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New results concerning the identification of neutral inhomogeneities in plane elasticity

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Using the concept of a spring-layer (imperfect) interface, we develop the series methods to determine the corresponding interface functions which ensure the (stress) neutrality of an elastic inhomogeneity. We assume that the inhomogeneity occupies a simply-connected domain with a regular boundary and that the inhomogeneity-matrix system is subjected to linear plane deformations. Of particular interest is the fact that the prescribed stress field inside the matrix is assumed to be non-uniform.

1. Introduction

In [1], Mansfield showed that the local stress concentration caused by the introduction of a hole into an elastic sheet could be eliminated entirely by reinforcing the hole with a particular stiffener or liner. Mansfield called these holes "neutral holes". The analogous problem of a 'neutral elastic inhomogeneity' in which the introduction of the inhomogeneity into an elastic body (of a different material), does not disturb the original stress field in the uncut body, was first studied by Ru [2]. Ru proved that neutral elastic inhomogeneities cannot exist under the assumption of a conventional perfectly bonded material interface. Moreover, Ru showed that by using an established spring-layer model of an imperfect interface (see, for example, [3] and the extensive bibliography therein), it was possible to control the corresponding interface functions (describing the material properties of the interface layer) in such a way as to achieve the desired stress neutrality for inhomogeneities of various shapes. The authors also acknowledge the important contributions made by many researchers to the area of neutrality and imperfectly bonded inclusions (see, for example, [4–7]).

The neutral inhomogeneities considered in [2], however, assume the existence of a *uniform* stress field in the surrounding matrix. Of a more practical interest is the case when the prescribed stress field in the matrix is *nonuniform*. This case has been considered for anti-plane shear deformations in two recent papers by Van Vliet *et al.* [8] and Schiavone [9]. The corresponding problem of plane deformations, however, has received very little attention in the literature, despite its importance to the design of composite materials and structures. This can be attributed to the relatively complicated nature of the equations describing the corresponding boundary value problems. Recently, however, Vasudevan and Schiavone [10] have used complex variable techniques to determine the corresponding interface functions for certain specialized (nonuniform) stress fields in the matrix under the assumption of linear plane deformations.

In this paper, we continue the work started in [10] and develop the methods based on complex series representations to extend the results in [10] to more general states of nonuniform stresses in the matrix. After some brief formalities concerning the formulation of the basic problem of a neutral inhomogeneity in plane elasticity, we develop (in Sec. 4) a general formalism for identifying neutral inhomogeneities of arbitrary (smooth, simply-connected) shapes. We use this formalism in Sec. 5 to construct neutral circular inhomogeneities for various non-uniform stress fields in the matrix and to discuss various results concerning nonexistence. In Sec. 6 we discuss the case of the neutral elliptic inhomogeneity and show that for general (polynomial) stress fields in the matrix, there are, in fact, no non-trivial solutions describing neutral elliptic inhomogeneities in the case of plane deformations. The generality of our conclusions is discussed in Sec. 7 where we show that our method does in fact produce the most general solutions with the desired physical constraints on the interface parameters. We end our paper with some concluding remarks presented in Sec. 8.

2. Formulation

We begin with a homogeneous and isotropic linearly elastic body, finite or infinite in extent, simply- or multiply-connected, subjected to a given state of stress under a prescribed loading system. We assume that the same elastic body is then cut into a number of simply-connected sub-domains. Each sub-domain is now filled with a different linearly homogeneous and isotropic elastic material and subsequently referred to as an inhomogeneity. Here we are concerned with the design of the material interface between any single inhomogeneity and the elastic body such that the corresponding inhomogeneity is 'neutral', what means that, it does not disturb the original prescribed stress field in the uncut elastic body.

Throughout this paper, we denote by (x, y) a generic point in $\mathbb{R}^2$ (referred to a Cartesian coordinate system), by $z = x + iy = re^{i\theta}$ the complex coordinate, and by D_2 and D_1 – the domains occupied by any single elastic inhomogeneity and the surrounding matrix, respectively. The interface between D_2 and D_1 is denoted by Γ. Finally, the subscripts 1 and 2 refer to the domains D_1 and D_2, respectively.

For plane deformations, it is well known that the components σ_{xx}, σ_{xy}, and σ_{yy} of stress and u_x and u_y of displacement can be represented in terms of two analytic functions $\phi(z)$ and $\psi(z)$ as follows [11]:

$$2\mu(u_x + iu_y) = \left[\kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}\right], \tag{2.1}$$

$$\begin{aligned} \sigma_{xx} + \sigma_{yy} &= 2\left[\phi'(z) + \overline{\phi'(z)}\right], \\ \sigma_{xx} - i\sigma_{xy} &= \phi'(z) + \overline{\phi'(z)} - \left[\overline{z}\phi''(z) + \psi'(z)\right]. \end{aligned} \tag{2.2}$$

Here $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (2 - \nu)/(1 + \nu)$ for plane stress, ν is Poisson's ratio, and μ is the shear modulus. Consequently, the boundary tractions and displacements in normal and tangential components are given by

$$2\mu(u_n + iu_t) = e^{-iN(z)}\left[\kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}\right], \tag{2.3}$$

$$\sigma_{nn} - i\sigma_{nt} = \phi'(z) + \overline{\phi'(z)} - e^{2iN(z)}\left[\overline{z}\phi''(z) + \psi'(z)\right], \tag{2.4}$$

where $e^{iN(z)}$ denotes (in complex form) the outward unit normal to Γ.

Across the interface Γ, the inhomogeneity is assumed to be bonded to the cut elastic body by the imperfect interface described in [2]. This means that across Γ, the tractions are continuous and jumps in displacement are proportional to their respective traction components in terms of the normal and tangential interface parameters $m(z)$ and $n(z)$. Thus

$$[[\sigma_{nn} - i\sigma_{nt}]] = 0, \tag{2.5}$$

$$\sigma_{nn} = m(z)[[u_n]], \qquad \sigma_{nt} = n(z)[[u_t]], \tag{2.6}$$

where $[[*]] = (*)_1 - (*)_2$ denotes the jump across Γ. Traction-free interface conditions are given by $m(z) = n(z) = 0$, and a perfectly bonded interface corresponds to the case $m(z) = n(z) \to \infty$. As shown in [2], for a single interfacial layer, $m(z)/n(z)$ is usually a material constant which is required to be greater than unity. In addition, the interface parameters $m(z)$ and $n(z)$ are also required to be nonnegative. Thus, to be physically realistic, we require the following necessary conditions on the interface parameters m and n:

$$m \geq 0\,, \qquad n \geq 0\,, \qquad m \geq n\,. \tag{2.7}$$

For general, possibly non-uniform, stress fields characterized by the functions $\phi(z)$ and $\psi(z)$, the interfacial conditions (2.6) give

$$\phi_2(z) = \phi_1(z) + iA_0 z\,, \qquad \psi_2(z) = \psi_1(z) + B_0\,,$$

where $A_0 \in \mathbb{R}$, and $B_0 \in \mathbb{C}$. The interfacial condition (2.5) now takes the form

$$\begin{aligned} \phi_1'(z) + \overline{\phi_1'(z)} - e^{2iN(z)}\left[\overline{z}\phi''(z) + \psi'(z)\right] &= \frac{m(z)+n(z)}{4}\left[\frac{e^{iN(z)}}{\mu_1}\left(\kappa_1\overline{\phi_1} - \overline{z}\phi_1' - \psi_1\right)\right. \\ &\quad \left. - \frac{e^{iN(z)}}{\mu_2}\left(\kappa_2\overline{\phi_1} - i\kappa_1 A_0\overline{z} - \overline{z}\phi_1' - iA_0\overline{z} - \psi_1 - B_0\right)\right] \\ &\quad + \frac{m(z)-n(z)}{4}\left[\frac{e^{-iN(z)}}{\mu_1}\left(\kappa_1\phi_1 - z\overline{\phi_1'} - \overline{\psi_1}\right)\right. \\ &\quad \left. - \frac{e^{-iN(z)}}{\mu_2}\left(\kappa_2(\phi_1 + iA_0 z) - z(\overline{\phi_1'} - iA_0) - \overline{\psi_1} - \overline{B_0}\right)\right]. \end{aligned} \tag{2.8}$$

This is the governing equation for interface design corresponding to a neutral elastic inhomogeneity in plane deformations, when a possibly non-uniform stress field is present in the matrix. The constants A_0 and B_0 should be chosen so as to eliminate any rigid body displacement between the elastic body and the inhomogeneity. In particular, if the inhomogeneity has two mutually orthogonal axes of symmetry and these are chosen as the coordinate axes, then $A_0 = B_0 = 0$. In the case of perfect bonding, that is, with m and n both infinite, (2.8) reduces to

$$\frac{1}{\mu_1}\left(\kappa_1\overline{\phi_1} - \overline{z}\phi_1' - \psi_1\right) = \frac{1}{\mu_2}\left(\kappa_2\overline{\phi_1} - i\kappa_1 A_0\overline{z} - \overline{z}\phi_1' - iA_0\overline{z} - \psi_1 - B_0\right). \tag{2.9}$$

It is clear that (2.9) cannot be satisfied for any Γ unless the inhomogeneity and the elastic body are made of identical materials. Hence, there is no neutral elastic inhomogeneity in plane deformations when a conventional perfectly bonded interface is assumed and a non-uniform stress field is present in the matrix.

3. Interface parameters of a neutral inhomogeneity of given shape

We confine our attention to inhomogeneities with two mutually orthogonal axes of symmetry. As mentioned earlier, by choosing the coordinate system so that the coordinate axes coincide with these two axes of symmetry, we can set $A_0 = B_0 = 0$.

By separating the real and imaginary parts of (2.8), we obtain expressions for m and n as follows:

$$\begin{aligned}\frac{2}{m(z)}\left[2\phi_1' + 2\overline{\phi_1'} - e^{2iN(z)}\left(\bar{z}\phi_1'' + \psi_1'\right) - e^{-2iN(z)}\left(z\overline{\phi_1''} + \overline{\psi_1'}\right)\right] &= e^{-iN(z)}\left[(\eta+\lambda)\phi_1 - \lambda\left(z\overline{\phi_1'} + \overline{\psi_1}\right)\right] \\ &\quad + e^{iN(z)}\left[(\eta+\lambda)\overline{\phi_1} - \lambda\left(\bar{z}\phi_1' + \psi_1\right)\right],\end{aligned} \tag{3.1}$$

$$\begin{aligned}\frac{2}{n(z)}\left[e^{2iN(z)}\left(\bar{z}\phi_1'' + \psi_1'\right) - e^{-2iN(z)}\left(z\overline{\phi_1''} + \overline{\psi_1'}\right)\right] &= e^{-iN(z)}\left[(\eta+\lambda)\phi_1 - \lambda\left(z\overline{\phi_1'} + \overline{\psi_1}\right)\right] \\ &\quad - e^{iN(z)}\left[(\eta+\lambda)\overline{\phi_1} - \lambda\left(\bar{z}\phi_1' + \psi_1\right)\right],\end{aligned} \tag{3.2}$$

where

$$\lambda = \frac{1}{\mu_1} - \frac{1}{\mu_2}, \qquad \eta = \frac{\kappa_1 - 1}{\mu_1} - \frac{\kappa_2 - 1}{\mu_2}. \tag{3.3}$$

Note that for typical materials, both λ and η have the same sign [2]. Of course, the conditions (2.7) must also be satisfied for any suitable pair (m, n) satisfying (3.1) and (3.2).

4. General formalism for arbitrary simply-connected neutral inhomogeneities

In this section we present a method for constructing a neutral (simply-connected) inhomogeneity with a smooth non-self-intersecting oriented boundary. Suppose that the boundary Γ of the inhomogeneity in the z-plane is mapped conformally onto the unit circle S^1 in the ξ-plane by the mapping function $z = w(\xi)$.

Next, on S^1 [12]

$$e^{iN(w(\xi))} = \xi \frac{w'(\xi)}{|w'(\xi)|}.$$

Also, since for $\xi \in S^1$, $|\xi| = 1$, $\overline{\xi} = \xi^{-1}$ and we have

$$e^{-iN(w(\xi))} = \frac{1}{\xi}\frac{\overline{w'(\overline{\xi})}}{|w'(\xi)|}\,.$$

Expressing (3.1) and (3.2) in the ξ-plane we can write expressions for the interface parameters m and n as

$$m(w(\xi)) = \frac{4|w'|^2[\phi_1' + \overline{\phi_1'}] - 2(w')^2[\overline{z}\phi_1'' + \psi_1'] - 2(\overline{w'})^2[z\overline{\phi_1''} + \overline{\psi_1'}]}{|w'|\{\overline{w'}[(\eta+\lambda)\phi_1 - \lambda(z\overline{\phi_1'} + \overline{\psi_1'})] + w'[(\eta+\lambda)\overline{\phi_1} - \lambda(\overline{z}\phi_1' + \psi_1)]\}}, \tag{4.1}$$

and

$$n(w(\xi)) = \frac{2(w')^2[\overline{z}\phi_1'' + \psi_1'] - 2(\overline{w'})^2[z\overline{\phi_1''} - \overline{\psi_1'}]}{|w'|\{\overline{w'}[(\eta+\lambda)\phi_1 - \lambda(z\overline{\phi_1'} + \overline{\psi_1'})] - w'[(\eta+\lambda)\overline{\phi_1} - \lambda(\overline{z}\phi_1' + \psi_1)]\}}, \tag{4.2}$$

where primes on ϕ and ψ denote derivatives with respect to z, followed by substituting $z = w(\xi)$, and primes on w denote derivatives with respect to ξ.

We can expand the numerators and denominators in each of these expressions separately into Laurent series in ξ as follows:

$$m(w(\xi)) = \frac{\sum_{n=-\infty}^{\infty} E_n}{|w'|\sum_{m=-\infty}^{\infty} F_m}, \qquad n(w(\xi)) = \frac{\sum_{n=-\infty}^{\infty} G_n}{|w'|\sum_{m=-\infty}^{\infty} H_m}, \tag{4.3}$$

where the Laurent coefficients can be evaluated using the method of residues by contour integration on S^1 in the ξ plane. For example,

$$F_n = \int\limits_{S^1} \frac{d\xi}{\xi^{n+1}} \left\{\overline{w'}[(\eta+\lambda)\phi_1 - \lambda(z\overline{\phi_1'} + \overline{\psi_1'})] + w'[(\eta+\lambda)\overline{\phi_1} - \lambda(\overline{z}\phi_1' + \psi_1)]\right\}.$$

Now, introducing the consistency conditions (2.7) to the expressions (4.3) for the interface parameters, we obtain relations between the physical properties of the material and the inhomogeneity, as well as descriptions of the stress fields that allow for the existence of a neutral inhomogeneity with the prescribed shape (assuming that such a solution exists). One possible solution (we will discuss later why this is much more general than it appears) is the following. Let m_0 and n_0 be two real numbers (constants) such that

$$m_0 > 0, \qquad n_0 > 0, \qquad m_0 \geq n_0\,.$$

Now, the conditions (2.7) are satisfied if the following ansatz is adopted

$$E_n = m_0 \, F_n \, , \qquad G_n = n_0 \, H_n \, , \qquad \forall \, n \in \mathbb{Z} \, . \tag{4.4}$$

If this system of equations can be solved consistently to yield constraints describing possible physical values of the physical parameters, then a neutral elastic inhomogeneity is possible with interface parameters given by

$$m(w(\xi)) = \frac{m_0}{|w'(\xi)|} \, , \qquad n(w(\xi)) = \frac{n_0}{|w'(\xi)|} \, . \tag{4.5}$$

The system of equations (4.4) may or may not be consistent, and even if it is consistent, it may not correspond to physically meaningful situations. The coefficients E_n, F_n, G_n, and H_n, are clearly expressed in terms of the parameters η and λ, as well as in terms of the coefficients of the Laurent expansions for the stress fields ϕ_1 and ψ_1. Thus, (4.4) is a system of recursion relations between these Laurent coefficients of the stress fields. The great simplification that has already been achieved, that is unlike the general highly nonlinear system of equations (3.1) and (3.2) describing the interface parameters, we have reduced the problem to solving a system of *linear* equations in Laurent coefficients. Not only these are in ideal form for numerical analysis, some fairly general situations are even analytically tractable in this formalism (unlike in [10], where the full nonlinear equations were studied, thereby resulting in solutions in very specific cases only).

We will now show how this method works in detail for circular and elliptic inhomogeneities, though the method is far more general since all we require is that the domain should have a smooth simple orientable boundary. Our solutions for the circle will include the solutions obtained in [10] as special cases, and we will also generalize the discussion initiated in [10] for the ellipse.

5. Neutral circular inhomogeneities

We consider a circle of radius R centered at the origin. In this case the conformal mapping w to the ξ-plane is trivially given by

$$z = w(\xi) = R\xi \, .$$

As a simple case, to recapture the solutions in [10], we assume that the stress fields take the form characterized by the functions

$$\phi_1(z) = A_p z^p \, , \qquad \psi_1(z) = B_q z^q \, ,$$

where p and q are non-negative integers (though the case of negative integers can be treated, as well as non-integral real numbers, though we have to carefully pick the correct branch of the Riemann surface describing our multi-valued solutions). In this case

$$
\begin{aligned}
E_{p-1} &= 2p(3-p)A_pR^{p-1}\,,\\
E_{q+1} &= -2qB_qR^{q-1}\,,\\
F_{p-1} &= (\eta+\lambda)A_pR^p - \lambda p A_pR^p\,,\\
F_{q+1} &= -\lambda B_qR^{q-1}\,,
\end{aligned}
\tag{5.1}
$$

and their complex conjugates, since $E_{-n} = \overline{E_n}$, and similarly for the F_n's. All the other E_n's and F_n's vanish. Similarly,

$$
\begin{aligned}
G_{p-1} &= 2p(p-1)A_pR^{p-1}\,,\\
G_{q+1} &= -2qB_qR^{q-1}\,,\\
H_{p-1} &= (\eta+\lambda)A_pR^p + \lambda p A_pR^p\,,\\
H_{q+1} &= -\lambda B_qR^{q-1}\,,
\end{aligned}
\tag{5.2}
$$

and their complex conjugates again, with other coefficients vanishing.

Using our ansatz, the solutions are

$$
m(w(\xi)) = \frac{m_0}{R}\,, \qquad n(w(\xi)) = \frac{n_0}{R}\,, \tag{5.3}
$$

with

$$
m_0 = \frac{2p(3-p)}{R[\eta+(1-p)\lambda]}\,, \qquad n_0 = \frac{2p(p-1)}{R[\eta+(1+p)\lambda]}\,, \tag{5.4}
$$

if $p \neq 0$, or if $p = 0$ and $q \neq 0$ then

$$
m_0 = \frac{2q}{\lambda}\,, \qquad n_0 = \frac{2q}{\lambda}\,, \tag{5.5}
$$

with a constraint that the two expressions must equal each other if both p and q are non-zero.

This solution is subject to the constraints on the parameters from the system of equations (4.4) given by

$$
\begin{aligned}
\lambda &> 0\,,\\
\eta + \lambda(1-p) &> 0\,,\\
\eta &= \frac{\lambda}{qR}\Big[p(3-p) - R(1-p)q\Big].
\end{aligned}
\tag{5.6}
$$

This solution and set of constraints agree with the special solutions obtained in [10] for the case of either $A_p = 0$ or $B_q = 0$, and a more detailed description of specific scenarios can be found there.

For the case when both A_p and B_q are non-zero and both p and q are positive integers, this describes a special solution not known earlier. This solution is unique in that it describes interface parameters which are constant up to a conformal factor (of course, for the circle the conformal factor is just the radius, which results in the parameters themselves being constant). Solutions of this type are very easy to generate (assuming they exist and are consistent and meaningful) using this method for arbitrary shapes, as seen from (4.5).

We will illustrate a few more situations, one where a conformally constant new solution does exist, and another where there are inconsistencies and thus no such new solutions exist.

Now consider the case when

$$\phi_1(z) = \sum_{n=1}^{p} A_n z^n\,, \quad \psi_1 = 0\,, \tag{5.7}$$

where as before we note that A_0 can be freely set to zero by a proper choice of coordinates. As before, by going to the ξ-plane using the conformal mapping $R = w(\xi) = R\xi$ and using Laurent series expansions, we have

$$\begin{aligned}
E_n &= 2(n+1)A_{n+1}R^n(2-n)\,, && n = 0, ..., p-1\,,\\
F_n &= [(\eta+\lambda) - (n+1)\lambda]A_{n+1}R^{n+1}\,, && n = 0, ..., p-1\,,\\
G_n &= 2n(n+1)A_{n+1}R^n\,, && n = 0, ..., p-1\,,\\
H_n &= [(\eta+\lambda) + (n+1)\lambda]A_{n+1}R^{n+1}\,, && n = 0, ..., p-1\,,
\end{aligned} \tag{5.8}$$

and their complex conjugates since, as before $E_{-n} = \overline{E_n}$, and similarly for the F_n's, G_n's, and H_n's. All other coefficients vanish.

As earlier, we use the ansatz (4.4) with non-negative constants m_0 and n_0 such that $m_0 \geq n_0$ to generate a solution of the form

$$m(w(\xi)) = \frac{m_0}{R}, \qquad n(w(\xi)) = \frac{n_0}{R}. \tag{5.9}$$

However, for the general stress fields (5.7), the system of equations (4.4) with the Laurent coefficients (5.8) is inconsistent, except for the special case when all the stress coefficients A_n vanish except for A_1 and A_p. That is, a solution can be found only in the case

$$\phi_1(z) = A_1 z + A_p z^p\,, \qquad \psi_1 = 0\,.$$

In this case we can write the solution explicitly:

$$E_0 = 4A_1, \qquad F_0 = \eta A_1 R,$$

$$E_{p-1} = 2pA_p R^{p-1}(3-p), \qquad F_{p-1} = (\eta + (1-p)\lambda)A_p R^p,$$

and applying our ansatz (4.4) we obtain the solution (4.5) with

$$m_0 = \frac{4}{\eta R}, \qquad n_0 = \frac{2p(p-1)}{R(\eta + (p+1)\lambda)}.$$

Now requiring (2.7) to be true imposes the following conditions on the physical parameters in order for the solution to exist:

$$\eta > 0, \qquad \lambda = \frac{\eta[p(3-p)-2]}{2(1-p)},$$

which can correspond to physically meaningful values of the parameters, and thus a neutral circular inhomogeneity is possible in this case.

Now consider the case where the stress fields are given by

$$\phi_1(z) = 0, \qquad \psi_1(z) = \sum_{n=1}^{M} B_n z^n, \tag{5.10}$$

where, as before, we use the fact that B_0 can be set to zero by the proper choice of axes. We will work first with the situation for the interface parameter $m(z)$. In this case, following the usual procedure

$$\begin{aligned} E_n &= -2(n-1)B_{n-1}R^{n-2}, \qquad && n = 2, ..., M+1, \\ F_n &= -\lambda B_{n-1}R^{n-1}, && n = 2, ..., M+1, \end{aligned} \tag{5.11}$$

and their complex conjugates and all other coefficients vanishing. Now applying (4.4) gives

$$m_0 = \frac{2(n-1)}{\lambda R}, \tag{5.12}$$

where this equation holds for all n with $B_{n-1} \neq 0$. This is obviously an inconsistent system of equations if there are any two B_n that are non-zero. Thus a consistent solution for the interface problem in this case cannot be found, unless there is only one non-zero B_n, i.e. for a pure power law. We have already treated this case and found the consistent solutions.

6. Neutral elliptic inhomogeneities

Consider an elliptic inhomogeneity, centered at the origin, with axes of lengths a and b, with $a \neq b$, coincident with the x- and y-axes, respectively. The boundary of the ellipse can be conformally mapped onto the unit circle in the ξ plane using the conformal mapping [11]:

$$z = w(\xi) = R\left[\xi + \frac{k^2}{\xi}\right], \qquad k \in (0,1)\,, R > 0\,.$$

Then

$$w'(\xi) = R(1 - k^2\xi^{-2})\,, \qquad \overline{w'(\overline{\xi})} = R(1 - k^2\xi^2)\,.$$

Thus with our procedure, if any solutions can be found, they take the form

$$m(w(\xi)) = \frac{m_0}{|1 - k^2\xi^2|}\,, \qquad n(w(\xi)) = \frac{n_0}{|1 - k^2\xi^2|}\,.$$

Before discussing the general case, we present an example that illustrates the main features. Consider the situation with

$$\phi_1(z) = A_2 z^2\,, \qquad \psi_1(z) = 0\,.$$

Again, we consider the interface parameter $m(z)$ first. Expanding in Laurent series, as before, we obtain

$$\begin{aligned} E_5 &= -4\overline{A_2}R^3k^4\,, \\ E_3 &= -4\overline{A_2}R^3(-2k^2 + 2k^4 + k^6) - 8R^3A_2k^2\,, \\ E_1 &= -4R^3\overline{A_2}(1 - 2k^4 + 2k^6) - 4R^3A_2(-2 + k^2)\,, \end{aligned} \tag{6.1}$$

and their complex conjugates with $E_{-n} = \overline{E_n}$, and all other E_n's vanishing. Also,

$$\begin{aligned} F_4 &= R^3\overline{A_2}k^4[\eta + 3\lambda]\,, \\ F_3 &= -2R^3A_2\lambda k^2\,, \\ F_2 &= R^3A_2(\eta + \lambda)[1 - k^2 - 2k^4] + R^3\overline{A_2}k^6[\lambda - \eta]\,, \\ F_0 &= -2R^3\overline{A_2}\lambda(1 + k^4) - 2R^3A_2\lambda(1 + k^4)\,, \end{aligned} \tag{6.2}$$

and again, their complex conjugates with $F_{-n} = \overline{F_n}$, and all other F_n's vanishing.

We immediately see a problem with imposing our solution method here, specifically with imposing (4.4) consistently. For instance, there are no $\xi^{\pm 5}$ terms in the denominator. Consequently, the only consistent way to impose $E_5 = m_0F_5$

is to take $E_5 = 0$. But this implies $A_2 = 0$, which is the trivial solution. A similar analysis can be repeated for a field of the form $\phi_1(z) = A_p z^p$ for arbitrary p, and also stress fields of the general form

$$\phi_1(z) = \sum_{n=1}^{p} A_n z^n\,, \qquad \psi_1(z) = \sum_{n=1}^{q} B_n z^n$$

It follows that, in each of these cases, no conformally constant solutions exist for the elliptic inhomogeneity in plane elasticity. Results like this were also observed earlier in [10].

This might appear to be a limitation of the procedure used here. However, such non-existence results obtained by using these methods are far more general statements. In fact, we can say that there are no non-trivial solutions describing neutral elliptic inhomogeneities in the case of plane deformations for stress fields of the very general form

$$\phi_1(z) = \sum_{n=1}^{p} A_n z^n\,, \qquad \psi_1(z) = \sum_{n=1}^{q} B_n z^n\,. \tag{6.3}$$

7. Generality of conformally constant solutions

Consider the general expression for the interface parameters in the ξ-plane

$$m(w(\xi)) = \frac{\sum_{n=-\infty}^{\infty} E_n}{|w'| \sum_{m=-\infty}^{\infty} F_m}\,, \qquad n(w(\xi)) = \frac{\sum_{n=-\infty}^{\infty} G_n}{|w'| \sum_{m=-\infty}^{\infty} H_m}\,.$$

At this stage, this is simply a rewriting of the basic Eq. (2.8) in the ξ-plane without any assumptions, and prior to imposing our ansatz. The Laurent coefficients are evaluated from the form of the conformal mapping $w(\xi)$, and the Laurent expansions for the stress fields ϕ_1 and ψ_1 by means of the residue theorem.

Since in the ξ-plane the boundary is a unit circle S^1, we can write

$$\xi = e^{i\theta} = \cos(\theta) + i\sin(\theta)\,,$$

and rewriting in terms of θ, we have

$$m(\theta) = \frac{\sum_{n=1}^{\infty} \mathrm{re}(E_n)\cos(n\theta)}{|w'| \sum_{n=1}^{\infty} \mathrm{re}(F_n)\cos(n\theta)}\,, \qquad n(\theta) = \frac{\sum_{n=1}^{\infty} \mathrm{im}(E_n)\sin(n\theta)}{|w'| \sum_{n=1}^{\infty} \mathrm{im}(F_n)\sin(n\theta)}\,.$$

Note that this form follows explicitly from the structure of (2.8), after some simple yet tedious manipulations. In general there would be cross-terms of the form $\cos(p\theta)\sin(q\theta)$ in the expansion, but these disappear in the cases of interest here (essentially since we have the real and imaginary parts of a single equation).

In situations where the stress fields have a finite power series expansion, such as (6.3), the Laurent series also terminate and we have the form

$$m(\theta) = \frac{\sum_{n=1}^{M} \mathrm{re}(E_n)\cos(n\theta)}{|w'|\sum_{n=1}^{N} \mathrm{re}(F_n)\cos(n\theta)}, \qquad n(\theta) = \frac{\sum_{n=1}^{P} \mathrm{im}(E_n)\sin(n\theta)}{|w'|\sum_{n=1}^{Q} \mathrm{im}(F_n)\sin(n\theta)}.$$

In general, it is very difficult to ensure positivity of such expressions since they involve trigonometric functions of different periods, and in general will have domains of θ, where they will be positive or negative. In the very exceptional case involving at most trigonometric functions of only one period and a constant term, positivity may be ensured by choosing the constant to be positive and larger than the coefficient in front of the trigonometric function (if possible). However, even in this situation positivity cannot be always ensured, since in choosing the coefficients, we may require values of the material constants which make no sense physically. For non-uniform stress field prescriptions, such as (6.3) positivity can only be ensured by choosing the coefficients such that the entire expression is a positive number (up to the factor of $|w'|$ which is positive anyway, though not necessarily constant on S^1).

Thus, in fact the ansatz adopted in this paper (4.4) produces the most general solutions with the desired constraints on the interface parameters. Note that many general non-constant, or even conformal constant, solutions to (2.8) may exist. It is the positivity constraint that eliminates most of these and leaves the one picked out by our formalism (assuming that such a solution exists and is consistent).

8. Conclusions

We consider the problem of a single elastic inhomogeneity embedded within an infinite elastic matrix subjected to plane deformations. In particular, we examine the (stress) neutrality of this inhomogeneity when a nonuniform stress field is prescribed in the surrounding matrix. The inhomogeneity is assumed to be imperfectly bonded to the matrix through an interphase region modelled by a spring-layer interface as in [2]. We present a formalism to analyze the design of an arbitrary simply-connected inhomogeneity with a smooth boundary. We extend the solutions and discussion of [10] by discussing, in detail, the practicality of designing circular and elliptical inhomogeneities in the case of plane deformations, for several non-uniform states of stress in the matrix. Our formalism establishes a general framework for treating arbitrary smooth neutral inhomogeneities in the context of planar deformations. The results obtained here illustrate the generality of the method, and also encompass all the previously known solutions for neutral inhomogeneities with a smooth simple boundary. The impossibility of design of a neutral elliptic inhomogeneity in the presence of the general finite non-uniform stress fields (6.3), is certainly noteworthy.

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