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Onset of surface-tension-driven convection in superposed layers of fluid and saturated porous medium

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The onset of surface-tension-driven convection is studied in a two-layer system comprising an incompressible fluid-saturated porous layer over which lies a layer of the same fluid. The lower rigid surface of the porous layer is either perfectly heat conducting or insulating, while the upper heat insulating fluid boundary is free and at which the surface tension effects are allowed for. At the contact surface between the fluid-saturated porous medium and the adjacent bulk fluid, both Beavers-Joseph and the Jones conditions are employed. The resulting eigenvalue problem is solved exactly. Besides, analytical expression for the critical Marangoni number is obtained for insulating boundaries by using regular perturbation technique. The effect of variation of different physical parameters on the onset of Marangoni convection is investigated in detail. It is found that the parameter ζ, the ratio of the thickness of the fluid layer to that of the porous layer, has a profound effect on the stability of the system.

1. Introduction

One of the convective instabilities that has been studied extensively by many researchers over the years is the onset of convection in a thin layer of fluid heated from below. The instability in such a fluid layer may either be due to buoyancy driven convection, known as Rayleigh–Benard convection, or may be due to surface tension driven convection at the upper free surface, referred to as Marangoni convection. The Benard or Marangoni or coupled Benard–Marangoni type of convective instability in a fluid layer has been studied in detail because of its applications in many scientific and engineering problems (see Pearson [1], Nield [2], Rudraiah *et al.* [3], Char and Chen [4] and references therein). The convective instability in a porous layer due to buoyancy, known by Darcy–Benard convection, has also received equal interest as found in the literature owing to its natural occurrence and relevance in various fields such as chemical engineering, industrial engineering, soil mechanics, geophysics, to mention a few (for details see Nield and Bejan [5] and Vafai [6]). A limited effort has also been put into understand Marangoni convection in porous media, but it is still in an infancy stage (Hennenberg *et al.*, [7], and Rudraiah and Prasad [8]).

However, rapid developments in modern technology during the recent years have posed challenges in studying convective instability problems in more complicated two- and multilayer fluid dynamical systems. The use of such composite systems can be found, for example, in the following applications: manufacture of composite materials used in aircraft structures and automobile industries, geophysics, bioconvection, nuclear reactors, solid-matrix heat exchangers, directional solidification of alloys and electronics cooling. There are several investigations pertaining to convective instability in a two-layer system composed of a fluid-saturated porous layer over which lies a fluid layer and also different systems of superposed porous and fluid layers.

Nield [9] was the first to study the onset of convection in superposed fluid and porous layers when the system is bounded by two horizontal heat insulting boundaries. The thermal stability for different systems of superposed porous and fluid regions has also been analysed by Taslim and Narusawa [10]. McKay [11] has considered the onset of buoyancy-driven convection in superposed reacting fluid and porous layers. Straughan [12, 13] has studied a fundamental model for convection in a porous-fluid layer system developed originally by Nield [9]. He has obtained the eigenvalues and eigenfunctions numerically by utilizing the Chebyshev tau method. In particular, the effect of surface tension is also allowed for in the former paper, while in the latter, the effect of variation of properties of relevant fluid and porous material on the control of convection is discussed by considering the upper surface to be fixed or stress-free. Chen [14] has investigated the effect of throughflow on the onset of thermal convection in a fluid layer overlying a porous layer, with an idea of understanding the control of convective instability by the adjustment of throughflow. Khalili *et al.*, [15] have obtained the closed-form solution for Chen's model by considering upper and lower boundaries as heat insulating. The coupled capillary and gravity-driven instability in a fluid layer overlying a porous layer has been studied by Desaive *et al.*, [16] using Brinkman's model to describe the flow in the porous medium. Recently, Carr [17] has studied penetrative convection via internal heating in a two-layer system in which a layer of fluid overlies and saturates the porous medium.

In the present study, the onset of convection in a composite porous-fluid layers system heated from below is considered only due to temperature-dependent surface tension forces at the upper free surface of the fluid layer and neglecting the effect of buoyancy forces. The problem under investigation helps in better understanding of the Marangoni convection in a fluid-saturated porous layer, as pointed out by Nield [18] apart from its importance in many practical applications mentioned earlier. The flow in the fluid layer is governed by Navier–Stokes equation, while the Darcy equation is used to describe the flow regime in the porous medium. The upper heat insulating fluid surface is open to free atmosphere and convection driven by surface tension is allowed for. The bottom boundary of the porous layer is considered to be rigid and either perfectly heat conducting or insulating. The contentious issue here is about the use of proper conditions at the contact surface between the fluid-saturated porous medium and the adjacent bulk fluid, which depend on the type of equation used to describe the flow in the porous medium. If the flow in the porous medium is governed by Darcy equation, then the well-established classical fluid slip velocity condition postulated by BEAVERS and JOSEPH [19] is proved to be more successful. For a Brinkman-type equation, however, the presence of viscous stress term does not allow to formulate the proper compatibility conditions for stresses at the contact surface of fluid-saturated porous medium and the bulk fluid. One of the main problems of the compatibility conditions on such a boundary is to determine viscous interactions of the bulk fluid with the porous skeleton and fluid filling its pores during its flow tangent to the boundary surface. This interesting problem has been analyzed recently by CIESZKO and KUBIK [20]. They have derived compatibility conditions for the tangential components of relative fluid flow velocities at the contact surface. It is shown that the results obtained from their linear compatibility conditions compare better with the experimental results than those obtained using the Beavers-Joseph slip condition.

Since we have adopted Darcy's law in the porous medium, both the Beavers–Joseph and the JONES [21] conditions are used in the present study. These conditions can be succinctly written in the following form:

$$\frac{\partial u}{\partial z} + \chi \frac{\partial w}{\partial x} = \frac{\alpha}{\sqrt{k}} \left(u - u_m \right), \tag{1.1}$$

$$\frac{\partial v}{\partial z} + \chi \frac{\partial w}{\partial y} = \frac{\alpha}{\sqrt{k}} \left(v - v_m \right), \tag{1.2}$$

where u and v are respectively the horizontal x and y components of the fluid velocity, while u_m and v_m are the equivalent components in the porous medium, α is the dimensionless material constant called the slip parameter and χ is a constant taking the value 0 for the Beavers–Joseph condition and 1 for the Jones condition. The linearized stability equations with accompanying boundary conditions are solved exactly. The critical Marangoni number and the corresponding wave number are determined over a wide range of physical parameters of the system. Besides, the solution to the eigenvalue problem is also sought for by regular perturbation technique in the case of insulating boundaries.

The paper is organized as follows. Secion 2 is devoted to the formulation of the problem. The methods of solution to the resulting eigenvalue problem are discussed in Sec. 3 and the results are discussed in Sec. 4.

2. Formulation of the problem

Let us consider the physical configuration consisting of an infinite horizontal incompressible fluid-saturated porous layer of thickness d_m underlying a layer of the same fluid of thickness d, as shown in Fig.1.

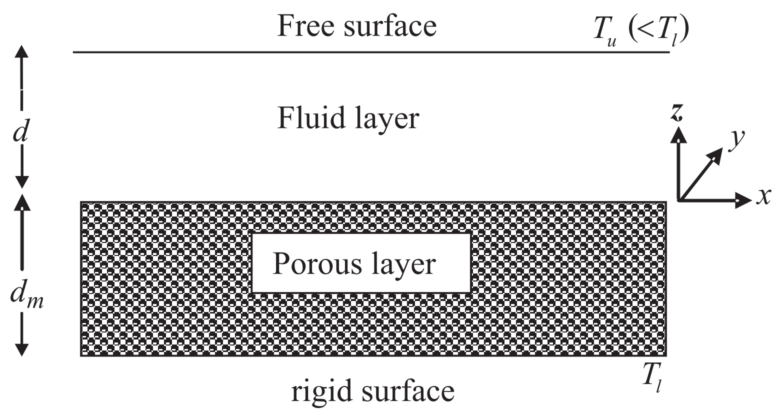

Fig. 1. Geometrical configuration of the system.

The bottom surface of the porous layer is rigid while the upper fluid surface is free to the atmosphere, with a linear dependence of the surface tension on temperature in the form [1]

$$\sigma = \sigma_0 - \sigma_T(T - T_0), \tag{2.1}$$

where σ_0 is the surface tension of the fluid at temperature T_0 and the constant rate of change of surface tension with temperature, σ_T, is assumed to be positive. Also, the free surface has been assumed to remain flat (undeformed). A Cartesian coordinate system $(x,\ y,\ z)$ is chosen such that the origin is at the interface between the bulk fluid layer and the fluid-saturated porous layer and the z-axis points vertically upwards. The temperatures of the lower and upper boundaries are taken to be uniform and equal to T_l and T_u, respectively, with $T_l > T_u$.

The governing equations for the fluid layer are:

$$\nabla \cdot \mathbf{q} = 0, \tag{2.2}$$

$$\rho_0 \left[\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla)\mathbf{q} \right] = -\nabla p + \mu \nabla^2 \mathbf{q}, \tag{2.3}$$

$$\frac{\partial T}{\partial t} + (\mathbf{q} \cdot \nabla)\, T = \kappa \nabla^2 T \tag{2.4}$$

and those for porous layer they are:

$$\nabla_m \cdot \mathbf{q}_m = 0, \tag{2.5}$$

$$\frac{\rho_0}{\phi} \frac{\partial \mathbf{q}_m}{\partial t} = -\nabla_m p_m - \frac{\mu}{\mathrm{K}} \mathbf{q}_m, \tag{2.6}$$

$$S \frac{\partial T_m}{\partial t} + (\mathbf{q}_m . \nabla_m)\, T_m = \kappa_m \nabla_m^2 T_m. \tag{2.7}$$

In the above equations, $\mathbf{q} = (u, v, w)$ is the velocity vector, T is the temperature, μ is the dynamic viscosity, p is the pressure, ρ_0 is the fluid density, K is the permeability of the porous medium, κ is the thermal diffusivity of the fluid, φ is the porosity of the porous medium, $S = (\rho_0 c_p)_m / (\rho_0 c_p)_f = [\phi(\rho_0 c_p)_f + (1-\phi)\,(\rho_0 c_p)_s] / (\rho_0 c_p)_f$ is the ratio of heat capacities of the fluid-saturated porous medium to that of the fluid, c_p is the specific heat and the subscripts m, f and s refer to the porous medium, fluid and solid, respectively.

The basic steady state is assumed to be quiescent and we consider the solution of the form

$$(u,\ v,\ w,\ p,\ T) = [0,\ 0,\ 0,\ p_b(z),\ T_b(z)]$$

in the fluid layer, and

$$(u_m, \nu_m, w_m, p_m, T_m) = [0, 0, 0,\ p_{mb}(z_m), T_{mb}(z_m)],$$

in the porous layer, where the subscript b denotes the basic state. The temperature distributions $T_b\,(z)$ and $T_{mb}\,(z_m)$ are found to be

$$T_b\,(z) = T_0 - \frac{(T_0 - T_u)\ z}{d} \qquad 0 \le z \le d, \tag{2.8}$$

$$T_{mb}\,(z_m) = T_0 - \frac{(T_\ell - T_0)\, z_m}{d_m} \qquad -d_m \le z_m \le 0, \tag{2.9}$$

where $T_0 = (\kappa_m T_\ell d + \kappa T_u d_m) \,/\, (\kappa_m d + \kappa\, d_m)$ is the interface temperature.

In order to investigate the stability of the basic solution, infinitesimal disturbances are introduced in the form

$$(u,\ v,\ w,\ p,\ T) = [0,\ 0,\ 0,\ p_b(z),\ T_b(z)] + \left(u',\ v',\ w',\ p',\ T'\right), \tag{2.10}$$

$$(u_m,\ v_m,\ w_m,\ p_m,\ T_m) = [0,\ 0,\ 0,\ p_{mb}(z_m),\ T_{mb}(z_m)] + \left(u'_m,\ v'_m,\ w'_m,\ p'_m,\ T'_m\right), \tag{2.11}$$

where the primed quantities are the perturbed ones over their equilibrium counterparts.

Now Eqs. (2.10) and (2.11) are substituted in Eqs. (2.2)–(2.7) and linearized in the usual manner. Next, the pressure term is eliminated from Eqs. (2.3) and (2.6) by taking curl twice on these two equations and only the vertical component is retained. The variables are then nondimensionalized using $d, d^2/\kappa, \kappa/d$ and $T_0 - T_u$ as the units of length, time, velocity and temperature in the fluid layer and $d_m, d_m^2/\kappa_m, \kappa_m/d_m$ and $T_\ell - T_0$ as the corresponding characteristic quantities in the porous layer. Note that separate length scales are chosen for the two layers so that each layer is of unit depth. In this manner, the detailed flow fields in both the fluid and porous layers can be clearly discerned for all depth ratios, $\zeta = d/d_m$. The dimensionless equations for the perturbed variables are given by

$$\left[\frac{1}{\mathrm{Pr}}\frac{\partial}{\partial t} - \nabla^2\right]\nabla^2 w = 0, \tag{2.12}$$

$$\left[\frac{\partial}{\partial t} - \nabla^2\right] T = w, \tag{2.13}$$

$$\left[\frac{\mathrm{Da}}{\mathrm{Pr}_m}\frac{\partial}{\partial t} + 1\right]\nabla_m^2 w_m = 0, \tag{2.14}$$

$$\left[S\frac{\partial}{\partial t} - \nabla_m^2\right] T_m = w_m. \tag{2.15}$$

For the fluid layer, $\mathrm{Pr} = v/\kappa$ is the Prandtl number, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplacian operator and for the porous layer $\mathrm{Pr}_m = v/\kappa_m\phi$ is the Prandtl number, $\mathrm{Da} = K/d_m^2$ is the Darcy number and $\nabla_m^2 = \partial^2/\partial x_m^2 + \partial^2/\partial y_m^2 + \partial^2/\partial z_m^2$. To solve the set of Eqs. (2.12)–(2.15), we need ten boundary conditions, which are detailed below.

The bottom boundary of the porous layer ($z_m = -1$) is assumed to be rigid and either perfectly heat–conducting or insulating, so that

$$w_m = 0, \qquad T_m = 0 \qquad \text{at} \quad z_m = -1, \tag{2.16}$$

or

$$w_m = 0, \qquad \frac{\partial T_m}{\partial z_m} = 0 \qquad \text{at} \quad z_m = -1. \tag{2.17}$$

The upper free surface of the fluid layer ($z = 1$) at which convection driven by temperature-dependent surface tension forces is allowed for and considered to be insulating, so that

$$w = 0, \qquad \frac{\partial T}{\partial z} = 0 \qquad \text{at} \quad z = 1 \tag{2.18}$$

and the balance between shear stresses and surface tension gradients, written in the non-dimensional form, give

$$\left.\begin{aligned} \frac{\partial u}{\partial z} &= -M\frac{\partial T}{\partial x} \\ \frac{\partial v}{\partial z} &= -M\frac{\partial T}{\partial y} \end{aligned}\right\} \qquad \text{at} \quad z = 1, \tag{2.19}$$

where $M = \sigma_T\,(T_0 - T_u)\,d/\mu\kappa$ is the Marangoni number. Equations (2.19) are differentiated partially with respect to x and$y$ respectively, and the results are added to get, after using Eq. (2.2), the following condition:

$$\frac{\partial^2 w}{\partial z^2} = M\nabla_h^2 T \qquad \text{at} \quad z = 1, \tag{2.20}$$

where $\nabla_h^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal Laplacian operator.

At the interface (i.e., at $z = 0$) the normal component of velocity, temperature and heat flux are continuous, which yield

$$w = \frac{\zeta}{\varepsilon_T} w_m, \tag{2.21}$$

$$T = \frac{\varepsilon_T}{\zeta} T_m, \tag{2.22}$$

$$\frac{\partial T}{\partial z} = \frac{\partial T_m}{\partial z_m}. \tag{2.23}$$

We note that two more conditions are required at $z = 0$. To derive these conditions we start with Eqs. (1.1) and (1.2). Differentiating Eqs. (1.1) and (1.2) partially with respect to x and y respectively, adding the resulting equations and using Eqs. (2.2) and (2.5), the condition in non-dimensional form is

$$\frac{\partial^2 w}{\partial z^2} - \chi\,\nabla_h^2 w = \frac{\alpha\zeta}{\sqrt{\mathrm{Da}}}\frac{\partial w}{\partial z} - \frac{\alpha\,\zeta^3}{\varepsilon_T}\left[\frac{1}{\sqrt{\mathrm{Da}}}\right]\frac{\partial w_m}{\partial z_m} \qquad \text{at} \quad z = 0, \tag{2.24}$$

where $\varepsilon_T = \kappa/\kappa_m$ is the ratio of thermal diffusivities. Further, the continuity of normal stress at the interface requires that

$$p_m = p - 2\mu \frac{\partial w}{\partial z} \quad \text{at} \quad z = 0. \tag{2.25}$$

We differentiate Eq. (2.25) partially with respect to x and y and substitute for $\partial p/\partial x$ and $\partial p/\partial y$ from Eq. (2.3), and for $\partial p_m/\partial x_m$ and $\partial p_m/\partial y_m$ from Eq. (2.6). After using Eqs. (2.2) and (2.5), the following interface condition is obtained in non-dimensional form:

$$\left(3\nabla_h^2 + \frac{\partial^2}{\partial z^2}\right)\frac{\partial w}{\partial z} - \frac{1}{\mathrm{Pr}}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial z}\right) = -\frac{\zeta^4}{\varepsilon_T}\left[\frac{1}{\mathrm{Da}}\right]\frac{\partial w_m}{\partial z_m} - \frac{1}{\mathrm{Pr}_m}\frac{\partial}{\partial t}\left(\frac{\partial w_m}{\partial z_m}\right) \quad \text{at} \quad z = 0. \tag{2.26}$$

We make the normal mode expansion and seek solutions for the dependent variables in the fluid and porous layers according to

$$(w, T) = [W(z), \Theta(z)] \exp\left[i(\ell x + my) + \omega t\right], \tag{2.27}$$

$$(w_m, T_m) = [W_m(z_m), \Theta_m(z_m)] \exp\left[i\left(\tilde{\ell} x_m + \tilde{m} y_m\right) + \omega_m t\right], \tag{2.28}$$

where ℓ and m are the wave numbers in the x and y directions respectively, in the fluid layer, while $\tilde{\ell}$ and $\tilde{m}$ are the corresponding wave numbers in the porous layer and $\omega = \omega_r + i\omega_i (= \varepsilon_T/\zeta^2 \omega_m)$ is the growth parameter of the disturbances. Here, ω_r is the growth rate of the instability and ω_i is the frequency. If $\omega_r > 0$, the infinitesimal disturbance grows and the system becomes unstable with respect to that disturbance. If $\omega_r < 0$, the infinitesimal disturbance decays and the system becomes stable. The convective motions in the neutral state are divided into the stationary state ($\omega_i = 0$) and the oscillatory state ($\omega_i \neq 0$). In the linear stability theory, the principle of exchange of stabilities often plays an important role and is assumed to be valid; that is, if $\omega_r = 0$, then $\omega_i = 0$ automatically. This principle has been proved to be valid (i.e., instability occurs via stationary convection) for the Marangoni convective instability in a single fluid layer [1, 22], in a porous layer [7, 8] and also in the composite fluid-porous layers system for the heated-below situation considered here (for details see [9] and [12]). We follow these earlier analyses and conveniently take $\omega = 0$ in our further discussion.

Substituting Eqs. (2.27) and (2.28) in Eqs. (2.12)–(2.15) and setting $\omega = 0$, we obtain the following equations relevant to neutral stability:

$$\left(D^2 - a^2\right)^2 W = 0, \tag{2.29}$$

$$\left(D^2 - a^2\right) \ \Theta = -W, \tag{2.30}$$

$$\left(D_m^2 - a_m^2\right) \ W_m = 0, \tag{2.31}$$

$$\left(D_m^2 - a_m^2\right) \ \Theta_m = -W_m, \tag{2.32}$$

where D and D_m denote differentiation with respect to z and z_m respectively, $a = \sqrt{\ell^2 + m^2}$ and $a_m = \sqrt{\tilde{\ell}^2 + \tilde{m}^2}$ are correspondingly the overall horizontal wave numbers in the fluid and porous layers. If matching of the solutions in the two layers is to be possible, the wave numbers must be the same for the fluid and porous layers, so that we have $a/d = a_m/d_m$, and hence $\zeta = a/a_m$.

The ten boundary conditions given by Eqs. (2.16) or (2.17), (2.18), (2.20)–(2.23), (2.24) and (2.26), using Eqs. (2.27) and (2.28), now take the form

$$W_m = 0, \qquad \Theta_m = 0 \qquad \text{at} \quad z_m = -1 \tag{2.33}$$

or

$$W_m = 0, \qquad D_m\Theta_m = 0 \qquad \text{at} \quad z_m = -1, \tag{2.34}$$

$$W = 0, \qquad D\Theta = 0 \qquad \text{at} \quad z = 1, \tag{2.35}$$

$$D^2W + Ma^2\Theta = 0 \qquad \text{at} \quad z = 1 \tag{2.36}$$

and those at the interface (i.e. at $z = 0$) become

$$\begin{aligned} W &= \frac{\zeta}{\varepsilon_T} W_m, \\ D\Theta &= D_m\Theta_m, \\ \Theta &= \frac{\varepsilon_T}{\zeta}\Theta_m, \\ \left[D^2 + \chi a^2 - \frac{\alpha\zeta\, D}{\sqrt{\text{Da}}}\right] W &= -\frac{\alpha\zeta^3}{\varepsilon_T\sqrt{\text{Da}}} D_m W_m, \\ \left[D^2 - 3a^2\right] DW &= -\frac{\zeta^4}{\varepsilon_T \text{Da}} D_m W_m. \end{aligned} \tag{2.37}$$

Thus the convective instability problem has now been reduced to an eigenvalue problem consisting of a sixth order ordinary differential equation in the fluid layer and a fourth order ordinary differential equation in the porous layer subject to five boundary as well as five interface conditions.

3. Methods of solution

The resulting eigenvalue problem is solved exactly, in general with M as an eigen-value. Besides, an analytical expression for the critical Marangoni number is also obtained by regular perturbation technique with wave number a as a perturbation parameter for the case of insulating boundaries.

3.1. Exact solution

Since Eqs. (2.29) and (2.31) are independent of Θ and Θ_m, they can be directly solved to get the general solution in the form

$$W = A_1 \cosh(az) + A_2 \sinh(az) + A_3 z \cosh(az) + A_4 z \sinh(az), \tag{3.1}$$

$$W_m = A_{m1} \sinh(a_m z_m) + A_{m2} \cosh(a_m z_m), \tag{3.2}$$

where $A_1 - A_4$, A_{m1}, and A_{m2} are constants to be determined. Using the boundary conditions $(2.33)_1$, $(2.35)_1$, $(2.37)_1$, $(2.37)_4$ and $(2.37)_5$ in Eqs. (3.1) and (3.2), we obtain

$$W = A_1 \left[\cosh(az) + \Delta_1 \sinh(az) + \Delta_2 z \cosh(az) + \Delta_3 z \sinh(az)\right], \tag{3.3}$$

$$W_m = A_1 \frac{\varepsilon_T}{\zeta} \left[\coth a_m \sinh(a_m z_m) + \cosh(a_m z_m)\right], \tag{3.4}$$

where

$$\Delta_1 = \frac{\zeta^3 a_m \coth a_m}{2a^3 Da},$$

$$\Delta_2 = -\frac{2a(\cosh a + \Delta_1 \sinh a) - \sinh a\left(a^2(\gamma+1) + \dfrac{\alpha\zeta^2 \coth a_m}{\sqrt{\mathrm{Da}}}\right) - \dfrac{\alpha\zeta a \Delta_1}{\sqrt{\mathrm{Da}}}}{2a\cosh a + \alpha\zeta \sinh a \Big/ \sqrt{\mathrm{Da}}},$$

$$\Delta_3 = -(1+\Delta_2)\coth a - \Delta_1.$$

The heat Eqs. (2.30) and (2.32) have now to be solved defining their right-hand sides by the expressions given by Eqs. (3.3) and (3.4), respectively. As mentioned earlier, two types of temperature boundary conditions are considered.

The solution obtained for Θ and Θ_m using the boundary conditions $(2.34)_2$, $(2.35)_2$, $(2.37)_2$ and $(2.37)_3$ (i.e., for the case when the lower boundary of the porous layer and upper boundary of the fluid layer are insulating), is found to be

$$\Theta = \frac{A}{4a^2}\left[K_1 \sinh(az) + L_1 \cosh(az) + f(z)\right], \tag{3.5}$$

$$\begin{aligned}\Theta_m = A\Bigg\{&\left[\frac{aK_1 - \lambda_4}{a_m} - \frac{\varepsilon_T z_m}{2\zeta a_m}\right]\sinh(a_m z_m)\\ &+\left[\frac{\varepsilon_T L_1}{\zeta} - \frac{\varepsilon_T \coth(a_m z_m)}{2\zeta a_m}\right]\cosh(a_m z_m)\Bigg\}\end{aligned} \tag{3.6}$$

while the solution obtained for Θ and Θ_m using the boundary conditions $(2.33)_2$, $(2.35)_2$, $(2.37)_2$ and $(2.37)_3$ (i.e., for the case when the lower boundary of the porous layer is isothermal and upper boundary of the fluid layer is insulating), is

$$\Theta = \frac{A}{4a^2}\left[K_1 \sinh(az) + L_1 \cosh(az) + f(z)\right], \tag{3.7}$$

$$\begin{aligned}\Theta_m = A\Bigg\{&\left[\frac{aK_2 - \lambda_4}{a_m} - \frac{\varepsilon_T z_m}{2\zeta a_m}\right]\sinh(a_m z_m)\\ &+\left[\frac{\varepsilon_T L_2}{\zeta} - \frac{\varepsilon_T \coth(a_m z_m)}{2\zeta a_m}\right]\cosh(a_m z_m)\Bigg\}.\end{aligned} \tag{3.8}$$

Here

$$K_i = (\lambda_1 \cosh a + \lambda_2 \sinh a - aL_i \sinh a)/a\cosh a, \quad (i = 1, 2),$$

$$L_1 = \frac{\cosh a(\lambda_3 - \lambda_4 \cosh a_m) + \cosh a_m(\lambda_1 \cosh a + \lambda_2 \sinh a)}{a_m\zeta \cosh a\ \sinh a_m/\varepsilon_T + a\cosh a_m \sinh a},$$

$$L_2 = -\frac{\lambda_5 a_m \cosh a + \sinh a_m\left[(\lambda_4 - \lambda_1)\cosh a - \lambda_2 \sinh a\right]}{a_m\zeta \cosh a\ \cosh a_m/\varepsilon_T + a\cosh a\ \cosh a_m},$$

$$f(z) = \left[(\Delta_3 - 2a)z - \Delta_2 a z^2\right]\sinh(az) + \left[\Delta_3 a z^2 + (\Delta_2 - 2a)z\right]\cosh(az),$$

with

$$\lambda_1 = \frac{1}{4a^2}\left\{2a^2 + (a^2 - 1)\Delta_2 + 2\Delta_1 a + a\Delta_3\right\},$$

$$\lambda_2 = \frac{1}{4a^2}\left\{2a + 2a^2\,\Delta_1 + a\,\Delta_2 + (a^2 - 1)\Delta_3\right\},$$

$$\lambda_3 = \frac{\varepsilon_T}{2\zeta a_m}\left\{ (a_m - \coth a_m)\cosh a_m - (a_m \coth a_m - 1)\sinh a_m \right\},$$

$$\lambda_4 = \frac{1}{4a^2}\left\{2a\Delta_1 - \Delta_2 - 2a\Delta_3\right\},$$

$$\lambda_5 = \frac{\varepsilon_T}{2\zeta a_m}\left(\coth a_m \cosh a_m - \sinh a_m\right).$$

Substituting Eqs. (3.3) and (3.5) or (3.7) suitably in the coupled boundary condition (2.36), we obtain an analytical expression for the Marangoni number M which can be conveniently written, for both types of temperature boundary conditions, in the form

$$M_i = \frac{4a\left[(a + \Delta_2 a + 2\Delta_3)\cosh a + (2\Delta_2 + \Delta_1 a + \Delta_3 a)\sinh a\right]}{K_i \sinh a + L_i \cosh a + f(1)}, \quad (i = 1, 2), \tag{3.9}$$

where M_1 is the Marangoni number corresponding to the case when the lower boundary of the porous layer and upper boundary of the fluid layer are insulating, while M_2 is the Marangoni number for lower isothermal and upper insulating boundaries. The critical Marangoni numbers M_{1c} and M_{2c} are obtained numerically by minimizing M_1 and M_2 respectively, with respect to the wave number a for various fixed values of $\zeta, \varepsilon_T, \mathrm{Da}$ and α.

3.2. Solution by regular perturbation technique for insulating boundaries

From the exact analysis carried out in the former section, it is observed that the onset of Marangoni convection corresponds to a vanishingly small wave number in the case of insulating boundaries. This fact has been exploited here to solve the eigenvalue problem for the assumed temperature boundary conditions by employing regular perturbation technique with wave number a as a perturbation parameter. Such a study helps in not only knowing the accuracy of the results obtained by this technique, but also provides a justification for using this technique, to solve those convective instability problems in general for which the critical stability parameter has to be found numerically.

The dependent variables in both the fluid and porous layers are now expanded in powers of a^2 in the form

$$(W_m, \Theta_m) = \sum_{i=0}^{N} \left(a^2\right)^i (W_i, \Theta_i), \tag{3.10}$$

$$(W_m, \Theta_m) = \sum_{i=0}^{N} \left(\frac{a^2}{\zeta^2}\right)^i (W_{mi}, \Theta_{mi}). \tag{3.11}$$

Substitution of Eqs. (3.10) and (3.11) into Eqs. (2.29)–(2.32) and the boundary conditions (2.34), (2.35), (2.36) and (2.37) yields a sequence of equations for the unknown functions $W_i(z)$, $\Theta_i(z)$, $W_{mi}(z)$ and $\Theta_{m\,i}(z)$ for $i = 0, 1, 2, \ldots$.

At the leading order in (a^2) Eqs. (2.29)–(2.32) become, respectively,

$$D^4 W_0 = 0, \tag{3.12}$$

$$D^2 \Theta_0 = -W_0, \tag{3.13}$$

$$D_m^2 W_{m_0} = 0, \tag{3.14}$$

$$D_m^2 \Theta_{m_0} = -\,W_{m_0} \tag{3.15}$$

and the boundary conditions (2.34), (2.35), (2.36) and (2.37) become

$$W_{m_0} = 0, \qquad D_m \Theta_{m_0} = 0, \qquad \text{at} \quad z_m = -1, \tag{3.16}$$

$$W_0 = 0, \qquad D\Theta_0 = 0, \qquad D^2 W_0 = 0, \qquad \text{at} \quad z = 1 \tag{3.17}$$

and at the interface (i.e. $z = 0$)

$$\begin{aligned} W_0 &= \frac{\zeta}{\varepsilon_T} W_{m_0}, \\ \Theta_0 &= \frac{\varepsilon_T}{\zeta} \Theta_{m_0}, \\ D\Theta_0 &= D_m \Theta_{m_0}, \\ D^2 W_0 - \frac{\alpha\zeta}{\sqrt{\mathrm{Da}}} D W_0 &= -\frac{\alpha\zeta^3}{\varepsilon_T \sqrt{\mathrm{Da}}} D_m W_{m_0}, \\ D^3 W_0 &= -\frac{\zeta^4}{\varepsilon_T \mathrm{Da}} D_m W_{m_0}. \end{aligned} \tag{3.18}$$

The solution to the zeroth order Eqs. (3.12)–(3.15) is given by

$$W_0 = 0, \qquad \Theta_0 = \frac{\varepsilon_T}{\zeta}, \tag{3.19}$$

$$W_{m_0} = 0, \qquad \Theta_{m_0} = 1. \tag{3.20}$$

At the first order in (a^2), Eqs. (2.29)–(2.32) then reduce to

$$D^4 W_1 = 0, \tag{3.21}$$

$$D^2\Theta_1 - \frac{\varepsilon_T}{\zeta} = -W_1, \tag{3.22}$$

$$D_m^2 W_{m_1} = 0, \tag{3.23}$$

$$D_m^2\Theta_{m_1} - 1 = -W_{m_1}, \tag{3.24}$$

the boundary conditions (2.34), (2.35), (2.36) and (2.37) become

$$W_{m_1} = 0, \qquad D_m\Theta_{m_1} = 0 \qquad \text{at} \quad z_m = -1, \tag{3.25}$$

$$W_1 = 0, \qquad D\Theta_1 = 0, \qquad D^2W_1 + M\frac{\varepsilon_T}{\zeta} = 0, \qquad \text{at} \quad z = 1 \tag{3.26}$$

and at the interface (i.e. $z = 0$)

$$\begin{aligned} W_1 &= \frac{1}{\zeta\,\varepsilon_T}W_{m_1}, \\ \Theta_1 &= \frac{\varepsilon_T}{\zeta^3}\Theta_{m_1}, \\ D\Theta_1 &= \frac{1}{\zeta^2}D_m\Theta_{m_1}, \\ D^2W_1 - \frac{\alpha\zeta^2}{\sqrt{\text{Da}}}DW_1 &= -\frac{\alpha\zeta}{\varepsilon_T\sqrt{\text{Da}}}D_mW_{m_1}, \\ D^3W_1 &= \frac{-\zeta^2}{\varepsilon_T\text{Da}}D_mW_{m_1}. \end{aligned} \tag{3.27}$$

Integrating Eq. (3.22) between $z = 0$ and 1, Eq. (3.24) between $z_m = -1$ and 0, using the boundary conditions $(3.25)_2$, $(3.26)_2$, $(3.27)_3$ and adding the resulting equations, we obtain the following solvability condition:

$$\int_0^1 W_1 dz + \frac{1}{\zeta^2}\int_{-1}^0 W_{m_1} dz_m = \frac{\varepsilon_T}{\zeta} + \frac{1}{\zeta^2}. \tag{3.28}$$

The general solution of Eqs.(3.21) and (3.23) are respectively given by

$$W_1 = M\left[C_1 + C_2 z + C_3 z^2 + C_4 z^3\right] \tag{3.29_1}$$

and

(3.29)$_2$ $$W_{m_1} = M\left[C_5 + C_6 z_m\right].$$

The constants $C_1 - C_6$ are determined using the boundary conditions $(3.25)_1$, $(3.26)_{1,3}$, $(3.27)_1$, $(3.27)_4$ and $(3.27)_5$, and are found to be

$$C_1 = \frac{\sqrt{\text{Da}}\varepsilon_T/\alpha\zeta^2 + \varepsilon_T/2\zeta}{1 + \zeta^2/\alpha\sqrt{\text{Da}} + \zeta + \zeta^3/3\text{Da}},$$

$$C_2 = \left(\frac{\zeta^2}{\alpha\sqrt{\text{Da}}} + \zeta\right) C_1 - \frac{\sqrt{\text{Da}}\varepsilon_T}{\alpha\zeta^2},$$

$$C_3 = \frac{\zeta^2 C_1}{2\text{Da}} - \frac{\varepsilon_T}{2\zeta},$$

$$C_4 = -\frac{\zeta^3 C_1}{6\text{Da}},$$

$$C_5 = C_6 = C_1\,\zeta\,\varepsilon_T.$$

Substituting for W_1 and W_{m_1} the values given by Eqs. $(3.29)_1$ and $(3.29)_2$ respectively, in Eq. (3.28) and performing the integration, we obtain an expression for the critical Marangoni number M_c in the form

(3.30) $$M_c = \frac{\left(\dfrac{\varepsilon_T}{\zeta} + \dfrac{1}{\zeta^2}\right)}{C_1 + C_2/2 + C_3/3 + C_4/4 + C_5/2\zeta^2}.$$

The expression for M_c thus obtained is evaluated numerically for different values of physical parameters ζ, Da, ε_T and the results are discussed in the following section.

4. Results and discussion

The onset of surface-tension-driven convection in a two-layer system consisting of a fluid layer overlying a porous layer saturated by the same fluid is investigated theoretically. The eigenvalue problem is solved exactly and an analytical expression for the Marangoni number is obtained for two types of temperature boundary conditions, viz. (i) lower rigid surface of the porous layer and upper free surface of the fluid layer are insulating, and (ii) lower rigid surface of the porous layer is isothermal and upper free surface of the fluid layer is insulating.

The critical Marangoni numbers for these two temperature boundary conditions are denoted by M_{1c} and M_{2c}, respectively. For the case of insulating temperature boundary conditions, the resulting eigenvalue problem is also solved by regular perturbation technique to obtain the critical Marangoni number. It is assumed that the porous medium consists of 3 mm diameter glass beads ran-

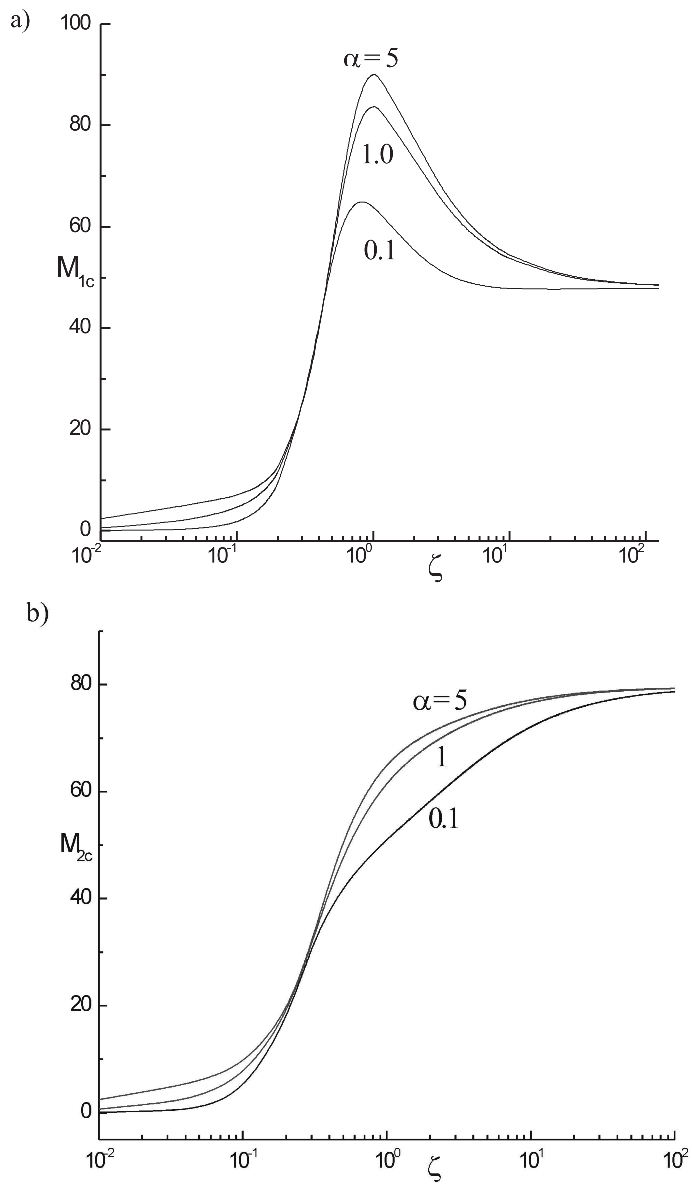

Fig. 2. Variation of a) M_{1c} and b) M_{2c} with ζ for different values of α when $\varepsilon_T = 0.725$ and Da $= 0.003$.

domly packed and depth of the porous layer being 3 cm, resulting in the Darcy number $\sqrt{\mathrm{Da}} = 0.003$ and $\varepsilon_T = 0.725$ [14]. The values of slip parameter α chosen here are 0.1, 1 and 5 which are the representative values for aloxite and foam-metals. In addition to these values, we have also used different values of Da and ε_T to know their effect on the critical Marangoni numbers M_{1c} and M_{2c}. As men-

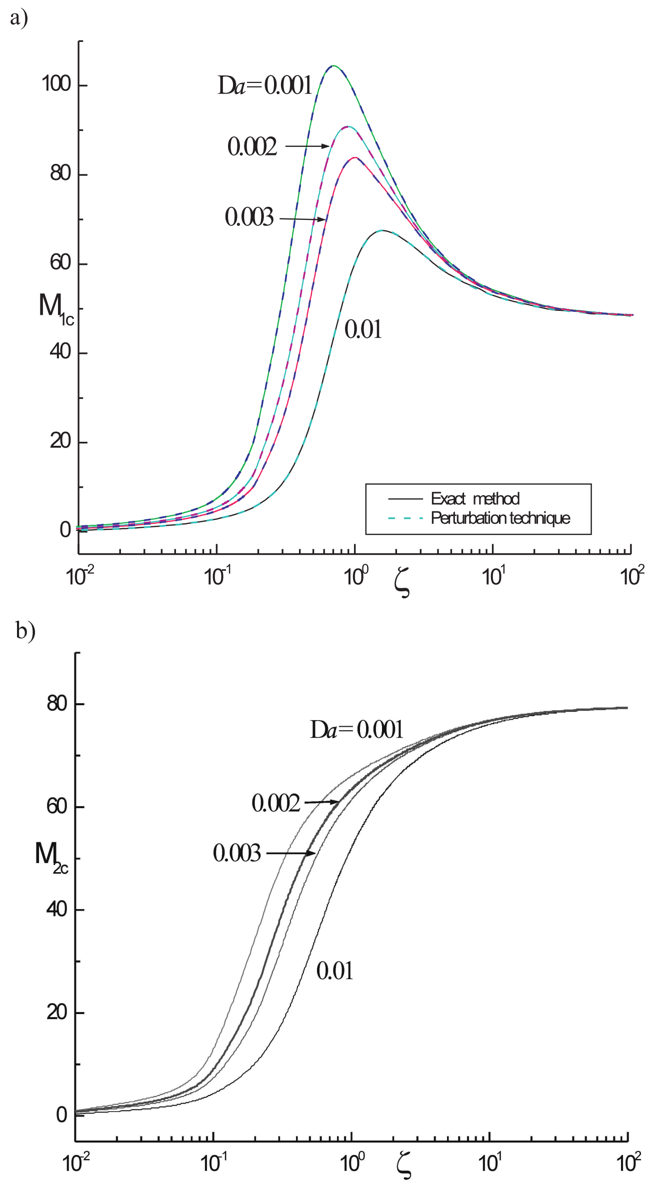

FIG. 3. Variation of a) M_{1c} and b) M_{2c} with ζ for different values of Da when $\varepsilon_T = 0.725$ and $\alpha = 1.0$.

tioned earlier, at the contact interface between fluid – saturated porous medium and the adjacent bulk fluid, both the Beavers–Joseph and the Jones conditions are used to examine their influence on the onset of Marangoni convection. The numerically evaluated critical Marangoni numbers and wave numbers for the Beavers–Joseph slip condition are presented graphically in Figs. 2–5.

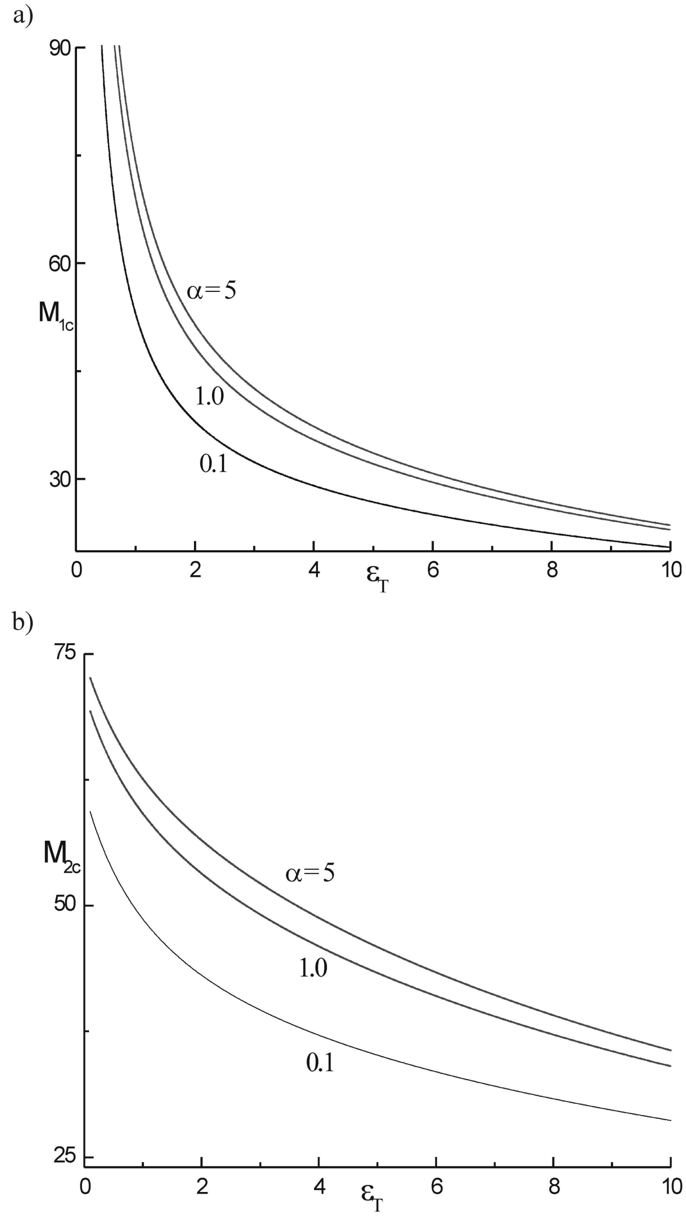

FIG. 4. Variation of a) M_{1c} and b) M_{2c} with ε_T for different values of α when and $\zeta = 1$.

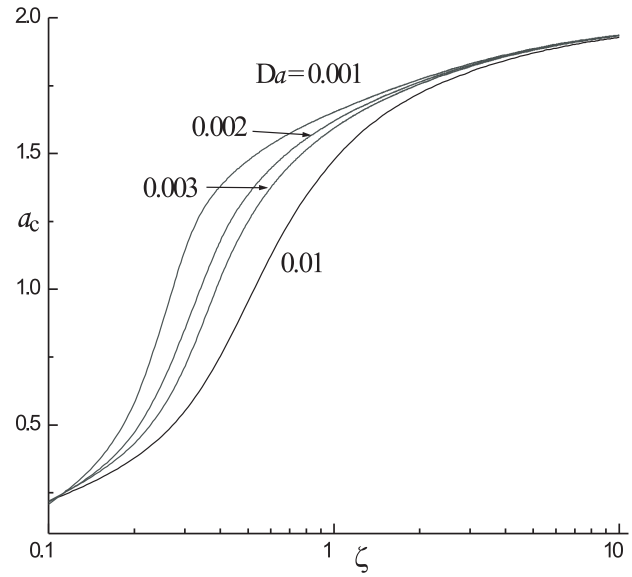

FIG. 5. Variation of a_c with ζ for different values of Da when $\alpha = 1.0$ and $\varepsilon_T = 0.725$.

The curves in Fig. 2a show the evolution of M_{1c} with the depth ratios (ζ) for three values of slip parameter $\alpha = 0.1$, 1 and 5 when $\varepsilon_T = 0.725$ and $\mathrm{Da} = 0.003$. As a result of the variation in the depth ratio, three distinguishable regions are evident from the figure for all the values of α considered. We note that M_{1c} increases only negligibly with ζ for small values of ζ; increases significantly with ζ up to 0.95 or 1, at which maximum values of M_{1c} are reached and then decreasing trends are found for ζ values above 1. That is, the presence of thin layers in the extreme cases of $\zeta \ll 1$ and $\zeta \gg 1$ is found to have a destabilizing influence on the system. Also, all the curves of α merge into one as ζ becomes very large. Moreover, the variations in M_{1c} with α are found to be noticeable for values of $\zeta < 0.2$ and also for $\zeta > 0.5$. This is to say that Marangoni convection is important when the depth of the fluid layer is relatively large or also when the depth of the fluid layer is relatively thin. Nonetheless, the situation observed for the lower isothermal and upper insulating temperature boundary conditions is quite different as far as the effect of ζ on M_{2c} is concerned and the same is evident from Fig. 2b. From this figure we note that M_{2c} increases, for all values of α considered, continuously with ζ without showing any decreasing trend as noticed in the case of insulating boundaries (see Fig. 2a). This may be due to the asymmetric temperature boundary conditions considered at the lower porous and upper fluid surfaces. However, α plays again a similar role in the stability of the system with ζ akin to the previous case. As $\zeta \to \infty$, both M_{1c} and M_{2c} attain different constant values 48 and 79.6, respectively, which are the exact values known for the case of single fluid layer [1, 22].

To analyze the influence of permeability of the porous layer on the onset of Marangoni convection, we have plotted in Figs. 3a and 3b the critical values M_{1c} and M_{2c} respectively, as a function of ζ for different values of Da when $\varepsilon_T = 0.725$ and $\alpha = 0.1$. It is seen that the decrease in Da increases the critical Marangoni numbers in both cases and thus making the system more stable. This is because low permeable porous layer dampens the fluid motion, requiring an increased critical Marangoni number to have surface tension-driven convective instability in the system. Also, the variation in Da has no significant effect on the onset of convection for values of $\zeta \ll 1$, while the curves of different Da merge into one for $\zeta \gg 1$ as expected. The critical Marangoni numbers obtained by regular perturbation technique are also shown in Fig. 3a by dotted lines. We note that the results are in excellent agreement with those obtained by the exact method. This also suggests that the regular perturbation technique with wave number a as a perturbation parameter can conveniently be used in solving convective instability problems in the case of insulating boundaries.

The variations in M_{1c} and M_{2c} as a function of thermal diffusivity ratio ε_T are shown in Figs. 4a and 4b, respectively for $\mathrm{Da} = 0.003$ and $\alpha = 0.1$, 1 and 5 when the depth of the fluid layer is the same as the depth of the porous layer (i.e., $\zeta = 1$). From the figures, it is obvious that an increase in the value of ε_T is to decrease the critical Marangoni number and thus having a destabilizing effect on the system. Nonetheless, increase in the value of slip parameter α increases the critical Marangoni numbers and hence its effect is to delay the onset of convection.

The critical wave numbers are found to be vanishingly small (i. e., $a_c \approx 0$) for various values of physical parameters chosen in the case of insulating boundaries. However, for lower isothermal and upper insulating surfaces, the critical wave number is found to be varying with physical parameters. Figure 5 shows variation in a_c with ζ for four different values of $\mathrm{Da} = 0.01$, 0.003, 0.002 and 0.001 when $\alpha = 1$ and $\varepsilon_T = 0.725$. We observe that an increase in ζ and decrease in Da is to increase a_c and hence their effect is to reduce the size of the convection cells. Also, $a_c \sim 2$ as $\zeta \to \infty$, which is the critical value observed in a single fluid layer [1].

The critical Marangoni numbers M_{1c} and M_{2c} obtained by employing both the Beavers–Joseph and the Jones interface conditions for different values of ζ when $\alpha = 1$, $\varepsilon_T = 0.725$ and $\mathrm{Da} = 0.001$ are presented in Table 1 for comparison.

Table 1. Comparison of critical Marangoni numbers for Beavers–Joseph (BJ) and Jones conditions.

ζ	M_{1c} (BJ)	M_{1c} (Jones)	Regular Perturbation Technique(M_c)	M_{2c} (BJ)	M_{2c} (Jones)
0.2	23.419	23.421	23.421	34.370	34.382
0.4	77.872	77.951	77.940	54.763	54.629
0.6	103.405	103.170	103.148	61.065	61.002
0.8	103.841	103.758	103.683	64.161	64.130

From the Table it can be seen that there is a negligible difference in the critical Marangoni numbers whether Beavers–Joseph or the more generalized Jones condition is being used. Thus the use of Beavers–Joseph slip condition is justified in trickling flow situations as we have considered the onset of convection only due to variation of surface tension with temperature. In the Table, the critical Marangoni numbers (M_c) obtained by regular perturbation technique are also given. Again, we note that M_c values are in very good agreement with those obtained by the exact method.

From the scenario envisaged, it is evident that it is possible to control the Marangoni convection effectively in a composite fluid and porous layers system by appropriately choosing the values of ζ, α, Da and ε_T.

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