Arch. Mech., **58**, 6, pp. 597–618, Warszawa 2006

On the extension of Lie group analysis to functional differential equations[*)]

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In the present paper the classical point symmetry analysis is extended from partial differential to functional differential equations. In order to perform the group analysis and deal with the functional derivatives, we extend the quantities such as infinitesimal transformations, prolongations and invariant solutions. For the sake of example, the procedure is applied to the functional formulation of the Burgers equation. The method can further lead to important applications in continuum mechanics.

1. Introduction

The purpose of the symmetry analysis based on the Lie group theory is to analyse, simplify and find solutions of partial differential equations (PDE) (cf. [1, 2, 3, 4]). The method gives a deep insight into the underlying physical problems described by PDE. Examples of its applications include problems of fluid mechanics, where a broad range of invariant solutions for turbulence statistics were found [5, 6]. The group method is also used to analyse and construct new turbulence models satisfying the required symmetries of the Navier–Stokes equations [7].

The present work concerns an application of the Lie group theory to functional differential equations. The description of turbulence in terms of functional equations has been first introduced in a seminal work of E. Hopf [8] and later presented in the book of Monin and Yaglom [9]. In statistical mechanics the system of N particles can be described by the probability density function $P(\mathbf{x}_1, \cdots, \mathbf{x}_N, \mathbf{v}_1, \cdots, \mathbf{v}_N, t)$, where the phase space of this function contains the positions $\mathbf{x}$ and velocities $\mathbf{v}$ of all the particles. In the continuum limit (e.g., in hydromechanics), the elements of the phase space $\mathbf{v}_1, \cdots, \mathbf{v}_N$ become a contin-

[*)] Paper presented at the 17–th Conference on Fluid Mechanics held in Bełchatów, September 17–21, 2006.

uous function of spatial variable $\mathbf{v} = \mathbf{v}(\mathbf{x})$. In this case the system is described by the probability density functional $P([\mathbf{v}(\mathbf{x})], t)$.

So far little attention has been paid to the symmetry analysis of functional equations. Some of the previous works concern integro-differential equations ([10, 11]), integro-differential equations with moving range of integration ([12, 13]) and delay differential equations ([14, 15]). However, to the best of the authors' knowledge, the equations with functional derivatives considered in the present paper, have not been studied so far in terms of the symmetry analysis. Hence, in the present work we extend the classical symmetry analysis to functional equations which contain functional derivatives. The new method is further applied to the Hopf formulation of the Burgers equation. The Burgers equation is often considered as a "toy model" of the Navier–Stokes equations. Functional formulation of the Burgers equation is also studied in [16] and [17]. From the transformation groups we find a particular, invariant solution of this equation. Analogous solution was found by HOPF in [8] for the functional formulation of the Navier–Stokes equations in the inviscid and stationary case. In our work, however, the group theory allows us to find solutions by a strictly determined procedure where no guessing is necessary. The new method can lead to significant applications in fluid dynamics, e.g. finding new invariant solutions for multipoint turbulence statistics. The method can also be applied in other areas of physics, where the description in terms of functional equations is used.

The structure of the paper is the following: first, we introduce the necessary notation to study functional differential equations. Next, we present the application of functional theory to describe turbulence, as it was introduced by HOPF [8]. In the third section we extend classical Lie group methods to functional differential equations by extending quantities such as infinitesimal transformations, prolongations or invariant solutions. Next, as an example, the new method is applied to the Hopf formulation of the Burgers equation. This leads to the transformation groups as well as to the invariant solution of the considered functional equation. Finally, we present conclusions and perspectives for a future work.

2. Functional equations

2.1. Notations

In the paper we consider such functional differential equations which can be regarded as the extensions of partial differential equations. The key idea is that the discrete set of independent variables $(v_1, v_2, \ldots, v_n)$ in a partial equation is replaced by a continuous set of variables denoted by $[v(x)]$, [18]; partial derivatives over v_i are replaced by functional derivatives, denoted by $\delta/\delta v(x)$. In order to illustrate this extension we introduce the example

$$\frac{\partial f}{\partial t} = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial v_i}, \tag{2.1}$$

where $f = f(v_1, v_2, \ldots, v_n, t)$. Taking the continuum limit we obtain

$$\frac{\partial f}{\partial t} = \int\limits_{l} v(x) \frac{\delta f}{\delta v(x)}\, dx, \tag{2.2}$$

where $f = f([v(x)], t)$ is a given functional. The exact definition of the functional derivative is presented, e.g., in [18, 19]; this definition can be also written in another form, which is particularly suitable for calculations

$$\frac{\delta f([v(x)])}{\delta v(x')} = \frac{\partial f([v(x)])}{\partial v(x')\, dx'} = \lim_{\epsilon \to 0} \frac{f([v(x) + \epsilon\, \delta(x - x')]) - f([v(x)])}{\epsilon}, \tag{2.3}$$

where $f([v(x)])$ is a functional and $\delta(x - x')$ is the Dirac delta. For the sake of example we use the above definition to compute the derivative of the following functional $f([v(x)]) = \int A(x)v(x)dx$ where $A(x)$ is a given function. According to Eq. (2.3), the functional derivative of f reads

$$\frac{\delta f([v(x)])}{\delta v(x')} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\int A(x)(v(x) + \epsilon\, \delta(x - x'))dx - \int A(x)v(x)dx \right] = A(x'). \tag{2.4}$$

In the present work we study the extension of the partial differential equation for a scalar function $\Phi(v_1, \ldots, v_n, t)$ of $n+1$ independent variables. The general form of the differential equation describing Φ reads:

$$F(v_1, \ldots, v_n, t, \Phi, \underset{1}{\Phi}, \underset{2}{\Phi}, \ldots, \underset{q}{\Phi}) = 0, \tag{2.5}$$

where $\underset{k}{\Phi}$ denote the k-th derivatives of the function Φ with respect to any possible combination of independent variables and q is the highest order of derivative present in Eq. (2.5). In the continuum limit v becomes a function of the continuous variable x.

The following considerations can be generalised for the vector forms of v, x and t. However, we do not present this general case here, in order to keep the notation as simple as possible.

The considered function Φ becomes a functional $\Phi = \Phi([v(x)], t)$ and the partial differential equation (2.5) becomes a functional differential equation:

$$F([v(x)], t, \Phi, \underset{1}{\Phi}, \underset{2}{\Phi}, \ldots, \underset{q}{\Phi}) = 0; \tag{2.6}$$

here again, $\underset{k}{\Phi}$ denotes all possible derivatives of order k, which can include partial derivatives with respect to t and functional derivatives with respect to $v(x)$. The following, equivalent notation will be used for the first functional derivatives:

$$\Phi_{,v(x)} = \frac{\delta\Phi}{\delta v(x)} = \frac{\partial\Phi}{\partial v(x) dx}. \tag{2.7}$$

Higher order derivatives can be expressed in an analogous way. The latter notations was originally used by HOPF [8] and can be more convenient in some situations, e.g., to denote a second order derivative, partial with respect to t and functional with respect to $v(x)$

$$\Phi_{,v(x)t} = \frac{\partial^2\Phi}{\partial v(x) dx\, \partial t}. \tag{2.8}$$

For further purposes we solve a simple functional equation by the method of characteristics. For the sake of clarity we will first present the necessary formulae for the partial differential equation (2.5) and introduce their counterparts in the continuum limit (2.6). The two approaches will also be called "classical" and "continuum" formulation, respectively. Let us consider the following hyperbolic equation in a classical and continuum formulation:

$$\Phi\frac{\partial F}{\partial \Phi} + \sum_{i=1}^{n} \frac{\partial F}{\partial v_i} = 0 \quad \rightarrow \quad \Phi\frac{\partial F}{\partial \Phi} + \int_a^b \frac{\delta F}{\delta v(x')}\, dx' = 0 \tag{2.9}$$

where $F = F(\Phi, v_1, \ldots, v_n)$ in the classical formulation and $F = F(\Phi, [v(x)])$ in the continuum limit; v_i and $v(x)$ constitute sets of independent variables and Φ is a dependent variable: $\Phi = \Phi(v_1, \ldots, v_n)$ or $\Phi = \Phi([v(x)])$.

The characteristic equations of (2.9)

$$\frac{d\Phi}{\Phi} = dv_1 = \cdots = dv_n \quad \rightarrow \quad \frac{\delta\Phi}{\Phi} = \delta v(x) \quad \text{for each} \quad x \in (a, b) \tag{2.10}$$

determine n integration constants C_i in the classical formulation and an infinite set of integration constants $C(x)$ in the continuum formulation. The constants can be employed as new (dependent and independent) variables of F. Corresponding solutions of Eqs. (2.9) have the forms $F = F(C_1, \ldots, C_n)$ and

$F = F(C_1, [C(x)])$. An example of a possible solution of the characteristic system (2.10) is presented below. We can, e.g., consider equations

$$\frac{d\Phi}{\Phi} = dv_1, \qquad dv_1 = dv_2, \qquad dv_2 = dv_3, \quad \ldots, \quad dv_{n-1} = dv_n \tag{2.11}$$

to obtain the following integration constants:

$$\begin{aligned} &C_1 = \Phi \exp[-v_1], \qquad C_2 = v_1 - v_2, \qquad C_3 = v_2 - v_3, \ldots, \\ &C_n = v_{n-1} - v_n, \end{aligned} \tag{2.12}$$

which have their counterparts in the continuum formulation,

$$C_1 = \Phi \exp[-v(x_1)], \qquad C(x) = \frac{dv(x)}{dx}\, dx \tag{2.13}$$

where x_1 is a fixed point in the domain $x \,\epsilon\, (a, b)$.

Hence, a functional that constitutes a solution of Eq. (2.9) in its continuum limit may be written, e.g., as

$$F = F\left(\Phi \exp[-v(x_1)], \left[\frac{dv(x)}{dx}\, dx\right]\right). \tag{2.14}$$

2.2. Description of turbulence in terms of the characteristic functional

Many attempts have been made to describe and to model the phenomenon of turbulence. One of the possible approaches is to treat the turbulent velocity as a random field and describe it in terms of probability density functions (pdf's). If a one-point pdf of velocity is considered, the expression $P(\mathbf{v}, \mathbf{x}, t)d\mathbf{v}$ denotes the probability that the velocity vector $\mathbf{u}(\mathbf{x}, t)$ is contained within the bounds

$$\mathbf{v} \leq \mathbf{u} \leq \mathbf{v} + d\mathbf{v}. \tag{2.15}$$

Above, the elements of the sample space are denoted by $\mathbf{v}$ to distinguish them from the physical variable $\mathbf{u}$. One-point moments (or ensemble averages) of any order can be computed from the pdf by integrating proper formulae over the sample space. As an example, the triple correlation $\langle u_\alpha u_\beta u_\gamma \rangle$ at point $\mathbf{x}$ and for time t is found from

$$\langle u_\alpha u_\beta u_\gamma \rangle = \int\limits_{\Omega} v_\alpha v_\beta v_\gamma P(\mathbf{v}, \mathbf{x}, t) d\mathbf{v}, \tag{2.16}$$

where $\int_\Omega$ denotes an integration over the whole sample space. However, to fully describe the turbulent velocity which is a random vector field, one must also take into account correlations between different points in space. The two-point pdf $P(\mathbf{v}_1, \mathbf{v}_2, \mathbf{x}_1, \mathbf{x}_2, t)$ carries information about the two-point one-time velocity statistics, e.g., the correlation $\langle u_\alpha(\mathbf{x}_1, t) u_\beta(\mathbf{x}_2, t)\rangle$ is given by the formula

$$\langle u_\alpha(\mathbf{x}_1, t) u_\beta(\mathbf{x}_2, t)\rangle = \int_\Omega v_{\alpha 1} v_{\beta 1} P(\mathbf{v}_1, \mathbf{v}_2, \mathbf{x}_1, \mathbf{x}_2, t) d\mathbf{v}_1 d\mathbf{v}_2. \tag{2.17}$$

However, this information content is still insufficient to fully characterise the turbulent flow. By increasing the number of points in the pdf, the amount of data carried by this function increases. In the continuum limit the finite set of the sample space variables becomes a continuous function of the spatial variable $\mathbf{x}$ and we deal with the probability density functional $P([\mathbf{v}(\mathbf{x})], t)$. The average of any function of velocity is defined by the functional path integral [20] with respect to $\mathbf{v}(\mathbf{x})$

$$\langle F(\mathbf{u})\rangle = \int_\Omega F(\mathbf{v}) P([\mathbf{v}(\mathbf{x})], t) D\mathbf{v}(\mathbf{x}). \tag{2.18}$$

Since the above formula is cumbersome for a practical use, the characteristic functional Φ is introduced

$$\Phi([\mathbf{y}(\mathbf{x})], t) = \int_\Omega \exp^{i(\mathbf{y},\mathbf{v})} P([\mathbf{v}(\mathbf{x})], t) D\mathbf{v}(\mathbf{x}) = \langle \exp^{i(\mathbf{y},\mathbf{u})}\rangle \tag{2.19}$$

where $(\mathbf{y}, \mathbf{u}) = \int_D u_\alpha y_\alpha d\mathbf{x}$ is the scalar product of two vector fields; integration is performed over the entire flow domain D. The characteristic functional is a functional analogue of the characteristic function of a finite-dimensional probability distribution [21]. Solutions of Φ are admitted only if at any time t the following conditions are fulfilled $\Phi^*([y(x)], t) = \Phi(-[y(x)], t)$, $\Phi(0, t) = 1$ and $|\Phi([y(x)], t)| \leq 1$. These conditions follow from the properties of probability density functional, which is strictly positive $P([v(x)], t) \geq 0$ and its integral over the entire sample space equals 1. Multipoint moments of velocity of any order are computed by taking the successive functional derivatives evaluated at $\mathbf{y} = 0$. The first functional derivative of Φ is given by

$$\frac{\delta \Phi([\mathbf{y}(\mathbf{x})], t)}{\delta y_\alpha(\mathbf{x})} = \int_\Omega i v_\alpha(\mathbf{x}) \exp^{i(\mathbf{y},\mathbf{v})} P([\mathbf{v}(\mathbf{x})], t) D\mathbf{v}(\mathbf{x}) = i\langle u_\alpha(\mathbf{x}) \exp^{i(\mathbf{y},\mathbf{u})}\rangle \tag{2.20}$$

hence, at $\mathbf{y} = 0$

$$\left.\frac{\delta \Phi([\mathbf{y}(\mathbf{x})], t)}{\delta y_\alpha(\mathbf{x})}\right|_{\mathbf{y}=0} = i\langle u_\alpha(\mathbf{x}, t)\rangle. \tag{2.21}$$

The k-th order derivatives of Φ give

$$\left.\frac{\delta^k \Phi([\mathbf{y}(\mathbf{x})],t)}{\delta y_\alpha(\mathbf{x}_1)\delta y_\beta(\mathbf{x}_2)\cdots\delta y_\gamma(\mathbf{x}_n)}\right|_{\mathbf{y}=0} = i^k\langle u_\alpha(\mathbf{x}_1,t)u_\beta(\mathbf{x}_2,t)\cdots u_\gamma(\mathbf{x}_n,t)\rangle. \tag{2.22}$$

We now consider the case where the arguments of the functional Φ constitute a sample space of the Fourier-transformed velocity $\hat{\mathbf{v}}(\mathbf{k},t)$ in the unbounded flow domain, as it is done in Sec. 4.1. for the 1D Burgers equation. The sample space function will be denoted by $\mathbf{z}(\mathbf{k})$ to distinguish it from the Fourier-transformed velocity $\hat{\mathbf{v}}(\mathbf{k},t)$.

The multipoint velocity correlations are then found from the formula

$$\begin{aligned} i^k\langle u_\alpha(\mathbf{x}_1,t)u_\beta(\mathbf{x}_2,t)\cdots u_\gamma(\mathbf{x}_n,t)\rangle \\ = \int\limits_{k_1}\int\limits_{k_2}\cdots\int\limits_{k_n} \frac{\delta^n\Phi([\mathbf{z}(\mathbf{k})],t)}{\delta z_\alpha(\mathbf{k}_1)\delta z_\beta(\mathbf{k}_2)\cdots\delta z_\gamma(\mathbf{k}_n)} \\ \left.\exp\big[-i(\mathbf{k}_1\mathbf{x}_1+\mathbf{k}_2\mathbf{x}_2+\cdots+\mathbf{k}_n\mathbf{x}_n)\big]d\mathbf{k}_1 d\mathbf{k}_2\cdots d\mathbf{k}_n\right|_{\mathbf{z}=0}. \end{aligned} \tag{2.23}$$

The characteristic functional Φ fully describes the random field as all the multipoint statistics can be computed from it. The time evolution of Φ is described by a functional differential equation. We do not present here the derivation of this equation, an interested reader is referred to the papers [8, 9]. The functional equation corresponding to the Burgers equation is considered in Sec. 4.1. Although our final goal is to consider characteristic functional of turbulent velocity, we have first chosen the Burgers equation due to its simplicity in comparison to the Navier–Stokes equations. We believe that this simplified example will better illustrate an application of the extended group method which is presented in the following section.

3. Extension of the Lie group analysis

3.1. Finite and infinitesimal transformations

In this section we recall the classical symmetry method ([1, 2, 3]) which can be used to analyse the partial differential equation (2.5) and present its continuum extension for the functional differential equation (2.6). By "symmetry transformation" we understand such transformation of variables which does not change the functional form of the considered equation. This means that for example, Eq. (2.6), in the old variables Φ, t, $v(x)$ and the same equation written in new, transformed variables $\bar{\Phi}$, $\bar{t}$, $\bar{v}(x)$

$$F\big([\bar{v}(x)],\bar{t},\bar{\Phi},\underset{1}{\bar{\Phi}},\underset{2}{\bar{\Phi}},\ldots,\underset{q}{\bar{\Phi}}\big)=0 \tag{3.1}$$

are equivalent. Note that x is not transformed in the present approach since it constitutes a continuous "counting" parameter, such as a summation index in the classical counterpart. Here, we consider only such transformations of variables which constitute Lie groups, i.e. they depend on a continuous parameter ε and satisfy group properties (such as closure, associativity and the existence of the unitary and inverse elements). Table 1 presents the comparison of a finite one-parameter Lie point transformation for the classical and continuum formulation.

Table 1. Comparison of one-parameter Lie point transformation for the classical and continuum formulation.

	classical formulation	continuum formulation
i.	$\bar{\Phi} = \psi(\Phi, v_1, \dots, v_n, t, \varepsilon)$	$\bar{\Phi} = \psi(\Phi, [v(x)], t, \varepsilon)$
ii.	$\bar{v}_1 = \phi_1(\Phi, v_1, \dots, v_n, t, \varepsilon)$	$\bar{v}(x) = \phi_x(\Phi, [v(x')], x, t, \varepsilon), \quad x \in G$
	$\vdots \quad \vdots$	
	$\bar{v}_n = \phi_n(\Phi, v_1, \dots, v_n, t, \varepsilon)$	
iii.	$\bar{t} = \phi_t(\Phi, v_1, \dots, v_n, t, \varepsilon)$	$\bar{t} = \phi_t(\Phi, [v(x)], t, \varepsilon)$

As it can be seen, the transformed variables $\bar{\Phi}$, $\bar{v}(x)$, $\bar{t}$ become functionals in the continuum limit and depend on the infinite set of variables $[v(x)]$. It should also be noted that instead of the finite set $\bar{v}_1 \dots \bar{v}_n$, in the continuum formulation we define $\bar{v}(x) = \Phi_x$, which is an explicit function of the variable x, since $\bar{v}$ defines a new variable at each point x. This has important consequences for further considerations. For the subsequent purpose of symmetry analysis all variables of equation (2.5), i.e. the sets $t, v_1, \dots, v_n$, as well as Φ and all its possible derivatives of any order will be treated as independent variables. Now, the following differential operators are introduced:

$$\frac{\mathcal{D}}{\mathcal{D}t} = \frac{\partial}{\partial t} + \Phi_{,t}\frac{\partial}{\partial \Phi} + \Phi_{,tt}\frac{\partial}{\partial \Phi_{,t}} + \sum_{j=1}^{n} \Phi_{,tv_j}\frac{\partial}{\partial \Phi_{,v_j}} + \cdots, \tag{3.2}$$

$$\frac{\mathcal{D}}{\mathcal{D}v_i} = \frac{\partial}{\partial v_i} + \Phi_{,v_i}\frac{\partial}{\partial \Phi} + \sum_{j=1}^{n} \Phi_{,v_i v_j}\frac{\partial}{\partial \Phi_{,v_j}} + \Phi_{,v_i t}\frac{\partial}{\partial \Phi_{,t}} + \cdots, \quad i = 1, \dots, n. \tag{3.3}$$

The partial derivatives, e.g., of the form $\partial/\partial t$ act only on the terms which depend explicitly on t. Analogous definitions will be applied in the continuum limit (2.6). The differential operators (3.2) and (3.3) have the following counterparts:

$$\frac{\mathcal{D}}{\mathcal{D}t} = \frac{\partial}{\partial t} + \Phi_{,t}\frac{\partial}{\partial \Phi} + \Phi_{,tt}\frac{\partial}{\partial \Phi_{,t}} + \int\limits_G dx\, \Phi_{,tv(x)}\frac{\delta}{\delta \Phi_{,v(x)}} + \cdots, \tag{3.4}$$

$$\frac{\mathcal{D}}{\mathcal{D}v(x)dx} = \frac{\delta}{\delta v(x)} + \Phi_{,v(x)}\frac{\partial}{\partial\Phi} + \Phi_{,v(x)\,t}\frac{\partial}{\partial\Phi_{,t}} + \int_G dx'\Phi_{,v(x)v(x')}\frac{\delta}{\delta\Phi_{,v(x')}} + \cdots, \quad x \in G \tag{3.4 [cont.]}$$

and the derivatives of Φ in terms of the new differential operators read:

$$\begin{aligned} &\Phi_{,t} = \frac{\mathcal{D}\Phi}{\mathcal{D}t}, \qquad \Phi_{,v(x)} = \frac{\mathcal{D}\Phi}{\mathcal{D}v(x)dx}, \qquad \Phi_{,v(x)v(x')} = \frac{\mathcal{D}}{\mathcal{D}v(x)dx}\frac{\mathcal{D}\Phi}{\mathcal{D}v(x')dx'}, \\ &\Phi_{,v(x)t} = \frac{\mathcal{D}}{\mathcal{D}v(x)dx}\frac{\mathcal{D}\Phi}{\mathcal{D}t}, \qquad \Phi_{,tt} = \frac{\mathcal{D}}{\mathcal{D}t}\frac{\mathcal{D}\Phi}{\mathcal{D}t}. \end{aligned} \tag{3.5}$$

Note that all derivatives in Eq. (3.5) are commutative. It is also important to distinguish between x and x' which denote different integration indices such as i and j in two consecutive summations. In the Lie group method, the quantities given by formulae (i), (ii), (iii) and derivatives of $\bar{\Phi}$ are written in a Taylor series expansion about $\varepsilon = 0$. Their infinitesimal forms, after neglecting terms of order $O(\varepsilon^2)$ are given in Table 2.

Table 2. Comparison of infinitesimal transformations for the classical and continuum formulation.

	classical formulation	continuum formulation
iv.	$\bar{\Phi} = \Phi + \eta(\Phi, v_1, ..., v_n, t)\,\varepsilon$	$\bar{\Phi} = \Phi + \eta(\Phi, [v(x)], t)\,\varepsilon$
v.	$\bar{v}_1 = v_1 + \xi_1(\Phi, v_1, ..., v_n, t)\,\varepsilon$	$\bar{v}(x) = v(x)$
	$\vdots \quad \vdots$	$+\xi_x(\Phi, [v(x')], x, t)\,\varepsilon, \quad x \in G$
	$\bar{v}_n = v_n + \xi_n(\Phi, v_1, ..., v_n, t)\,\varepsilon$	
vi.	$\bar{t} = t + \xi_t(\Phi, v_1, ..., v_n, t)\,\varepsilon$	$\bar{t} = t + \xi_t(\Phi, [v(x)], t)\,\varepsilon$
vii.	$\bar{\Phi}_{,\bar{t}} = \Phi_{,t} + \zeta_{;t}(\Phi, v_1, ..., v_n, t)\,\varepsilon$	$\bar{\Phi}_{,\bar{t}} = \Phi_{,t} + \zeta_{;t}(\Phi, [v(x')], t)\,\varepsilon$
viii.	$\bar{\Phi}_{,\bar{v}_1} = \Phi_{,v_1} + \zeta_{;v_1}(\Phi, v_1, ..., v_n, t)\,\varepsilon$	$\bar{\Phi}_{,\bar{v}(x)} = \Phi_{,v(x)} + \zeta_{;v(x)}(\Phi, [v(x')], x, t)\,\varepsilon$
	$\vdots \quad \vdots$	$x \in G$
	$\bar{\Phi}_{,\bar{v}_n} = \Phi_{,v_n} + \zeta_{;v_n}(\Phi, v_1, ..., v_n, t)\,\varepsilon$	
	$\vdots \quad \vdots$	$\bar{\Phi}_{,\bar{v}(x)\bar{v}(x')} = \Phi_{,v(x)v(x')}$
ix.	$\bar{\Phi}_{,\bar{v}_i\bar{v}_j} = \Phi_{,v_iv_j}$	$+\zeta_{;v(x)v(x')}(\Phi, [v(x'')], x, x', t)\,\varepsilon$
	$\vdots \quad + \zeta_{;v_iv_j}(\Phi, v_1, ..., v_n, t)\,\varepsilon$	$x, x' \in G$

A few notation particularities should be mentioned. Indices of ζ are separated by a semicolon to distinguish them from derivatives. Functional ξ_x which denotes infinitesimals corresponding to u is an explicit function of x. The same dependence holds true for infinitesimals corresponding to the functional derivatives of Φ, such as $\zeta_{;v(x)}$. In these cases the index of the set $[v(x)]$ has been given a different name such as $[v(x')]$ to avoid confusion with the parameter x. The remaining infinitesimals do not depend on x explicitly. The key property of the Lie group method is that the finite transformations, given by the formulae (i)–(iii), can be computed from their infinitesimal forms (iv)–(vi) (cf. [22]). According to the first Lie theorem, the finite form of the transformation can be obtained by integrating the first-order system of equations. For the continuum formulation this system takes the form:

$$\frac{d\bar{\Phi}}{d\varepsilon}, \qquad \frac{d\bar{\Phi}}{d\varepsilon}, \qquad \frac{d\bar{v}(x)}{d\varepsilon}, \quad x \in G, \tag{3.6}$$

where the latter equations should be integrated with the initial condition

$$\varepsilon = 0: \ \bar{\Phi} = \Phi, \qquad \bar{t} = t, \qquad \bar{v}(x) = v(x). \tag{3.7}$$

Now, to calculate the new, transformed variables $\bar{\Phi}$, $\bar{v}(x)$, $\bar{t}$ from Eq. (3.6) it is first necessary to find the infinitesimal forms η, ξ_t, ξ_x. To do this we should first express the infinitesimals ζ in terms of η, ξ_t, ξ_x and independent variables $t, [v(x)], \Phi$. For this purpose, in the classical formulation, the differential operators (3.2) and (3.3) are applied to the transformed variable $\bar{\Phi} = \psi(\Phi, v_1, \cdots, v_n, t, \epsilon)$

$$\frac{\mathcal{D}\psi}{\mathcal{D}v_i} = \sum_{k=1}^{n} \frac{\mathcal{D}\phi_k}{\mathcal{D}v_i}\frac{\bar{\mathcal{D}}\bar{\Phi}}{\bar{\mathcal{D}}\bar{v}_k} + \frac{\mathcal{D}\phi_t}{\mathcal{D}v_i}\frac{\bar{\mathcal{D}}\bar{\Phi}}{\bar{\mathcal{D}}\bar{t}} = \sum_{k=1}^{n} \bar{\Phi}_{,\bar{v}_k}\frac{\mathcal{D}\phi_k}{\mathcal{D}v_i} + \bar{\Phi}_{,\bar{t}}\frac{\mathcal{D}\phi_t}{\mathcal{D}v_i}; \tag{3.8}$$

a continuum counterpart of the above relation becomes

$$\frac{\mathcal{D}\psi}{\mathcal{D}v(x)dx} = \int\limits_G \bar{\Phi}_{,v(x')}\frac{\mathcal{D}\phi_{x'}(\Phi,[v(x)],x',t,\varepsilon)}{\mathcal{D}v(x)dx}dx' + \bar{\Phi}_{,\bar{t}}\frac{\mathcal{D}\phi_t}{\mathcal{D}v(x)dx}. \tag{3.9}$$

When the infinitesimal forms (iv)–$(viii)$, are introduced into equation (3.9) we obtain:

$$\frac{\mathcal{D}(\Phi+\eta\varepsilon)}{\mathcal{D}v(x)dx} = \int\limits_G \left(\Phi_{,v(x')} + \zeta_{;v(x')}\varepsilon\right)\frac{\mathcal{D}(v(x')+\xi_{x'}\varepsilon)}{\mathcal{D}v(x)dx}dx' + \left(\Phi_{,t} + \zeta_{;t}\varepsilon\right)\frac{\mathcal{D}(t+\xi_t\varepsilon)}{\mathcal{D}v(x)dx}. \tag{3.10}$$

Equation (3.10) can be further split into two equations, containing terms $O(1)$ and $O(\varepsilon)$, respectively. The first of the two gives the identity

$$\frac{\mathcal{D}\Phi}{\mathcal{D}v(x)dx} = \Phi_{,v(x)}. \tag{3.11}$$

From $O(\varepsilon)$ we obtain a formula for the infinitesimal $\zeta_{;v(x)}$

$$\zeta_{;v(x)} = \frac{\mathcal{D}\eta}{\mathcal{D}v(x)dx} - \int\limits_G \Phi_{,v(x')}\frac{\mathcal{D}\xi_{x'}}{\mathcal{D}v(x)dx}dx' - \Phi_{,t}\frac{\mathcal{D}\xi_t}{\mathcal{D}v(x)dx}. \tag{3.12}$$

By analogy, formula for the infinitesimal $\zeta_{;t}$ can be found:

$$\zeta_{;t} = \frac{\mathcal{D}\eta}{\mathcal{D}t} - \int\limits_G \phi_{,v(x')}\frac{\mathcal{D}\xi_{x'}}{\mathcal{D}t}dx' - \Phi_{,t}\frac{\mathcal{D}\xi_t}{\mathcal{D}t}. \tag{3.13}$$

The infinitesimals of higher orders will follow from the recursive formulae:

$$\frac{\mathcal{D}\psi_{,v(x^{(1)})\ldots v(x^{(s-1)})}}{\mathcal{D}v(x^{(s)})dx^{(s)}} = \int\limits_G \bar{\bar{\Phi}}_{,v(x^{(1)})\ldots v(x^{(s-1)})v(x)}\frac{\mathcal{D}\phi_x}{\mathcal{D}v(x^{(s)})dx^{(s)}}\,dx + \bar{\bar{\Phi}}_{,v(x^{(1)})\ldots v(x^{(s-1)})t}\frac{\mathcal{D}\phi_t}{\mathcal{D}v(x^{(s)})dx^{(s)}}. \tag{3.14}$$

or

$$\frac{\mathcal{D}\psi_{,v(x^{(1)})\ldots v(x^{(s-1)})}}{\mathcal{D}t} = \int\limits_G \bar{\bar{\Phi}}_{,v(x^{(1)})\ldots v(x^{(s-1)})v(x)}\frac{\mathcal{D}\phi_x}{\mathcal{D}t}\,dx + \bar{\bar{\Phi}}_{,v(x^{(1)})\ldots v(x^{(s-1)})t}\frac{\mathcal{D}\phi_t}{\mathcal{D}t}. \tag{3.15}$$

Hence, at this point, we have derived the formulae which express the infinitesimals $\zeta_{;t}$, $\zeta_{;v(x)}$ etc. in terms of η, ξ_x, ξ_t and t, $[v(x)]$, Φ. These formulae will be used in the following section; the whole procedure will finally lead to an equation, which will determine the forms of infinitesimal transformations η, ξ_x, ξ_t.

3.2. Generator X and its prolongations

Once all the necessary infinitesimal forms are obtained, they can be substituted into the Eqs. (2.5) or (2.6) written in the transformed variables. In order to simplify notation we will assume that Eqs. (2.5) and (2.6) contain derivatives up to the second order only. Generalization of the following relations to the case of higher-order derivatives is straightforward. After expansion of Eq. (3.1) in Taylor series about $\varepsilon = 0$ in both the classical and continuum formulations, the expanded equation has the form:

$$F + \varepsilon X^{(2)}F + \frac{\varepsilon^2}{2}\left[X^{(2)}\right]^2 F + O(\varepsilon^3) = 0 \tag{3.16}$$

where $X^{(2)}$ is called the prolongation of the generator X of the second order. In the classical formulation the generator X is given by

$$X = \eta \frac{\partial}{\partial \Phi} + \xi_t \frac{\partial}{\partial t} + \sum_{j=1}^{n} \xi_j \frac{\partial}{\partial v_j}. \tag{3.17}$$

Here, we consider only the prolongation of the second order because, as it was assumed, Eqs. (2.5) and (2.6) contain derivatives up to the second order only. In the classical formulation $X^{(2)}$ is given by the formula

$$X^{(2)} = \eta \frac{\partial}{\partial \Phi} + \xi_t \frac{\partial}{\partial t} + \sum_{j=1}^{n} \xi_j \frac{\partial}{\partial v_j} + \zeta_{;t} \frac{\partial}{\partial \Phi_{,t}} + \sum_{j=1}^{n} \zeta_{;v_j} \frac{\partial}{\partial \Phi_{,v_j}} + \zeta_{;tt} \frac{\partial}{\partial \Phi_{,tt}} + \sum_{j=1}^{n} \zeta_{;v_j t} \frac{\partial}{\partial \Phi_{,v_j t}} + \sum_{j=1}^{n} \sum_{k=1}^{n} \zeta_{;v_j v_k} \frac{\partial}{\partial \Phi_{,v_j v_k}}. \tag{3.18}$$

The corresponding formulae for the continuum limit are given by

$$X = \eta \frac{\partial}{\partial \Phi} + \xi_t \frac{\partial}{\partial t} + \int_G dx\, \xi_x \frac{\delta}{\delta v(x)} \tag{3.19}$$

and

$$X^{(2)} = \eta \frac{\partial}{\partial \Phi} + \xi_t \frac{\partial}{\partial t} + \int_G dx\, \xi_x \frac{\delta}{\delta v(x)} + \zeta_{;t} \frac{\partial}{\partial \Phi_{,t}} + \int_G dx\, \zeta_{;v(x)} \frac{\delta}{\delta \Phi_{,v(x)}} + \zeta_{;tt} \frac{\partial}{\partial \Phi_{,tt}} + \int_G dx\, \zeta_{;v(x)t} \frac{\delta}{\delta \Phi_{,v(x)t}} + \int_G \int_G dx\, dx' \zeta_{;v(x)v(x')} \frac{\delta}{\delta \Phi_{,v(x)v(x')}}. \tag{3.20}$$

The first term in Eq. (3.16) equals zero, as follows from the Eqs. (2.5) or (2.6). All the remaining terms $[X^{(2)}]^n$ in (3.16), representing a successive application of $X^{(2)}$ will be zero if the following relation holds

$$X^{(2)} F = 0. \tag{3.21}$$

In order to find the infinitesimal transformations we use the condition

$$\left[X^{(2)} F \right] \Big|_{F=0} = 0 \tag{3.22}$$

where, in the continuum formulation, the prolongation $X^{(2)}$ is expressed by formula (3.20) and the forms of infinitesimals ζ are found from relations (3.12)–(3.15). The resulting condition constitutes an overdetermined system of linear differential equations. In the continuum limit we obtain a set of functional differential equations. This system can be finally solved for the infinitesimals η, ξ_t and ξ_x.

3.3. Invariant solutions

If the functional differential equation (2.6) admits a symmetry given by the generator (3.19), then a solution $\Phi = \Theta(t, [v(x)])$ of this equation is called an invariant solution if it satisfies the relation

$$X\left[\Phi - \Theta\left(t, [v(x)]\right)\right] = 0. \tag{3.23}$$

After employing (3.19) and expanding the derivatives, the following, hyperbolic functional equation is obtained from (3.23)

$$\xi_t \frac{\partial \Theta}{\partial t} + \int\limits_G \xi_x \frac{\delta \Theta}{\delta v(x)}\, dx = \eta. \tag{3.24}$$

This equation can be solved by the method of characteristics. The corresponding system of equations reads

$$\frac{dt}{\xi_t} = \frac{d\Phi}{\eta} = \frac{\delta v(x)}{\xi_x} \qquad \text{for each} \qquad x \in G. \tag{3.25}$$

Above, Θ has been replaced by Φ. Note that the last term in fact corresponds to an infinite set of equations for each point in G. The infinite set of constants, which is a solution of the above system, can be employed as new variables in Eq. (2.6). In the considered case one of them, say C_1, will be a dependent variable and the rest will constitute a set of independent variables; the following relation holds

$$C_1 = H(C_2, [C(x)]), \tag{3.26}$$

where H is an arbitrary functional. After the process of solving the characteristic system for a partial differential equation with a finite set of variables, the number of independent variables is reduced by one. In the case of functional differential equations in formula (3.26), one variable will be excluded from a set $(C_1, C_2, [C(x)])$, $x \in G/\{x_1\}$ however, the total number of variables remains infinite.

4. Application of the extended group method

4.1. Group analysis of the Hopf formulation of the Burgers equation in the inviscid case

The extended Lie group method will now be applied to the functional formulation of the Burgers equation. The 1D Burgers equation in the physical space reads

$$\frac{\partial u(x,t)}{\partial t} + u(x,t)\frac{\partial u(x,t)}{\partial x} = \mu \frac{\partial^2 u(x,t)}{\partial x^2}. \tag{4.1}$$

In the present paper we consider the Fourier transform of Eq. (4.1) in the infinite domain

$$\frac{\partial \hat{v}(k,t)}{\partial t} = i \int\limits_{k'+k''=k} k'' \hat{v}(k',t)\hat{v}(k'',t)\, dk' - \mu k^2 \hat{v}(k,t), \tag{4.2}$$

where $\hat{v}(k,t)$ is a Fourier transform of a real variable $u(x,t)$.

We further assume that $\mu = 0$ and consider the inviscid case only. The Hopf functional formulation (cf. [8]) of the Burgers equation reads

$$\frac{\partial \Phi}{\partial t} = \int\limits_{k'}\int\limits_{k''} k'' z(k'+k'') \frac{\delta^2 \Phi}{\delta z(k')\delta z(k'')}\, dk'\, dk'' - \mu \int\limits_{k} k^2 z(k) \frac{\delta \Phi}{\delta z(k)} dk, \tag{4.3}$$

where $\Phi([z(k)],t)$ is the characteristic functional which depends on the infinite set of variables $[z(k)]$ (a sample space of the Fourier-transformed velocity $\hat{v}(k,t)$) and time.We will exclude here the mode $z(k=0)$ from the considerations. Such assumption was also made by Hopf [8]; in the case of Navier–Stokes equations this corresponds to the zero mean velocity in the domain. In the continuum formulation the generator (cf. Eq. (3.17)) applied to Eq. (4.3) has the form

$$X = \eta \frac{\partial}{\partial \Phi} + \xi_t \frac{\partial}{\partial t} + \int\limits_{k} dk\, \xi_k \frac{\delta}{\delta z(k)}, \tag{4.4}$$

where the infinitesimals η, ξ_t and ξ_k are functionals; ξ_k is also an explicit function of k. The prolongation of the generator $X^{(2)}$ (cf. Eq. (3.20)) for the considered functional equation is given by

$$X^{(2)} = \eta \frac{\partial}{\partial \Phi} + \xi_t \frac{\partial}{\partial t} + \int\limits_{k} dk\, \xi_k \frac{\delta}{\delta z(k)} + \zeta_{;t} \frac{\partial}{\partial \Phi_{,t}} + \int\limits_{k} dk\, \zeta_{;z(k)} \frac{\delta}{\delta \Phi_{,z(k)}} + \int\limits_{k} dk \int\limits_{k'} dk'\, \zeta_{;z(k)z(k')} \frac{\delta}{\delta \Phi_{,z(k)z(k')}}, \tag{4.5}$$

where any unneeded ζ has been omitted. Applying (4.5) to (4.3), the first two terms of (4.5) have no effect, the fourth one acts on $\Phi_{,t}$ to lead to $\zeta_{;t}$, while the third and the last term in (4.5) act on the integral term in Eq. (4.3). As a result we obtain

$$\zeta_{;t} - \int\limits_{k}\int\limits_{k'} k\, \xi_{k+k'}\, \Phi_{,z(k)z(k')} dk\, dk' - \int\limits_{k}\int\limits_{k'} k\, z(k+k') \zeta_{;z(k)z(k')} dk\, dk' = 0. \tag{4.6}$$

Into the above equation we substitute the infinitesimals $\zeta_{;t}$, $\zeta_{;z(k)z(k')}$ which can be found from Eqs. (3.12)–(3.14) and the differential operators $\mathcal{D}/\mathcal{D}t$ and

$\mathcal{D}/\mathcal{D}z(k)dk$ from Eqs. (3.4). One also makes use of the considered equation (4.3) by substituting for $\Phi_{,t}$

$$\Phi_{,t} = \int\limits_{k}\int\limits_{k'} kz(k+k')\Phi_{,z(k)z(k')}\,dk\,dk'. \tag{4.7}$$

The final form of the equation is lengthy and is not given here. We only note that the terms η, ξ_t, ξ_x do not depend on the derivatives of Φ, hence from the equation the following system of differential equations can be obtained, where on the left-hand side the coefficient function is written:

$$\Phi_{,z(k)t}\Phi_{,z(k')} : \quad \frac{\partial \xi_t}{\partial \Phi} = 0, \tag{4.8}$$

$$\Phi_{,z(k)t} : \quad \frac{\delta \xi_t}{\delta z(k)} = 0, \quad \text{for each} \quad k, \tag{4.9}$$

$$\Phi_{,z(k)z(k')}\Phi_{,z(k'')} : \quad \frac{\partial \xi_k}{\partial \Phi} = 0, \quad \text{for each} \quad k, \tag{4.10}$$

$$\Phi_{,z(k)}\Phi_{,z(k')} : \quad \frac{\partial^2 \eta}{\partial \Phi^2} = 0, \tag{4.11}$$

$$\begin{aligned} \Phi_{,z(k)z(k')} : \quad & -(k+k')z(k+k')\,\frac{\partial \xi_t}{\partial t} - (k+k')\,\xi_{k+k'} \\ & + \int\limits_{k''}(k+k'')z(k+k'')\frac{\delta \xi_{k'}}{\delta z(k'')}\,dk'' + \int\limits_{k''}(k'+k'')z(k'+k'')\frac{\delta \xi_k}{\delta z(k'')}\,dk'' = 0, \\ & \text{for each} \quad k \text{ and } k', \end{aligned} \tag{4.12}$$

$$\begin{aligned} \Phi_{,z(k)} : \quad & \frac{\partial \xi_k}{\partial t} + \int\limits_{k'}(k+k')z(k+k')\frac{\partial^2 \eta}{\partial \Phi \partial\, z(k')\, dk'}dk' \\ & - \int\limits_{k'}\int\limits_{k''} k''z(k'+k'')\frac{\delta^2 \xi_k}{\delta z(k')\delta z(k'')}\,dk'\,dk'' = 0, \\ & \text{for each } k, \end{aligned} \tag{4.13}$$

$$1 : \quad \frac{\partial \eta}{\partial t} - \int\limits_{k}\int\limits_{k'} kz(k+k')\frac{\delta^2 \eta}{\delta z(k)\delta z(k')}\,dk\,dk' = 0. \tag{4.14}$$

We note that relations (4.8)–(4.11) give:

$$\xi_t = \xi_t(t), \tag{4.15}$$

$$\xi_k = \xi_k([z(k')], k, t), \tag{4.16}$$

$$\eta = f_1([z(k)], t)\,\Phi + f_2([z(k)], t). \tag{4.17}$$

It is left for further considerations to find a complete, general solution of the system (4.8)–(4.14). So far, we have found a number of particular solutions for the infinitesimals η, ξ_t and ξ_k; here we restrict ourselves to one of them which is of our interest here:

$$\xi_t = 0, \tag{4.18}$$

$$\xi_k = kz(k), \tag{4.19}$$

$$\eta = 0. \tag{4.20}$$

From these infinitesimals we find a particular solution of the considered functional equation (4.3).

4.2. Invariant solutions of the Hopf formulation of the Burgers equation

As it was presented in Sec. 3.3, an invariant solution of a functional differential equation (in our case $\Phi = \Theta(t, [z(k)])$) can be determined from a hyperbolic functional equation (3.24) which can be solved by the method of characteristics with the corresponding system of equations (3.25). This is analogous to the procedure used in the group analysis of partial differential equations (Refs. [1, 2, 3]). In our case, for the infinitesimals (4.18)–(4.20) where $\xi_t = 0$ and $\eta = 0$, the system (3.25) is written in the following, shorthand notation

$$\frac{\delta t}{0} = \frac{\delta z(k)}{k\,z(k)} = \frac{\delta\Phi}{0} \quad \text{for each} \quad k, \tag{4.21}$$

which is equivalent, e.g., to following equations

$$\delta t = 0, \tag{4.22}$$

$$\delta\Phi = 0, \tag{4.23}$$

$$\frac{\delta z(k)}{k\,z(k)} = \frac{\delta z(k_1)}{k_1\,z(k_1)} \quad \text{for each} \quad k \neq k_1, \tag{4.24}$$

where k_1 can be, e.g., a selected point in the wavenumber space. However, an interesting solution is obtained if we choose $k_1 = -k$. In such a case we solve the system for the pairs k and $-k$

$$\frac{\delta z(k)}{k\, z(k)} = \frac{\delta z(-k)}{-k\, z(-k)} \quad \text{for} \quad k > 0 \tag{4.25}$$

which further leads to

$$\ln[z(k)] = -\ln[z(k)] + A(k), \tag{4.26}$$

where $A(k)$ is an infinite set of integration constants. The above formula can be rearranged to give a new set of variables $C(k)$

$$C(k) = z(k)z(-k). \tag{4.27}$$

The remaining integration constants of Eqs. (4.22) and (4.23) are $\tau = t$ and $\mathit{\Phi} = \Phi$. We note that in a discrete case, e.g. with $2N+1$ variables $z_{-N}, \cdots, z_{-1}$, z_0, $z_1, \cdots, z_N$, solving (4.25) would provide only N new variables.

Hence, additional equations would be necessary, so that the total number of variables would be reduced by one after solving the hyperbolic system. However, we did not consider so far any restrictions on $z(k)$; as $z(k)$ is a Fourier transform of a real field, the condition $z(-k) = z^*(k)$ must be satisfied. Hence, the variables $z(k)$ and $z(-k)$ are not independent and we argue that $C(k)$ and τ fully parametrise the functional Φ.

The integration constant $\mathit{\Phi}$ is considered as a new dependent variable, while the remaining ones constitute a set of independent variables. Hence, the invariant solution will be of the form

$$\Phi = F([C(k)], \tau) = F([z(k)z^*(k)], \tau) = F([z(k)z(-k)], \tau) \tag{4.28}$$

which will now be substituted back into Eq. (4.3). Before it is done we first compute the second functional derivative of F with respect to $z(k)$ and $z(k')$. The first derivative of F reads

$$\frac{\delta F}{\delta z(k)} = \int\limits_{k''} \frac{\delta F}{\delta C(k'')} \frac{\delta C(k'')}{\delta z(k')} dk'' = \int\limits_{k''} \frac{\delta F}{\delta C(k'')} \big[z(-k'')\,\delta(k-k'') + z(k'')\,\delta(-k-k'')\big]\, dk'' = 2\frac{\delta F}{\delta C(k)} z(-k).$$

After the second functional differentiation with respect to $z(k')$ we obtain

$$\frac{\delta^2 F}{\delta z(k)\,\delta z(k')} = 4z(-k)z(-k')\frac{\delta^2 F}{\delta C(k)\,\delta C(k')} + 2\delta(k+k')\frac{\delta F}{\delta C(k')}.$$

Hence, Eq. (4.3) becomes

$$\frac{\partial F}{\partial t} = \int_k \int k' k\, z(k+k') \left[4z(-k)z(-k') \frac{\delta^2 F}{\delta C(k)\, \delta C(k')} + 2\delta(k+k') \frac{\delta F}{\delta C(k')} \right] dk dk'$$

the second term on the RHS is zero as $z(k=0)$ is excluded from the considerations. After the change of variables $(k \to -k)$ and $(k' \to -k')$, the above formula reads

$$\frac{\partial F}{\partial t} = -4 \int_k \int_{k'} k\, z(-k-k') z(k) z(k') \frac{\delta^2 F}{\delta C(k)\, \delta C(k')} dk dk' \tag{4.29}$$

where we also have used the fact that $C(k) = C(-k)$ as follows from (4.27). We can further write the RHS integral as a surface integral over k and k' such that $k + k' + k'' = 0$

$$\begin{aligned} \frac{\partial F}{\partial t} &= -4 \int \int_{k+k'+k''=0} k\, z(k'') z(k) z(k') \frac{\delta^2 F}{\delta C(k)\, \delta C(k')} d\mathcal{S} \\ &= -4 \int \int_{k+k'+k''=0} k''\, z(k) z(k'') z(k') \frac{\delta^2 F}{\delta C(k'')\, \delta C(k')} d\mathcal{S} \end{aligned} \tag{4.30}$$

where $d\mathcal{S} = dk\, dk' = dk\, dk'' = dk''\, dk'$. The second equality is obtained by the change of the variables $k \to k''$. It can be shown that the following relation is also true:

$$2\frac{\partial F}{\partial t} = 4 \int \int_{k+k'+k''=0} k''\, z(k'') z(k) z(k') \frac{\delta^2 F}{\delta C(k)\, \delta C(k')}\, d\mathcal{S}. \tag{4.31}$$

The above can be derived by changing the variables $k \to k'$ in Eq. (4.29) and adding the resulting formula to (4.29). After adding both Eqs. (4.30) and (4.31) we arrive at

$$\begin{aligned} 3\frac{\partial F}{\partial t} = 4 \int \int_{k+k'+k''=0} k''\, z(k'') z(k) z(k') \Bigg(& \frac{\delta^2 F}{\delta C(k)\, \delta C(k')} \\ & - \frac{\delta^2 F}{\delta C(k'')\, \delta C(k')} \Bigg)\, d\mathcal{S}. \end{aligned} \tag{4.32}$$

If the invariant solution substituted into Eq. (4.3) leads to the reduction of variables, Eq. (4.32) should be written in terms of $C(k) = z(k)z(-k)$. This is not the case in the above example unless the bracketed term inside the integral is zero. If we exclude a linear solution of F which is not of our interest here, we have

$$\frac{\delta F}{\delta C(k)} = \frac{\delta F}{\delta C(k'')} \tag{4.33}$$

which is true provided that F is a functional with the argument $\int C(k)dk$. If the RHS of (4.32) is zero it follows that also the time derivative of the functional F is zero, hence, we obtain a solution for the stationary case. Its final form reads

$$\Phi = F\left(\int C(k)dk\right) = F\left(\int z(k)z(-k)dk\right). \tag{4.34}$$

We note here that the same form of solution was found by Hopf [8] for the functional formulation of the Navier–Stokes equations in the inviscid and stationary case. Here, we have shown that this is also an invariant solution for the Hopf formulation of the Burgers equation which can be derived from its symmetries. We note that the solution was obtained by a strictly determined procedure where no guessing of its form was necessary. This is one of the advantages of the symmetry analysis which can be used as a powerful mathematical tool for analysing different types of equations.

By looking for the solution of the Hopf formulation of an equation, we have to take into account the restrictions imposed on the characteristic functional, i.e. $\Phi^*([z(k)]) = \Phi([-z(k)])$, $|\Phi| \leq 1$ and $\Phi(0) = 1$. An example of the invariant solution which satisfies those restrictions may be given by

$$\Phi = \exp\left(-\frac{\kappa^2}{2}\int z(k)z(-k)dk\right) = \exp\left(-\frac{\kappa^2}{2}\int z(k)z^*(k)dk\right) \tag{4.35}$$

where κ is a given constant. This solution represents a characteristic functional of a Gauss distribution [8], where the fluctuations at different points of the field are independent. If formula (2.23) is applied to find the first and second-order statistics of the field $u(x)$, we obtain

$$\langle u(x)\rangle = 0, \tag{4.36}$$

$$\langle u(x)u(x')\rangle = \kappa^2 \int\limits_k \exp\left[-ik(x-x')\right]dk = \kappa^2\delta(x-x'). \tag{4.37}$$

It is seen from the second equation that the spectral energy density has a constant value κ^2 for all wavenumbers k; this may be a consequence of neglecting viscosity in Eq. (4.3). The two-point correlations become a Dirac delta function; this also proves that the fluctuations of the considered field $u(x)$ are independent if $x \neq x'$.

Unfortunately, such solution is rather irrelevant from the physical point of view. So far, we have only illustrated how the extended group method can be applied to a functional differential equation. We have shown that it leads to the new forms of variables, cf. Eq. (4.27), and to the analytical solution of the equation. We hope to find more interesting, physical results by considering the Burgers equation with viscosity, as well as by considering the Hopf formulation of the Navier–Stokes equations.

5. Conclusions and perspectives

In the present paper the classical, point-symmetry group analysis is extended from partial differential equations to their counterparts in the continuum limit. In particular, we introduce the procedure of applying symmetry analysis to the case when functional derivatives are present in the equation.

As an example we consider the Hopf functional formulation of the Burgers equation, Eq. (4.3). We perform the symmetry analysis extended for the functional differential equations and find the symmetry transformations Eqs. (4.18)–(4.20). By solving the hyperbolic equation with the derived infinitesimals, interesting invariant solutions of the equation can be found.

The most significant result of the paper consists in the demonstration that a particular solution of the functional equation found by Hopf [8] for the stationary and inviscid case, Eq. (4.34), can also be derived from the symmetry analysis by a strictly determined way of reasoning where no guessing is necessary. The solution represents a characteristic functional of a Gaussian distribution. The spectral energy density of the field has a constant value for all wavenumbers. We expect to obtain more relevant, physical results from the group analysis when viscosity is included in the functional equation. This is a perspective for the future work.

The Burgers equation is sometimes considered as a simplified model of the Navier–Stokes equations. Hence, another perspective is to investigate the Hopf formulation of the Navier–Stokes equations by the symmetry method. We hope to find information about the moments of the solutions of the Navier–Stokes equations by considering the invariant solutions of the corresponding functional. The presented extension of the Lie groups can be a useful tool for the analysis of functional equations. The method can be further extended to the case where both types of derivatives: functional with respect to a given $v(x)$ and partial with respect to x, are present in the equation. This is also left for future work. We believe that the new approach is highly relevant to a variety of important functional differential equations (FDE) in physics, especially in the classical and quantum field theories [23]. Generally speaking, very little is known on how to analytically treat and solve FDEs (numerical treatment is difficult anyway because of the high dimensionality). Hence, the methods may give a chance to treat equations which so far have been put aside because of the missing analytical methods. In fact, the benefit is twofold since the symmetries not only allow for analytical solutions but are also useful in themselves, since symmetries illuminate the properties of the physical model equations.

Acknowledgments

The authors are thankful to V. N. Grebenev for his useful comments and discussions concerning the paper.

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Received April 21, 2006; revised version October 26, 2006.