a_1, a_2, a_3, a_5, a_7, a_8, a_9)$ and $(\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_5, \bar{a}_7, \bar{a}_8, \bar{a}_9)$ (lacking only the two 'shear-like' coefficients a_4, a_6 and $\bar{a}_4, \bar{a}_6$), form the symmetric dual inverse pair:

$$\begin{aligned}
\mathbb{A} &= a_2\,\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I} + a_1\,\mathbf{I}\otimes\mathbf{I} + a_3\,\mathbf{w}\otimes\mathbf{w} + a_5\,\mathbf{w}^2\otimes\mathbf{w}^2 \\
&\quad + a_7\,(\mathbf{w}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}) + a_8\,(\mathbf{w}^2\otimes\mathbf{w} + \mathbf{w}\otimes\mathbf{w}^2) + a_9\,(\mathbf{w}^2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}^2), \\
\bar{\mathbb{A}} &= \bar{a}_2\,\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I} + \bar{a}_1\,\mathbf{I}\otimes\mathbf{I} + \bar{a}_3\,\bar{\mathbf{w}}\otimes\bar{\mathbf{w}} + \bar{a}_5\,\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}^2 \\
&\quad + \bar{a}_7\,(\bar{\mathbf{w}}\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{w}}) + \bar{a}_8\,(\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}} + \bar{\mathbf{w}}\otimes\bar{\mathbf{w}}^2) + \bar{a}_9\,(\bar{\mathbf{w}}^2\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{w}}^2)\,,
\end{aligned} \tag{3.9}$$

with

$$\bar{a}_2 = \frac{1}{a_2}, \qquad \bar{a}_1 = \frac{\bar{n}_{21}}{a_2\, \bar{d}_2}, \qquad \bar{a}_3 = \frac{\bar{n}_{23}}{a_2\, \bar{d}_2}\,, \qquad \bar{a}_5 = \frac{\bar{n}_{25}\, {}^{\mathrm{w}}I_3^{\,2}}{a_2\, \bar{d}_2}\,, \tag{3.10}$$
$$\bar{a}_7 = \frac{\bar{n}_{27}}{a_2\, \bar{d}_2}\,, \qquad \bar{a}_8 = \frac{\bar{n}_{28}\, {}^{\mathrm{w}}I_3}{a_2\, \bar{d}_2}\,, \qquad \bar{a}_9 = -\frac{\bar{n}_{29}\, {}^{\mathrm{w}}I_3}{a_2\, \bar{d}_2}\,,$$

where quantities $\bar{d}_2$, $\bar{n}_{21}$, $\bar{n}_{23}$, $\bar{n}_{25}$, $\bar{n}_{27}$, $\bar{n}_{28}$, $\bar{n}_{29}$ represent lengthy expressions that are reported separately in Eqs. (A.1)–(A.7) of Appendix A.1.

In principle, this solution can be derived following arguments similar to those already presented in deriving Eqs. (3.1)–(3.2). The representations for $\mathbb{A}$ ad $\bar{\mathbb{A}}$ are taken without 'shear-like' terms attached to a_4, a_6 and $\bar{a}_4$, $\bar{a}_6$, which are set to zero, and with $\bar{a}_2 = 1/a_2$; the product $\mathbb{A} : \bar{\mathbb{A}}$ is formed by tensor multiplication; the 'non-shear' coefficients with bars are identified by a system of 6×6 linear equations that arises by imposing the equality $\mathbb{A} : \bar{\mathbb{A}} = \mathbf{I} \underline{\overline{\otimes}} \mathbf{I}$ at any damage state. On the other hand, to perform such a task in practice, the procedures that are explained in [20] are employed, which go through matrix representations in the principal axes of damage and are implemented with the help of a mathematical symbolic software.

Selected reduced particular cases of this general solution based on both the five and four 'non-shear' coefficients (including also non-symmetric instances that are not comprised in the general relations provided here) are given in [20]. Further solution instances of this type could be obtained as well as particular cases of *Solution (2.6)* but would be given by even lengthier expressions. From the general solution it can also be checked by inspection that there are no particular cases of this general solution that hold without constraints on the coefficients (besides of course the isotropic *Solution (2.1)* and the degenerate instance containing only the 'shear-like' tensor generators $a_2\, \mathbf{I} \underline{\overline{\otimes}} \mathbf{I}$, $\bar{a}_2\, \mathbf{I} \underline{\overline{\otimes}} \mathbf{I}$ alone). Thus, the family based on *Solution (2.6)* is not explored further.

Four additional significative cases based on 'shear-like' generators $a_6\, \mathbf{w} \underline{\overline{\otimes}} \mathbf{w}$, $\bar{a}_6\, \bar{\mathbf{w}} \underline{\overline{\otimes}} \bar{\mathbf{w}}$ can be reported below (including two non-symmetric ones):

SOLUTION (6.0). The 'shear-like' generators attached to a_6 and $\bar{a}_6$ taken alone (i.e. *zero* 'non-shear' coefficients):

$$\mathbb{A} = a_6\, \mathbf{w} \underline{\overline{\otimes}} \mathbf{w}, \qquad \bar{\mathbb{A}} = \bar{a}_6\, \bar{\mathbf{w}} \underline{\overline{\otimes}} \bar{\mathbf{w}}\,, \tag{3.11}$$

with

$$\bar{a}_6 = \frac{1}{a_6}\,. \tag{3.12}$$

This case is remarkable since it renders, through Eqs. (2.2), (2.4), the Valanis-type compliance and stiffness [22, 25] in which the inverse integrity tensor

$\boldsymbol{\phi}=\bar{\boldsymbol{\phi}}^{-1}$ and integrity tensor $\bar{\boldsymbol{\phi}}$ just replace the identity $\mathbf{I}$ in the original isotropic compliance and stiffness (2.2). Indeed, taking $a_6 = \bar{a}_6 = 1$, i.e. by assuming the so-called 'basic' damage-effect tensors $\mathbb{A}_{\mathrm{bas}} = \boldsymbol{\phi}^{1/2} \,\overline{\underline{\otimes}}\, \boldsymbol{\phi}^{1/2}$ and $\bar{\mathbb{A}}_{\mathrm{bas}} = \bar{\boldsymbol{\phi}}^{1/2} \,\overline{\underline{\otimes}}\, \bar{\boldsymbol{\phi}}^{1/2}$, inverse of each other, Valanis-type secant compliance and stiffness are recovered [7]: $\mathbb{C} = (1+\nu_0)/E_0 \, \boldsymbol{\phi} \,\overline{\underline{\otimes}}\, \boldsymbol{\phi} - \nu_0/E_0 \, \boldsymbol{\phi} \,\overline{\underline{\otimes}}\, \boldsymbol{\phi}$, $\mathbb{E} = 2G_0 \, \bar{\boldsymbol{\phi}} \,\overline{\underline{\otimes}}\, \bar{\boldsymbol{\phi}} + \Lambda_0 \, \bar{\boldsymbol{\phi}} \,\overline{\underline{\otimes}}\, \bar{\boldsymbol{\phi}}$.

Solution (6.1). The implications of *Solution (6.0)* suggest that the symmetric Valanis-type structure of compliance and stiffness could be taken by itself to express the damage-effect tensors, i.e. by keeping only the dual two-coefficients sets (a_6, a_3) and $(\bar{a}_6, \bar{a}_3)$ in the representations of $\mathbb{A}$ and $\bar{\mathbb{A}}$:

$$\mathbb{A} = a_6 \, \mathbf{w} \,\overline{\underline{\otimes}}\, \mathbf{w} + a_3 \, \mathbf{w} \otimes \mathbf{w}, \qquad \bar{\mathbb{A}} = \bar{a}_6 \, \bar{\mathbf{w}} \,\overline{\underline{\otimes}}\, \bar{\mathbf{w}} + \bar{a}_3 \, \bar{\mathbf{w}} \otimes \bar{\mathbf{w}} \, , \tag{3.13}$$

with

$$\bar{a}_6 = \frac{1}{a_6} \, , \qquad \bar{a}_3 = -\frac{a_3}{a_6 \, (3a_3 + a_6)} \, . \tag{3.14}$$

Notice that the 'isotropic' inversion relation between a_3 and $\bar{a}_3$ in Eq. $(3.14)_2$ holds similarly to what is displayed by *Solution (2.1)*, Eq. $(3.8)_2$. Clearly, the arising secant compliance and stiffness are no longer of the Valanis-type. This solution represents the first symmetric generalization of *Solution (6.0)* based on *single* additional 'non-shear' coefficients $a_3, \bar{a}_3$. Further symmetric generalizations containing more 'non-shear' coefficients are pursued systematically in Sec. 4.

Two supplemental *non-symmetric* instances based respectively on *one* and *four* 'non-shear' coefficients are given instead below:

Solution (6.1ns). A non-symmetric solution instance still based on the same 'shear-like' generators attached to $a_6, \bar{a}_6$ and in a sense similar to *Solution (6.1)*, is given by the dual non-symmetric damage-effect tensors embedding just two-coefficients sets (a_6, a_{92}) and $(\bar{a}_6, \bar{a}_{91})$ (in the commented 12-coefficients non-symmetric counterpart of representation (2.5)):

$$\mathbb{A} = a_6 \, \mathbf{w} \,\overline{\underline{\otimes}}\, \mathbf{w} + a_{92} \, \mathbf{I} \otimes \mathbf{w}^2, \qquad \bar{\mathbb{A}} = \bar{a}_6 \, \bar{\mathbf{w}} \,\overline{\underline{\otimes}}\, \bar{\mathbf{w}} + \bar{a}_{91} \, \bar{\mathbf{w}}^2 \otimes \mathbf{I} \, , \tag{3.15}$$

with

$$\bar{a}_6 = \frac{1}{a_6}, \qquad \bar{a}_{91} = -\frac{a_{92}}{a_6 \, (3a_{92} + a_6)} \, . \tag{3.16}$$

The arising secant compliance and stiffness still belong to the above-mentioned Valanis-type structure, but include modified elastic parameters replacing undamaged ones and embed a convenient volumetric/deviatoric decomposition of the damage properties allowing to assign different weights to bulk and shear damage components ('extended' formulation, see [8] for the details).

An additional non-symmetric case still based on the same 'shear-like' coefficients $a_6, \bar{a}_6$ and generalizing the 'extended' model that may be derived from *Solution (6.1ns)* can also be deduced as reported in [20]. This solution case includes four 'non-shear' coefficients. It is remarkable because it comprises previous *Solutions (6.0), (6.1), (6.1ns)* and holds as well *without constraints on the coefficients*:

SOLUTION (6.4ns). Five-coefficients sets $(a_6, a_3, a_{72}, a_{82}, a_{92})$ and $(\bar{a}_6, \bar{a}_3, \bar{a}_{71}, \bar{a}_{81}, \bar{a}_{91})$ give rise to the non-symmetric dual inverse pair (based on *four* 'non-shear' coefficients):

$$\begin{aligned} \mathbb{A} &= a_6\ \mathbf{w} \underline{\overline{\otimes}}\, \mathbf{w} + a_3\ \mathbf{w} \otimes \mathbf{w} + a_{72}\ \mathbf{I} \otimes \mathbf{w} + a_{82}\ \mathbf{w} \otimes \mathbf{w}^2 + a_{92}\ \mathbf{I} \otimes \mathbf{w}^2, \\ \bar{\mathbb{A}} &= \bar{a}_6\ \bar{\mathbf{w}} \underline{\overline{\otimes}}\, \bar{\mathbf{w}} + \bar{a}_3\ \bar{\mathbf{w}} \otimes \bar{\mathbf{w}} + \bar{a}_{71}\ \bar{\mathbf{w}} \otimes \mathbf{I} + \bar{a}_{81}\ \bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}} + \bar{a}_{91}\ \bar{\mathbf{w}}^2 \otimes \mathbf{I}\,, \end{aligned} \tag{3.17}$$

with

$$\begin{aligned} \bar{a}_6 &= \frac{1}{a_6}, \qquad \bar{a}_3 = -\frac{a_6 a_3 + 3(a_3 a_{92} - a_{72} a_{82})}{a_6\ \bar{d}_{ns}}, \\ \bar{a}_{71} &= -\frac{a_6 a_{82} - {}^{\mathrm{w}}I_2/{}^{\mathrm{w}}I_3 (a_3 a_{92} - a_{72} a_{82})}{a_6\ \bar{d}_{ns}}, \\ \bar{a}_{91} &= -\frac{a_6 a_{92} + 3(a_3 a_{92} - a_{72} a_{82})}{a_6\ \bar{d}_{ns}}, \\ \bar{a}_{81} &= -\frac{a_6 a_{72} - {}^{\mathrm{w}}I_1 (a_3 a_{92} - a_{72} a_{82})}{a_6\ \bar{d}_{ns}}, \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} \bar{d}_{ns} = a_6(a_6 + 3a_3 + 3a_{92} + a_{72}{}^{\mathrm{w}}I_2/{}^{\mathrm{w}}I_3 + a_{82}{}^{\mathrm{w}}I_1) \\ + (a_3 a_{92} - a_{72} a_{82})(9 - {}^{\mathrm{w}}I_1\ {}^{\mathrm{w}}I_2/{}^{\mathrm{w}}I_3)\,. \end{aligned} \tag{3.19}$$

Solution (6.1ns) is recovered from *Solution (6.4ns)* by setting consistently $a_3 = a_{72} = a_{82} = 0$, $\bar{a}_3 = \bar{a}_{72} = \bar{a}_{82} = 0$ in Eqs. (3.17)–(3.19). Notice that *'twins'* of *Solutions (6.1ns)* and *(6.4ns)* may be obtained as well just by inverting the roles between coefficients with and without bars in Eqs. (3.15) and (3.17).

4. A family of symmetric dual damage-effect tensors

In this section the complete solution family based on 'shear-like' tensor generators $a_6 \mathbf{w} \underline{\overline{\otimes}}\, \mathbf{w}$, $\bar{a}_6 \bar{\mathbf{w}} \underline{\overline{\otimes}}\, \bar{\mathbf{w}}$ is derived. The more general solution including all 'non-shear' coefficients is presented first. Then, all particular cases endowed with the dual structure are systematically located by inspection and worked out in full invariant form. One notable case with three 'non-shear' coefficients holds without constraints on the coefficients (*Solution (6.3))*.

Solution (6.6). A full solution instance based on 'shear-like' coefficients $a_6, \bar{a}_6$ generalizing *Solutions (6.0), (6.1)* and containing all *six* 'non-shear' terms (thus without constraints on the coefficients) can be obtained by taking symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the seven-coefficients sets $(a_1, a_3, a_5, a_6, a_7, a_8, a_9)$ and $(\bar{a}_1, \bar{a}_3, \bar{a}_5, \bar{a}_6, \bar{a}_7, \bar{a}_8, \bar{a}_9)$ (lacking only the two 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$):

$$
\begin{aligned}
&\mathbb{A} = a_6 \mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{w} + a_1\, \mathbf{I} \otimes \mathbf{I} + a_3\, \mathbf{w} \otimes \mathbf{w} + a_5\, \mathbf{w}^2 \otimes \mathbf{w}^2 \\
&\quad + a_7\, (\mathbf{w} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}) + a_8\, (\mathbf{w}^2 \otimes \mathbf{w} + \mathbf{w} \otimes \mathbf{w}^2) + a_9\, (\mathbf{w}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}^2), \\
&\bar{\mathbb{A}} = \bar{a}_6\, \bar{\mathbf{w}} \,\underline{\overline{\otimes}}\, \bar{\mathbf{w}} + \bar{a}_1\, \mathbf{I} \otimes \mathbf{I} + \bar{a}_3\, \bar{\mathbf{w}} \otimes \bar{\mathbf{w}} + \bar{a}_5\, \bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}}^2 \\
&\quad + \bar{a}_7\, (\bar{\mathbf{w}} \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}) + \bar{a}_8\, (\bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}} + \bar{\mathbf{w}} \otimes \bar{\mathbf{w}}^2) + \bar{a}_9\, (\bar{\mathbf{w}}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}^2)\,,
\end{aligned}
\tag{4.1}
$$

with

$$
\begin{aligned}
&\bar{a}_6 = \frac{1}{a_6}\,; \qquad \bar{a}_1 = -\frac{\bar{n}_{61}}{a_6\, \bar{d}_6}\,, \qquad \bar{a}_3 = -\frac{\bar{n}_{63}}{a_6\, \bar{d}_6}\,, \qquad \bar{a}_5 = \frac{\bar{n}_{65}\; {}^{\mathrm{w}}I_3{}^2}{a_6\, \bar{d}_6}\,, \\
&\qquad \bar{a}_7 = -\frac{\bar{n}_{67}}{a_6\, \bar{d}_6}\,, \qquad \bar{a}_8 = \frac{\bar{n}_{68}\; {}^{\mathrm{w}}I_3}{a_6\, \bar{d}_6}\,, \qquad \bar{a}_9 = \frac{\bar{n}_{69}\; {}^{\mathrm{w}}I_3}{a_6\, \bar{d}_6}\,,
\end{aligned}
\tag{4.2}
$$

where quantities $\bar{d}_6$, $\bar{n}_{61}$, $\bar{n}_{63}$, $\bar{n}_{65}$, $\bar{n}_{67}$, $\bar{n}_{68}$, $\bar{n}_{69}$ are given in Eqs. (A.8)–(A.14) of Appendix A.2.

Although still given by quite lengthy expressions, *Solution (6.6)* looks simpler than its counterpart *Solution (2.6)* based on 'shear-like' generators $a_2\, \mathbf{I} \,\underline{\overline{\otimes}}\, \mathbf{I}$, $\bar{a}_2\, \mathbf{I} \,\underline{\overline{\otimes}}\, \mathbf{I}$ (Sec. 3) and originates further interesting particular cases, which are explored systematically in the sequel. As stated right after Eq. (A.7), this solution has been worked-out algebraically by the procedures reported in [20], which correspond to impose the duality relation $\mathbb{A}{:}\bar{\mathbb{A}}{=}\mathbf{I} \,\underline{\overline{\otimes}}\, \mathbf{I}$ at any damage state, leading to a linear system of 6×6 equations (recall that the 3×3 solution of 'shear-like' coefficients has been handled independently at the beginning of Sec. 3).

We start listing below the six solution cases that contain *five* of the 'non-shear' coefficients and are obtained by eliminating in turn one of the 'non-shear' coefficients from *Solution (6.6)*. The *missing* 'non-shear' coefficient is indicated by the third label digit.

SOLUTION (6.5.1). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the six-coefficients sets $(a_3, a_5, a_6, a_7, a_8, a_9)$ and $(\bar{a}_3, \bar{a}_5, \bar{a}_6, \bar{a}_7, \bar{a}_8, \bar{a}_9)$ (lacking only 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and 'non-shear' coefficients a_1 and $\bar{a}_1$) form the dual inverse pair:

$$\begin{aligned}
&\mathbb{A} = a_6 \mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{w} + a_3\, \mathbf{w} \otimes \mathbf{w} + a_5\, \mathbf{w}^2 \otimes \mathbf{w}^2 \\
&\quad + a_7\,(\mathbf{w} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}) + a_8\,(\mathbf{w}^2 \otimes \mathbf{w} + \mathbf{w} \otimes \mathbf{w}^2) + a_9\,(\mathbf{w}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}^2), \\
&\bar{\mathbb{A}} = \bar{a}_6\, \bar{\mathbf{w}} \,\underline{\overline{\otimes}}\, \bar{\mathbf{w}} + \bar{a}_3\, \bar{\mathbf{w}} \otimes \bar{\mathbf{w}} + \bar{a}_5\, \bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}}^2 \\
&\quad + \bar{a}_7\,(\bar{\mathbf{w}} \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}) + \bar{a}_8\,(\bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}} + \bar{\mathbf{w}} \otimes \bar{\mathbf{w}}^2) + \bar{a}_9\,(\bar{\mathbf{w}}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}^2)\,,
\end{aligned} \tag{4.3}$$

provided that

$$\begin{aligned}
a_3 &= \frac{n^\star_{13}}{3a_5 a_6\, {}^{\mathrm{w}}\!I_3{}^2 - 2a_9^2({}^{\mathrm{w}}\!I_2{}^2 - 3\,{}^{\mathrm{w}}\!I_1\, {}^{\mathrm{w}}\!I_3)}; \\
\bar{a}_3 &= \frac{\bar{n}^\star_{13}}{3\bar{a}_5 \bar{a}_6\, {}^{\bar{\mathrm{w}}}\!I_3{}^2 - 2\bar{a}_9^2({}^{\bar{\mathrm{w}}}\!I_2{}^2 - 3\,{}^{\bar{\mathrm{w}}}\!I_1\, {}^{\bar{\mathrm{w}}}\!I_3)},
\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
n^\star_{13} &= a_6\, {}^{\mathrm{w}}\!I_3 \Big(a_9^2\, {}^{\mathrm{w}}\!I_1 - 2(a_5 a_7 - a_8 a_9)\, {}^{\mathrm{w}}\!I_2 - (a_5 a_6 - 3a_8^2)\, {}^{\mathrm{w}}\!I_3 \Big) \\
&\qquad + \Big(2a_5 a_7^2 + a_9(a_6 a_9 - 4a_7 a_8) \Big)\, ({}^{\mathrm{w}}\!I_2{}^2 - 3\,{}^{\mathrm{w}}\!I_1\, {}^{\mathrm{w}}\!I_3), \\
\bar{n}^\star_{13} &= \bar{a}_6\, {}^{\bar{\mathrm{w}}}\!I_3 \Big(\bar{a}_9^2\, {}^{\bar{\mathrm{w}}}\!I_1 - 2(\bar{a}_5 \bar{a}_7 - \bar{a}_8 \bar{a}_9)\, {}^{\bar{\mathrm{w}}}\!I_2 - (\bar{a}_5 \bar{a}_6 - 3\bar{a}_8^2)\, {}^{\bar{\mathrm{w}}}\!I_3 \Big) \\
&\qquad + \Big(2\bar{a}_5 \bar{a}_7^2 + \bar{a}_9(\bar{a}_6 \bar{a}_9 - 4\bar{a}_7 \bar{a}_8) \Big)\, ({}^{\bar{\mathrm{w}}}\!I_2{}^2 - 3\,{}^{\bar{\mathrm{w}}}\!I_1\, {}^{\bar{\mathrm{w}}}\!I_3)\,,
\end{aligned} \tag{4.5}$$

with

$$\begin{aligned}
&\bar{a}_6 = \frac{1}{a_6}\,; \qquad \bar{a}_3 = \frac{\bar{n}_{13}}{a_6\, \bar{d}_1^2}\,, \qquad \bar{a}_5 = \frac{\bar{n}_{15}\, {}^{\mathrm{w}}\!I_3{}^2}{a_6\, \bar{d}_1^2}\,, \\
&\bar{a}_7 = \frac{a_9^2({}^{\mathrm{w}}\!I_2{}^2 - 2\,{}^{\mathrm{w}}\!I_1\, {}^{\mathrm{w}}\!I_3) - (a_5 a_6\, {}^{\mathrm{w}}\!I_3 + (a_5 a_7 - a_8 a_9)\, {}^{\mathrm{w}}\!I_2)\, {}^{\mathrm{w}}\!I_3}{a_6\, \bar{d}_1}\,, \\
&\bar{a}_8 = \frac{\bar{n}_{18}\, {}^{\mathrm{w}}\!I_3}{a_6\, \bar{d}_1^2}\,, \qquad \bar{a}_9 = \frac{3a_5 a_7\, {}^{\mathrm{w}}\!I_3 - a_9(a_9\, {}^{\mathrm{w}}\!I_2 + 3a_8\, {}^{\mathrm{w}}\!I_3)}{a_6\, \bar{d}_1}\, {}^{\mathrm{w}}\!I_3\,,
\end{aligned} \tag{4.6}$$

where quantities $\bar{d}_1$, $\bar{n}_{13}$, $\bar{n}_{15}$, $\bar{n}_{18}$ are given in Eqs. (A.15)–(A.18) of Appendix A.3. *Solution (6.5.1)* is obtained as a particular case of *Solution (6.6)* by setting $a_1 = 0$, $\bar{a}_1 = 0$, which leads to the constraint (4.4).

Notice that constraints (4.4) must necessarily hold to make (4.3) a dual pair through either the correspondence $\mathbb{A} \rightarrow \bar{\mathbb{A}}$, as presented here, or $\bar{\mathbb{A}} \rightarrow \mathbb{A}$. On the other hand, Eq. $(4.6)_2$ gives the expression of $\bar{a}_3$ in terms of $\mathbf{w}$ as obtained from the solution in the correspondence $\mathbb{A} \rightarrow \bar{\mathbb{A}}$, according to the following reasoning: given $\mathbb{A}$ (in terms of coefficients a_i and tensor $\mathbf{w}$, with constraint $(4.4)_1$ holding on a_3, find out $\bar{\mathbb{A}}$ that is the dual of $\mathbb{A}$, thus all its coefficients $\bar{a}_i$ in terms of a_i and $\mathbf{w}$). One may check that Eq. $(4.6)_2$, in terms of $\mathbf{w}$, turns out to be consistent with Eq. $(4.4)_2$, in terms of $\bar{\mathbf{w}}$.

SOLUTION (6.5.3). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the six-coefficients sets $(a_1, a_5, a_6, a_7, a_8, a_9)$ and $(\bar{a}_1, \bar{a}_5, \bar{a}_6, \bar{a}_7, \bar{a}_8, \bar{a}_9)$ (lacking only 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and 'non-shear' coefficients a_3 and $\bar{a}_3$) form the dual inverse pair:

$$
\begin{aligned}
\mathbb{A} &= a_6 \mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{w} + a_1 \mathbf{I} \otimes \mathbf{I} + a_5 \mathbf{w}^2 \otimes \mathbf{w}^2 \\
&\quad + a_7 (\mathbf{w} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}) + a_8 (\mathbf{w}^2 \otimes \mathbf{w} + \mathbf{w} \otimes \mathbf{w}^2) + a_9 (\mathbf{w}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}^2), \\
\bar{\mathbb{A}} &= \bar{a}_6 \bar{\mathbf{w}} \,\underline{\overline{\otimes}}\, \bar{\mathbf{w}} + \bar{a}_1 \mathbf{I} \otimes \mathbf{I} + \bar{a}_5 \bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}}^2 \\
&\quad + \bar{a}_7 (\bar{\mathbf{w}} \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}) + \bar{a}_8 (\bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}} + \bar{\mathbf{w}} \otimes \bar{\mathbf{w}}^2) + \bar{a}_9 (\bar{\mathbf{w}}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}^2),
\end{aligned}
\tag{4.7}
$$

provided that

$$
\begin{aligned}
a_1 &= \frac{1}{a_8^2} \, \frac{n_{31}^\star}{9{}^{\mathrm{w}}I_3{}^2 - ({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)}; \\
\bar{a}_1 &= \frac{1}{\bar{a}_8^2} \, \frac{\bar{n}_{31}^\star}{9{}^{\bar{\mathrm{w}}}I_3{}^2 - ({}^{\bar{\mathrm{w}}}I_1{}^2 - 2{}^{\bar{\mathrm{w}}}I_2)({}^{\bar{\mathrm{w}}}I_2{}^2 - 2{}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_3)},
\end{aligned}
\tag{4.8}
$$

where

$$
\begin{aligned}
n_{31}^\star &= a_6 a_8 \Big(6a_7 + a_8({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)\Big){}^{\mathrm{w}}I_3{}^2 + a_6 a_7^2({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) \\
&\qquad - a_7(a_5 a_7 - 2a_8 a_9)\Big(9{}^{\mathrm{w}}I_3{}^2 - ({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big), \\
\bar{n}_{31}^\star &= \bar{a}_6 \bar{a}_8 \Big(6\bar{a}_7 + \bar{a}_8({}^{\bar{\mathrm{w}}}I_1{}^2 - 2{}^{\bar{\mathrm{w}}}I_2)\Big){}^{\bar{\mathrm{w}}}I_3{}^2 + \bar{a}_6 \bar{a}_7^2({}^{\bar{\mathrm{w}}}I_2{}^2 - 2{}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_3) \\
&\qquad - \bar{a}_7(\bar{a}_5 \bar{a}_7 - 2\bar{a}_8 \bar{a}_9)\Big(9{}^{\bar{\mathrm{w}}}I_3{}^2 - ({}^{\bar{\mathrm{w}}}I_1{}^2 - 2{}^{\bar{\mathrm{w}}}I_2)({}^{\bar{\mathrm{w}}}I_2{}^2 - 2{}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_3)\Big),
\end{aligned}
\tag{4.9}
$$

with

$$\bar{a}_6 = \frac{1}{a_6}\,; \qquad \bar{a}_1 = \frac{\bar{n}_{31}}{a_6\,\bar{d}_3^2}\,, \qquad \bar{a}_5 = -\frac{\bar{n}_{35}\,{}^{\mathrm{w}}I_3{}^2}{a_6\,\bar{d}_3^2}\,,$$

(4.10)
$$\bar{a}_7 = -a_8\,\frac{a_7({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) + 3a_8{}^{\mathrm{w}}I_3{}^2}{a_6\,\bar{d}_3}\,,$$

$$\bar{a}_8 = a_8\,\frac{3a_7 + a_8({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)}{a_6\,\bar{d}_3}\,{}^{\mathrm{w}}I_3{}^2\,, \qquad \bar{a}_9 = \frac{\bar{n}_{39}\,{}^{\mathrm{w}}I_3}{a_6\,\bar{d}_3^2}\,,$$

where quantities $\bar{d}_3$, $\bar{n}_{31}$, $\bar{n}_{35}$, $\bar{n}_{39}$ are given in Eqs. (A.19)–(A.22) of Appendix A.4. *Solution (6.5.3)* is obtained as a particular case of *Solution (6.6)* by setting $a_3 = 0$, $\bar{a}_3 = 0$, which leads to the constraint (4.8).

Solution (6.5.5). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the six-coefficients sets $(a_1, a_3, a_6, a_7, a_8, a_9)$ and $(\bar{a}_1, \bar{a}_3, \bar{a}_6, \bar{a}_7, \bar{a}_8, \bar{a}_9)$ (lacking only 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and 'non-shear' coefficients a_5 and $\bar{a}_5$) form the dual inverse pair:

(4.11)
$$\begin{aligned}\mathbb{A} &= a_6\,\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} + a_1\,\mathbf{I}\otimes\mathbf{I} + a_3\,\mathbf{w}\otimes\mathbf{w}\\ &\quad + a_7\,(\mathbf{w}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}) + a_8\,(\mathbf{w}^2\otimes\mathbf{w} + \mathbf{w}\otimes\mathbf{w}^2) + a_9\,(\mathbf{w}^2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}^2),\\ \bar{\mathbb{A}} &= \bar{a}_6\,\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}} + \bar{a}_1\,\mathbf{I}\otimes\mathbf{I} + \bar{a}_3\,\bar{\mathbf{w}}\otimes\bar{\mathbf{w}}\\ &\quad + \bar{a}_7\,(\bar{\mathbf{w}}\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{w}}) + \bar{a}_8\,(\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}} + \bar{\mathbf{w}}\otimes\bar{\mathbf{w}}^2) + \bar{a}_9\,(\bar{\mathbf{w}}^2\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{w}}^2)\,,\end{aligned}$$

provided that

(4.12)
$$\begin{aligned}a_3 &= -\frac{a_6\Big(a_1a_6 - 3a_7^2 + 2(a_1a_8 - a_7a_9){}^{\mathrm{w}}I_1 - a_9^2({}^{\mathrm{w}}I_1{}^2 - 2\,{}^{\mathrm{w}}I_2)\Big)}{3a_1a_6 - 2a_9^2({}^{\mathrm{w}}I_1{}^2 - 3\,{}^{\mathrm{w}}I_2)}\\ &\qquad + \frac{2a_8(a_1a_8 - 2a_7a_9)({}^{\mathrm{w}}I_1{}^2 - 3\,{}^{\mathrm{w}}I_2)}{3a_1a_6 - 2a_9^2({}^{\mathrm{w}}I_1{}^2 - 3\,{}^{\mathrm{w}}I_2)},\\ \bar{a}_3 &= -\frac{\bar{a}_6\Big(\bar{a}_1\bar{a}_6 - 3\bar{a}_7^2 + 2(\bar{a}_1\bar{a}_8 - \bar{a}_7\bar{a}_9){}^{\bar{\mathrm{w}}}I_1 - \bar{a}_9^2({}^{\bar{\mathrm{w}}}I_1{}^2 - 2\,{}^{\bar{\mathrm{w}}}I_2)\Big)}{3\bar{a}_1\bar{a}_6 - 2\bar{a}_9^2({}^{\bar{\mathrm{w}}}I_1{}^2 - 3\,{}^{\bar{\mathrm{w}}}I_2)}\\ &\qquad + \frac{2\bar{a}_8(\bar{a}_1\bar{a}_8 - 2\bar{a}_7\bar{a}_9)({}^{\bar{\mathrm{w}}}I_1{}^2 - 3\,{}^{\bar{\mathrm{w}}}I_2)}{3\bar{a}_1\bar{a}_6 - 2\bar{a}_9^2({}^{\bar{\mathrm{w}}}I_1{}^2 - 3\,{}^{\bar{\mathrm{w}}}I_2)},\end{aligned}$$

with

$$\bar{a}_6 = \frac{1}{a_6}; \qquad \bar{a}_1 = \frac{\bar{n}_{51}}{a_6\,\bar{d}_5^2}\,, \qquad \bar{a}_3 = \frac{\bar{n}_{53}}{a_6\,\bar{d}_5^2}\,, \qquad \bar{a}_7 = -\frac{\bar{n}_{57}}{a_6\,\bar{d}_5^2}\,,$$

$$\bar{a}_8 = -\frac{a_1\,(a_6 + a_8\,{}^{\mathrm{w}}I_1) - a_9\left(a_7\,{}^{\mathrm{w}}I_1 + a_9\,({}^{\mathrm{w}}I_1{}^2 - 2\,{}^{\mathrm{w}}I_2)\right)}{a_6\,\bar{d}_5}\,{}^{\mathrm{w}}I_3\,, \tag{4.13}$$

$$\bar{a}_9 = \frac{3\,a_1\,a_8 - a_9\,(3\,a_7 + a_9{}^{\mathrm{w}}I_1)}{a_6\,\bar{d}_5}\,{}^{\mathrm{w}}I_3\,,$$

where quantities $\bar{d}_5$, $\bar{n}_{51}$, $\bar{n}_{53}$, $\bar{n}_{57}$ are given in Eqs. (A.23)–(A.26) of Appendix A.5. *Solution (6.6)* is obtained as a particular case of *Solution (6.5.5)* by setting $a_5 = 0$, $\bar{a}_5 = 0$, which leads to the constraint (4.12).

SOLUTION (6.5.7). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the six-coefficients sets $(a_1, a_3, a_5, a_6, a_8, a_9)$ and $(\bar{a}_1, \bar{a}_3, \bar{a}_5, \bar{a}_6, \bar{a}_8, \bar{a}_9)$ (lacking only 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and 'non-shear' coefficients a_7 and $\bar{a}_7$) form the dual inverse pair:

$$\begin{aligned}\mathbb{A} &= a_6\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} + a_1\,\mathbf{I}\otimes\mathbf{I} + a_3\,\mathbf{w}\otimes\mathbf{w} + a_5\,\mathbf{w}^2\otimes\mathbf{w}^2\\ &\qquad + a_8\,(\mathbf{w}^2\otimes\mathbf{w} + \mathbf{w}\otimes\mathbf{w}^2) + a_9\,(\mathbf{w}^2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}^2),\\ \bar{\mathbb{A}} &= \bar{a}_6\,\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}} + \bar{a}_1\,\mathbf{I}\otimes\mathbf{I} + \bar{a}_3\,\bar{\mathbf{w}}\otimes\bar{\mathbf{w}} + \bar{a}_5\,\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}^2\\ &\qquad + \bar{a}_8\,(\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}} + \bar{\mathbf{w}}\otimes\bar{\mathbf{w}}^2) + \bar{a}_9\,(\bar{\mathbf{w}}^2\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{w}}^2)\,,\end{aligned} \tag{4.14}$$

provided that

$$\begin{aligned}a_5 &= \frac{1}{a_3}\,\frac{n^\star_{75}}{a_6{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3{}^2 + a_1({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^2{}^{\mathrm{w}}I_3 - 3{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3)},\\ \bar{a}_5 &= \frac{1}{\bar{a}_3}\,\frac{\bar{n}^\star_{75}}{\bar{a}_6{}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_3{}^2 + \bar{a}_1({}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_2{}^2 - 2{}^{\bar{\mathrm{w}}}I_1{}^2{}^{\bar{\mathrm{w}}}I_3 - 3{}^{\bar{\mathrm{w}}}I_2{}^{\bar{\mathrm{w}}}I_3)}\,,\end{aligned} \tag{4.15}$$

where

$$\begin{aligned}n^\star_{75} &= a_1a_6a_8({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) - a_3a_6a_9{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3 + a_6a_8(a_6 + 3a_9 + a_8{}^{\mathrm{w}}I_1){}^{\mathrm{w}}I_3{}^2\\ &\qquad + (a_1a_8^2 + a_3a_9^2)({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^2{}^{\mathrm{w}}I_3 - 3{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3),\\ \bar{n}^\star_{75} &= \bar{a}_1\bar{a}_6\bar{a}_8({}^{\bar{\mathrm{w}}}I_2{}^2 - 2{}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_3) - \bar{a}_3\bar{a}_6\bar{a}_9{}^{\bar{\mathrm{w}}}I_2{}^{\bar{\mathrm{w}}}I_3 + \bar{a}_6\bar{a}_8(\bar{a}_6 + 3\bar{a}_9 + \bar{a}_8{}^{\bar{\mathrm{w}}}I_1){}^{\bar{\mathrm{w}}}I_3{}^2\\ &\qquad + (\bar{a}_1\bar{a}_8^2 + \bar{a}_3\bar{a}_9^2)({}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_2{}^2 - 2{}^{\bar{\mathrm{w}}}I_1{}^2{}^{\bar{\mathrm{w}}}I_3 - 3{}^{\bar{\mathrm{w}}}I_2{}^{\bar{\mathrm{w}}}I_3),\end{aligned} \tag{4.16}$$

with

$$\bar a_6 = \frac{1}{a_6}\,; \qquad \bar a_1 = \frac{(a_3 a_9 {}^{\mathrm w}I_2 - a_6 a_8 {}^{\mathrm w}I_3){}^{\mathrm w}I_3 - a_1 a_8({}^{\mathrm w}I_2{}^2 - 2{}^{\mathrm w}I_1{}^{\mathrm w}I_3)}{a_6\,\bar d_{71}}\,,$$

(4.17)
$$\bar a_3 = -a_3\,\frac{a_1({}^{\mathrm w}I_2{}^2 - 2{}^{\mathrm w}I_1{}^{\mathrm w}I_3) + (a_6+3a_9){}^{\mathrm w}I_3{}^2}{a_6\,\bar d_{73}}\,, \qquad \bar a_5 = \frac{\bar n_{75}\ {}^{\mathrm w}I_3{}^2}{a_6\,\bar d_{71}\,\bar d_{73}}\,,$$

$$\bar a_8 = a_3\,\frac{a_1{}^{\mathrm w}I_2 + a_9{}^{\mathrm w}I_1{}^{\mathrm w}I_3}{a_6\,\bar d_{73}}\ {}^{\mathrm w}I_3\,, \qquad \bar a_9 = \frac{3a_1 a_8 - a_3 a_9{}^{\mathrm w}I_1}{a_6\,\bar d_{71}}\ {}^{\mathrm w}I_3{}^2\,,$$

where quantities $\bar d_{71}$, $\bar d_{73}$, $\bar n_{75}$ are given in Eqs. (A.27)–(A.29) of Appendix A.6. *Solution (6.5.7)* is obtained as a particular case of *Solution (6.6)* by setting $a_7 = 0$, $\bar a_7 = 0$, which leads to the constraint (4.15).

Solution (6.5.8). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the six-coefficients sets $(a_1, a_3, a_5, a_6, a_7, a_9)$ and $(\bar a_1, \bar a_3, \bar a_5, \bar a_6, \bar a_7, \bar a_9)$ (lacking only 'shear-like' coefficients a_2, a_4 and $\bar a_2, \bar a_4$, and 'non-shear' coefficients a_8 and $\bar a_8$) form the dual inverse pair:

(4.18)
$$\begin{aligned}\mathbb{A} &= a_6\ \mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} + a_1\,\mathbf{I}\otimes\mathbf{I} + a_3\,\mathbf{w}\otimes\mathbf{w} + a_5\,\mathbf{w}^2\otimes\mathbf{w}^2\\ &\qquad + a_7\,(\mathbf{w}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}) + a_9\,(\mathbf{w}^2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}^2)\,,\\ \bar{\mathbb{A}} &= \bar a_6\ \bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}} + \bar a_1\,\mathbf{I}\otimes\mathbf{I} + \bar a_3\,\bar{\mathbf{w}}\otimes\bar{\mathbf{w}} + \bar a_5\,\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}^2\\ &\qquad + \bar a_7\,(\bar{\mathbf{w}}\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{w}}) + \bar a_9\,(\bar{\mathbf{w}}^2\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{w}}^2)\,,\end{aligned}$$

provided that

(4.19)
$$\begin{aligned}a_1 &= \frac{1}{a_3}\ \frac{n^\star_{81}}{a_6{}^{\mathrm w}I_2 + a_5({}^{\mathrm w}I_1{}^2\,{}^{\mathrm w}I_2 - 2{}^{\mathrm w}I_2{}^2 - 3{}^{\mathrm w}I_1{}^{\mathrm w}I_3)}\,,\\ \bar a_1 &= \frac{1}{\bar a_3}\ \frac{\bar n^\star_{81}}{\bar a_6{}^{\bar{\mathrm w}}I_2 + \bar a_5({}^{\bar{\mathrm w}}I_1{}^2\,{}^{\bar{\mathrm w}}I_2 - 2{}^{\bar{\mathrm w}}I_2{}^2 - 3{}^{\bar{\mathrm w}}I_1{}^{\bar{\mathrm w}}I_3)}\,,\end{aligned}$$

where

(4.20)
$$\begin{aligned}n^\star_{81} &= a_6^2 a_7{}^{\mathrm w}I_3 + (a_3 a_9^2 + a_5 a_7^2)({}^{\mathrm w}I_1{}^2\,{}^{\mathrm w}I_2 - 2{}^{\mathrm w}I_2{}^2 - 3{}^{\mathrm w}I_1{}^{\mathrm w}I_3)\\ &\quad + a_6\Big[a_7^2{}^{\mathrm w}I_2 - a_3 a_9{}^{\mathrm w}I_1{}^{\mathrm w}I_3 + a_7\Big(3a_9 + a_5({}^{\mathrm w}I_1{}^2 - 2{}^{\mathrm w}I_2)\Big){}^{\mathrm w}I_3\Big]\,,\end{aligned}$$

$$\bar{n}^{\star}_{81} = \bar{a}_6^2\bar{a}_7{}^{\bar{w}}I_3 + (\bar{a}_3\bar{a}_9^2 + \bar{a}_5\bar{a}_7^2)({}^{\bar{w}}I_1{}^2\,{}^{\bar{w}}I_2 - 2\,{}^{\bar{w}}I_2{}^2 - 3\,{}^{\bar{w}}I_1{}^{\bar{w}}I_3)$$
$$+ \bar{a}_6\Big[\bar{a}_7^2\,{}^{\bar{w}}I_2 - \bar{a}_3\bar{a}_9{}^{\bar{w}}I_1{}^{\bar{w}}I_3 + \bar{a}_7\Big(3\bar{a}_9 + \bar{a}_5({}^{\bar{w}}I_1{}^2 - 2\,{}^{\bar{w}}I_2)\Big){}^{\bar{w}}I_3\Big]\,, \tag{4.20 [cont.]}$$

with

$$\bar{a}_6 = \frac{1}{a_6}; \qquad \bar{a}_1 = \frac{\bar{n}_{81}}{a_6\,\bar{d}_{83}\,\bar{d}_{85}}\,,$$
$$\bar{a}_3 = -a_3\,\frac{a_6 + 3a_9 + a_5({}^{w}I_1{}^2 - 2\,{}^{w}I_2)}{a_6\,\bar{d}_{83}}\,{}^{w}I_3\,,$$
$$\bar{a}_5 = -\frac{a_6a_7 - a_3a_9{}^{w}I_1 + a_5a_7({}^{w}I_1{}^2 - 2\,{}^{w}I_2)}{a_6\,\bar{d}_{85}}\,{}^{w}I_3{}^2\,, \tag{4.21}$$
$$\bar{a}_7 = a_3\,\frac{a_5{}^{w}I_1{}^{w}I_3 + a_9{}^{w}I_2}{a_6\,\bar{d}_{83}}\,, \qquad \bar{a}_9 = -\frac{a_3a_9{}^{w}I_2 - 3a_5a_7{}^{w}I_3}{a_6\,\bar{d}_{85}}\,{}^{w}I_3\,,$$

where quantities $\bar{d}_{83}$, $\bar{d}_{85}$, $\bar{n}_{81}$ are given in Eqs. (A.30)–(A.32) of Appendix A.7. *Solution (6.5.8)* is obtained as a particular case of *Solution (6.6)* by setting $a_8 = 0$, $\bar{a}_8 = 0$, which leads to the constraint (4.19).

Solution (6.5.9). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the six-coefficients sets $(a_1, a_3, a_5, a_6, a_7, a_8)$ and $(\bar{a}_1, \bar{a}_3, \bar{a}_5, \bar{a}_6, \bar{a}_7, \bar{a}_8)$ (lacking only 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and 'non-shear' coefficients a_9 and $\bar{a}_9$) form the dual inverse pair:

$$\mathbb{A} = a_6\,\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} + a_1\,\mathbf{I}\otimes\mathbf{I} + a_3\,\mathbf{w}\otimes\mathbf{w} + a_5\,\mathbf{w}^2\otimes\mathbf{w}^2$$
$$+ a_7\,(\mathbf{w}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}) + a_8\,(\mathbf{w}^2\otimes\mathbf{w} + \mathbf{w}\otimes\mathbf{w}^2)\,,$$
$$\bar{\mathbb{A}} = \bar{a}_6\,\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}} + \bar{a}_1\,\mathbf{I}\otimes\mathbf{I} + \bar{a}_3\,\bar{\mathbf{w}}\otimes\bar{\mathbf{w}} + \bar{a}_5\,\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}^2 \tag{4.22}$$
$$+ \bar{a}_7\,(\bar{\mathbf{w}}\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{w}}) + \bar{a}_8\,(\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}} + \bar{\mathbf{w}}\otimes\bar{\mathbf{w}}^2)\,,$$

provided that

$$a_3 = \frac{a_1a_6a_8{}^{w}I_2 + (a_1a_8^2 + a_5a_7^2)({}^{w}I_1{}^{w}I_2 - 9\,{}^{w}I_3)}{a_1a_5({}^{w}I_1{}^{w}I_2 - 9\,{}^{w}I_3)} \tag{4.23}$$
$$+ \frac{a_6(3a_1a_5 + 3a_7a_8 + a_5a_7{}^{w}I_1){}^{w}I_3}{a_1a_5({}^{w}I_1{}^{w}I_2 - 9\,{}^{w}I_3)}\,,$$

(4.23) [cont.]
$$\bar{a}_3 = \frac{\bar{a}_1\bar{a}_6\bar{a}_8\,{}^{\bar{\mathrm{w}}}I_2 + (\bar{a}_1\bar{a}_8^2 + \bar{a}_5\bar{a}_7^2)({}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_2 - 9{}^{\bar{\mathrm{w}}}I_3)}{\bar{a}_1\bar{a}_5({}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_2 - 9{}^{\bar{\mathrm{w}}}I_3)}, + \frac{\bar{a}_6(3\bar{a}_1\bar{a}_5 + 3\bar{a}_7\bar{a}_8 + \bar{a}_5\bar{a}_7{}^{\bar{\mathrm{w}}}I_1){}^{\bar{\mathrm{w}}}I_3}{\bar{a}_1\bar{a}_5({}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_2 - 9{}^{\bar{\mathrm{w}}}I_3)},$$

with

(4.24)
$$\bar{a}_6 = \frac{1}{a_6}; \qquad \bar{a}_1 = -a_5\,\frac{a_1{}^{\mathrm{w}}I_2 + 3a_7{}^{\mathrm{w}}I_3}{a_6\,\bar{d}_{91}},$$
$$\bar{a}_3 = -\frac{\bar{n}_{93}}{a_6\,\bar{d}_{91}\,\bar{d}_{95}}, \qquad \bar{a}_5 = -a_1\,\frac{3a_8 + a_5{}^{\mathrm{w}}I_1}{a_6\,\bar{d}_{95}}\,{}^{\mathrm{w}}I_3{}^2,$$
$$\bar{a}_7 = a_5\,\frac{3a_1 + a_7{}^{\mathrm{w}}I_1}{a_6\,\bar{d}_{91}}\,{}^{\mathrm{w}}I_3, \qquad \bar{a}_8 = a_1\,\frac{a_8{}^{\mathrm{w}}I_2 + 3a_5{}^{\mathrm{w}}I_3}{a_6\,\bar{d}_{95}}\,{}^{\mathrm{w}}I_3,$$

where quantities $\bar{d}_{91}$, $\bar{d}_{95}$, $\bar{n}_{93}$ are given in Eqs. (A.33)–(A.35) of Appendix A.8. *Solution (6.5.9)* is obtained as a particular case of *Solution (6.6)* by setting $a_9 = 0$, $\bar{a}_9 = 0$, which leads to the constraint (4.23). *Solutions (6.5.7), (6.5.8), (6.5.9)* display a similar degree of complexity, which appears to be lower than that shown by *Solutions (6.5.1), (6.5.3), (6.5.5).* These, in turn, look even more involved than the source *Solution (6.6)* itself.

We consider now the only three dual solution instances that are based on *four* 'non-shear' coefficients. The two *missing* coefficients are indicated by the last two label digits.

SOLUTION (6.4.17). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the five-coefficients sets $(a_3, a_5, a_6, a_8, a_9)$ and $(\bar{a}_3, \bar{a}_5, \bar{a}_6, \bar{a}_8, \bar{a}_9)$ (lacking only 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and 'non-shear' coefficients a_1, a_7 and $\bar{a}_1, \bar{a}_7$) form the dual inverse pair:

(4.25)
$$\mathbb{A} = a_6\;\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} + a_3\;\mathbf{w}\otimes\mathbf{w} + a_5\;\mathbf{w}^2\otimes\mathbf{w}^2 + a_8\;(\mathbf{w}^2\otimes\mathbf{w} + \mathbf{w}\otimes\mathbf{w}^2) + a_9\;(\mathbf{w}^2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}^2);$$
$$\bar{\mathbb{A}} = \bar{a}_6\;\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}} + \bar{a}_3\;\bar{\mathbf{w}}\otimes\bar{\mathbf{w}} + \bar{a}_5\;\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}^2 + \bar{a}_8\;(\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}} + \bar{\mathbf{w}}\otimes\bar{\mathbf{w}}^2) + \bar{a}_9\;(\bar{\mathbf{w}}^2\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{w}}^2),$$

provided that

$$(4.26)\qquad \begin{aligned} a_3 &= \frac{a_6 a_8}{a_9}\,\frac{{}^{\mathrm{w}}I_3}{{}^{\mathrm{w}}I_2}\,, \qquad a_5 = a_9\,\frac{a_8\,{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3 + a_9\,({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)}{a_6\,{}^{\mathrm{w}}I_3{}^2}\,,\\ \bar a_3 &= \frac{\bar a_6 \bar a_8}{\bar a_9}\,\frac{{}^{\bar{\mathrm{w}}}I_3}{{}^{\bar{\mathrm{w}}}I_2}\,, \qquad \bar a_5 = \bar a_9\,\frac{\bar a_8\,{}^{\bar{\mathrm{w}}}I_2{}^{\bar{\mathrm{w}}}I_3 + \bar a_9\,({}^{\bar{\mathrm{w}}}I_2{}^2 - 2{}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_3)}{\bar a_6\,{}^{\bar{\mathrm{w}}}I_3{}^2}\,, \end{aligned}$$

with

$$(4.27)\qquad \begin{aligned} &\bar a_6 = \frac{1}{a_6}; \qquad \bar a_3 = -\frac{1}{a_6}\,\frac{a_8\,{}^{\mathrm{w}}I_3}{3a_8{}^{\mathrm{w}}I_3 + a_9{}^{\mathrm{w}}I_2},\\ &\bar a_5 = a_9^2\,\frac{2a_8({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_3 + a_9({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_2}{a_6\,(3a_9 + a_6)^2\,(3a_8{}^{\mathrm{w}}I_3 + a_9{}^{\mathrm{w}}I_2)},\\ &\bar a_8 = \frac{a_8\,a_9{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3}{a_6\,(3a_9 + a_6)\,(3a_8{}^{\mathrm{w}}I_3 + a_9{}^{\mathrm{w}}I_2)}, \qquad \bar a_9 = -\frac{a_9}{a_6\,(3a_9 + a_6)}. \end{aligned}$$

Solution (6.4.17) is obtained as a particular case of *Solution (6.6)* by setting $a_1 = a_7 = 0$, $\bar a_1 = \bar a_7 = 0$, which leads to the two constraints (4.26). If constraint $(4.26)_1$ is placed on a_9 instead on a_3, relation $(4.27)_2$ for $\bar a_3$ transforms to the 'isotropic' inversion relation $\bar a_3 = -a_3/[a_6(3a_3 + a_6)]$ displayed by *Solution (6.1)*, Eq. $(3.14)_2$. Notice also that a similar 'isotropic' inversion relation holds in Eq. $(4.27)_5$ between coefficients a_9 and $\bar a_9$.

Solution (6.4.58). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the five-coefficients sets $(a_1, a_3, a_6, a_7, a_9)$ and $(\bar a_1, \bar a_3, \bar a_6, \bar a_7, \bar a_9)$ (lacking only 'shear-like' coefficients a_2, a_4 and $\bar a_2, \bar a_4$, and 'non-shear' coefficients a_5, a_8 and $\bar a_5, \bar a_8$) form the dual inverse pair:

$$(4.28)\qquad \begin{aligned} \mathbb{A} &= a_6\,\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w} + a_1\,\mathbf{I}\otimes\mathbf{I} + a_3\,\mathbf{w}\otimes\mathbf{w} + a_7\,(\mathbf{w}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w})\\ &\qquad + a_9\,(\mathbf{w}^2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}^2)\,,\\ \bar{\mathbb{A}} &= \bar a_6\,\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}} + \bar a_1\,\mathbf{I}\otimes\mathbf{I} + \bar a_3\,\bar{\mathbf{w}}\otimes\bar{\mathbf{w}} + \bar a_7\,(\bar{\mathbf{w}}\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{w}})\\ &\qquad + \bar a_9\,(\bar{\mathbf{w}}^2\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{w}}^2)\,, \end{aligned}$$

provided that

$$(4.29)\qquad \begin{aligned} a_1 &= a_9\,\frac{a_7{}^{\mathrm{w}}I_1 + a_9({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)}{a_6}\,, \qquad a_3 = \frac{a_6\,a_7}{a_9\,{}^{\mathrm{w}}I_1}\,,\\ \bar a_1 &= \bar a_9\,\frac{\bar a_7{}^{\bar{\mathrm{w}}}I_1 + \bar a_9({}^{\bar{\mathrm{w}}}I_1{}^2 - 2{}^{\bar{\mathrm{w}}}I_2)}{\bar a_6}\,, \qquad \bar a_3 = \frac{\bar a_6\,\bar a_7}{\bar a_9\,{}^{\bar{\mathrm{w}}}I_1}\,, \end{aligned}$$

with

$$\bar{a}_6 = \frac{1}{a_6}; \qquad \bar{a}_1 = a_9^2\,\frac{2a_7({}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) + a_9{}^{\mathrm{w}}I_1({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)}{a_6\,(3a_9 + a_6)^2\,(3a_7 + a_9{}^{\mathrm{w}}I_1)\,{}^{\mathrm{w}}I_3{}^2},$$

$$\bar{a}_3 = -\frac{a_7}{a_6\,(3a_7 + a_9{}^{\mathrm{w}}I_1)}, \tag{4.30}$$

$$\bar{a}_7 = \frac{a_7\,a_9\,{}^{\mathrm{w}}I_2}{a_6\,(3a_9 + a_6)\,(3a_7 + a_9{}^{\mathrm{w}}I_1)\,{}^{\mathrm{w}}I_3}, \qquad \bar{a}_9 = -\frac{a_9}{a_6\,(3a_9 + a_6)}.$$

Solution (6.4.58) is obtained as a particular case of *Solution (6.6)* by setting $a_5 = a_8 = 0$, $\bar{a}_5 = \bar{a}_8 = 0$, which leads to the two constraints (4.29). Once again, if constraint $(4.29)_2$ is placed on a_7 instead on a_3, Eq. $(4.30)_3$ for $\bar{a}_3$ transforms to the 'isotropic' inversion relation $\bar{a}_3 = -a_3/[a_6(3a_3 + a_6)]$ displayed by *Solution (6.1)*. A similar 'isotropic' inversion relation also holds in Eq. $(4.30)_5$ between a_9 and $\bar{a}_9$.

SOLUTION (6.4.78). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the five-coefficients sets $(a_1, a_3, a_5, a_6, a_9)$ and $(\bar{a}_1, \bar{a}_3, \bar{a}_5, \bar{a}_6, \bar{a}_9)$ (lacking only 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and 'non-shear' coefficients a_7, a_8 and $\bar{a}_7, \bar{a}_8$) form the dual inverse pair:

$$\begin{aligned} \mathbb{A} &= a_6\,\mathbf{w}\,\underline{\otimes}\,\mathbf{w} + a_1\,\mathbf{I}\otimes\mathbf{I} + a_3\,\mathbf{w}\otimes\mathbf{w} + a_5\,\mathbf{w}^2\otimes\mathbf{w}^2 \\ &\qquad + a_9\,(\mathbf{w}^2\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{w}^2)\,, \\ \bar{\mathbb{A}} &= \bar{a}_6\,\bar{\mathbf{w}}\,\underline{\otimes}\,\bar{\mathbf{w}} + \bar{a}_1\,\mathbf{I}\otimes\mathbf{I} + \bar{a}_3\,\bar{\mathbf{w}}\otimes\bar{\mathbf{w}} + \bar{a}_5\,\bar{\mathbf{w}}^2\otimes\bar{\mathbf{w}}^2 \\ &\qquad + \bar{a}_9\,(\bar{\mathbf{w}}^2\otimes\mathbf{I} + \mathbf{I}\otimes\bar{\mathbf{w}}^2)\,, \end{aligned} \tag{4.31}$$

provided that

$$\begin{aligned} a_1 &= -a_9\,\frac{{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3}{{}^{\mathrm{w}}I_2}\,, \qquad & a_5 &= -a_9\,\frac{{}^{\mathrm{w}}I_2}{{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3} \\ \bar{a}_1 &= -\bar{a}_9\,\frac{{}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_3}{{}^{\bar{\mathrm{w}}}I_2}\,, \qquad & \bar{a}_5 &= -\bar{a}_9\,\frac{{}^{\bar{\mathrm{w}}}I_2}{{}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_3}\,, \end{aligned} \tag{4.32}$$

with

$$\begin{aligned} &\bar{a}_6 = \frac{1}{a_6}; \qquad \bar{a}_1 = \frac{a_9\,{}^{\mathrm{w}}I_2{}^2}{a_6\,\bar{d}_{78}}\,, \qquad \bar{a}_3 = -\frac{a_3}{a_6\,(3a_3 + a_6)}\,, \\ &\bar{a}_5 = \frac{a_9{}^{\mathrm{w}}I_1{}^2\,{}^{\mathrm{w}}I_3{}^2}{a_6\,\bar{d}_{78}}\,, \qquad \bar{a}_9 = -\frac{a_9\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3}{a_6\,\bar{d}_{78}}\,, \end{aligned} \tag{4.33}$$

where

$$\bar{d}_{78} = a_6 \, {}^{\rm w}I_1 \, {}^{\rm w}I_2 \, {}^{\rm w}I_3 - 2\, a_9 \left({}^{\rm w}I_2{}^2 \, ({}^{\rm w}I_1{}^2 - {}^{\rm w}I_2) - {}^{\rm w}I_1 \, {}^{\rm w}I_3 \, ({}^{\rm w}I_1{}^2 + 3\, {}^{\rm w}I_2) \right) . \tag{4.34}$$

Solution (6.4.78) is obtained as a particular case of *Solution (6.6)* by setting $a_7 = a_8 = 0$, $\bar{a}_7 = \bar{a}_8 = 0$, which leads to the two constraints (4.32). Notice the 'isotropic' inversion relation between a_3 and $\bar{a}_3$ in Eq. $(4.33)_3$ as it appears in *Solution (6.1)*.

The only particular solution which is based on *three* 'non-shear' coefficients is considered next:

Solution (6.3). Symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with the four-coefficients sets (a_1, a_5, a_6, a_9) and $(\bar{a}_1, \bar{a}_5, \bar{a}_6, \bar{a}_9)$ (lacking 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and 'non-shear' coefficients a_3, a_7, a_8 and $\bar{a}_3, \bar{a}_7, \bar{a}_8$) form the dual inverse pair:

$$\begin{aligned} \mathbb{A} &= a_6 \, \mathbf{w} \,\underline{\overline{\otimes}}\, \mathbf{w} + a_1 \, \mathbf{I} \otimes \mathbf{I} + a_5 \, \mathbf{w}^2 \otimes \mathbf{w}^2 + a_9 \, (\mathbf{w}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}^2) , \\ \bar{\mathbb{A}} &= \bar{a}_6 \, \bar{\mathbf{w}} \,\underline{\overline{\otimes}}\, \bar{\mathbf{w}} + \bar{a}_1 \, \mathbf{I} \otimes \mathbf{I} + \bar{a}_5 \, \bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}}^2 + \bar{a}_9 \, (\bar{\mathbf{w}}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}^2) , \end{aligned} \tag{4.35}$$

with

$$\begin{aligned} \bar{a}_6 &= \frac{1}{a_6} \, ; \qquad \bar{a}_1 = \frac{\left(a_9{}^2 - a_1 a_5\right) \, ({}^{\rm w}I_2{}^2 - 2\, {}^{\rm w}I_1 \, {}^{\rm w}I_3) - a_5 a_6 \, {}^{\rm w}I_3{}^2}{a_6 \, \bar{d}} , \\ \bar{a}_5 &= -\frac{a_1 a_6 - \left(a_9{}^2 - a_1 a_5\right) \, ({}^{\rm w}I_1{}^2 - 2\, {}^{\rm w}I_2)}{a_6 \, \bar{d}} \, {}^{\rm w}I_3{}^2 , \\ \bar{a}_9 &= -\frac{a_6 a_9 + 3(a_9{}^2 - a_1 a_5)}{a_6 \, \bar{d}} \, {}^{\rm w}I_3{}^2 , \end{aligned} \tag{4.36}$$

where

$$\begin{aligned} \bar{d} = {} & \left(a_6(a_6 + 6a_9) + 9(a_9{}^2 - a_1 a_5) + a_5 a_6 \, ({}^{\rm w}I_1{}^2 - 2\, {}^{\rm w}I_2)\right) \, {}^{\rm w}I_3{}^2 \\ & + \left(a_1 a_6 - \left(a_9{}^2 - a_1 a_5\right) \, ({}^{\rm w}I_1{}^2 - 2\, {}^{\rm w}I_2)\right) \, ({}^{\rm w}I_2{}^2 - 2\, {}^{\rm w}I_1 \, {}^{\rm w}I_3) . \end{aligned} \tag{4.37}$$

Solution (6.3) is obtained from *Solution (6.6)* just by setting consistently $a_3 = a_7 = a_8 = 0$, $\bar{a}_3 = \bar{a}_7 = \bar{a}_8 = 0$ and holds *without constraints on the coefficients*.

The two solutions embedding the *two* 'non-shear' coefficients that are *indicated* by the last two label digits are now reported:

Solution (6.2.59). Another solution instance arises as a particular case of *Solution (6.3)* above by setting $a_1 = 0$, $\bar{a}_1 = 0$, namely by taking symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with only the three-coefficients sets (a_5, a_6, a_9) and $(\bar{a}_5, \bar{a}_6, \bar{a}_9)$ (lacking 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and 'non-shear' coefficients a_1, a_3, a_7, a_8 and $\bar{a}_1, \bar{a}_3, \bar{a}_7, \bar{a}_8$):

$$\text{(4.38)} \qquad \begin{aligned} \mathbb{A} &= a_6\ \mathbf{w} \underline{\overline{\otimes}}\, \mathbf{w} + a_5\ \mathbf{w}^2 \otimes \mathbf{w}^2 + a_9\ (\mathbf{w}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}^2)\,, \\ \bar{\mathbb{A}} &= \bar{a}_6\ \bar{\mathbf{w}} \underline{\overline{\otimes}}\, \bar{\mathbf{w}} + \bar{a}_5\ \bar{\mathbf{w}}^2 \otimes \bar{\mathbf{w}}^2 + \bar{a}_9\ (\bar{\mathbf{w}}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}^2)\,, \end{aligned}$$

provided that

$$\text{(4.39)} \qquad a_5 = \frac{a_9^2\,({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)}{a_6\ {}^{\mathrm{w}}I_3{}^2}\,; \qquad \bar{a}_5 = \frac{\bar{a}_9^2\,({}^{\bar{\mathrm{w}}}I_2{}^2 - 2{}^{\bar{\mathrm{w}}}I_1{}^{\bar{\mathrm{w}}}I_3)}{\bar{a}_6\ {}^{\bar{\mathrm{w}}}I_3{}^2}\,,$$

with

$$\text{(4.40)} \qquad \bar{a}_6 = \frac{1}{a_6}\,; \qquad \bar{a}_5 = \frac{a_9^2\,({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)}{a_6\,(3a_9 + a_6)^2}\,, \qquad \bar{a}_9 = -\frac{a_9}{a_6\,(3a_9 + a_6)}\,.$$

Notice that the conditions $a_1 = 0$, $\bar{a}_1 = 0$ imposed onto *Solution (6.3)* lead to the constraint (4.39).

Solution (6.2.19). A further solution instance which is an alternative particular case of *Solution (6.3)* is obtained by setting $a_5 = 0$, $\bar{a}_5 = 0$, namely by taking symmetric damage-effect tensors $\mathbb{A}$ and $\bar{\mathbb{A}}$ with only the three-coefficients sets (a_1, a_6, a_9) and $(\bar{a}_1, \bar{a}_6, \bar{a}_9)$ (lacking 'shear-like' coefficients a_2, a_4 and $\bar{a}_2, \bar{a}_4$, and 'non-shear' coefficients a_3, a_5, a_7, a_8 and $\bar{a}_3, \bar{a}_5, \bar{a}_7, \bar{a}_8$):

$$\text{(4.41)} \qquad \begin{aligned} \mathbb{A} &= a_6\ \mathbf{w} \underline{\overline{\otimes}}\, \mathbf{w} + a_1\ \mathbf{I} \otimes \mathbf{I} + a_9\ (\mathbf{w}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{w}^2)\,, \\ \bar{\mathbb{A}} &= \bar{a}_6\ \bar{\mathbf{w}} \underline{\overline{\otimes}}\, \bar{\mathbf{w}} + \bar{a}_1\ \mathbf{I} \otimes \mathbf{I} + \bar{a}_9\ (\bar{\mathbf{w}}^2 \otimes \mathbf{I} + \mathbf{I} \otimes \bar{\mathbf{w}}^2)\,, \end{aligned}$$

provided that

$$\text{(4.42)} \qquad a_1 = \frac{a_9^2\,({}^{\mathrm{w}}I_1{}^2 - 2\,{}^{\mathrm{w}}I_2)}{a_6}\,; \qquad \bar{a}_1 = \frac{\bar{a}_9^2\,({}^{\bar{\mathrm{w}}}I_1{}^2 - 2\,{}^{\bar{\mathrm{w}}}I_2)}{\bar{a}_6}\,,$$

with

$$\text{(4.43)} \qquad \bar{a}_6 = \frac{1}{a_6}\,; \qquad \bar{a}_1 = \frac{a_9^2\,({}^{\mathrm{w}}I_2{}^2 - 2\,{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)}{a_6\,(3a_9 + a_6)^2\ {}^{\mathrm{w}}I_3{}^2}\,, \qquad \bar{a}_9 = -\frac{a_9}{a_6\,(3a_9 + a_6)}\,.$$

Notice that the conditions $a_5 = 0$, $\bar{a}_5 = 0$ imposed onto *Solution (6.3)* lead to the constraint (4.42). The presence of coefficients $a_9, \bar{a}_9$ is common to both *Solutions (6.2.59)* and *(6.2.19)*. Also, coefficients $a_9, \bar{a}_9$ correspond to each other with the same 'isotropic' inversion relation $(4.40)_3$ or $(4.43)_3$, which holds as well

Table 1. Solution instances of dual symmetric damage-effect tensors Solution instances of dual symmetric damage-effect tensors based on 'shear-like' generator $a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}$, *Solution (6.0)*, missing two 'shear-like' coefficients ($a_2 = a_4 = 0$, $a_6 \neq 0$). *Solution (6.6)* embeds all the six 'non-shear' coefficients $a_1, a_3, a_5, a_7, a_8, a_9$ and misses only the two 'shear-like' coefficients a_2, a_4 from the general orthotropic representation. It contains all the following solutions as particular cases, which are obtained by suppressing sets of the 'non-shear' coefficients. Six-, five- and three-coefficients solutions hold through constraints on the coefficients. Four-coefficients *Solution (6.3)* and two-coefficients *Solution (6.1)* hold without constraints on the coefficients and are obtained from *Solution (6.6)* just by setting consistently $a_3 = a_7 = a_8 = 0$ and $a_1 = a_5 = a_7 = a_8 = a_9 = 0$, respectively.

Solution	Damage-effect tensor	Coefficients set	Constraints
(6.6)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_3\,\mathbf{w}\otimes\mathbf{w}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2+a_7\,(\mathbf{w}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w})+a_8\,(\mathbf{w}^2\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{w}^2)+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	7: $(a_1, a_3, a_5, a_6, a_7, a_8, a_9)$	*none*
(6.5.1)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_3\,\mathbf{w}\otimes\mathbf{w}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2+a_7\,(\mathbf{w}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w})+a_8\,(\mathbf{w}^2\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{w}^2)+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	6: $(a_3, a_5, a_6, a_7, a_8, a_9)$	1 on a_3
(6.5.3)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2+a_7\,(\mathbf{w}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w})+a_8\,(\mathbf{w}^2\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{w}^2)+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	6: $(a_1, a_5, a_6, a_7, a_8, a_9)$	1 on a_1
(6.5.5)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_3\,\mathbf{w}\otimes\mathbf{w}+a_7\,(\mathbf{w}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w})+a_8\,(\mathbf{w}^2\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{w}^2)+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	6: $(a_1, a_3, a_6, a_7, a_8, a_9)$	1 on a_3
(6.5.7)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_3\,\mathbf{w}\otimes\mathbf{w}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2+a_8\,(\mathbf{w}^2\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{w}^2)+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	6: $(a_1, a_3, a_5, a_6, a_8, a_9)$	1 on a_5
(6.5.8)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_3\,\mathbf{w}\otimes\mathbf{w}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2+a_7\,(\mathbf{w}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w})+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	6: $(a_1, a_3, a_5, a_6, a_7, a_9)$	1 on a_1
(6.5.9)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_3\,\mathbf{w}\otimes\mathbf{w}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2+a_7\,(\mathbf{w}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w})+a_8\,(\mathbf{w}^2\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{w}^2)$	6: $(a_1, a_3, a_5, a_6, a_7, a_8)$	1 on a_3
(6.4.17)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_3\,\mathbf{w}\otimes\mathbf{w}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2+a_8\,(\mathbf{w}^2\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{w}^2)+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	5: $(a_3, a_5, a_6, a_8, a_9)$	2 on a_3, a_5
(6.4.58)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_3\,\mathbf{w}\otimes\mathbf{w}+a_7\,(\mathbf{w}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w})+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	5: $(a_1, a_3, a_6, a_7, a_9)$	2 on a_1, a_3
(6.4.78)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_3\,\mathbf{w}\otimes\mathbf{w}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	5: $(a_1, a_3, a_5, a_6, a_9)$	2 on a_1, a_5
(6.3)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	4: (a_1, a_5, a_6, a_9)	*none*
(6.2.59)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_5\,\mathbf{w}^2\otimes\mathbf{w}^2+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	3: (a_5, a_6, a_9)	1 on a_5
(6.2.19)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_1\,\mathbf{I}\otimes\mathbf{I}+a_9\,(\mathbf{w}^2\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{w}^2)$	3: (a_1, a_6, a_9)	1 on a_1
(6.1)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}+a_3\,\mathbf{w}\otimes\mathbf{w}$	2: (a_3, a_6)	*none*
(6.0)	$\mathbb{A}=a_6\,\mathbf{w}\,\overline{\underline{\otimes}}\,\mathbf{w}$	1: (a_6)	*none*

for *Solutions (6.4.17)* and (6.4.58), Eqs. $(4.27)_5$ and $(4.30)_5$, and, similarly, for non-symmetric *Solution (6.1ns)*, Eq. $(3.16)_2$. Notice once more the similarity between this relation and the relations that hold between a_1 and $\bar{a}_1$ in *Solution (2.1)*, see Eq. $(3.8)_2$, and between a_3 and $\bar{a}_3$ in *Solutions (6.1), (6.4.78)*, see Eqs. $(3.14)_2$, $(4.33)_3$, and in *Solutions (6.4.17), (6.4.58)*, see comments following Eqs. (4.27), (4.30).

The remaining solution instances of the family that embed respectively only one and none of the 'non-shear' coefficients are already given in *Solutions (6.1), (6.0)*, Sec. 3. The various solution instances of the family based on 'shear-like' generators $a_6\,\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w}$, $\bar{a}_6\,\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}}$ are finally summarized in Table 1.

5. Conclusions

A complete family of symmetric orthotropic fourth-order damage-effect tensors with dual structures has been derived in full invariant form. These instances complement those already presented in a companion paper [20]. The solution family is based on 'shear-like' generators $a_6\,\mathbf{w}\,\underline{\overline{\otimes}}\,\mathbf{w}$, $\bar{a}_6\,\bar{\mathbf{w}}\,\underline{\overline{\otimes}}\,\bar{\mathbf{w}}$ and includes *fifteen* solution instances (*seven* of which were not presented previously: *Solutions (6.5.1), (6.5.3), (6.5.7), (6.5.8), (6.5.9), (6.4.17), (6.2.59)*), starting with a more general *Solution (6.6)*, which includes all the six 'non-shear' coefficients. Most of the obtained solutions are particular cases of others and hold *through constraints on the coefficients*. However, *Solutions (6.6), (6.3), (6.1), (6.0)* succeed in reaching the dual structure *without constraints on the coefficients*. The new general solution based on 'shear-like' generators $a_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}$, $\bar{a}_2\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}$ and including all the 'non-shear' terms (*Solution (2.6)*) is derived as well in the paper in full invariant form. Particular instances of such a case are available in [20].

While from the algebraic point of view the problem posed in the paper looks conceptually simple, the solution turns out to be non-trivial. Despite that the target and content of the paper are mainly algebraic, the motivations and possible use of this work lay entirely in the task of formulating constitutive relations of anisotropic elastic damage in quasi-brittle materials that can be conveniently handled and implemented, ideally embedding a dual attitude towards either stress- or strain-based derivations. Thus, we hope that the solutions advanced in the paper should be of interest also to other researchers dealing with similar constitutive derivations in the theory of elasticity.

These solution instances represent new propositions of damage-effect tensors (or, directly, of damaged compliance and stiffness tensors) allowing for dual compliance- and stiffness-based derivations of the constitutive relations in orthotropic damage. These damage-effect tensors should lead to damaged stiffness and increased compliance embedding less restricted forms of orthotropic damage than that of Valanis-type, with a complexity which increases with the number of additional coefficients that are kept in the representation. The ultimate convenience of any of the damage-effect tensors advanced here in the final development and implementation of a constitutive model of orthotropic elastic damage remains to be validated on physical grounds and explored both analytically and numerically.

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Appendix A. Expressions of the solution terms

For the ease of reading, this Appendix gathers the explicit expressions of the various quantities entering *Solutions (2.6), (6.6), (6.5.1), (6.5.3), (6.5.5), (6.5.7), (6.5.8), (6.5.9).*

A.1. Terms entering *Solution (2.6)*

$$
\begin{aligned}
\text{(A.1)}\qquad \bar{d}_2 = a_2^3 + \Big(a_1a_3a_5 - a_1a_8^2 - a_5a_7^2 - a_9(a_3a_9 - 2a_7a_8) \\
+ a_2(a_3a_5 - a_8^2)\Big) \cdot ({}^{\mathrm{w}}I_1{}^2\,{}^{\mathrm{w}}I_2{}^2 - 4\,{}^{\mathrm{w}}I_2{}^3 - 4\,{}^{\mathrm{w}}I_1{}^3\,{}^{\mathrm{w}}I_3 18\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3 - 27\,{}^{\mathrm{w}}I_3{}^2) \\
+ a_2^2\Big(3a_1 + 2a_7\,{}^{\mathrm{w}}I_1 + (a_3 + 2a_9)({}^{\mathrm{w}}I_1{}^2 - 2\,{}^{\mathrm{w}}I_2) + 2a_8({}^{\mathrm{w}}I_1{}^3 - 3\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2 + 3\,{}^{\mathrm{w}}I_3) \\
+ a_5({}^{\mathrm{w}}I_1{}^4 - 4\,{}^{\mathrm{w}}I_1{}^2\,{}^{\mathrm{w}}I_2 + 2\,{}^{\mathrm{w}}I_2{}^2 + 4\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)\Big) \\
+ 2a_2\Big[(a_1a_5 - a_9^2)({}^{\mathrm{w}}I_1{}^4 - 4\,{}^{\mathrm{w}}I_1{}^2\,{}^{\mathrm{w}}I_2 + {}^{\mathrm{w}}I_2{}^2 + 6\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3) \\
+ \Big(a_1a_3 - a_7^2 + (a_1a_8 - a_7a_9)\,{}^{\mathrm{w}}I_1 + (a_5a_7 - a_8a_9)\,{}^{\mathrm{w}}I_3\Big)({}^{\mathrm{w}}I_1{}^2 - 3\,{}^{\mathrm{w}}I_2) \\
+ \Big(a_1a_8 - a_7a_9 + (a_5a_7 - a_8a_9)\,{}^{\mathrm{w}}I_2 + (a_3a_5 - a_8^2)\,{}^{\mathrm{w}}I_3\Big)({}^{\mathrm{w}}I_1{}^3 - 4\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2 + 9\,{}^{\mathrm{w}}I_3) \\
- (a_3a_9 - a_7a_8)({}^{\mathrm{w}}I_1{}^2\,{}^{\mathrm{w}}I_2 - 4\,{}^{\mathrm{w}}I_2{}^2 + 3\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)\Big],
\end{aligned}
$$

$$\text{(A.2)}\qquad \bar{n}_{21} = \Big(a_5 a_7^2 - a_1(a_3 a_5 - a_8^2) + a_9(a_3 a_9 - 2a_7 a_8)\Big) \cdot ({}^{\mathrm{w}}I_1{}^2\,{}^{\mathrm{w}}I_2{}^2 - 2\,{}^{\mathrm{w}}I_2{}^3 - 4\,{}^{\mathrm{w}}I_1{}^3\,{}^{\mathrm{w}}I_3 + 10\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3 - 9\,{}^{\mathrm{w}}I_3{}^2)$$

$$- a_2^2\Big[a_1 + 2a_7\,{}^{\mathrm{w}}I_1 + a_3\,{}^{\mathrm{w}}I_1{}^2 + \Big(2a_9 + 2a_8\,{}^{\mathrm{w}}I_1 + a_5({}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2)\Big)({}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2)\Big]$$

$$- a_2\Big[2(a_1 a_3 - a_7^2)({}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2) + (a_1 a_5 - a_9^2)(2\,{}^{\mathrm{w}}I_1{}^4 - 4\,{}^{\mathrm{w}}I_1{}^2\,{}^{\mathrm{w}}I_2 + {}^{\mathrm{w}}I_2{}^2 + 4\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)$$

$$+ 2\,(a_1 a_8 - a_7 a_9)(2\,{}^{\mathrm{w}}I_1{}^3 - 3\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2 + 3\,{}^{\mathrm{w}}I_3)$$

$$- 2\,(a_3 a_9 - a_7 a_8)({}^{\mathrm{w}}I_1{}^2\,{}^{\mathrm{w}}I_2 - 2\,{}^{\mathrm{w}}I_2{}^2 + 3\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)$$

$$+ 2\,(a_5 a_7 - a_8 a_9)({}^{\mathrm{w}}I_1{}^3\,{}^{\mathrm{w}}I_2 - 2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2{}^2 + {}^{\mathrm{w}}I_1{}^2\,{}^{\mathrm{w}}I_3 + 3\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3)$$

$$+ \Big(a_3 a_5\,{}^{\mathrm{w}}I_1{}^2 - a_8^2({}^{\mathrm{w}}I_1{}^2 - 2\,{}^{\mathrm{w}}I_2)\Big)({}^{\mathrm{w}}I_2{}^2 - 2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)$$

$$- 2\,\Big(a_8^2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3 + a_3 a_5({}^{\mathrm{w}}I_2{}^2 - 3\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)\Big)\,{}^{\mathrm{w}}I_2\Big]\,,$$

$$\text{(A.3)}\qquad \bar{n}_{23} = \Big(2a_5 a_7^2 - (2a_1 + a_2)(a_3 a_5 - a_8^2) + 2a_9(a_3 a_9 - 2a_7 a_8)\Big) \cdot ({}^{\mathrm{w}}I_2{}^4 - 4\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2{}^2\,{}^{\mathrm{w}}I_3 + {}^{\mathrm{w}}I_1{}^2\,{}^{\mathrm{w}}I_3{}^2 + 6\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3{}^2)$$

$$+ a_2\,\Big\{\Big(3a_7^2\,{}^{\mathrm{w}}I_2 - (a_3 a_5 - a_8^2)({}^{\mathrm{w}}I_2{}^3 - 2\,{}^{\mathrm{w}}I_3{}^2) + 2(a_3 a_9 - a_7 a_8)(2\,{}^{\mathrm{w}}I_2{}^2 - {}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)\Big)\,{}^{\mathrm{w}}I_2$$

$$- (3a_1 + a_2)\Big[a_3\,{}^{\mathrm{w}}I_2{}^2 + \Big(a_5({}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3) + 2a_8\,{}^{\mathrm{w}}I_2\Big)({}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3)\Big]$$

$$+\Big(6a_7 a_9\,{}^{\mathrm{w}}I_2 - 2(a_5 a_7 - a_8 a_9)(2\,{}^{\mathrm{w}}I_2{}^2 - {}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3) + 3a_9^2({}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3)\Big)({}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3)\Big\},$$

$$\text{(A.4)}\qquad \bar{n}_{25} = 2\Big(a_5 a_7^2 + a_9(a_3 a_9 - 2a_7 a_8) - (a_1 + a_2)(a_3 a_5 - a_8^2)\Big)({}^{\mathrm{w}}I_2{}^2 - 3\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)$$

$$+ a_2\Big[3a_7^2 + 3a_9(2a_7 + a_9\,{}^{\mathrm{w}}I_1)\,{}^{\mathrm{w}}I_1 - (3a_1 + a_2)\Big(a_3 + (2a_8 + a_5\,{}^{\mathrm{w}}I_1)\,{}^{\mathrm{w}}I_1\Big)$$

$$- 4\Big(a_7(a_8 + a_5\,{}^{\mathrm{w}}I_1) - a_9(a_3 + a_8\,{}^{\mathrm{w}}I_1)\Big)\,{}^{\mathrm{w}}I_2 - 4(a_3 a_5 - a_8^2)\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3\Big],$$

(A.5)
$$\bar{n}_{27} = \Big((a_1+a_2)(a_3a_5-a_8^2) - a_5a_7^2 - a_9(a_3a_9-2a_7a_8)\Big)$$
$$\cdot\Big({}^{\mathrm{w}}I_1({}^{\mathrm{w}}I_2{}^3 + 6{}^{\mathrm{w}}I_3{}^2) - {}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3(4{}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2)\Big)$$
$$+ a_2^2\Big[(a_7 + a_3{}^{\mathrm{w}}I_1){}^{\mathrm{w}}I_2 + \Big(a_9 + a_5({}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2)\Big)({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3)$$
$$- a_8({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3 - 2{}^{\mathrm{w}}I_1{}^2{}^{\mathrm{w}}I_2 + {}^{\mathrm{w}}I_2{}^2)\Big]$$
$$+ a_2\Big[2\Big(a_1a_3 - a_7^2 + (a_5a_7 - a_8a_9){}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2\Big){}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2$$
$$+ (a_1a_5 - a_9^2)({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3)(2{}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2)$$
$$- \Big(a_5a_7 - a_8a_9 - (a_3a_5 - a_8^2){}^{\mathrm{w}}I_1\Big)(2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - 3{}^{\mathrm{w}}I_3){}^{\mathrm{w}}I_3$$
$$- (a_1a_8 - a_7a_9)(2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3 - 4{}^{\mathrm{w}}I_1{}^2{}^{\mathrm{w}}I_2 + {}^{\mathrm{w}}I_2{}^2)$$
$$- (a_3a_9 - a_7a_8)(2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 + {}^{\mathrm{w}}I_3){}^{\mathrm{w}}I_2\Big]\,,$$

(A.6)
$$\bar{n}_{28} = \Big((a_1+a_2)(a_3a_5-a_8^2) - a_3a_9^2 - a_7(a_5a_7-2a_8a_9)\Big)$$
$$\cdot(2{}^{\mathrm{w}}I_2{}^3 - 7{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3 + 9{}^{\mathrm{w}}I_3{}^2) + a_2\Big[(3a_1 + a_2)\Big(a_3{}^{\mathrm{w}}I_2 + a_5{}^{\mathrm{w}}I_1({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3)\Big)$$
$$- 3(a_7^2 - a_9^2{}^{\mathrm{w}}I_1{}^2){}^{\mathrm{w}}I_2 - 2(a_5a_7 - a_8a_9){}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3 - 4(a_3a_5 - a_8^2){}^{\mathrm{w}}I_3{}^2$$
$$- \Big(3a_7a_9 - a_8(3a_1 + a_2) + 3a_9^2{}^{\mathrm{w}}I_1 - 2(a_3a_5 - a_8^2){}^{\mathrm{w}}I_3\Big)(2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - {}^{\mathrm{w}}I_3)$$
$$- \Big((a_3a_9 - a_7a_8) - (a_5a_7 - a_8a_9){}^{\mathrm{w}}I_1\Big)(4{}^{\mathrm{w}}I_2{}^2 - {}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big]$$

(A.7)
$$\bar{n}_{29} = \Big((a_1+a_2)(a_3a_5-a_8^2) - a_5a_7^2 - a_9(a_3a_9-2a_7a_8)\Big)$$
$$\cdot({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^2 - 4{}^{\mathrm{w}}I_1{}^2{}^{\mathrm{w}}I_3 + 3{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3)$$
$$+ a_2^2\Big(a_7 + (a_3 + a_9){}^{\mathrm{w}}I_1 + a_5({}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_1 + a_8(2{}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2)\Big)$$
$$+ a_2\Big[\Big(2(a_1a_3 - a_7^2) + 2(a_3a_5 - a_8^2){}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3 + (a_1a_5 - a_9^2)(2{}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2)$$
$$+ (a_5a_7 - a_8a_9)(2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 + {}^{\mathrm{w}}I_3)\Big){}^{\mathrm{w}}I_1$$
$$+ (a_1a_8 - a_7a_9)(4{}^{\mathrm{w}}I_1{}^2 - {}^{\mathrm{w}}I_2) - (a_3a_9 - a_7a_8)(2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 + 3{}^{\mathrm{w}}I_3)\Big]\,.$$

A.2. Terms entering *Solution (6.6)*

(A.8)
$$\bar{d}_6 = a_6\Big(- 9a_1a_5 + 3a_3a_6 + (a_6 + 3a_9)^2 + 2a_6a_8{}^{w}I_1 + a_5a_6({}^{w}I_1{}^2 - 2{}^{w}I_2) + 2(a_3a_5 - a_8^2)({}^{w}I_1{}^2 - 3{}^{w}I_2)\Big){}^{w}I_3{}^2 - a_6\Big[2(a_3a_9 - a_7a_8)({}^{w}I_1{}^{w}I_2 - 9{}^{w}I_3) - 2\Big(a_6a_7 - 3(a_1a_8 - a_7a_9)\Big){}^{w}I_2 + 2(a_1a_3 - a_7^2){}^{w}I_1 - 2(a_5a_7 - a_8a_9)\Big({}^{w}I_2({}^{w}I_1{}^2 - 2{}^{w}I_2) - 3{}^{w}I_1{}^{w}I_3\Big)\Big]{}^{w}I_3 + a_6\Big(a_1a_6 + 2(a_1a_3 - a_7^2) + 2(a_1a_8 - a_7a_9){}^{w}I_1 + (a_1a_5 - a_9^2)({}^{w}I_1{}^2 - 2{}^{w}I_2)\Big)\,({}^{w}I_2{}^2 - 2{}^{w}I_1{}^{w}I_3) - \Big(a_1a_8^2 - a_5(a_1a_3 - a_7^2) + a_9(a_3a_9 - 2a_7a_8)\Big)\cdot \cdot\,({}^{w}I_1{}^2{}^{w}I_2{}^2 - 4{}^{w}I_2{}^3 - 4{}^{w}I_1{}^3{}^{w}I_3 + 18{}^{w}I_1{}^{w}I_2{}^{w}I_3 - 27{}^{w}I_3{}^2)\,,$$

(A.9)
$$\bar{n}_{61} = a_6\Big[(a_1a_5 - a_9^2){}^{w}I_1 + 2(a_5a_7 - a_8a_9){}^{w}I_2 + \Big(a_5(3a_3 + a_6) - 3a_8^2\Big){}^{w}I_3\Big]{}^{w}I_3 - \Big[2a_1a_8^2 - a_5\Big(a_1(2a_3 + a_6) - 2a_7^2\Big) + a_9\Big(a_9(2a_3 + a_6) - 4a_7a_8\Big)\Big]\cdot \cdot\,({}^{w}I_2{}^2 - 3{}^{w}I_1{}^{w}I_3)\,,$$

(A.10)
$$\bar{n}_{63} = a_6\Big(a_3a_6 + 6(a_3a_9 - a_7a_8) + (a_3a_5 - a_8^2)({}^{w}I_1{}^2 - 2{}^{w}I_2)\Big){}^{w}I_3{}^2 - \Big(a_1a_8^2 - a_5(a_1a_3 - a_7^2) + a_9(a_3a_9 - 2a_7a_8)\Big) \cdot\Big(({}^{w}I_1{}^2 - 2{}^{w}I_2)({}^{w}I_2{}^2 - 2{}^{w}I_1{}^{w}I_3) - 9{}^{w}I_3{}^2\Big) + a_6(a_1a_3 - a_7^2)({}^{w}I_2{}^2 - 2{}^{w}I_1{}^{w}I_3)\,,$$

(A.11)
$$\bar{n}_{65} = a_6\Big(3a_7^2 - a_1(3a_3 + a_6) - 2(a_1a_8 - a_7a_9){}^{w}I_1 - (a_1a_5 - a_9^2)({}^{w}I_1{}^2 - 2{}^{w}I_2)\Big) + 2\Big(a_1a_8^2 - a_5(a_1a_3 - a_7^2) + a_9(a_3a_9 - 2a_7a_8)\Big)({}^{w}I_1{}^2 - 3{}^{w}I_2)\,,$$

$$\bar{n}_{67} = a_6 a_8 (a_6 + 3a_9 + a_8 {}^{w}I_1){}^{w}I_3{}^2 + a_6(a_1 a_8 - a_7 a_9)({}^{w}I_2{}^2 - 2{}^{w}I_1{}^{w}I_3) + \Big(a_5 a_7^2 + a_9(a_3 a_9 - 2a_7 a_8) - a_1(a_3 a_5 - a_8^2)\Big) \cdot \Big({}^{w}I_1({}^{w}I_2{}^2 - 2{}^{w}I_1{}^{w}I_3) - 3{}^{w}I_2{}^{w}I_3\Big) - a_6{}^{w}I_3\Big(a_7(3a_5{}^{w}I_3 - a_8{}^{w}I_2) + a_3(a_5{}^{w}I_1{}^{w}I_3 + a_9{}^{w}I_2)\Big)\ , \tag{A.12}$$

$$\bar{n}_{68} = a_6(a_1 a_3 - a_7^2){}^{w}I_2 + a_6\Big[3a_1 a_8 - a_7(a_6 + 3a_9) - a_7\Big(a_8{}^{w}I_1 + a_5({}^{w}I_1{}^2 - 2{}^{w}I_2)\Big) + a_9\Big(a_3{}^{w}I_1 + a_8({}^{w}I_1{}^2 - 2{}^{w}I_2)\Big)\Big]{}^{w}I_3 - \Big(a_1 a_8^2 - a_5(a_1 a_3 - a_7^2) + a_9(a_3 a_9 - 2a_7 a_8)\Big)\Big({}^{w}I_2({}^{w}I_1{}^2 - 2{}^{w}I_2) - 3{}^{w}I_1{}^{w}I_3\Big), \tag{A.13}$$

$$\bar{n}_{69} = a_6(a_1 a_8 - a_7 a_9){}^{w}I_2 + a_6\Big(3(a_1 a_5 + a_7 a_8) - a_9(3a_3 + a_6 + 3a_9) + (a_5 a_7 - a_8 a_9){}^{w}I_1\Big){}^{w}I_3 + \Big(a_1 a_8^2 - a_5(a_1 a_3 - a_7^2) + a_9(a_3 a_9 - 2a_7 a_8)\Big)({}^{w}I_1{}^{w}I_2 - 9{}^{w}I_3)\ . \tag{A.14}$$

A.3. Terms entering *Solution (6.5.1)*

$$\bar{d}_1 = a_6(3a_8 + a_5{}^{w}I_1){}^{w}I_3{}^2 + a_9{}^{w}I_3(a_6{}^{w}I_2 - a_8{}^{w}I_1{}^{w}I_2 + 9a_8{}^{w}I_3) - a_9^2({}^{w}I_1{}^{w}I_2{}^2 - 2{}^{w}I_1{}^2{}^{w}I_3 - 3{}^{w}I_2{}^{w}I_3) + a_7\Big(a_5({}^{w}I_1{}^{w}I_2 - 9{}^{w}I_3){}^{w}I_3 - 2a_9({}^{w}I_2{}^2 - 3{}^{w}I_1{}^{w}I_3)\Big), \tag{A.15}$$

$$\bar{n}_{13} = 2a_7{}^{w}I_3\Big[a_5^2 a_6{}^{w}I_2{}^{w}I_3{}^2({}^{w}I_1{}^2 - 2{}^{w}I_2) + 2a_8 a_9(a_6 + 3a_9){}^{w}I_3({}^{w}I_2{}^2 - 3{}^{w}I_1{}^{w}I_3) + a_5\Big(a_6{}^{w}I_3{}^2(a_6{}^{w}I_2 + 9a_8{}^{w}I_3) - a_9{}^{w}I_3(a_8{}^{w}I_1{}^2{}^{w}I_2{}^2 - 2a_8{}^{w}I_2{}^3 - 6a_6{}^{w}I_2{}^{w}I_3 - 27a_8{}^{w}I_3{}^2)\Big)\Big] + a_9^2\Big(2a_8 a_9{}^{w}I_2{}^{w}I_3 - 2a_5{}^{w}I_3(a_7{}^{w}I_2 + a_6{}^{w}I_3) + a_9^2({}^{w}I_2{}^2 - 2{}^{w}I_1{}^{w}I_3)\Big) \cdot ({}^{w}I_1{}^2{}^{w}I_2{}^2 - 2{}^{w}I_2{}^3 - 2{}^{w}I_1^3{}^{w}I_3 + 4{}^{w}I_1{}^{w}I_2{}^{w}I_3 - 9{}^{w}I_3^2) + a_6^2\Big(a_5 a_6 - 3a_8^2 + a_5^2({}^{w}I_1{}^2 - 2{}^{w}I_2)\Big){}^{w}I_3{}^4 - 2a_6 a_9{}^{w}I_3{}^3\Big[a_6(a_8{}^{w}I_2 - 3a_5{}^{w}I_3) + a_8\Big(a_5({}^{w}I_1{}^2 - 2{}^{w}I_2){}^{w}I_2 + 9a_8{}^{w}I_3\Big)\Big] \tag{A.16}$$

(A.16) [cont.]
$$+ (a_5^2 a_7^2 + a_8^2 a_9^2)({}^{\mathrm{w}}I_1{}^2\,{}^{\mathrm{w}}I_2{}^2 - 2\,{}^{\mathrm{w}}I_2{}^3 - 27\,{}^{\mathrm{w}}I_3{}^2)\,{}^{\mathrm{w}}I_3{}^2$$
$$- a_7^2\Big(12a_5a_9\,{}^{\mathrm{w}}I_3{}^2({}^{\mathrm{w}}I_2{}^2 - 3\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)$$
$$+ 2a_9^2({}^{\mathrm{w}}I_2{}^2 - 3\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)({}^{\mathrm{w}}I_2{}^2 - 2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3) - a_5a_6\,{}^{\mathrm{w}}I_2{}^2\,{}^{\mathrm{w}}I_3{}^2\Big)$$
$$- a_6a_9^2\,{}^{\mathrm{w}}I_3{}^2\Big((a_6 + 6a_9)({}^{\mathrm{w}}I_2{}^2 - 2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3) + 3\,{}^{\mathrm{w}}I_3(4a_8\,{}^{\mathrm{w}}I_2 + 3a_5\,{}^{\mathrm{w}}I_3)\Big),$$

(A.17)
$$\bar{n}_{15} = 3a_7a_5\,{}^{\mathrm{w}}I_3{}^2\Big(a_6(3a_7 + 2a_9\,{}^{\mathrm{w}}I_1) + 2(a_5a_7 - 2a_8a_9)({}^{\mathrm{w}}I_1{}^2 - 3\,{}^{\mathrm{w}}I_2)\Big)$$
$$+ a_9^2\Big[a_5a_6\,{}^{\mathrm{w}}I_1{}^2\,{}^{\mathrm{w}}I_3{}^2 - 2a_7(3a_7 + 2a_9\,{}^{\mathrm{w}}I_1)$$
$$\cdot({}^{\mathrm{w}}I_2{}^2 - 3\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3) - 2a_9^2({}^{\mathrm{w}}I_2{}^3 - {}^{\mathrm{w}}I_1{}^3\,{}^{\mathrm{w}}I_3)$$
$$- 2\Big(2a_5a_7\,{}^{\mathrm{w}}I_2 - a_8(2a_9\,{}^{\mathrm{w}}I_2 + 3a_8\,{}^{\mathrm{w}}I_3)\Big)({}^{\mathrm{w}}I_1{}^2 - 3\,{}^{\mathrm{w}}I_2)\,{}^{\mathrm{w}}I_3\Big],$$

(A.18)
$$\bar{n}_{18} = a_5(a_6a_9^2 - 3a_5a_7^2)({}^{\mathrm{w}}I_1{}^2\,{}^{\mathrm{w}}I_2 - 2\,{}^{\mathrm{w}}I_2{}^2 - 3\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)\,{}^{\mathrm{w}}I_3{}^2$$
$$- a_7^2\Big(3a_5a_6\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3{}^2 - 2a_5a_9\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3({}^{\mathrm{w}}I_2{}^2 - 3\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3) - 2a_9^2({}^{\mathrm{w}}I_2{}^3 - 3\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3)\Big)$$
$$- 2a_9^2\,{}^{\mathrm{w}}I_3(a_9a_8 - a_5a_7)(2\,{}^{\mathrm{w}}I_1{}^2\,{}^{\mathrm{w}}I_2{}^2 - 4\,{}^{\mathrm{w}}I_2{}^3 - 3\,{}^{\mathrm{w}}I_1{}^3\,{}^{\mathrm{w}}I_3 + 3\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3)$$
$$+ a_9\Big(3a_8(2a_5a_7 - a_8a_9)\,{}^{\mathrm{w}}I_3{}^2 - a_9^3({}^{\mathrm{w}}I_2{}^2 - 2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)\Big)({}^{\mathrm{w}}I_1{}^2\,{}^{\mathrm{w}}I_2 - 2\,{}^{\mathrm{w}}I_2{}^2 - 3\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)$$
$$+ a_6a_9\Big[\Big(3a_8\,{}^{\mathrm{w}}I_1(a_8 + a_5\,{}^{\mathrm{w}}I_1) - a_5(a_6\,{}^{\mathrm{w}}I_1 + 6a_8\,{}^{\mathrm{w}}I_2)\Big)\,{}^{\mathrm{w}}I_3{}^3$$
$$+ a_9\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3\Big(2a_8\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3 + a_9({}^{\mathrm{w}}I_2{}^2 - 2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)\Big)\Big]$$
$$- a_7\,{}^{\mathrm{w}}I_3\Big[a_6\Big(3a_5(a_6 + a_8\,{}^{\mathrm{w}}I_1 + a_5\,{}^{\mathrm{w}}I_1{}^2 - 2a_5\,{}^{\mathrm{w}}I_2)\,{}^{\mathrm{w}}I_3{}^2 + a_5a_9\,{}^{\mathrm{w}}I_3(2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2 + 9\,{}^{\mathrm{w}}I_3)\Big)$$
$$- 2a_9^2(a_6 + 3a_9 - a_8\,{}^{\mathrm{w}}I_1)({}^{\mathrm{w}}I_2{}^2 - 3\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)\Big].$$

A.4. Terms entering *Solution (6.5.3)*

(A.19)
$$\bar{d}_3 = a_6\Big(a_7({}^{\mathrm{w}}I_2{}^2 - 2\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3) + 3a_8\,{}^{\mathrm{w}}I_3{}^2\Big)$$
$$+ a_8\Big(a_7({}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2{}^2 - 2\,{}^{\mathrm{w}}I_1{}^2\,{}^{\mathrm{w}}I_3 - 3\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3) - a_8\,{}^{\mathrm{w}}I_3({}^{\mathrm{w}}I_1{}^2\,{}^{\mathrm{w}}I_2 - 2\,{}^{\mathrm{w}}I_2{}^2 - 3\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_3)\Big)$$
$$+ (a_5a_7 - a_8a_9)({}^{\mathrm{w}}I_1{}^2\,{}^{\mathrm{w}}I_2{}^2 - 2\,{}^{\mathrm{w}}I_2{}^3 - 2\,{}^{\mathrm{w}}I_1{}^3\,{}^{\mathrm{w}}I_3 + 4\,{}^{\mathrm{w}}I_1\,{}^{\mathrm{w}}I_2\,{}^{\mathrm{w}}I_3 - 9\,{}^{\mathrm{w}}I_3{}^2)\,,$$

(A.20)
$$\bar{n}_{31} = (a_5 a_7 - a_8 a_9)\Big[a_8\Big(2a_8 {}^{w}I_2 {}^{w}I_3 + a_9({}^{w}I_2{}^2 - 2{}^{w}I_1 {}^{w}I_3)\Big) - a_5 a_7({}^{w}I_2{}^2 - 2{}^{w}I_1 {}^{w}I_3)\Big] \cdot ({}^{w}I_1{}^2 {}^{w}I_2{}^2 - 2{}^{w}I_2{}^3 - 2{}^{w}I_1{}^3 {}^{w}I_3 + 4{}^{w}I_1 {}^{w}I_2 {}^{w}I_3 - 9{}^{w}I_3{}^2) + a_8^2\Big[2a_7\Big(6a_8 {}^{w}I_3{}^2 + a_7({}^{w}I_2{}^2 - 2{}^{w}I_1 {}^{w}I_3)\Big)({}^{w}I_2{}^2 - 3{}^{w}I_1 {}^{w}I_3) - a_8^2 {}^{w}I_3{}^2({}^{w}I_1{}^2 {}^{w}I_2{}^2 - 2{}^{w}I_2{}^3 - 27{}^{w}I_3{}^2)\Big] - a_5 a_6\Big(3a_8 {}^{w}I_3{}^2 + a_7({}^{w}I_2{}^2 - 2{}^{w}I_1 {}^{w}I_3)\Big)^2,$$

(A.21)
$$\bar{n}_{35} = (a_5 a_7 - a_8 a_9)\Big[a_7\Big(2(a_6 + a_8 {}^{w}I_1) + a_5({}^{w}I_1{}^2 - 2{}^{w}I_2)\Big) - a_8 a_9({}^{w}I_1{}^2 - 2{}^{w}I_2)\Big] \cdot ({}^{w}I_1{}^2 {}^{w}I_2{}^2 - 2{}^{w}I_2{}^3 - 2{}^{w}I_1{}^3 {}^{w}I_3 + 4{}^{w}I_1 {}^{w}I_2 {}^{w}I_3 - 9{}^{w}I_3{}^2) + 3a_7\Big[a_5 a_6\Big(3a_7 + 2a_8({}^{w}I_1{}^2 - 2{}^{w}I_2)\Big) + 4a_8^2\Big({}^{w}I_1(a_6 - a_8 {}^{w}I_1) + 3a_8 {}^{w}I_2\Big)\Big]{}^{w}I_3{}^2 + a_7^2 a_8\Big(2a_6 {}^{w}I_1({}^{w}I_2{}^2 - 2{}^{w}I_1 {}^{w}I_3) + a_8({}^{w}I_1{}^2 {}^{w}I_2{}^2 - 2{}^{w}I_1{}^3 {}^{w}I_3 - 27{}^{w}I_3{}^2)\Big) + a_8^2\Big[a_6\Big(2a_8 {}^{w}I_1 + a_5({}^{w}I_1{}^2 - 2{}^{w}I_2)\Big) - 2a_8^2({}^{w}I_1{}^2 - 3{}^{w}I_2)\Big]({}^{w}I_1{}^2 - 2{}^{w}I_2){}^{w}I_3{}^2 + a_6^2\Big(6a_7 a_8 {}^{w}I_3{}^2 + a_8^2({}^{w}I_1{}^2 - 2{}^{w}I_2){}^{w}I_3{}^2 + a_7^2({}^{w}I_2{}^2 - 2{}^{w}I_1 {}^{w}I_3)\Big),$$

(A.22)
$$\bar{n}_{39} = \Big[3a_5^2 a_7^2 {}^{w}I_3 + a_5 a_7 a_8\Big(a_7 {}^{w}I_2 - (6a_9 + a_8 {}^{w}I_1){}^{w}I_3\Big) - a_8^2\Big(a_7(a_9 {}^{w}I_2 + 3a_8 {}^{w}I_3) - a_9(a_6 + 3a_9 + a_8 {}^{w}I_1){}^{w}I_3\Big)\Big] \cdot ({}^{w}I_1{}^2 {}^{w}I_2{}^2 - 2{}^{w}I_2{}^3 - 2{}^{w}I_1{}^3 {}^{w}I_3 + 4{}^{w}I_1 {}^{w}I_2 {}^{w}I_3 - 9{}^{w}I_3{}^2) + \Big(a_8^2({}^{w}I_1 {}^{w}I_2 - 9{}^{w}I_3) + a_6(a_8 {}^{w}I_2 + 3a_5 {}^{w}I_3)\Big) \cdot \Big(6a_7 a_8 {}^{w}I_3{}^2 + a_8^2({}^{w}I_1{}^2 - 2{}^{w}I_2){}^{w}I_3{}^2 + a_7^2({}^{w}I_2{}^2 - 2{}^{w}I_1 {}^{w}I_3)\Big).$$

A.5. Terms entering *Solution (6.5.5)*

(A.23)
$$\bar{d}_5 = a_1\Big(a_6 {}^{w}I_2 + a_8({}^{w}I_1 {}^{w}I_2 - 9{}^{w}I_3)\Big) + a_6(3a_7 + a_9 {}^{w}I_1){}^{w}I_3 - a_7 a_9({}^{w}I_1 {}^{w}I_2 - 9{}^{w}I_3) - 2a_8 a_9({}^{w}I_1{}^2 - 3{}^{w}I_2){}^{w}I_3 - a_9^2({}^{w}I_1{}^2 {}^{w}I_2 - 2{}^{w}I_2{}^2 - 3{}^{w}I_1 {}^{w}I_3),$$

$$
\text{(A.24)}\qquad
\begin{aligned}
\bar{n}_{51} = a_1\Big(& a_6(3a_8{}^{\mathrm{w}}I_3 + a_9{}^{\mathrm{w}}I_2)^2 \\
& + 2a_8(3a_1a_8 - 6a_7a_9 - 2a_9^2{}^{\mathrm{w}}I_1)({}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big) \\
& + 2a_9^2\Big(a_7(3a_7 + 2a_9{}^{\mathrm{w}}I_1)({}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) \\
& - a_8(3a_8{}^{\mathrm{w}}I_3 + 2a_9{}^{\mathrm{w}}I_2)({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_3 + a_9^2({}^{\mathrm{w}}I_2{}^3 - {}^{\mathrm{w}}I_1{}^3{}^{\mathrm{w}}I_3)\Big)\,,
\end{aligned}
$$

$$
\text{(A.25)}\qquad
\begin{aligned}
\bar{n}_{53} = \Big[& a_1a_8(a_1a_8 - 2a_7a_9) - a_9^2\Big(2a_1a_6 - a_7^2 + 2(a_1a_8 - a_7a_9){}^{\mathrm{w}}I_1 \\
& - a_9^2({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)\Big)\Big] \cdot ({}^{\mathrm{w}}I_1{}^2{}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^3{}^{\mathrm{w}}I_3 - 27{}^{\mathrm{w}}I_3{}^2) \\
& + a_1\Big\{a_6^2(a_6 + 6a_9 + 2a_8{}^{\mathrm{w}}I_1){}^{\mathrm{w}}I_3{}^2 - 12a_8^2a_9({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_3{}^2 \\
& + a_6\Big[a_8(12a_9 + a_8{}^{\mathrm{w}}I_1){}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3{}^2 + 9(2a_7a_8 - a_9^2){}^{\mathrm{w}}I_3{}^2 \\
& + \Big(a_1a_6 + 2(a_1a_8 - a_7a_9){}^{\mathrm{w}}I_1\Big)({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big]\Big\} \\
& - a_7\Big[(a_6 + 6a_9)\Big(3a_6a_7 + 2a_9(a_6{}^{\mathrm{w}}I_1 - a_8{}^{\mathrm{w}}I_1{}^2 + 3a_8{}^{\mathrm{w}}I_2)\Big) \\
& - 2a_6a_8a_9({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2)\Big]{}^{\mathrm{w}}I_3{}^2 \\
& - a_9^2\Big(6a_6a_9 + a_6^2 + 2a_8^2({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2)\Big)({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_3{}^2 \\
& - 2a_9^2\Big(2a_1a_6 + 2(a_1a_8 - a_7a_9){}^{\mathrm{w}}I_1 - a_9^2({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)\Big) \\
& \cdot (9{}^{\mathrm{w}}I_3{}^2 - {}^{\mathrm{w}}I_2{}^3 + 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3)\,,
\end{aligned}
$$

$$
\text{(A.26)}\qquad
\begin{aligned}
\bar{n}_{57} = a_1\Big[& a_6^2(3a_8{}^{\mathrm{w}}I_3 + a_9{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_3 - 2a_8^2a_9({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3 \\
& + a_6\Big(a_8(3a_8{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3 + 2a_9{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 + 9a_9{}^{\mathrm{w}}I_3){}^{\mathrm{w}}I_3 \\
& -3a_7(a_9{}^{\mathrm{w}}I_2{}^2 - a_8{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3 - 2a_9{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big)\Big] \\
& + 3a_1^2a_6a_8({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) \\
& - 2a_9^2(a_1a_8 - a_7a_9)(2{}^{\mathrm{w}}I_1{}^2{}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_2{}^3 - 4{}^{\mathrm{w}}I_1{}^3{}^{\mathrm{w}}I_3 + 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3) \\
& + \Big(3a_1^2a_8^2 + 3a_7^2a_9^2 - a_1a_9(6a_7a_8 + a_6a_9) + a_9^4({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)\Big) \\
& \cdot ({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^2{}^{\mathrm{w}}I_3 - 3{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3)
\end{aligned}
$$

(A.26) [cont.]
$$+ a_9{}^{\mathrm{w}}I_3\Big\{2a_8a_9({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2)(a_7{}^{\mathrm{w}}I_2 - a_8{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3 - 3a_9{}^{\mathrm{w}}I_3)$$
$$- a_6\Big[a_7(3a_7 + 2a_9{}^{\mathrm{w}}I_1){}^{\mathrm{w}}I_2 + a_9\Big(2a_8({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_3$$
$$+ a_9({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_2\Big)\Big]\Big\}.$$

A.6. Terms entering *Solution (6.5.7)*

(A.27)
$$\bar{d}_{71} = \Big(a_3(a_6 + 3a_9){}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3 - (a_3a_9{}^{\mathrm{w}}I_2 - a_6a_8{}^{\mathrm{w}}I_3)({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)\Big){}^{\mathrm{w}}I_3$$
$$+ a_1\Big(a_3({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^2{}^{\mathrm{w}}I_3 - 3{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3)$$
$$+ a_8({}^{\mathrm{w}}I_1{}^2{}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_2{}^3 - 2{}^{\mathrm{w}}I_1{}^3{}^{\mathrm{w}}I_3 + 4{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3 - 9{}^{\mathrm{w}}I_3{}^2)\Big),$$

(A.28)
$$\bar{d}_{73} = \Big[a_6(a_6 + 3a_9 + a_8{}^{\mathrm{w}}I_1){}^{\mathrm{w}}I_3 + a_3\Big(3(a_6 + 3a_9){}^{\mathrm{w}}I_3 - a_9{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2\Big)\Big]{}^{\mathrm{w}}I_3$$
$$+ a_1\Big(2a_3({}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) + a_6({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)$$
$$+ a_8({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^2{}^{\mathrm{w}}I_3 - 3{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3)\Big),$$

(A.29)
$$\bar{n}_{75} = a_3a_9^2\Big(2a_3({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2) + a_6({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)\Big){}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3{}^2$$
$$- a_1{}^{\mathrm{w}}I_3\Big\{a_6a_8(a_6 + 3a_9 + a_8{}^{\mathrm{w}}I_1)({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_3$$
$$+ a_3^2\Big(3a_6{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3 - 2a_9({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_2\Big)$$
$$+ a_3\Big[a_6^2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3 + 6a_8a_9({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_3$$
$$+ a_6\Big(2a_8(2{}^{\mathrm{w}}I_1{}^2{}^{\mathrm{w}}I_3 - 3{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3) - a_9({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_2\Big)\Big]\Big\}$$
$$- a_1^2\Big[a_6a_8({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)$$
$$+ 2a_3a_8(2{}^{\mathrm{w}}I_1{}^2{}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_2{}^3 - 4{}^{\mathrm{w}}I_1{}^3{}^{\mathrm{w}}I_3 + 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3)$$
$$+ \Big(3a_3^2 + a_3a_6 + a_8^2({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)\Big)({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^2{}^{\mathrm{w}}I_3 - 3{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3)\Big].$$

A.7. Terms entering *Solution (6.5.8)*

$$\text{(A.30)}\qquad \bar{d}_{83} = a_6^2{}^{\mathrm{w}}I_3 + a_5a_7({}^{\mathrm{w}}I_1{}^2{}^{\mathrm{w}}I_2 - 2{}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) + a_3\Big(2a_5({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2){}^{\mathrm{w}}I_3 - a_9({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - 9{}^{\mathrm{w}}I_3)\Big) + a_6\Big[a_7{}^{\mathrm{w}}I_2 + \Big(3a_3 + 3a_9 + a_5({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)\Big){}^{\mathrm{w}}I_3\Big]\,,$$

$$\text{(A.31)}\qquad \bar{d}_{85} = a_5a_7({}^{\mathrm{w}}I_1{}^2{}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_2{}^3 - 2{}^{\mathrm{w}}I_1{}^3{}^{\mathrm{w}}I_3 + 4{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3 - 9{}^{\mathrm{w}}I_3{}^2) + a_6\Big(a_3{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3 + a_7({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big) + a_3\Big(a_5{}^{\mathrm{w}}I_3({}^{\mathrm{w}}I_1{}^2{}^{\mathrm{w}}I_2 - 2{}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) - a_9({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^2{}^{\mathrm{w}}I_3 - 3{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3)\Big)\,,$$

$$\text{(A.32)}\qquad \bar{n}_{81} = a_3a_9^2{}^{\mathrm{w}}I_2\Big(2a_3({}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) + a_6({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big) - a_5^2\Big\{a_7{}^{\mathrm{w}}I_3\Big[a_6({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) + 2a_3\Big(2{}^{\mathrm{w}}I_2{}^2({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2) - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3({}^{\mathrm{w}}I_1^2 - {}^{\mathrm{w}}I_2)\Big)\Big] + \Big(a_3(3a_3 + a_6){}^{\mathrm{w}}I_3{}^2 + a_7^2({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big)({}^{\mathrm{w}}I_1{}^2{}^{\mathrm{w}}I_2 - 2{}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big\} - a_5\Big\{2a_3a_9(3a_7 - a_3{}^{\mathrm{w}}I_1)({}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3){}^{\mathrm{w}}I_3 + a_6^2{}^{\mathrm{w}}I_3\Big(a_3{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3 + a_7({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big) + a_6\Big[a_7{}^{\mathrm{w}}I_3\Big((4a_3 + 3a_9){}^{\mathrm{w}}I_2{}^2 - 6(a_3 + a_9){}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3\Big) + a_7^2{}^{\mathrm{w}}I_2({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) + a_3{}^{\mathrm{w}}I_3\Big(3a_3{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3 - a_9{}^{\mathrm{w}}I_1({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big)\Big]\Big\}\,.$$

A.8. Terms entering *Solution (6.5.9)*

$$\text{(A.33)}\qquad \bar{d}_{91} = a_7\Big(3a_6 + 2a_5({}^{\mathrm{w}}I_1{}^2 - 3{}^{\mathrm{w}}I_2)\Big){}^{\mathrm{w}}I_3 + a_1\Big[\Big(a_6 + {}^{\mathrm{w}}I_1(a_8 + a_5{}^{\mathrm{w}}I_1) - 2a_5{}^{\mathrm{w}}I_2\Big){}^{\mathrm{w}}I_2 - 3(3a_8 + a_5{}^{\mathrm{w}}I_1){}^{\mathrm{w}}I_3\Big]\,,$$

$$\text{(A.34)}\qquad \bar{d}_{95} = \Big(3a_6a_8{}^{\mathrm{w}}I_3 + a_5(a_7{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - 9a_7{}^{\mathrm{w}}I_3 + a_6{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3)\Big){}^{\mathrm{w}}I_3 + a_1\Big(2a_8({}^{\mathrm{w}}I_2{}^2 - 3{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) + a_5({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{2\,\mathrm{w}}I_3 - 3{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3)\Big)\,,$$

$$\begin{aligned}\text{(A.35)}\qquad \bar{n}_{93} = a_1{}^{\mathrm{w}}I_3\Bigg(& a_7(3a_8 + a_5{}^{\mathrm{w}}I_1)\Big(a_6 + a_5({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)\Big){}^{\mathrm{w}}I_2{}^2 \\ &- \Big\{2a_7(3a_8 + a_5{}^{\mathrm{w}}I_1)(a_6 + a_5{}^{\mathrm{w}}I_1{}^2){}^{\mathrm{w}}I_1 \\ &- \Big[2a_5a_7(3a_8+2a_5{}^{\mathrm{w}}I_1){}^{\mathrm{w}}I_1 + a_6a_8\Big(a_6+(a_8+a_5{}^{\mathrm{w}}I_1){}^{\mathrm{w}}I_1\Big)\Big]{}^{\mathrm{w}}I_2 \\ &+ 2a_5a_6a_8{}^{\mathrm{w}}I_2{}^2\Big\}{}^{\mathrm{w}}I_3 - 3\Big[3a_6a_8^2 - a_5(a_6^2 + 9a_7a_8) \\ &+ a_5^2\Big(3a_7{}^{\mathrm{w}}I_1 - a_6({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)\Big)\Big]{}^{\mathrm{w}}I_3{}^2\Bigg) \\ + a_7\Big(a_6 + a_5({}^{\mathrm{w}}I_1{}^2 - 2{}^{\mathrm{w}}I_2)\Big)&\Big[3a_6a_8{}^{\mathrm{w}}I_3 + a_5\Big(a_6{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3 + a_7({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - 9{}^{\mathrm{w}}I_3)\Big)\Big]{}^{\mathrm{w}}I_3{}^2 \\ &+ a_1^2\Big[\Big(a_6(a_8{}^{\mathrm{w}}I_2 + 3a_5{}^{\mathrm{w}}I_3) + a_8^2({}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2 - 9{}^{\mathrm{w}}I_3)\Big)({}^{\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_3) \\ &+ a_5(a_8{}^{\mathrm{w}}I_2 + 3a_5{}^{\mathrm{w}}I_3)({}^{\mathrm{w}}I_1{}^{2\,\mathrm{w}}I_2{}^2 - 2{}^{\mathrm{w}}I_2{}^3 - 2{}^{\mathrm{w}}I_1{}^{3\,\mathrm{w}}I_3 + 4{}^{\mathrm{w}}I_1{}^{\mathrm{w}}I_2{}^{\mathrm{w}}I_3 - 9{}^{\mathrm{w}}I_3{}^2)\Big]\,.\end{aligned}$$

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