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Brief Note

Description of tetragonal pore space structure of porous materials

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The generalized macroscopic description is formulated in the paper for tetragonal pore space structure of porous materials. The anisotropic pore space is modelled as Minkowski space, the metric tensor of which plays the fundamental role in description of transport phenomena in such a medium. To describe the metric properties of the tetragonal space, the fourth order tensor with internal symmetries of the compliance tensor used in the linear theory of elasticity of anisotropic materials has been applied. Its reduction by the automorphisms group describing point symmetries of the square net gave the general metrics of tetragonal pore space containing only two scalar parameters, that represent tortuosities of the pore space in the main and diagonal directions.

1. Introduction

In the paper the macroscopic description of anisotropic properties of the pore space is presented for the new generation of porous materials with the tetragonal symmetry of pore structure. The anisotropic properties of such a medium are predicted by the theory presented in papers [1–3]. They concern the macroscopic modelling of a fluid motion in an undeformable porous material with anisotropic pore space structure, considered as a motion of material continuum in the Minkowski metric space. This approach results in a description in which anisotropic properties of the pore space are represented by the properties of its metric tensor. This tensor defines the value of the pore tortuosity parameter for each direction of the pore space, that play the fundamental role in dynamic behaviour of fluid in permeable porous material, (e.g. it strongly influences the velocity of wave propagation), characterizes anisotropic properties of viscous interaction between fluid and skeleton, [3], and also the resistance of electric current passage in an electrolyte filling the pore space of the non-conductive skeleton. In such a description, even slow processes of a fluid flow or a small amplitude wave propagation are described by equations nonlinearly dependent on the direction of that process. It is a direct consequence of independent modelling of the pore space structure, where the transport process takes place, and the mechanical behaviour of a fluid in that space. As a result, the linearization of the physical aspect of the problem does not influence the geometrical non-linearity caused by the metric tensor of the anisotropic pore space modelled as a Minkowski space, [1, 6, 7].

Such description is not consistent with the predictions presented e.g. in papers [4, 5], in which problems of mass, linear momentum and energy transport are considered in porous media that have the tetragonal symmetry of microscopic structure formed by a bundle of parallel cylindrical fibers arranged in the square net. It is proved that such a medium is isotropic for processes in the plane perpendicular to the fibers axes. It results from the assumption of linear dependence between the gradient of potential, which causes the transport process, and the vector of the flux density of the transported quantity. The coefficient that relates these two vectors in the general linear case is a symmetric, second order tensor. It imposes a strong restrictions on the character of anisotropy that can be described by such a law. Due to the tensor symmetry, in the most general case, it is orthotropy. Such description applied to the set of fibers arranged in the square net necessitates consideration of that medium as macroscopically isotropic in the plane perpendicular to fiber axes, and the whole medium as transversally isotropic.

This reasoning can be applied to every porous medium with the tetragonal symmetry of microscopic pore structure, independently of its complexity. It concerns also the media with a stochastic structure of microscopic pore space that have tetragonal symmetry of the pore structure on the macroscopic level.

The solution of the apparent contradiction between these two descriptions is the extension of the class of porous materials, the pore space of which is modelled as the Minkowski metric space. An example of such a medium is shown in Fig.1. In that case, the material consists of a bundle of parallel fibers with rhomboidal cross-section which are able to rotate free around the axis formed by one of their edges. These axes are ordered in the square net. We assume that during the process (e.g. a fluid flow) the fibers take position of balance minimizing the resistance to flow. For a transport process in the main directions of the square net (Fig. 1a, b) the configurations of fibers placement are the same. Therefore, the properties of such medium in both directions are identical. In turn, for a transport process in the diagonal direction of the square net (Fig. 1c), the configuration of fibers placement is essentially different from those for a transport process in the main direction. Taking into consideration the limit case of the square cross-section of the fibers, it is evident that such a medium must have anisotropic properties.

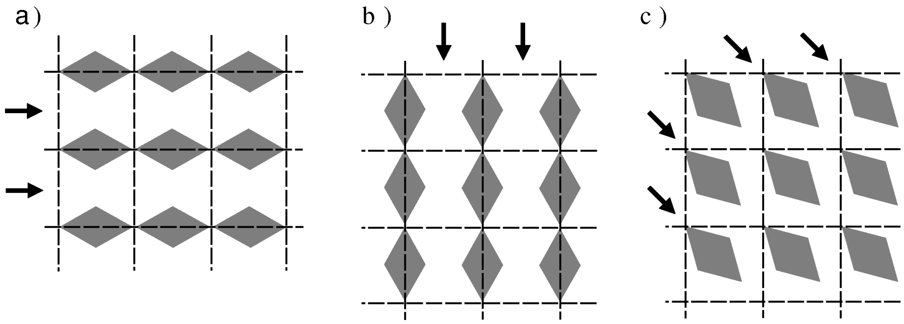

Fig. 1. Tetragonal architecture of porous medium with rotary rhomboidal fibers. Configurations of fibers for transport process in the main (a, b) and diagonal (c) directions of the pore space.

The purpose of this paper is to formulate the general form of a macroscopic metrics for the two-dimensional, tetragonal pore space of a porous material modelled as the anisotropic Minkowski space, the exemplary pore space of which is shown in Fig. 1.

To describe the metric properties of the tetragonal space, the fourth order tensor with internal symmetries of the compliance tensor used in the linear theory of elasticity of anisotropic materials has been applied. After spectral decomposition of this tensor and application of the automorphisms group describing point symmetries of the square net, the general metrics of tetragonal pore space was obtained. It contains only two scalar parameters that represent tortuosities of the pore space in the main and diagonal directions. The limit values of their ratio have been determined. They confine the domain for which the indicatrix of pore space is convex.

2. Minkowski metrics of the anisotropic pore space

To describe the macroscopic structure of the anisotropic pore space of permeable material, we assume that this space can be modelled as a Minkowski metric space (see Appendix). In such an approach the pore space forms the real space in which (e.g.) a fluid motion takes place and therefore, from the macroscopic point of view, it can be considered as a motion of material continuum in the anisotropic metric space.

In this section we formulate the general form of macroscopic metrics $L_A(\mathbf{r})$ for two-dimensional pore space $\mathcal{V}$ ($\mathbf{r} \in \mathcal{V}$; dim $(\mathcal{V}) = 2$). In these considerations we apply the fourth order tensor $\mathbf{C} \in \otimes\mathcal{V}^{\star 4}$ with internal symmetries of the compliance tensor used in the linear theory of elasticity of anisotropic solid materials. The metrics takes the form, [1, 2],

$$L_A^4(\mathbf{r}) = \binom{\mathbf{r}}{\mathbf{r}} : \mathbf{C} : \binom{\mathbf{r}}{\mathbf{r}} \tag{2.1}$$

and tensor $\mathbf{C}$ has the symmetries defined by the conditions

$$\binom{\mathbf{u}}{\mathbf{v}} : \mathbf{C} : \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} = \binom{\mathbf{v}}{\mathbf{u}} : \mathbf{C} : \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} = \binom{\mathbf{u}}{\mathbf{v}} : \mathbf{C} : \begin{pmatrix}\mathbf{y}\\ \mathbf{x}\end{pmatrix} = \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} : \mathbf{C} : \binom{\mathbf{u}}{\mathbf{v}} \tag{2.2}$$

that have to be satisfied for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y} \in \mathcal{V}$. The applied notations and basic definitions are contained in the Appendix.

Tensor $\mathbf{C}$ we will call the structure tensor of the anisotropic pore space. Components of this tensor, besides the conditions of symmetry (2.2), will also be restricted by assuming the function (2.1) to be positively definite and convex. The last condition is equivalent to requirement of convexity of the indicatrix generated by the metrics (2.1). Norm (2.1) uniquely defines the form of metric tensor $\mathbf{M}_A(\mathbf{r})$. We have, [1],

$$\begin{aligned}\mathbf{M}_A(\mathbf{r}) &= \frac{1}{2}\,\frac{\partial^2 L_A^2(\mathbf{r})}{\partial\mathbf{r}\,\partial\mathbf{r}} \\ &= \frac{1}{L_A^6(\mathbf{r})}\;\mathbf{r}\cdot\Big\{L_A^4(\mathbf{r})\,(2\,\mathbf{C}+\mathbf{C}^T) - 2\,\mathbf{C} : \binom{\mathbf{r}\,\mathbf{r}}{\mathbf{r}\,\mathbf{r}} : \mathbf{C}\Big\}\cdot\mathbf{r}\,.\end{aligned} \tag{2.3}$$

An interesting form of (2.1) we obtain for the spectral decomposition of tensor $\mathbf{C}$. For this purpose, in the space of symmetric second order tensors $S^\star = \mathrm{sym}\{\otimes\mathcal{V}^{\star 2}\}$ we define the scalar product $\mathbf{U}\diamond\mathbf{V}$ of any tensors $\mathbf{U}$, $\mathbf{V} \in \mathcal{S}^\star$. We take

$$\mathbf{U}\diamond\mathbf{V} = \mathbf{U} : \begin{pmatrix}\mathbf{M}^{-1}\\ \mathbf{M}^{-1}\end{pmatrix} : \mathbf{V}, \tag{2.4}$$

where $\mathbf{M}^{-1}$ is the metric tensor of dual vector space $\mathcal{V}^\star$. Space $\mathcal{S}^\star$ is three-dimensional. Therefore, any three linearly independent tensors of this space form its base. Since the scalar product has been defined in the space $\mathcal{S}^\star$, among all possible bases we can distinguish orthogonal bases, e.g. $\{\mathbf{K}_m\}$ $(m = 1, 2, 3)$ that satisfy the conditions

$$\mathbf{K}_m \diamond \mathbf{K}_n = \delta_{mn} \qquad (m, n = 1, 2, 3). \tag{2.5}$$

Then, any tensor $\mathbf{T} \in \mathcal{S}^\star$ has the representation

$$\mathbf{T} = T_m\mathbf{K}_m\,, \tag{2.6}$$

where $T_m = \mathbf{T}\diamond\mathbf{K}_m$ are components of tensor $\mathbf{T}$ in the base $\{\mathbf{K}_m\}$.

Using the space $\mathcal{S}^\star$ of second order symmetric tensors, we can define the nine-dimensional space $\mathcal{S}^\star \otimes \mathcal{S}^\star$ of fourth order tensors. Then, the structure tensor $\mathbf{C}$ will be an element of this space ($\mathbf{C} \in \mathcal{S}^\star \otimes \mathcal{S}^\star$) and can be represented in the form, [8, 9],

$$\mathbf{C} = \alpha_1 \mathbf{K}_1 \otimes \mathbf{K}_1 + \alpha_2 \mathbf{K}_2 \otimes \mathbf{K}_2 + \alpha_3 \mathbf{K}_3 \otimes \mathbf{K}_3\,, \tag{2.7}$$

if the base tensors $\mathbf{K}_n \in \mathcal{S}^\star$ $(n = 1, 2, 3)$ are the eigentensors of $\mathbf{C}$. In that case, coefficients in formula (2.7) are eigenvalues of tensor $\mathbf{C}$ and they satisfy the conditions

$$\mathbf{C} : \begin{pmatrix} \mathbf{M}^{-1} \\ \mathbf{M}^{-1} \end{pmatrix} : \mathbf{K}_n = \alpha_n \mathbf{K}_n \ . \tag{2.8}$$

Taking into consideration (2.7), definition of the metrics (2.1) reduces to the form

$$L_A^4(\mathbf{r}) = \alpha_1 \left(\mathbf{K}_1 : \left(\begin{smallmatrix} \mathbf{r} \\ \mathbf{r} \end{smallmatrix}\right)\right)^2 + \alpha_2 \left(\mathbf{K}_2 : \left(\begin{smallmatrix} \mathbf{r} \\ \mathbf{r} \end{smallmatrix}\right)\right)^2 + \alpha_3 \left(\mathbf{K}_3 : \left(\begin{smallmatrix} \mathbf{r} \\ \mathbf{r} \end{smallmatrix}\right)\right)^2 \ . \tag{2.9}$$

From the formula (2.9) it results that the norm $L_A^4(\mathbf{r})$ will be positively definite for all positive eigenvalues α_n.

Expression (2.9) define the general form of the norm $L_A(\mathbf{r})$ and due to (2.3), also of the metric tensor $\mathbf{M}_A(\mathbf{r})$ for the two-dimensional pore space in materials, the macroscopic pore structure of which can be described by the fourth order tensor.

From representation (2.7) of the structure tensor $\mathbf{C}$ it results that this tensor will define a metrics of the pore space with the given symmetry, if the distinguished base tensors $\mathbf{K}_n$ are invariant with respect to the group of rotation tensors characterizing symmetries of the pore space.

3. Symmetries of tetragonal pore space structure

To obtain the explicit form of the metrics (2.9) suitable for the anisotropic pore space with the tetragonal symmetries, we determine the group of orthogonal tensors that characterize the point symmetries of the square net. In considerations we apply the general representation of the orthogonal tensor $\mathbf{Q} \in \mathcal{V} \otimes \mathcal{V}^\star$ $((\mathbf{M} \cdot \mathbf{Q})^T = \mathbf{M} \cdot \mathbf{Q}^{-1})$ of the two-dimensional vector space $\mathcal{V}$ by the tensor

$$\mathbf{W} = \mathbf{E}_2 \cdot \mathbf{M} \quad \in \mathcal{V} \otimes \mathcal{V}^\star \tag{3.1}$$

where $\mathbf{E}_2 \in \mathcal{V}^2$ is the normed skew-symmetric tensor ($\mathbf{E}_2^T = -\mathbf{E}_2$) representing the surface element of unit area measured with respect to the Euclidean metrics, [1]. This representation has the form

$$\mathbf{Q} = \mathbf{I}\cos(\alpha) + \mathbf{W}\sin(\alpha)\,, \tag{3.2}$$

where $\mathbf{I} \in \mathcal{V} \otimes \mathcal{V}^{\star}$ is the identity tensor of the vector space $\mathcal{V}$, and α is the rotation angle of tensor $\mathbf{Q}$.

Denoting by $\mathbf{e}_1$ and $\mathbf{e}_2$ any two versors that form the orthonormal basis in the space $\mathcal{V}$ $(\mathbf{e}_i \cdot \mathbf{M} \cdot \mathbf{e}_j = \delta_{ij})$, tensor $\mathbf{W}$ can be represented as

$$\mathbf{W} = \mathbf{e}_2 \otimes \mathbf{e}^1 - \mathbf{e}_1 \otimes \mathbf{e}^2, \tag{3.3}$$

where $\mathbf{e}^i = \mathbf{M} \cdot \mathbf{e}_j$ are versors dual to $\mathbf{e}_i$.

Now, we assume that versors $\mathbf{e}_1$ and $\mathbf{e}_2$ correspond to principal axes of symmetry of the square net (Fig.1). Since the basic symmetry of the square net results from its invariance under rotation by an angle of $\pi/2$, from (3.2) we have

$$\mathbf{Q} = \mathbf{W}\,. \tag{3.4}$$

Taking into account that

$$\mathbf{W}^2 = -\mathbf{I}\,, \qquad \mathbf{W}^3 = -\mathbf{W}, \qquad \mathbf{W}^4 = \mathbf{I}\,,$$

the automorphisms

$$\mathbf{W}\,, \qquad \mathbf{I}\,, \qquad -\mathbf{I}\,, \tag{3.5}$$

will belong to the group of symmetries of the square net. Tensor $\mathbf{I}$ is the identity automorphism, whereas $-\mathbf{I}$ is the central symmetry of the square net.

The full group of symmetries of the square net we obtain determining additionally the automorphisms that represent its axial symmetries. To this end, we use the general form of tensor $\mathbf{V}_m$ representing axial symmetry in the two-dimensional space $\mathcal{V}$ with respect to an axis determined by versor $\mathbf{m}$. It can be written as

$$\mathbf{V}_m = \mathbf{m} \otimes \mathbf{m} \cdot \mathbf{M} - \mathbf{n} \otimes \mathbf{n} \cdot \mathbf{M} \qquad \in \mathcal{V} \otimes \mathcal{V}^{\star},$$

where $\mathbf{n} = \mathbf{W} \cdot \mathbf{m}$ is the versor orthogonal to $\mathbf{m}$. Then, tensor of symmetry with respect to the axis $\mathbf{e}_1$ has the form

$$\mathbf{V}_1 = \mathbf{e}_1 \otimes \mathbf{e}^1 - \mathbf{e}_2 \otimes \mathbf{e}^2\,,$$

and such tensor for axis $\mathbf{e}_2$ is given by

$$\mathbf{V}_2 = \mathbf{e}_2 \otimes \mathbf{e}^2 - \mathbf{e}_1 \otimes \mathbf{e}^1 = -\mathbf{V}_1\,. \tag{3.6}$$

Taking into account that versors:

$$\mathbf{k}_1 = \frac{(\mathbf{e}_1 + \mathbf{e}_2)}{\sqrt{2}}, \qquad \mathbf{k}_2 = \frac{(\mathbf{e}_2 - \mathbf{e}_1)}{\sqrt{2}}\,, \tag{3.7}$$

determine the diagonal axes of symmetry of the square net, the corresponding tensor $\mathbf{V}$ of symmetry takes the form

$$\mathbf{V} = \mathbf{e}_1 \otimes \mathbf{e}^2 + \mathbf{e}_2 \otimes \mathbf{e}^1 \,. \tag{3.8}$$

Tensors $\mathbf{V}$, $\mathbf{V}_1$ and (3.5) characterize the symmetry of the square net. However, the full group of symmetry is composed only of the following linearly independent elements:

$$\mathbf{I}\,, \quad \mathbf{V}\,, \quad \mathbf{V}_1\,, \quad \mathbf{W}\,, \tag{3.9}$$

and tensors $\mathbf{V}$ and $\mathbf{V}_1$ form the set of the group generators.

4. Macroscopic metrics of tetragonal pore space

We use tensors (3.9) that characterize symmetry of the square net to reduce the general form of metrics (2.9) to the form describing macroscopic structure of pore space with tetragonal symmetry. For this purpose we transform the tensors: $\mathbf{I}$, $\mathbf{V}$ and $\mathbf{V}_1$ to the symmetrical form and create an orthonormal base in the space $\mathcal{S}^\star$ of symmetric tensors. Multiplying $\mathbf{I}$, $\mathbf{V}$ and $\mathbf{V}_1$ by the metric tensor $\mathbf{M}$ we obtain the tensors:

$$\mathbf{M} = \mathbf{M}\cdot\mathbf{I}\,, \qquad \overline{\mathbf{V}} = \mathbf{M}\cdot\mathbf{V}\,, \qquad \overline{\mathbf{V}}_1 = \mathbf{M}\cdot\mathbf{V}_1\,, \tag{4.1}$$

that are elements of space $\mathcal{S}^\star$. Since this space is three-dimensional, and tensors $\mathbf{M}$, $\overline{\mathbf{V}}$, $\overline{\mathbf{V}}_1$ are linearly independent, they form a base in the space $\mathcal{S}^\star$. It is easy to verify that these tensors are orthogonal with respect to the scalar product defined by the formula (2.4),

$$\mathbf{M}\diamond\overline{\mathbf{V}}_1 = \mathbf{M}\diamond\overline{\mathbf{V}} = \overline{\mathbf{V}}_1\diamond\overline{\mathbf{V}} = 0\,.$$

However, tensors $\mathbf{M}$, $\overline{\mathbf{V}}$ and $\overline{\mathbf{V}}_1$ are not normed. Taking into account that

$$\mathbf{M}\diamond\mathbf{M} = \overline{\mathbf{V}}_1\diamond\overline{\mathbf{V}}_1 = \overline{\mathbf{V}}\diamond\overline{\mathbf{V}} = 2\,,$$

the orthonormal tensors $\mathbf{K}_i$, used in Sec. 2, can be represented by

$$\begin{aligned} \mathbf{K}_1 &= \frac{1}{\sqrt{2}}\,\mathbf{M} = \frac{1}{\sqrt{2}}\left(\binom{\mathbf{e}^1}{\mathbf{e}^1} + \binom{\mathbf{e}^2}{\mathbf{e}^2}\right), \\ \mathbf{K}_2 &= \frac{1}{\sqrt{2}}\,\overline{\mathbf{V}}_1 = \frac{1}{\sqrt{2}}\left(\binom{\mathbf{e}^1}{\mathbf{e}^1} - \binom{\mathbf{e}^2}{\mathbf{e}^2}\right), \\ \mathbf{K}_3 &= \frac{1}{\sqrt{2}}\,\overline{\mathbf{V}} = \frac{1}{\sqrt{2}}\left(\binom{\mathbf{e}^1}{\mathbf{e}^2} + \binom{\mathbf{e}^2}{\mathbf{e}^1}\right). \end{aligned} \tag{4.2}$$

These tensors make it possible to reduce the formula (2.9), defining metrics of the pore space, to the form suitable for macroscopic structure with the tetragonal symmetry. We have

$$L_A^4(\mathbf{r}) = p(\mathbf{r}\circ\mathbf{r})^2 + q\left((\mathbf{r}\circ\mathbf{e}_1)^4 + (\mathbf{r}\circ\mathbf{e}_2)^4\right), \tag{4.3}$$

where $\mathbf{u}\circ\mathbf{v} \equiv \mathbf{u}\cdot\mathbf{M}\cdot\mathbf{v}$ is the scalar product of vectors in the space $\mathcal{V}$, and

$$p = \alpha_3 + \frac{(\alpha_1-\alpha_2)}{2}, \qquad q = \alpha_2 - \alpha_3 . \tag{4.4}$$

Invariance of the metrics (4.3) at the transposition of versors $\mathbf{e}_1$ and $\mathbf{e}_2$ is showing symmetry of the assumed structure of pore space.

Scalar parameters p and q that appear in (4.3) can be represented by quantities of strictly geometrical meaning, if we take into consideration direct relation of the pore space metrics $L_A(\mathbf{r})$ with the tortuosity of pores $\delta(\mathbf{n})$ of the porous medium. This relation, for any direction determined by the Euclidean versor $\mathbf{n}$, is given by the formula, [1, 2],

$$\delta^2(\mathbf{n}) = \mathbf{n}\cdot\mathbf{M}_A(\mathbf{n})\cdot\mathbf{n} = L_A^2(\mathbf{n}) . \tag{4.5}$$

Therefore, metrics (4.3) can be transformed to the form

$$L_A^4(\mathbf{r}) = \delta_o^4\left((\mathbf{r}\circ\mathbf{e}_1)^2 - (\mathbf{r}\circ\mathbf{e}_2)^2\right)^2 + 4\,\delta_p^4(\mathbf{r}\circ\mathbf{e}_1)^2\,(\mathbf{r}\circ\mathbf{e}_2)^2 \tag{4.6}$$

or

$$L_A^4(\mathbf{r}) = \delta_o^4\left[(\mathbf{r}\circ\mathbf{r})^2 + 4(\alpha-1)(\mathbf{r}\circ\mathbf{e}_1)^2\,(\mathbf{r}\circ\mathbf{e}_2)^2\right] \tag{4.7}$$

where

$$\alpha = (\delta_p/\delta_o)^4 , \tag{4.8}$$

and δ_o and δ_p stand for parameters of tortuosity in the main directions of the pore space, defined by versors $\mathbf{e}_1$ and $\mathbf{e}_2$ and in the diagonal directions defined by (3.7), respectively (Fig. 1). From the norm (4.7) it results that it is positively definite for any values of parameters δ_o and δ_p. Parameter α that appears in formula (4.7) is a measure of anisotropy of the pore space structure. For isotropic space $\alpha = 1$, and the domain of this parameter is additionally restricted by the condition (A.3) of convexity of function $L_A(\mathbf{r})$. From this condition we obtain

$$1/2 \;<\; \alpha \;<\; 2 .$$

It means that the maximum degree of anisotropy of the pore space with tetragonal symmetry cannot exceed the value of $\sqrt[4]{2}$.

Using norm (4.7), the pore tortuosity $\delta(\mathbf{n})$ given by the formula (4.5) can be represented by expression

$$\delta^4(\mathbf{n}) = \delta_o^4 \left[1 + 4(\alpha - 1)(\mathbf{n} \circ \mathbf{e}_1)^2 (\mathbf{n} \circ \mathbf{e}_2)^2\right] , \tag{4.9}$$

and in the polar coordinates it reduces to the form

$$\delta^4 = \delta_o^4 \left[1 + (\alpha - 1)\sin^2(2\varphi)\right] . \tag{4.10}$$

where φ is an angle between versors $\mathbf{e}_1$ and $\mathbf{n}$.

Taking into consideration that vector

$$\mathbf{N}(\mathbf{n}) = \frac{\mathbf{n}}{\delta(\mathbf{n})} ,$$

has a unit length with respect to the Minkowski metrics, its measure with respect to the Euclidean metrics, i.e. the reciprocal of the pore tortuosity, will describe the indicatrix of the Minkowski space.

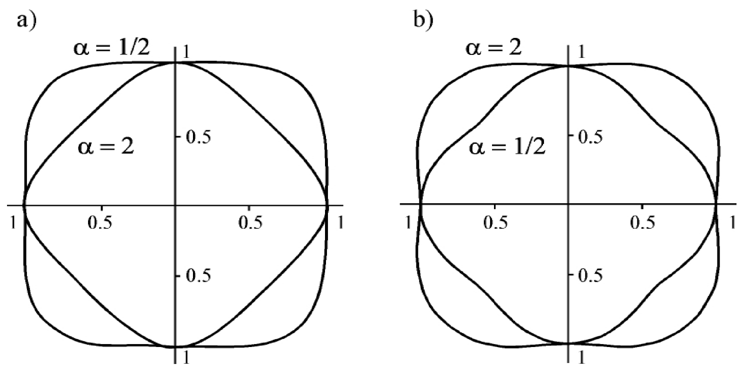

FIG. 2. Limit graphs of convex indicatrices (a) and their functions of pore tortuosity (b) for the anisotropic pore space with tetragonal symmetry of structure.

The graphs of indicatrices and their functions of pore tortuosity $\delta(\mathbf{n})$ for the limit values of parameter α are shown in Fig. 2a and Fig. 2b, respectively. From their form it is seen that convexity of the indicatrix of Minkowski metrics does not induce the convexity of the pore tortuosity function $\delta(\mathbf{n})$.

5. Final remarks

In the paper, the general form of the macroscopic metrics has been formulated for tetragonal pore space of permeable porous materials, described as the anisotropic Minkowski space. This metrics characterizes directional properties of the pore space and define its basic macroscopic parameters of structure: the pore tortuosity and the surface and volume porosity.

To describe the metric properties of the tetragonal pore space, the fourth order tensor was applied with symmetries of the compliance tensor used in the linear theory of elasticity of anisotropic materials. The obtained metrics is fully characterized by two scalar parameters representing the tortuosities of a tetragonal pore space in the main and diagonal directions. It was proved that macroscopic tetragonal symmetry has a porous material formed by a bundle of parallel (e.g.) rhomboidal fibres which can rotate around one of their edges, taking position dependent on the direction of process in such medium. These edges are arranged in the nodes of the square net.

The explicit form of the obtained metrics for a pore space with a given microscopic structure, e.g. as that shown in the Fig.1, needs determination of two parameters appearing in this metrics. It might be done experimentally or by numerical simulations of a current passage in an electrolyte filling a porous medium with a non-conductive skeleton. The numerical simulation allows one to determine the values of both parameters for a wide range of volume porosities. Conductometric measurements belong to the most common methods used for determination of the pore tortuosity parameter in porous materials. These problems will be presented in the separate paper.

Appendix. Notations and basic definitions

Vectors and tensors. We denote by $\mathcal{V}$ three-dimensional real vector space, and by $\mathcal{V}^\star$ the dual vector space of $\mathcal{V}$. If $\mathbf{u} \in \mathcal{V}$ and $\mathbf{v} \in \mathcal{V}^\star$, then the scalar $\mathbf{u} \cdot \mathbf{v}^\star = \mathbf{v}^\star \cdot \mathbf{u} \ \in R$ we will call the dual product of vector $\mathbf{u}$ and covector $\mathbf{v}^\star$. By a dot $(\cdot)$ we will denote the bilinear exterior operation defined on elements of spaces $\mathcal{V}$ and $\mathcal{V}^\star$, and will be called dual multiplication.

The multilinear transformations of vector spaces are called tensors. They are elements of linear spaces, which are tensor products of vector spaces. For example, tensor $\mathbf{A} \in \mathcal{V} \otimes \mathcal{V}^\star$ is an endomorphism of spaces $\mathcal{V}$ and $\mathcal{V}^\star$. For $\mathbf{u} \in \mathcal{V}$ and $\mathbf{v}^\star \in \mathcal{V}^\star$ we have: $\mathbf{A} \cdot \mathbf{u} \in \mathcal{V}$ and $\mathbf{v}^\star \cdot \mathbf{A} \in \mathcal{V}^\star$.

Since in our considerations we will have to do with tensors of order four, to simplify the operations on such objects we introduce the new, alternative notation for tensor products. For $\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y} \in \mathcal{V}$ we take

$$\binom{\mathbf{u}}{\mathbf{v}} \equiv \mathbf{u} \otimes \mathbf{v} \quad \in \mathcal{V} \otimes \mathcal{V} = \otimes \mathcal{V}^2 \,, \qquad \begin{pmatrix} \mathbf{u} & \mathbf{x} \\ \mathbf{v} & \mathbf{y} \end{pmatrix} \equiv \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{x} \otimes \mathbf{y} \quad \in \otimes \mathcal{V}^4 \,.$$

For example, for $\mathbf{A} \in \otimes \mathcal{V}^2$ and $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ we have the identity

$$\mathbf{A} : \binom{\mathbf{u}}{\mathbf{v}} \equiv \mathbf{u} \cdot \mathbf{A} \cdot \mathbf{v}.$$

The normed vector spaces. Vector space $\mathcal{V}$ is the normed space, if there is defined a real-valued function $u = L_A(\mathbf{u})$, which satisfies the following axioms, [6]:

$$L_A(\mathbf{u}) > 0 \qquad \text{for } \mathbf{u} \neq \mathbf{0} \quad \text{and} \quad L_A(\mathbf{0}) = 0\,, \tag{A.1}$$

$$L_A(k\,\mathbf{u}) = k\,L_A(\mathbf{u}) \qquad \text{for } k > 0\,, \tag{A.2}$$

$$L_A(\mathbf{u} + \mathbf{v}) < L_A(\mathbf{u}) + L_A(\mathbf{v}) \tag{A.3}$$

for linearly independent vectors $\mathbf{u}, \mathbf{v} \in \mathcal{V}$.

Function $L_A(\mathbf{u})$ with the above properties is called the norm of vector space $\mathcal{V}$ and due to its positive homogeneity (A.2), in general, it does not have to be symmetric $(L_A(-\mathbf{u}) \neq L_A(\mathbf{u}))$.

In many applications of the normed vector space it is easier to use its metric tensor instead of metrics $L_A(\mathbf{u})$. This tensor is defined by the formula, [8],

$$\mathbf{M}_A(\mathbf{u}) = \frac{1}{2}\frac{\partial^2 L_A^2(\mathbf{u})}{\partial \mathbf{u}\,\partial \mathbf{u}} \qquad \in\ \mathcal{V}^\star \otimes \mathcal{V}^\star\,, \tag{A.4}$$

and due to (A.2), it has the following properties:

$$\mathbf{u} \cdot \mathbf{M}_A(\mathbf{u}) \cdot \mathbf{u} = L_A^2(\mathbf{u})\,, \qquad \mathbf{M}_A(k\,\mathbf{u}) = \mathbf{M}_A(\mathbf{u}) \quad \text{for} \quad k > 0\,. \tag{A.5}$$

From property $(\mathrm{A.2})_2$ it results that, in general, the metric tensor $\mathbf{M}_A(\mathbf{u})$ depends on direction of vector $\mathbf{u}$ and is independent of its length. This property of tensor $\mathbf{M}_A(\mathbf{u})$ defines the anisotropic properties of the normed vector space $\mathcal{V}$.

Affine spaces. The pair $(P, \mathcal{V})$ composed of a point P and a vector space $\mathcal{V}$ we will identify with the affine point space. It is possible, because structures of both objects are isomorphic. Point P is called the reference point, and $\mathcal{V}$ is the space of placement vectors. Affine space $(P, \mathcal{V})$ is normed if the space $\mathcal{V}$ of placement vectors is normed.

Minkowski and Euclidean point spaces. The normed affine point space, the metrics of which is defined by expression $d(\mathbf{u}, \mathbf{v}) \equiv L_A(\mathbf{v} - \mathbf{u})$ and has general properties (A.1)–(A.3), is called Minkowski space, [7]. Its metric tensor is given by expression (A.4). This space is plane and has the anisotropic properties, and the possible lack of symmetry of distance determines the peculiar features of its internal, geometrical structure. Minkowski spaces have been used in papers [1–3] as a model of the anisotropic pore space of rigid porous materials.

The Euclidean point space is the simplest special case of the Minkowski space. The affine space is Euclidean when its norm is given by expression

$$L^2(\mathbf{u}) = \mathbf{u} \cdot \mathbf{M} \cdot \mathbf{u} = \mathbf{M} : \binom{\mathbf{u}}{\mathbf{u}}$$

where tensor $\mathbf{M} \in \mathcal{V}^\star \otimes \mathcal{V}^\star$ is independent of $\mathbf{u}$ and is called the metric tensor of the Euclidean space. Tensor $\mathbf{M}$ is a non-singular, symmetric and positively definite. The Euclidean space is plane, homogeneous and isotropic, and is applied as a model of the physical space in the classical mechanics.

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