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Continuum modelling of laminates with a slowly graded microstructure

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In Memory of Professor Henryk Zorski

The considerations are concerned with modelling and analysis of the dynamic response for micro-laminated two-phased solids. The main attention is focussed on modelling of the laminates which have a slowly graded microstructure in the direction normal to the layering (slowly graded laminates, SGL). Periodic and functionally graded laminates can be treated as special cases of SGL. A new mathematical model which couples micro- and macro-response of the linear elastic SGL is proposed. It is shown that for laminates with a weak transversal inhomogeneity, the derived model equations can be decomposed into asymptotic equations which describe the behavior of a laminate independently on the macro- and micro-level. This decomposition is estimated on the example of a specific vibration problem.

1. Introduction

The object of analysis is a dynamic response of two-phase multilayered solids with microstructure slowly varying in the direction normal to the layering. The above solids will be referred to as the slowly graded laminates (SGL). A fragment of a certain SGL from the macroscopic and microstructural viewpoints is shown in Fig. 1. The concept of SGL can be treated as a generalization of the concept of a functionally graded laminate, where we deal with a slow macroscopic passage between two distinct materials. It can be seen that in contrast to the functionally graded materials with irregular (stochastic) microstructure, the laminates under consideration have a deterministic structure.

Recent literature on the elastic response of the linear-elastic functionally graded solids is rather extensive. The simplest mathematical model is that of the locally homogenized medium with material properties described by effective moduli, slowly varying in space, [4, 5]. On the other hand, it is known that homogenized models are unable to describe many microstructural phenomena even for periodically laminated solids, Łaciński, [3]. That is why some higher-order models were proposed *by coupling the local (microstructural) and global*

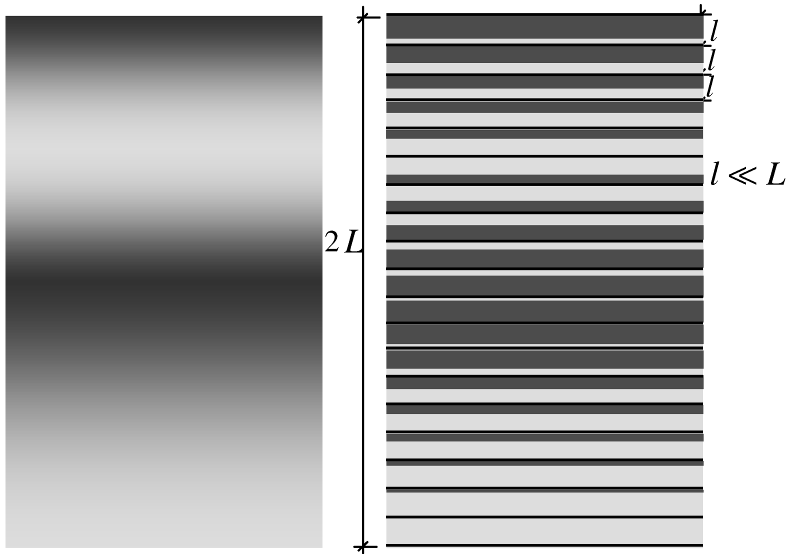

Fig. 1. A fragment of a certain SGL from the macroscopic and microstructural viewpoints.

(macrostructural) responses of functionally graded solids. We can mention here the models proposed by Aboudi *et al.*, [1]. However, the models proposed in [1] are rather complicated involving a large number of unknowns.

The aim of this contribution is twofold. First, we formulate a simple mathematical model involving two vector fields representing respectively macro- and micro-response of linear-elastic SGL. Second, for laminated media with a weak transversal inhomogeneity (which will be defined in this paper) we decompose the derived model equations into two asymptotic approximations. These approximations describe the dynamic response of a laminate independently on the macro- and micro-levels. The above decomposition will be estimated of a specific vibration problem of the periodically laminated solid.

Denotations

Small bold-face letters stand for vectors and points in the 3-space, capital bold-face letters represent the second-order tensors and block letters are used for the third- and fourth-order tensors. The scalar and double-scalar products of these objects are denoted by a dot or a double dot between letters, respectively. We introduce in the physical space the orthogonal Cartesian coordinate system $Ox_1x_2x_3$ with the x_3-axis perpendicular to the laminae interfaces. The partial derivatives with respect to x_i, $i = 1, 2, 3$, are denoted by ∂_i and the time derivate by the overdot. We also use differential operators $\nabla = (\partial_1, \partial_2, \partial_3)$, $\overline{\nabla} = (\partial_1, \partial_2, 0)$

and introduce the unit vector $\mathbf{e} = (0, 0, 1)$ together with notations $z = x_3$, $\mathbf{x} = (x_1, x_2)$. For an arbitrary integrable function f defined in $[-L, L]$ we define its mean value by

$$\langle f \rangle (z) = \frac{1}{l} \int\limits_{-l/2}^{l/2} f(z+y)\, dy, \qquad z \in \left[-L + \frac{l}{2},\, L - \frac{l}{2}\right],$$

where $l \ll L$ and f can also depend on $\mathbf{x} = (x_1, x_2)$ and time t.

2. Preliminaries

Let $\Omega = \Pi \times (-L, L)$, $\Pi \subset R^2$ stand for the region occupied in the physical space by the laminated solid in its natural configuration. We assume that this solid consists of a large number $2m$ of thin layers Λ_n with a constant thickness l; here and subsequently $n = -m, ..., -1, 1, ..., m$. Every layer Λ_n comprises two homogeneous laminae Λ'_n, Λ''_n of thicknesses l'_n, l''_n respectively. Laminae Λ'_n are made of a linear-elastic material with ρ', $\mathbb{C}'$ as constant mass density and tensor of the elastic moduli, respectively. Similarly, by ρ'', $\mathbb{C}''$ we denote the pertinent material characteristics for laminae Λ''_n. It is assumed that all adjacent laminae are perfectly bonded and material planes parallel to the interfaces between laminae are elastic-symmetry planes.

Let $\varepsilon > 0$ be a small parameter, $\varepsilon \ll 1$. The leading role in the definition of SGL play the smooth functions $\nu'(\cdot)$, $\nu''(\cdot)$ defined on $[-L, L]$, such that:

(i) $\nu'(z) + \nu''(z) = 1$, $z \in [-L, L]$,

(ii) $|\nu'(z) - l'_n/l| \le \varepsilon$ for every $z \in \Lambda'_n$, $n = -m, ..., -1, 1, ..., m$,

(iii) $l \left| \dfrac{d\nu'(z_1)}{dz} - \dfrac{d\nu'(z_2)}{dz} \right| \le \varepsilon$ for every $z_1, z_2 \in [-L, L]$ such that $|z_1 - z_2| \le l$.

Obviously, restrictions of the form (ii), (iii) are also imposed on function $\nu''(\cdot)$. Under conditions (i)–(iii) a laminated medium will be referred to as the slowly graded laminate (SGL). Functions ν', ν'' determine (with a tolerance ε) the distribution of mean volume fractions across thickness $2L$ of the SGL. Setting $\nu = \sqrt{\nu'\nu''}$ we introduce also function $\nu(\cdot)$ defined on $[-L, L]$, which represents the phase distribution in SGL.

3. Modelling concepts and assumptions

The general line of the proposed modelling technique is an extension of that based on the tolerance averaging, cf. WOŹNIAK and WIERZBICKI, [7], and the list of references therein. In contrast to the known homogenisation method, [2], the tolerance averaging results in a certain coupling macro- and micro-response of a composite solid. The basic concepts of the tolerance averaging technique are those of a slowly varying function and a fluctuation shape function, [7]. Let $F(\cdot)$ be an arbitrary function defined on $\overline{\Omega}\times[0,\infty)$ such that $F(\mathbf{x},\cdot,t)\in C^1([-L,L])$ for every $\mathbf{x}\in \Pi$, $t\geq 0$. Function $F(\mathbf{x},\cdot,t)$ will be called slowly varying (related to the length l within tolerance ε, $0<\varepsilon\ll 1$) if for every $|\Delta z|\leq l$ it satisfies the conditions

$$\begin{aligned} |F(\mathbf{x},z+\Delta z,t)-F(\mathbf{x},z,t)| &\leq \varepsilon F_0,\\ |\partial_3 F(\mathbf{x},z+\Delta z,t)-\partial_3 F(\mathbf{x},z,t)| &\leq \varepsilon F_1, \end{aligned}$$

where F_0, F_1 are certain known *a priori* unit measures for F and $\partial_3 F$, respectively. Under the above conditions we shall write $F(\mathbf{x},\cdot,t)\in SV^1_\varepsilon(l)$. It can be seen that $\nu',\nu''\in SV^1_\varepsilon(l)$. To every phase distribution function $\nu(\cdot)$ we assign a continuous function $g(\cdot)$ of argument z, $z\in[-L,L]$ such that

$$\begin{aligned} &g(nl)=l\sqrt{3}\nu(nl), \qquad g\left(nl+l'_n\right)=-l\sqrt{3}\nu(nl),\\ &g((n+1)l)=-l\sqrt{3}\nu((n+1)l), \qquad n=\pm1,\pm2,...,\pm m \end{aligned}$$

and linear in every interval

$$\left[ln, ln+l'_n\right] \quad \text{and} \quad \left[ln+l'_n,(n+1)l\right], \qquad n=\pm1,\pm2,...,\pm m.$$

The function $g(\cdot)$ is referred to as the fluctuation shape function for the SGL under consideration and it represents a generalization of the saw-like function, well-known in the tolerance averaging technique of periodic laminates, [7]. It can be seen that function g takes into account the microstructure length dimension.

Let us observe that for every slowly varying function $F\in C^1([-L,L])$, every fluctuation function g and every integrable function f defined in $[-L,L]$, we obtain

$$\begin{aligned} &\text{(i)} \qquad \langle fF\rangle(z)=\langle f\rangle(z)F(z)+O(\varepsilon),\\ &\text{(ii)} \quad \langle f\partial(gF)\rangle(z)=\langle f\partial g\rangle(z)F(z)+O(\varepsilon), \qquad z\in[-L,L]. \end{aligned}$$

The continuum model of SGL will be based on two assumptions. The first of them states that terms $O(\varepsilon)$ in the above formulas can be neglected. This assumption will be referred to as *the tolerance averaging approximation*, TAA.

Let us denote by $\mathbf{w}=\mathbf{w}(\mathbf{x},z,t)$, $\mathbf{x}\in\Pi$, $z\in[-L,L]$, the displacement field at time t. The second modelling assumption states that the distribution of displacement across the thickness of every lamina can be approximated by linear functions. Using the concepts of the slowly varying and fluctuation shape functions and applying TAA, the aforementioned assumption can be written in the form

$$\begin{aligned} &\mathbf{w}(\mathbf{x},z,t)=\mathbf{u}(\mathbf{x},z,t)+g(z)\mathbf{v}(\mathbf{x},z,t),\\ &\mathbf{u}(\mathbf{x},\cdot,t)\in SV^1_\varepsilon(l), \quad \mathbf{v}(\mathbf{x},\cdot,t)\in SV^1_\varepsilon(l), \quad \mathbf{x}\in\Pi, \quad z\in[-L,L] \end{aligned} \tag{3.1}$$

and will be referred to as the *kinematical assumption* (KA). Functions $\mathbf{u}$ and $\mathbf{v}$ are the basic kinematical unknowns. Using TAA we obtain from (3.1) that $\mathbf{u} = \langle \mathbf{w} \rangle$. Hence $\mathbf{u}$ is the averaged displacement and $g\mathbf{v}$ represents displacement fluctuations of the SGL solid. This is why $\mathbf{v}$ will be referred to as the fluctuation amplitude. Let us observe that in the case of limit passage with the cell dimension tending to zero, the function $\mathbf{v}$ plays a similar role as the gradient of displacement $\mathbf{u}$ in the asymptotic homogenization approach. Hence the components of field $\mathbf{v}$ are dimensionless.

It can be seen that the proposed line of modelling in fact causes an increase of the number of unknown functions, but the resulting equations do not involve highly oscillating and discontinuous coefficients in model equations. Moreover, this procedure – in contrast to the homogenization approach – enables us to describe the fluctuations of displacement field in the boundary zone.

4. Model equations

Now, we are going to present the averaged governing equations for $\mathbf{u}$ and $\mathbf{v}$. These equations will be derived from the principle of stationary action with the averaged Lagrangian given by

$$\langle \mathcal{L} \rangle = \frac{1}{2} \langle \rho \dot{\mathbf{w}} \cdot \dot{\mathbf{w}} \rangle - \frac{1}{2} \langle \mathbf{E} : \mathbb{C} : \mathbf{E} \rangle ,$$

where $\mathbf{E} = \frac{1}{2} \left(\nabla \mathbf{w} + (\nabla \mathbf{w})^T \right)$ and $\mathbf{w}$ is taken in the form (3.1). By applying (KA) and (TAA), and denoting

$$\begin{aligned}
&\langle \rho \rangle = \nu' (z) \rho' + \nu'' (z) \rho'', && \langle \mathbb{C} \rangle = \nu' (z) \mathbb{C}' + \nu'' (z) \mathbb{C}'', \\
&[\mathbb{C}] \equiv 2\sqrt{3} \nu (z) \left(\mathbb{C}'' - \mathbb{C}' \right) \cdot \mathbf{e}, && [\mathbb{C}]^T \equiv 2\sqrt{3} \nu (z) \mathbf{e} \cdot \left(\mathbb{C}'' - \mathbb{C}' \right), \\
&\{\mathbf{C}\} \equiv 12 \mathbf{e} \cdot \left(\mathbb{C}' \nu'' (z) + \mathbb{C}'' \nu' (z) \right) \cdot \mathbf{e}, && z \in [-L, L],
\end{aligned}$$

the Euler–Lagrange equations will take the form

$$\begin{aligned}
\langle \rho \rangle \ddot{\mathbf{u}} - \nabla \cdot \left(\langle \mathbb{C} \rangle : \nabla \mathbf{u} + [\mathbb{C}] \cdot \mathbf{v} \right) &= \mathbf{0}, \\
l^2 \nu^2 \langle \rho \rangle \ddot{\mathbf{v}} - l^2 \nu^2 \overline{\nabla} \cdot \left(\langle \mathbb{C} \rangle : \overline{\nabla} \mathbf{v} \right) + \{\mathbf{C}\} \mathbf{v} + [\mathbb{C}]^T : \nabla \mathbf{u} &= \mathbf{0}.
\end{aligned} \tag{4.1}$$

Equations (4.1) have to be considered together with the proper boundary and initial conditions imposed on fields $\mathbf{u}$ and $\mathbf{v}$. The boundary conditions for $\mathbf{u}$ have to be prescribed on $\partial \Pi \times (-L, L)$ and $\Pi \times \{-L, L\}$ while the boundary conditions for $\mathbf{v}$ – only on $\partial \Pi \times (-L, L)$. The form of boundary and initial conditions for Eqs. (4.1) is restricted by formulae (3.1) for total displacement $\mathbf{w}$. In order to simplify the considerations, we have neglected the body forces. For an arbitrary linear elastic SGL the above model equations, by means of conditions $\nu', \nu'' \in SV_\varepsilon^1(l)$, have slowly varying smooth coefficients. If volume fractions ν', ν'' are strongly monotone functions, then Eqs. (3.1), (4.1) describe the model of a functionally graded laminate. If ν', ν'' are constant then Eqs. (4.1) have constant coefficients and represent a model of the periodically laminated medium, [7]. In this contribution formulae (3.1), (4.1) are the starting point for the subsequent analysis.

5. Macro-micro coupling of the model equations

Now we shall transform Eqs. (4.1) to the form which couples the macro- and micro-response of the linear-elastic SGL. To this end we shall derive an alternative form of the model equations (4.1). In this form, instead of the fluctuation amplitude $\mathbf{v}$, we shall deal with a new kinematical unknown $\mathbf{r}$ defined by

$$\mathbf{r} = \{\mathbf{C}\}^{-1} \cdot [\mathbb{C}]^T : \nabla\mathbf{u} + \mathbf{v}, \tag{5.1}$$

where $\{\mathbf{C}\}^{-1}$ represents a transformation inverse to the non-singular linear transformation $\{\mathbf{C}\}$. Neglecting in Eqs. (4.1) the terms depending on the microstructure length l, we obtain $\mathbf{r} = \mathbf{0}$. This is why $\mathbf{r}$ is referred to as the intrinsic fluctuation amplitude, i.e., the amplitude independent of the averaged displacement field $\mathbf{u}$. At the same time, from (3.1) and (5.1) we obtain

$$\mathbf{w}(\mathbf{x}, z, t) = \mathbf{u}(\mathbf{x}, z, t) - g(z)\{\mathbf{C}\}^{-1} \cdot [\mathbb{C}]^T : \nabla\mathbf{u}(\mathbf{x}, z, t) + g(z)\,\mathbf{r}(\mathbf{x}, z, t) \tag{5.2}$$

where $g\mathbf{r}$ represents the intrinsic fluctuation of displacements. The part of response of the linear elastic SGL described by the averaged displacement $\mathbf{u}$ will be referred to as *the macro-response*, while that described by the intrinsic fluctuation amplitude $\mathbf{r}$ will be called *the micro-response*.

In order to formulate the governing equations for functions $\mathbf{u}$ and $\mathbf{r}$, we shall use the notion of the homogenized tensor of elastic moduli, defined by

$$\mathbb{C}^h \equiv \langle\mathbb{C}\rangle - [\mathbb{C}] \cdot \{\mathbf{C}\}^{-1} \cdot [\mathbb{C}]^T .$$

We also introduce the following differential operators:

$$\begin{aligned}
&A\mathbf{u} \equiv \langle\rho\rangle\,\ddot{\mathbf{u}} - \nabla\cdot\left(\mathbb{C}^h : \nabla\mathbf{u}\right),\\
&D\mathbf{r} \equiv l^2\nu^2\left[\langle\rho\rangle\,\ddot{\mathbf{r}} - \overline{\boldsymbol{\nabla}}\cdot\left(\langle\mathbb{C}\rangle : \overline{\nabla}\mathbf{r}\right)\right] + \{\mathbf{C}\}\cdot\mathbf{r},\\
&F\mathbf{u} \equiv l^2\nu^2\left[\langle\rho\rangle\{\mathbf{C}\}^{-1}\cdot[\mathbb{C}]^T : \nabla\ddot{\mathbf{u}} - \overline{\boldsymbol{\nabla}}\cdot\left(\langle\mathbb{C}\rangle : \overline{\nabla}\cdot\left(\{\mathbf{C}\}^{-1}\cdot[\mathbb{C}]^T : \nabla\mathbf{u}\right)\right)\right].
\end{aligned}$$

Combining Eqs. (4.1) with formula (5.1) we obtain finally the system of equations

$$\begin{aligned}
A\mathbf{u} &= [\mathbb{C}]^T : \nabla\mathbf{r},\\
D\mathbf{r} &= F\mathbf{u},
\end{aligned} \tag{5.3}$$

representing coupling between the macro- and micro-response of SGL. It has to be emphasized that Eqs. (5.3) have a physical sense only if $\mathbf{u}$ and $\mathbf{r}$ are slowly varying functions of argument z. From Eqs. (5.3) it follows that boundary conditions for $\mathbf{u}$ have to be prescribed both on $\partial\Pi \times (-L, L)$ and $\Pi \times \{-L, L\}$, while the boundary conditions for $\mathbf{r}$ – only on $\partial\Pi \times (-L, L)$. The form of boundary and initial conditions for Eqs. (5.3) is restricted by formulae (5.2) for total displacement $\mathbf{w}$.

6. Asymptotic approximations

Now we are to show that under certain conditions, the coupled macro-micro Equations (5.3) can be decomposed into approximate model equations describing independently the macro- and micro-response of the laminated solid.

Let us denote by $\|\cdot\|_n$ an arbitrary but fixed norm in the linear space of all n-th order tensors related to space E^3. We define

$$\eta = \sup_{z\in[-L,L]} \left(\frac{\|[\mathbb{C}]\|_3}{\|\langle\mathbb{C}\rangle\|_4} \right)$$

as a transversal inhomogeneity parameter of the laminated solid under consideration. This solid is said to have a weak transversal inhomogeneity provided that η satisfies the condition $0 < \eta \ll 1$. This kind of inhomogeneity takes place for laminae reinforced by long high-strength fibres. In this case, the components of the elastic moduli tensor $\mathbb{C}$ which are related to the Ox_1x_2-plane are strongly different in adjacent laminae; the remaining components attain only small jumps across the lamina interfaces. The above condition holds true for many laminated materials used in civil and mechanical engineering.

The subsequent analysis will be restricted to laminated solids with a weak transversal inhomogeneity, where η is treated as a certain small parameter. Notice that the values $[\mathbb{C}]^T : \nabla\mathbf{r}$ and $F\mathbf{u}$ are of orders $O(\mathbf{r}\eta)$, $O(\mathbf{u}\eta)$, respectively. Moreover, $A\mathbf{u}$ and $D\mathbf{r}$ are of the same order as $\mathbf{u}$ and $\mathbf{r}$, respectively. Let us assume that the solutions to Eqs. (5.3) can be represented in the form

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_0 + \mathbf{u}_\Delta, \\ \mathbf{r} &= \mathbf{r}_0 + \mathbf{r}_\Delta, \end{aligned} \tag{6.1}$$

where $\mathbf{u}_0 \in O(\eta^0)$, $\mathbf{r}_0 \in O(\eta^0)$, $\mathbf{u}_\Delta \in O(\eta)$, $\mathbf{r}_\Delta \in O(\eta)$. Bearing in mind (6.1) and applying the limit passage $\eta \to 0$ to Eqs. (5.3), we obtain the following system of equations for $\mathbf{u}_0$, $\mathbf{r}_0$:

$$\begin{aligned} A\mathbf{u}_0 &= \mathbf{0}, \\ D\mathbf{r}_0 &= \mathbf{0}. \end{aligned} \tag{6.2}$$

We shall assume that $\mathbf{u}_0$, $\mathbf{r}_0$ satisfy the boundary/initial conditions which coincide with those imposed on $\mathbf{u}$ and $\mathbf{r}$, respectively. From (5.3), (6.1), (6.2) we conclude that $\mathbf{u}_\Delta$, $\mathbf{r}_\Delta$ have to satisfy the equations

$$\begin{aligned} A\mathbf{u}_\Delta &= [\mathbb{C}]^T : \nabla\left(\mathbf{r}_0 + \mathbf{r}_\Delta\right), \\ D\mathbf{r}_\Delta &= F\left(\mathbf{u}_0 + \mathbf{u}_\Delta\right), \end{aligned} \tag{6.3}$$

as well as the corresponding homogeneous boundary/initial conditions. It has to be emphasized that Eq. $(6.2)_1$ represents the model obtained by the homogenization technique, which for periodic structures reduces to the form detailed in [2]. Equation $(6.2)_2$ describes the phenomena related to the fluctuations of boundary and initial displacements, which for periodic laminates were examined in [6]. Equations (6.2) will be referred to as *the first order approximation model* for slowly graded laminates with a weak transversal inhomogeneity. In the framework of this model the basic kinematic unknowns $\mathbf{u}$, $\mathbf{r}$ are approximated by $\mathbf{u}_0$, $\mathbf{r}_0$, respectively. In this case formula (6.1) yields

$$\mathbf{u} = \mathbf{u}_0 + O\left(\eta\right), \qquad \mathbf{r} = \mathbf{r}_0 + O\left(\eta\right)$$

i.e. we deal with an asymptotic approximation of order $O\left(\eta\right)$.

Now we assume that $\mathbf{u}_\Delta$, $\mathbf{r}_\Delta$ can be written in the form

$$\mathbf{u}_\Delta = \mathbf{u}_1 + o\left(\eta\right), \qquad \mathbf{r}_\Delta = \mathbf{r}_1 + o\left(\eta\right)$$

where $\mathbf{u}_1$, $\mathbf{r}_1$ are assumed to be linear functions of η. Applying the limit passage $\eta \to 0$ to Eqs. (6.3) we obtain the following system of equations for $\mathbf{u}_1$, $\mathbf{r}_1$:

$$\begin{aligned} A\mathbf{u}_1 &= [\mathbb{C}]^T : \nabla\mathbf{r}_0, \\ D\mathbf{r}_1 &= F\mathbf{u}_0. \end{aligned} \tag{6.4}$$

The above equations are assumed to hold together with homogeneous boundary and initial conditions. These conditions are assumed to have the same form as the pertinent homogeneous conditions for $\mathbf{u}_\Delta$, $\mathbf{r}_\Delta$, respectively. Equations (6.1) together with (6.3) will be referred to as *the second order approximation model*. In this case we deal with an asymptotic approximation of order $o\left(\eta\right)$ given by

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 + o\left(\eta\right), \qquad \mathbf{r} = \mathbf{r}_0 + \mathbf{r}_1 + o\left(\eta\right).$$

It can be seen that the right-hand sides of Eqs. (6.4) are known provided that the boundary/initial value problem for Eqs. (6.2) has been previously solved.

Summarizing the obtained results we state that model equations (5.3) for $\mathbf{u}$ and $\mathbf{r}$ can be decomposed to the simplified asymptotic form given by equations (6.2) and (6.4). It can be seen that the presented modelling line leads to the formulation of higher-order approximation models. These problems will be studied in a separate paper.

7. Illustrative example

The aim of this section is twofold. Firstly, we are to illustrate on a certain benchmark problem the general results obtained in Secs. 3–5. Secondly, for the above mentioned problem we are to compare the results obtained in the framework of model Eqs. (5.3) with those obtained by the first and the second order approximation models.

The object of our analysis is the plane strain problem of the infinite periodically laminated medium bonded by planes $x_2 = 0$, $x_2 = H$ normal to the interfaces between the laminae. A scheme of a fragment of the medium and a diagram of the shape function g are given in Fig. 2. The considerations will be restricted to the class of deformations of the form

$$\mathbf{w}\,(x_2, x_3, t) = \mathbf{u}\,(x_2, t) + g\,(x_3)\,\mathbf{v}\,(x_2, t)$$

where $u_1 = \mathrm{v}_1 = 0$.

For periodic laminates, Eqs. (4.1) yield

$$\begin{aligned}
\langle\rho\rangle\,\ddot{u}_3 - \langle C_{2323}\rangle\,\partial_2^2 u_3 - [C_{2323}]\,\partial_2 \mathrm{v}_2 &= 0,\\
l^2\,\langle\rho\rangle\,\ddot{\mathrm{v}}_2 - l^2\,\langle C_{2222}\rangle\,\partial_2^2 \mathrm{v}_2 + \{C_{2323}\}\,\mathrm{v}_2 + [C_{2323}]\,\partial_2 u_3 &= 0,\\
\langle\rho\rangle\,\ddot{u}_2 - \langle C_{2222}\rangle\,\partial_2^2 u_2 - [C_{2233}]\,\partial_2 \mathrm{v}_3 &= 0,\\
l^2\,\langle\rho\rangle\,\ddot{\mathrm{v}}_3 - l^2\,\langle C_{2323}\rangle\,\partial_2^2 \mathrm{v}_3 + \{C_{3333}\}\,\mathrm{v}_3 + [C_{2233}]\,\partial_2 u_2 &= 0.
\end{aligned}$$

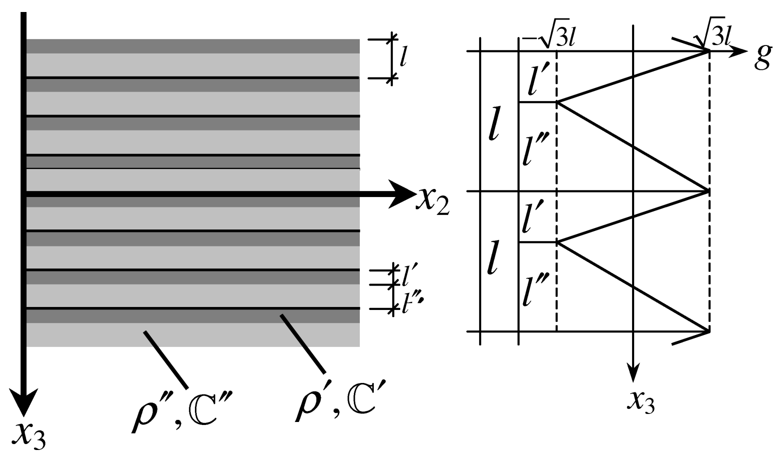

Fig. 2. A fragment of a periodic laminate and a shape function g.

It can be seen that for the above system of equations we can formulate two independent problems: the shear strain problem with unknowns u_3, v_2 and the longitudinal strain problem with u_2, $\mathtt{v}_3$. The subsequent analysis will be restricted to the shear strain problem. Denote $u = u_3$, $\mathtt{v} = \mathtt{v}_2$, $G = C_{2323}$, $E = C_{2222}$ and $(\cdot)' = \partial(\cdot)/\partial x$. Using these denotations, Eqs. (5.3) yield

$$\begin{aligned} &\langle\rho\rangle\,\ddot{u} - G^h u'' = [G]\,r', \\ (7.1)\qquad &l^2\left(\langle\rho\rangle\,\ddot{r} - \langle E\rangle\,r''\right) + \{G\}\,r = l^2\frac{[G]}{\{G\}}\left(\langle\rho\rangle\,\ddot{u}' - \langle E\rangle\,u'''\right). \end{aligned}$$

Let T be a certain time constant. Denoting $\xi = \dfrac{x}{H}$, $\tau = \dfrac{t}{T}$ and setting ul instead of u we arrive at the dimensionless form of Eqs. (7.1)

$$\begin{aligned} &\ddot{u} - \frac{G^h T^2}{\langle\rho\rangle\,H^2}u'' = \frac{[G]\,T^2}{Hl\,\langle\rho\rangle}r', \\ (7.2)\qquad &l^2\left(\frac{\langle\rho\rangle}{T^2}\ddot{r} - \frac{\langle E\rangle}{H^2}r''\right) + \{G\}\,r = l^3\frac{[G]}{\{G\}}\left(\frac{\langle\rho\rangle}{HT^2}\ddot{u}' - \frac{\langle E\rangle}{H^3}u'''\right). \end{aligned}$$

We shall investigate the problem of forced vibrations setting

$$u(\xi,\tau) = u(\xi)\cos\omega\tau, \qquad r(\xi,\tau) = r(\xi)\cos\omega\tau$$

with ω as a dimensionless vibration frequency. Hence from (7.2) we obtain

$$\begin{aligned} &u'' + \alpha^2\Omega^2 u = -\eta\beta r', \\ (7.3)\qquad &r'' - \kappa^2\left(1 - \Omega^2\right)r = \eta\gamma\Omega^2 u' + \eta\delta u''', \end{aligned}$$

where we have denoted

$$\alpha^2 = \frac{\{G\}\,H^2}{G^h l^2}, \qquad \eta = \frac{[G]}{\langle G\rangle}, \qquad \beta = \frac{\langle G\rangle\,H}{G^h l},$$

$$\kappa^2 = \frac{\{G\}\,H^2}{\langle E\rangle\,l^2}, \qquad \Omega = \frac{l}{H}\omega, \qquad \gamma = \frac{\langle G\rangle\,H}{\langle E\rangle\,l}, \qquad \delta = \frac{\langle G\rangle\,l}{\{G\}\,H}$$

and assumed that

$$\frac{H^2}{T^2} = \frac{\{G\}}{\langle\rho\rangle}.$$

Equations (7.3) have to be considered together with the boundary conditions in the form $u(0) = \overline{u}$, $u(1) = 0$, $r(0) = \overline{r}$, $r(1) = 0$. At the same time, from (6.2) we obtain

$$\begin{aligned} &u_0'' + \alpha^2\Omega^2 u_0 = 0, \\ (7.4)\qquad &r_0'' - \kappa^2\left(1 - \Omega^2\right)r_0 = 0, \end{aligned}$$

with the boundary conditions

$$u_0(0) = \overline{u}, \quad u_0(1) = 0,$$
$$r_0(0) = \overline{r}, \quad r_0(1) = 0,$$

and correspondingly, from (6.4)

$$\begin{aligned} &u_1'' + \alpha^2 \Omega^2 u_1 = -\eta \beta r_0', \\ &r_1'' - \kappa^2 \left(1 - \Omega^2\right) r_1 = \eta \gamma \Omega^2 u_0' + \eta \delta u_0''', \end{aligned} \tag{7.5}$$

with the homogeneous boundary conditions

$$u_1(0) = 0, \quad u_1(1) = 0,$$
$$r_1(0) = 0, \quad r_1(1) = 0.$$

Solutions to the boundary value problems (7.4) and (7.5) lead to the first and second order approximations of u, r given by $\widetilde{u}_1 = u_0$, $\widetilde{r}_1 = r_0$ and $\widetilde{u}_2 = u_0 + u_1$, $\widetilde{r}_2 = r_0 + r_1$, respectively. These results are numerically illustrated by diagrams. To this end we assume that

$$E = 3G, \quad l' = l'', \quad G' = 34/35G'', \quad H/l = 20, \quad \overline{u} = 1, \quad \overline{r} = 5.$$

Dimensionless vibration frequency Ω is taken as a parameter and argument $\xi \in [0.1001; 0.1002]$. Figures 3–8 represent exact solutions to the problem under considerations, together with the first and the second approximations. The plots in Figs. 3–8 are made for $\Omega = 0$, $\Omega = 0.1$, $\Omega = 1.1$. From the obtained diagrams it follows that for the stationary case and lower vibration frequencies, the averaged displacement u obtained in the framework of the second order model nearly coincides with that obtained from the coupled micro-macro model, whilst the first order model is unable to describe properly the problem under consideration. In the case of the intrinsic fluctuation amplitude r and lower vibration

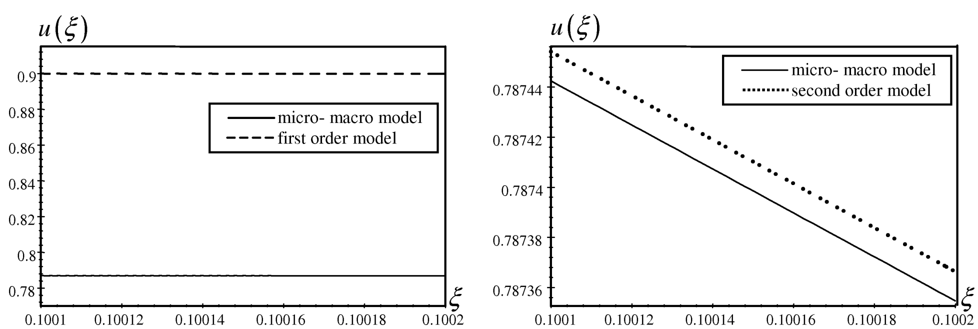

FIG. 3. The averaged displacement u; $\Omega = 0$.

frequencies, both the approximate models give satisfactory results. However, for higher vibration frequencies the asymptotic models may be not sufficient. It has to be underlined that the general considerations of this paper were carried out for the slowly graded laminates. The presented example has only an illustrative character and in order to simplify the calculations, we restricted ourselves to the periodic structure.

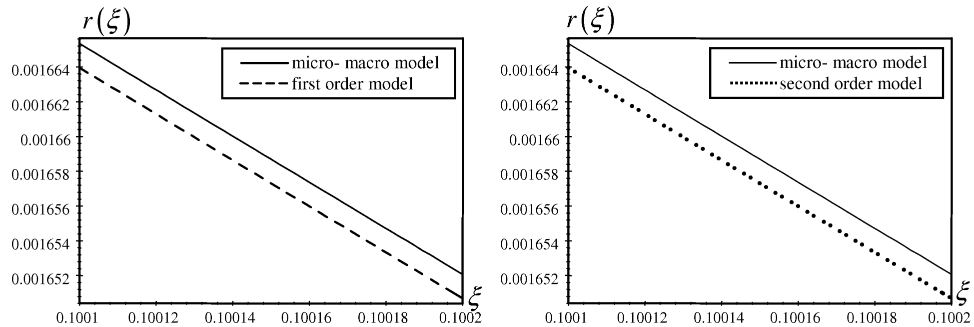

Fig. 4. The intrinsinc fluctuation amplitude r; $\Omega = 0$.

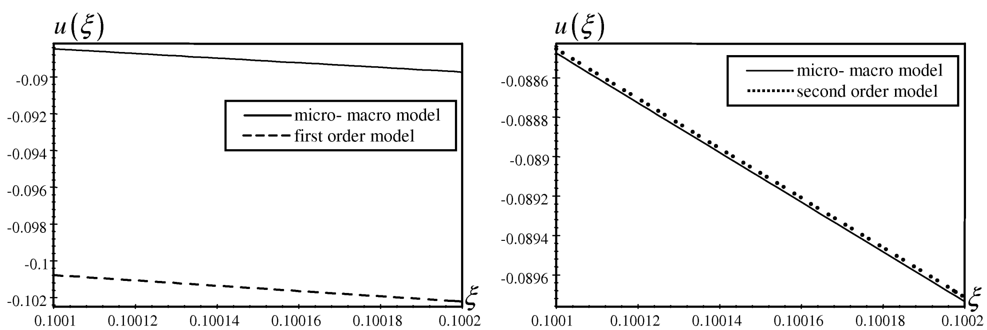

Fig. 5. The averaged displacement u; $\Omega = 0.1$.

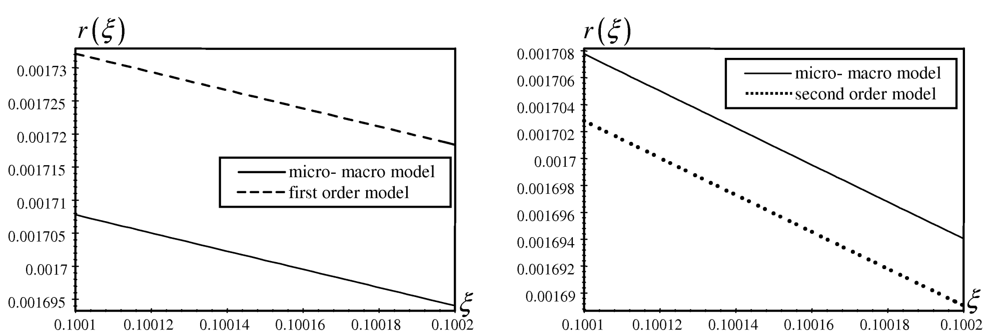

Fig. 6. The intrinsinc fluctuation amplitude r; $\Omega = 0.1$

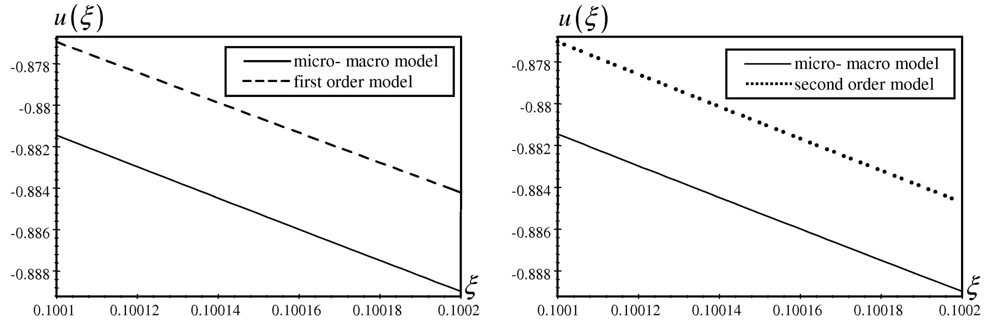

FIG. 7. The averaged displacement u; $\Omega = 1.1$.

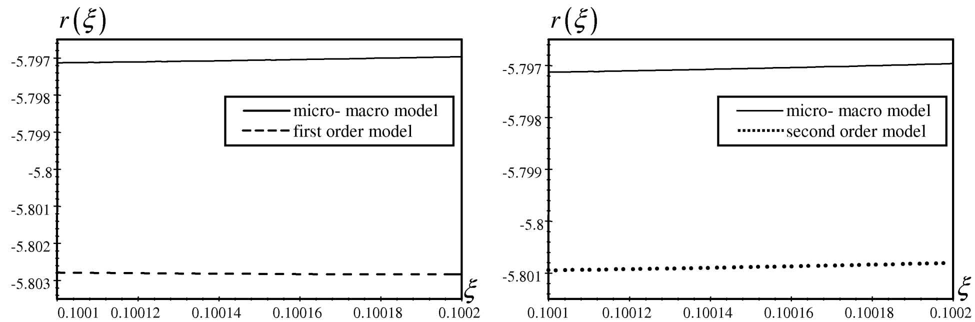

FIG. 8. The intrinsinc fluctuation amplitude r; $\Omega = 1.1$.

8. Conclusions

The following new results and information about SGL can be listed as follows:

1. The coupling of macrostructural and microstructural response for the two-phase linear elastic SGL was described in the framework of the proposed model equations. The form of these equations depends on the microstructure length parameter l. After neglecting the terms containing this parameter, the obtained model equations reduce to the equations of the locally homogenized laminated medium.

2. The obtained model equations have a rather simple form and involve only two independent kinematic vector fields: the macroscopic (averaged) displacement field and the intrinsic displacement fluctuation field. It is shown that in the framework of the homogenized model, the intrinsic fluctuation cannot be investigated.

3. The concept of laminates with a weak transversal inhomogeneity was introduced. This concept is based on the proposed definition of the transversal

inhomogeneity parameter η. This kind of inhomogeneity takes place in the case of laminae reinforced by long high-strength fibres.

4. It was shown that for laminates with a weak transversal inhomogeneity, the model equations (5.3) for $\mathbf{u}$ and $\mathbf{r}$ can be decomposed into successively independent Eqs. (6.2) and (6.4) for new unknowns $\mathbf{u}_0$, $\mathbf{r}_0$, $\mathbf{u}_1$, $\mathbf{r}_1$. Equations (6.2) for $\mathbf{u}_0$, $\mathbf{r}_0$ represent the first order approximation model. Equations (6.2) for $\mathbf{u}_0$, $\mathbf{r}_0$ together with Eqs. (6.4) for $\mathbf{u}_1$, $\mathbf{r}_1$ constitute the second order approximation model.

5. The general results have been illustrated by a simple benchmark problem of forced vibrations in a periodically laminated solid. Solutions obtained in the framework of the first and the second-order approximation models were numerically compared with the exact solution. It has been shown that for the averaged displacement and lower vibration frequencies the second order approximation model yields reliable results whilst the first-order approximation model is unable to describe properly the problem under consideration. On the other hand, for higher vibration frequencies both the first and the second order approximations may be not sufficient. In this case we have to apply the coupled macro-micro equations.

Possible applications of the proposed modeling approach to some engineering problems of functionally graded laminates will be presented in a separate paper.

References

1. J. Aboudi, M.-J. Pindera, S. M. Arnold, *Higher-order theory for functionally graded materials*, Composites: Part B, **30**, 777–832, 1999.
2. N. C. Bakhvalov, G. P. Panasenko, *Averaging processes in periodic media* [in Russian], Nauka, Moscow 1984.
3. Ł. Laciński, *Numerical verification of two mathematical models for the heat transfer in a laminated rigid conductor*, J. Theor. Appl. Mech., **2**, 43, 367–384, 2005.
4. T. Reiter, G. J. Dvorak, V. Tvergaard, *Micromechanical models for graded composite materials*, J. Mech. Phys. Solids, **45**, 1281–1302, 1997.
5. S. Suresh, A. Mortensen, *Fundamentals of functionally graded materials*, The University Press, Cambridge 1998.
6. E. Wierzbicki, Cz. Woźniak, L. Łacińska, *Boundary and initial fluctuation effect on dynamic behavior of a laminated solid*, Arch. Appl. Mech., **74**, 618–628, 2005.
7. Cz. Woźniak, E. Wierzbicki, *Averaging techniques in thermomechanics of composite solids*, Wyd. Politechniki Częstochowskiej, Częstochowa 2000.

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