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Boundary layer phenomena in elastodynamics of functionally graded laminates

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In Memory of Professor Henryk Zorski

THE OBJECT OF CONSIDERATIONS are micro-laminated two-phase solids having macroscopic properties continuously varying in the direction normal to the layering (functionally graded laminates, FGL). The main question posed in this contribution is how to investigate both micro- and macro- dynamic response of the linear-elastic FGL. To answer this question, a new approximate mathematical model of FGL is proposed. This model is represented by a certain refined homogenized equation of motion for averaged displacements and a boundary-layer equation for intrinsic displacement fluctuations. Main attention is given to the investigations of near-initial and near-boundary dynamic phenomena in FGL, which are related to the specific form of initial and boundary conditions. For a periodically laminated medium the obtained results reduce to those derived in [7].

1. Introduction

THE OBJECT OF CONSIDERATIONS is the dynamic behavior of multilayered laminated media made of two linear-elastic components. Every layer has the constant thickness l and consists of two homogeneous laminae. Moreover, thicknesses of laminae in different layers are different, so that the apparent (effective) material characteristics of the laminate can be treated on a macroscopic level as continuously varying in the direction normal to the layering. Under the above condition, the laminated solid will be referred to as the functionally graded laminate (FGL). Obviously, FGL represents a special case of a functionally graded material (FGM), cf. [5] and the extensive list of references therein. Fragments of two FGL solids from the macroscopic point of view are shown in Fig. 1.

It is known that due to a large number of interfaces between laminae, investigations of both periodic and functionally graded laminates on a microscopic level can lead to ill-conditioned and complicated computational problems (cf. [2], p.VII). Hence the first aim of this contribution is to formulate a certain

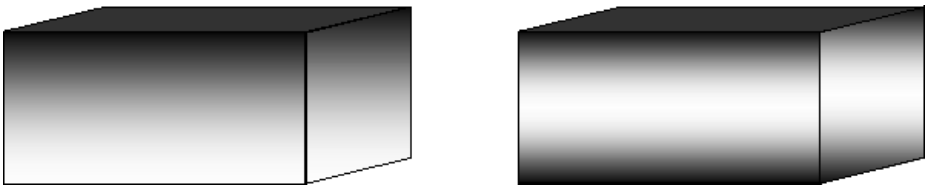

FIG. 1. Fragments of two FGL solids from the macroscopic point of view with different distributions of the effective material characteristics.

averaged mathematical model for the analysis of both macro- and micro-dynamic behavior of FGL. Here and subsequently, terms "macro" and "micro" are related to the overall behavior of FGL and to the behavior of its arbitrary single two-component layer, respectively. In contrast to the known asymptotic models, cf. [2–4], the modelling technique proposed in the contribution is rather simple and can be formulated on different levels of accuracy.

The second aim of the contribution is to apply the obtained mathematical model of FGL to the microvibration analysis. It will be shown that if vibration frequencies are not very high then the displacement fluctuations, caused by the heterogeneity of FGL, strongly decay in a certain near-boundary layer. This situation does not take place for higher vibration frequencies where we can also deal with a certain intrinsic resonance of displacement fluctuations. This problem for periodic laminates was studied in [7].

Considerations will be carried out in the framework of the linear elasticity theory. Moreover, lamina materials are assumed to have elastic symmetry planes parallel to the lamina interfaces. We also assume the perfect bonding between adjacent laminae.

Denotations. By $0x_1x_2x_3$ we denote the Carthesian orthogonal coordinate system in the physical space. Let $\Pi \times (-L, L)$, $\Pi \subset R^2$, be the region in this space occupied by the laminated solid in the reference configuration in which the x_3-axis is normal to the lamina interfaces. We denote $\mathbf{e} \equiv (0, 0, 1)$, $\mathbf{x} \equiv (x_1, x_2)$, $z \equiv x_3$ and t stands for the time coordinate. The partial differentiation with respect to arguments x_k, $k = 1, 2, 3$, is denoted by ∂_k and time differentiation by the overdot. We introduce gradient operators $\nabla = (\partial_1, \partial_2, \partial_3)$ and $\overline{\nabla} = (\partial_1, \partial_2, 0)$. Throughout Secs. 2 and 3 the tensor notation is used with "dot" and "double dot" as the scalar and the double scalar products, respectively. In Sec. 4 we apply the index notation where subscript k is related to the coordinate x_k, $k = 1, 2, 3$.

We assume that FGL is divided into $2m$ layers along its thickness $2L$. The thickness of every layer is the same and will be denoted by l. Hence $L = ml$, where m has to satisfy condition $m^{-1} \ll 1$, cf. Fig. 2. Thicknesses of lamina in the n-th

layer, $n = -m, ..., -1, 1, ..., m$, are denoted by l'_n, l''_n. A cross-section of the FGL solid and a fragment of its n-th layer are shown in Fig. 2, where ρ', ρ'' and $\mathbb{C}'$, $\mathbb{C}''$ stand for mass densities and tensors of elastic moduli in every pair of adjacent laminae, respectively. By $\nu'(\cdot)$, $\nu''(\cdot)$ we denote smooth functions defined on $[-L,\ L]$ representing distributions of mean volume fractions of lamina materials, $\nu'(z) + \nu''(z) = 1$, $z \in [-L, L]$. It means that $\nu'(z_n) = l'_n/l$ and $\nu''(z_n) = l''_n/l$ either for some $z_n \in [(n-1)\,l, nl]$, $n = 1, ..., m$ or for some $z_n \in [nl, (n+1)\,l]$, $n = -m, ..., -1$. Diagrams of mean volume fractions ν', ν'' related to two solids are shown in Fig. 3. We introduce also function $\nu(\cdot)$ defined on $[-L, L]$ in the form $\nu = \sqrt{\nu'\nu''}$ which will be called phase distribution function.

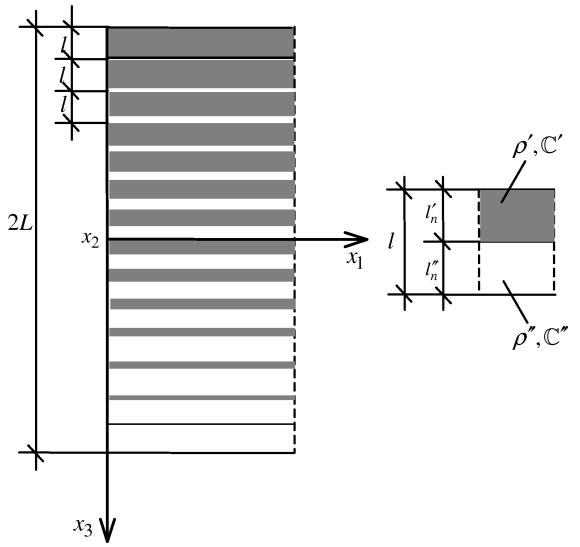

FIG. 2. A cross-section of the FGL solid and a fragment of its n-th layer.

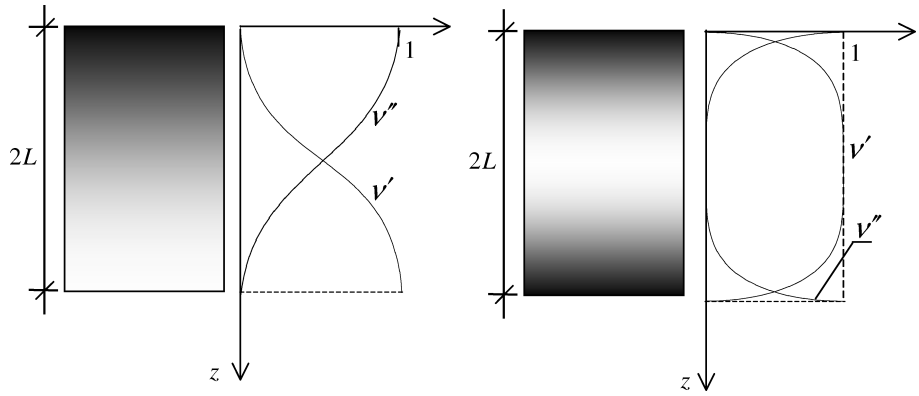

FIG. 3. Diagrams of mean volume fractions ν', ν'' of lamina materials for two FGL solids.

For an arbitrary integrable function f defined in $(-L, L)$ (f can also depend on $\mathbf{x}$ and t) *the averaging* of this function is denoted by

$$\langle f \rangle (z) = \frac{1}{l} \int_{z-l/2}^{z+l/2} f(\zeta)\, d\zeta.$$

Vectors and vector fields are denoted by small boldface letters, second-order tensors and tensor fields by capital boldface letters and higher-order tensors and tensor fields by block letters.

2. Modelling technique

The modelling question that we are going to answer in this section is how to describe dynamic behavior of the linear-elastic FGL by means of PDEs with smooth functional coefficients. Moreover, the proposed model has to describe this behavior also on the microstructural level by taking into account the effect of lamina thickness on the averaged characteristics of FGL. At last, the model equations should have a relatively simple form which makes it possible to obtain also analytical solutions to selected benchmark problems.

So far, modelling techniques for laminated media were restricted mainly to periodic laminates and two main lines of modelling can be mentioned. The first one is based on the asymptotic techniques, cf. [2, 3], which even for relatively simple micro-dynamic problems leads to rather lengthy computations, [4]. The second line of modelling takes into account certain heuristic assumptions; we can mention here the effective stiffness theories, [1], for the analysis of wave propagation problems and various applications of the tolerance averaging technique. The review of recent papers on this subject can be found in [6].

In this contribution we deal with non-periodic laminates and the proposed modelling technique will be based on a certain generalization of the procedure discussed in [6]. To this end we shall introduce two important notions.

Function $F \in C^1([-L, L])$ of argument $x_3 = z$ (F can also depend on $\mathbf{x}$ and t) will be called *slowly varying* (with respect to length l, $l \ll L$, and within tolerance ε assigned to F, $0 < \varepsilon \ll 1$) if functions $l\partial_3 F$ and εF are of the same order $O(\varepsilon)$ related to ε. If this condition holds also for all derivatives of F which occur in the problem under consideration then we shall write $F \in SV_\varepsilon(l)$, where ε is called a tolerance parameter. For a detailed discussion of this concept the reader is referred to [6].

By $g : [-L, L] \to R$ we denote a continuous function the diagram of which in every n-th interval $[(n-1)\, l, nl]$, $n = 1, ..., m$, is shown in Fig. 4. In intervals $[nl, (n+1)\, l]$, $n = -1, ..., -m$ the form of function $g(\cdot)$ is similar to that given

by Fig. 4. Function $g(\cdot)$ will be referred to as *the fluctuation shape function* and represents a certain generalization of the saw-like function, well known in modelling of periodic laminates, [6].

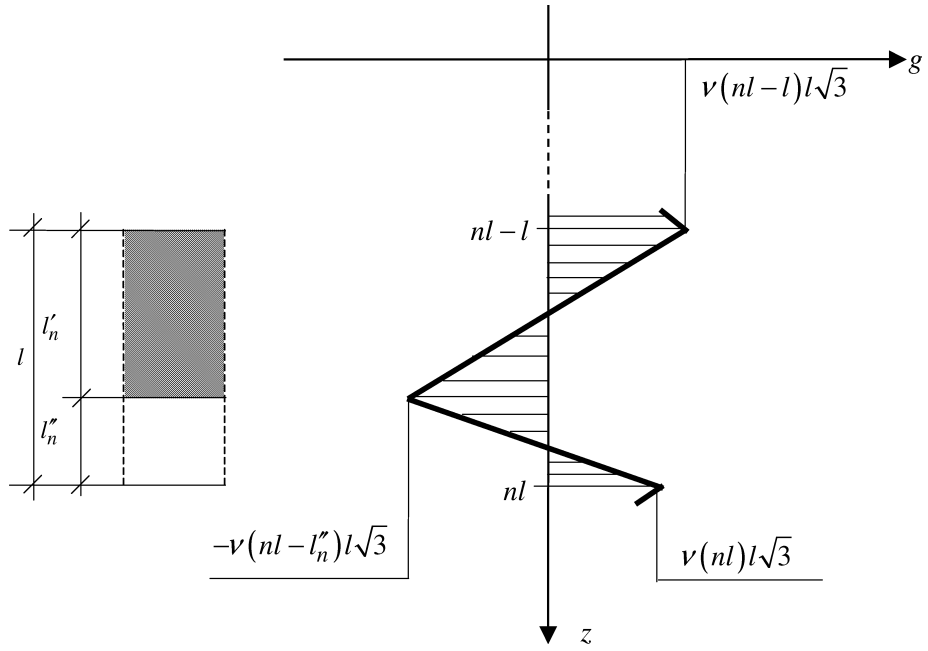

FIG. 4. A diagram of the fluctuation shape function in the n-th layer.

We begin the modelling procedure with *the statement that in every FGL mean volume fractions are slowly varying*, i. e. they are restricted by conditions $\nu'(\cdot) \in SV_\varepsilon(l)$, $\nu''(\cdot) \in SV_\varepsilon(l)$. We also assume that for every slowly varying function F, $F \in SV_\varepsilon(l)$, terms $l\partial_3 F$ which are of an order $O(\varepsilon F)$ can be neglected as small when compared to F. This assumption is referred to as *the tolerance approximation*.

Let us denote by $\mathbf{w}(\mathbf{x}, z, t)$, $\mathbf{x} = (x_1, x_2) \in \overline{\Pi}$, $z \in [-L, L]$ the displacement field at time t. The subsequent analysis will be restricted to problems in which *distributions of displacements* $\mathbf{w}(\mathbf{x}, \cdot, t)$ *across the thickness of every lamina can be approximated* (within a certain tolerance ε) *by linear functions*. Using the notion of the fluctuation shape function and that of the slowly-varying function, we conclude that the aforementioned restriction can be assumed in the form of the decomposition

$$\mathbf{w}(\mathbf{x}, z, t) = \mathbf{u}(\mathbf{x}, z, t) + g(z)\,\mathbf{v}(\mathbf{x}, z, t) \tag{2.1}$$

where $\mathbf{u}$, $\mathbf{v}$ are slowly varying functions of argument z:

$$\mathbf{u}(\mathbf{x},\cdot,t) \in SV_{\varepsilon}(l), \qquad \mathbf{v}(\mathbf{x},\cdot,t) \in SV_{\varepsilon}(l). \tag{2.2}$$

Applying the tolerance approximation we also obtain

$$\mathbf{u}(\mathbf{x},z,t) = \langle \mathbf{w} \rangle (\mathbf{x},z,t)$$

for every $(\mathbf{x},z) \in \Pi \times [-L+l/2, L-l/2]$ and every time t. It follows that $\mathbf{u}$ is *the averaged displacement* and $g\mathbf{v}$ represent fluctuations of displacements. That is why function $\mathbf{v}$ will be referred to as *the fluctuation amplitude*. Subsequently we are going to show that the above displacement fluctuations are caused by the heterogeneous structure of the solid under consideration.

Conditions (2.1), (2.2) constitute *the kinematic assumption* which introduces averaged displacement $\mathbf{u}$ and fluctuation amplitude $\mathbf{v}$ as the basic kinematic unknowns.

The governing equations for kinematic unknowns will be derived from the principle of stationary action. The Lagrange function is assumed in the form

$$\mathcal{L} = \frac{1}{2}\langle \rho \dot{\mathbf{w}} \cdot \dot{\mathbf{w}} \rangle - \frac{1}{2}\langle \nabla \mathbf{w} : \mathbb{C} : \nabla \mathbf{w} \rangle,$$

where displacement field $\mathbf{w}$ is restricted by conditions (2.1), (2.2). Using the tolerance approximation and recalling that $\mathbf{e} = (0,0,1)$, we shall approximate $\nabla \mathbf{w}$ by $\nabla \mathbf{u} + g'(z)\,\mathbf{e} \otimes \mathbf{v} + g(z)\,\overline{\nabla}\mathbf{v}$. Similarly we conclude that

$$\begin{aligned} \langle \rho \rangle &= \nu'(z)\,\rho' + \nu''(z)\,\rho'', \\ \langle \mathbb{C} \rangle &= \nu'(z)\,\mathbb{C}' + \nu''(z)\,\mathbb{C}''. \end{aligned} \tag{2.3}$$

We shall also introduce denotations

$$\begin{aligned} [\mathbb{C}] &\equiv 2\sqrt{3}\nu(z)\left(\mathbb{C}' - \mathbb{C}''\right)\cdot \mathbf{e}, \\ [\mathbb{C}]^T &\equiv 2\sqrt{3}\nu(z)\,\mathbf{e}\cdot\left(\mathbb{C}' - \mathbb{C}''\right), \\ \{\mathbf{C}\} &\equiv 12\mathbf{e}\cdot\left(\mathbb{C}'\nu''(z) + \mathbb{C}''\nu'(z)\right)\cdot \mathbf{e}. \end{aligned} \tag{2.4}$$

After rather lengthy manipulations, the Euler–Lagrange equations for $\mathcal{L}$ lead to the following *equations of motion*

$$\begin{aligned} \langle \rho \rangle \ddot{\mathbf{u}} - \nabla \cdot \mathbf{S} &= \mathbf{0}, \\ l^2\nu^2 \langle \rho \rangle \ddot{\mathbf{v}} - l^2\nu^2 \overline{\nabla} \cdot \left(\langle \mathbb{C} \rangle : \overline{\nabla}\mathbf{v}\right) + \mathbf{h} &= \mathbf{0} \end{aligned} \tag{2.5}$$

and *constitutive equations*

$$\begin{aligned} \mathbf{S} &= \langle \mathbb{C} \rangle : \nabla \mathbf{u} + [\mathbb{C}] \cdot \mathbf{v}, \\ \mathbf{h} &= \{\mathbf{C}\} \cdot \mathbf{v} + [\mathbb{C}]^T : \nabla \mathbf{u}. \end{aligned} \tag{2.6}$$

Equations (2.5), (2.6) with coefficients defined by (2.3), (2.4) and formulae (2.1), (2.2) represent an averaged mathematical model of the FGL solid under consideration. The basic unknowns are: averaged displacement $\mathbf{u}$ and fluctuation amplitude $\mathbf{v}$. Equations (2.5), (2.6) have to be considered together with the appropriate initial and boundary conditions for $\mathbf{u}$ and $\mathbf{v}$. It can be seen that for a homogeneous solid and under homogeneous initial-boundary conditions for $\mathbf{v}$, the model equations reduce to the well-known equations of the linear elasticity theory where $\mathbf{v} \equiv \mathbf{0}$.

The main characteristic features of the model equations (2.5), (2.6) are:

1° functional coefficients in these equations are slowly varying smooth functions of argument z;

2° the model equations depend on the microstructure size l;

3° solutions to model equations have a physical sense only if $\mathbf{u}$, $\mathbf{v}$ together with their derivatives are slowly varying functions of argument z;

4° the model equations are deprived of the second derivatives of $\mathbf{v}$ with respect to x_3 and hence the boundary conditions for $\mathbf{v}$ have to be formulated only on $\partial\Pi \times (-L, L)$.

Let us observe that after the formal limit passage $l \to 0$, from (2.5), (2.6) we obtain

$$\mathbf{v} = -\{\mathbf{C}\}^{-1} \cdot [\mathbb{C}]^T : \nabla \mathbf{u} \tag{2.7}$$

and after introducing the following tensor of effective elastic moduli

$$\mathbb{C}^h \equiv \langle \mathbb{C} \rangle - [\mathbb{C}] \cdot \{\mathbf{C}\}^{-1} \cdot [\mathbb{C}]^T \tag{2.8}$$

we obtain equation

$$\langle \rho \rangle \, \ddot{\mathbf{u}} - \nabla \cdot \Big(\mathbb{C}^h : \nabla \mathbf{u} \Big) = \mathbf{0}. \tag{2.9}$$

The above equations together with formulae (2.1), (2.2) where $\mathbf{v}$ is defined by (2.7), represent what will be called *the locally homogenized model* of the linear-elastic FGL.

Let us observe that for a periodically laminated solid, the coefficients in Eqs. (2.5)–(2.9) are constant. In this case we deal with equations derived and discussed in [6].

3. Boundary-layer equation

Following the approach proposed in [7] let us decompose the fluctuation amplitude $\mathbf{v}$ in Eqs. (2.5), (2.6) into the sum

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{r} + \mathbf{r}_*, \tag{3.1}$$

where $\mathbf{v}_0$ is defined by

$$\mathbf{v}_0 = -\left\{\mathbf{C}\right\}^{-1} \cdot \left[\mathbb{C}\right]^T : \nabla \mathbf{u} \tag{3.2}$$

and $\mathbf{r}$ is assumed to satisfy equation

$$l^2\nu^2 \left\langle \rho \right\rangle \ddot{\mathbf{r}} - l^2\nu^2 \overline{\nabla} \cdot \left(\left\langle \mathbb{C} \right\rangle : \overline{\nabla} \mathbf{r} \right) + \left\{\mathbf{C}\right\} \mathbf{r} = \mathbf{0} \tag{3.3}$$

together with initial conditions and boundary condition on $\partial \Pi \times (-L, L)$, which coincide with the boundary conditions imposed on $\mathbf{v}$. Substituting the right-hand side of (3.1) into Eqs. (2.5), (2.6) we conclude that

$$\left\langle \rho \right\rangle \ddot{\mathbf{u}} - \nabla \cdot \left(\mathbb{C}^h : \nabla \mathbf{u} \right) = \left[\mathbb{C}\right] : \nabla \left(\mathbf{r} + \mathbf{r}_* \right), \tag{3.4}$$

where $\mathbf{r}_*$ has to satisfy the equation

$$\begin{aligned} &l^2\nu^2 \left\langle \rho \right\rangle \ddot{\mathbf{r}}_* - l^2\nu^2 \overline{\nabla} \cdot \left(\left\langle \mathbb{C} \right\rangle : \overline{\nabla} \mathbf{r}_* \right) + \left\{\mathbf{C}\right\} \mathbf{r}_* \\ &\quad = l^2\nu^2 \left[\left\langle \rho \right\rangle \left\{\mathbf{C}\right\}^{-1} \cdot \left[\mathbb{C}\right]^T : \nabla \ddot{\mathbf{u}} - \overline{\nabla} \cdot \left(\left\langle \mathbb{C} \right\rangle : \overline{\nabla} \cdot \left(\left\{\mathbf{C}\right\}^{-1} \cdot \left[\mathbb{C}\right]^T : \nabla \mathbf{u} \right) \right) \right] \end{aligned} \tag{3.5}$$

together with the homogeneous initial conditions and homogeneous boundary conditions on $\partial \Pi \times (-L, L)$. The aforementioned requirement implies that $\mathbf{r}_* \in O\left(l^2\right)$. On the other hand we shall assume that in the general case, the pertinent initial and boundary conditions for $\mathbf{r}$ can be not homogeneous and hence their values for $t = 0$ and on $\partial \Pi \times (-L, L)$ are independent of l. In this case we shall introduce the extra approximation neglecting in (3.1) term $\mathbf{r}_*$ as small when compared to $\mathbf{r}$. Under the above approximation we obtain from (3.4) the simplified equation

$$\left\langle \rho \right\rangle \ddot{\mathbf{u}} - \nabla \cdot \left(\mathbb{C}^h : \nabla \mathbf{u} \right) = \left[\mathbb{C}\right] : \nabla \mathbf{r}, \tag{3.6}$$

which will be referred to as *the locally homogenized equation of motion*. Thus we jump to the conclusion that the system of Eqs. (2.5), (2.6) can be approximated by Eqs. (3.3) and (3.6). At the same time using (2.1), (3.1), (3.2) and neglecting in (3.1) term $\mathbf{r}_*$ as small when compared to $\mathbf{r}$, we obtain the formula

$$\mathbf{w}\left(\mathbf{x}, z, t\right) = \mathbf{u}\left(\mathbf{x}, z, t\right) - g\left(z\right) \left\{\mathbf{C}\right\}^{-1} \cdot \left[\mathbb{C}\right]^T : \nabla \mathbf{u}\left(\mathbf{x}, z, t\right) + g\left(z\right) \mathbf{r}\left(\mathbf{x}, z, t\right), \tag{3.7}$$

which determine distribution of displacements in terms of $\mathbf{u}$ and $\mathbf{r}$.

Equations (3.3) and (3.6) together with formula (3.7) constitute the proposed simplified mathematical model for the analysis of elastodynamic response of the functionally graded laminate. For periodically laminated media this model reduces to the form which was derived independently in [7]. Let us also observe that applying the formal asymptotic approximation $l \to 0$ to Eq. (3.3) we obtain $\mathbf{r} = \mathbf{0}$. In this case, for a periodically laminated medium, Eq. (3.6) coincides with the well-known homogenized equation where $\mathbb{C}^h$ is the effective tensor of elastic modulae, [6]. It follows that the characteristic feature of Eq. (3.3) is that it describes the effect of the period length on the behavior of the functionally graded laminates. In the subsequent section it will be shown that Eq. (3.3) also describes certain near-initial and near-boundary phenomena strictly related to the initial and boundary conditions (on the part of boundary intersecting interfaces between laminae) imposed on $\mathbf{r}$. That is why Eq. (3.3) will be referred to as *the boundary-layer equation* where the term "boundary" is related both to time and space. Following [7], unknown $\mathbf{r}$ will be called the intrinsic displacement amplitude.

4. Application to the microvibration analysis

For the sake of simplicity let us restrict consideration to the plane problem setting $\mathbf{u} = \mathbf{u}(x_2, x_3, t)$, $\mathbf{r} = \mathbf{r}(x_2, x_3, t)$, $x_2 \in [0, H]$, $x_3 \in [-L, L]$. Let us recall that material planes $x_3 = \text{const}$ are elastic symmetry planes, cf. Fig. 1. In this case shear problems, in which $u_2 = u_2(x_2, x_3, t)$, $r_2 = r_2(x_2, x_3, t)$ are the basic unknowns, are described by separate equations. After denotations $G \equiv C_{2323}$, $E \equiv C_{2222}$ and

$$[G] = 2\sqrt{3}\left(G' - G''\right)\nu(x_3),$$

$$\{G\} = 12\left(G'\nu''(x_3) + G''\nu'(x_3)\right),$$

$$G^h = \frac{G'G''}{G'\nu''(x_3) + G''\nu'(x_3)},$$

Eqs. (3.3), (3.6) for unknowns u_2, r_2 reduce to the form

$$\begin{gathered} \langle\rho\rangle\,\ddot{u}_2 - \partial_3\left(G^h\partial_3 u_2\right) = [G]\,\partial_3 r_2, \\ l^2\nu^2\langle\rho\rangle\,\ddot{r}_2 - l^2\nu^2\langle E\rangle\,\partial_2\partial_2 r_2 + \{G\}\,r_2 = 0. \end{gathered} \tag{4.1}$$

Let us observe that $r_2 = r_2(x_2, x_3, t)$ in Eqs. (4.1) depends on x_3 as a parameter. Equation $(4.1)_1$ has the well-known form of equation of motion for the linear-elastic heterogeneous solid with volume forces $-[G]\,\partial_3 r_2$. That is why we shall restrict subsequent analysis to Eq. $(4.1)_2$. Let us transform this equation

to the dimensionless form by introducing time constant T and dimensionless arguments

$$\eta = \frac{x_2}{H}, \qquad \xi = \frac{x_3}{L}, \qquad \tau = \frac{t}{T}.$$

Let us also define

$$\varphi(\eta, \xi, \tau) \equiv r_2(H\eta, L\xi, T\tau)$$

where $\eta \in [0,1]$, $\xi \in [-1,1]$ and introduce dimensionless parameters

$$\gamma \equiv \frac{H}{\nu l}, \qquad \kappa \equiv \frac{\{G\}}{\langle E \rangle}.$$

Under the assumption that

$$\frac{H^2}{T^2} = \frac{\{G\}}{\langle \rho \rangle}$$

and using the above denotations, the intrinsic fluctuation equation $(4.1)_2$ takes the form

$$\frac{\partial^2 \varphi}{\partial \tau^2} - \frac{1}{\kappa}\frac{\partial^2 \varphi}{\partial \eta^2} + \gamma^2 \varphi = 0, \tag{4.2}$$

where $\varphi = \varphi(\eta, \xi, \tau)$, $\eta \in [0,1]$ and where $\xi \in [-1,1]$ is a parameter. We shall investigate the problem of harmonic microvibrations by setting

$$\varphi = \Phi(\eta, \xi)\cos\omega\tau$$

with ω as a dimensionless vibration frequency. Hence

$$\frac{\partial^2 \Phi}{\partial \eta^2} - \kappa\left(\gamma^2 - \omega^2\right)\Phi = 0.$$

Let the boundary conditions for Φ have the form $\Phi(1,\xi) = 0$, $\Phi(0,\xi) = f(\xi)$. It can be shown that $f(\cdot)$ is a slowly varying function, $f \in SV_\varepsilon(l/L)$. The following special cases of microvibrations take place:

1. If $\omega = 0$ then we deal with a stationary problem and after setting $k^2 \equiv \kappa\gamma^2$ we obtain

$$\varphi(\eta, \xi, \tau) = f(\xi)\exp(-k\eta).$$

Because of $k \gg 1$ we deal here with strongly decaying intrinsic fluctuations.

2. If $0 < \omega \ll \gamma^2$ then under notation

$$k^2(\omega) \equiv \kappa\left(\gamma^2 - \omega^2\right)$$

we obtain

$$\varphi(\eta, \xi, \tau) = f(\xi)\exp(-k(\omega)\eta)\cos\omega\tau.$$

In this case microvibrations are strongly decaying.
3. If $0 \ll \omega < \gamma$ then

$$\varphi(\eta,\xi,\tau) = f(\xi)\left[\frac{\exp(-k(\omega)\eta)}{1-\exp(-2k(\omega))} + \frac{\exp(k(\omega)\eta)}{1-\exp(2k(\omega))}\right]\cos\omega\tau$$

and microvibrations are not decaying.
4. If $\omega = \gamma$ then

$$\varphi(\eta,\xi,\tau) = f(\xi)(1-\eta)\cos\omega\tau$$

and we deal with the linear distribution of microvibrations.
5. If $\omega > \gamma$ and

$$\lambda^2 \equiv \kappa\left(\omega^2-\gamma^2\right) \neq n^2\pi^2, \qquad n = 1,2,...$$

then

$$\varphi(\eta,\xi,\tau) = \frac{f(\xi)}{\sin\lambda}\sin\lambda(1-\eta)\cos\omega\tau.$$

We deal here with higher microvibration frequencies.
6. If $\omega > \gamma$ and

$$\lambda^2 \equiv \kappa\left(\omega^2-\gamma^2\right) = n^2\pi^2$$

then we obtain resonance microvibrations with resonance frequencies

$$\omega_n^2 = \frac{n^2\pi^2}{\kappa} + \gamma^2, \qquad n = 1,2,...$$

From the above results it follows that if microvibrations are strongly decaying then the values of function $\varphi(\cdot,\xi,\tau)$, $\xi \in [-1,1]$, for ξ not very small when compared to 1, can be neglected. It means that the intrinsic fluctuations $\mathbf{r}$ can be treated as equal to zero outside a certain narrow layer near boundary $x_2 = 0$. Outside this layer we can assume that $r_2 = 0$ and Eqs. (4.1) reduce to the asymptotic model equation.

5. Numerical results

In this section we present some numerical results. To this end we introduce dimensionless coefficients

$$\Gamma^2 = \kappa\gamma^2, \qquad \Omega^2 = \frac{\omega^2}{\gamma^2}.$$

It can be seen that the character of intrinsic microvibrations depends on the dimensionless vibration frequency Ω. Numerical results describing the boundary-value problem are computed for $\Gamma = 20$ and for vibration frequency $\Omega = 0,\ 0.01,$

0.99, 1, 1.1 in Fig. 5ab, 6ab, 7ab, 8ab, 9ab, respectively. It can be observed that if $0 \leq \Omega \ll 1$ then displacement fluctuations are not strongly decaying in a certain near-boundary layer (Fig. 5ab, 6ab). For $0 \ll \Omega < 1$ displacement fluctuations are not decaying so strongly (Fig. 7ab). If $\Omega \geq 1$ then fluctuation propagates into the layer $0 < x_2 < H$. For $\Omega = 1$ we obtain linear distribution of microvibrations (Fig. 8ab). It has to mentioned that for $0 \leq \Omega \ll 1$ intrinsic fluctuations decay so rapidly that differences between diagrams of the dimensionless fluctuation variable are not noticeable (Fig. 5ab, 6ab).

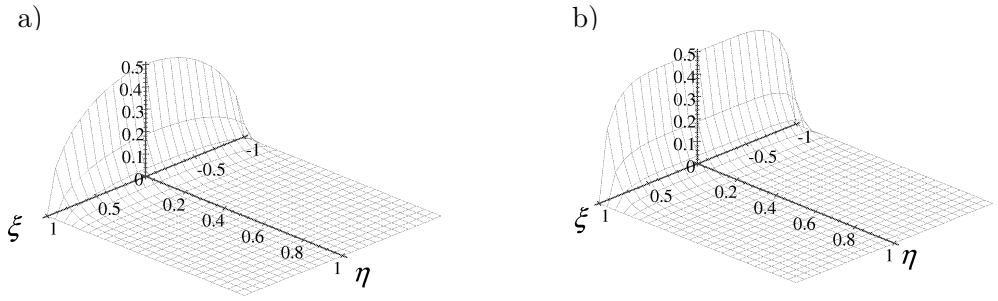

FIG. 5. Diagrams of the dimensionless intrinsic microvibration amplitude for $\Omega = 0$, $\Gamma = 20$ and $f(\xi) = 0.5\sqrt{1-\xi^2}$ (Fig. 5a), $f(\xi) = 0.5\sqrt{1-\xi^6}$ (Fig. 5b).

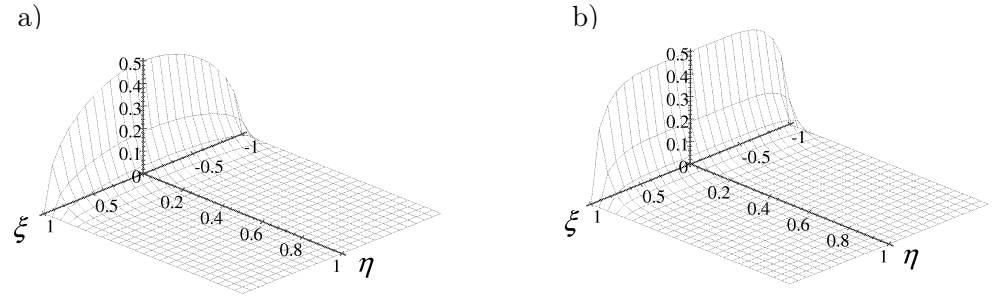

FIG. 6. Diagrams of the dimensionless intrinsic microvibration amplitude for $\Omega = 0.01$, $\Gamma = 20$ and $f(\xi) = 0.5\sqrt{1-\xi^2}$ (Fig. 6a), $f(\xi) = 0.5\sqrt{1-\xi^6}$ (Fig. 6b).

In contrast to the periodic laminated medium, [7], in FGL a very important role plays the function $f(\xi)$ occurring in the boundary conditions. This function was assumed in the form $f(\xi) = 0.5\sqrt{1-\xi^2}$ (Fig. 5a–9a) or $f(\xi) = 0.5\sqrt{1-\xi^6}$ (Fig. 5b–9b). It can be observed that the choice of function $f(\xi)$ has an important influence on the character of intrinsic microvibrations. It has to be emphasized that f has to be a slowly varying function, $f \in SV_\varepsilon(l/L)$.

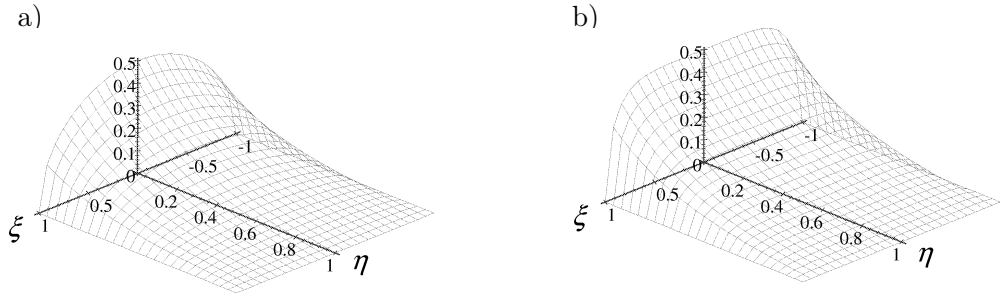

FIG. 7. Diagrams of the dimensionless intrinsic microvibration amplitude for $\Omega = 0.99$, $\Gamma = 20$ and $f(\xi) = 0.5\sqrt{1-\xi^2}$ (Fig. 7a), $f(\xi) = 0.5\sqrt{1-\xi^6}$ (Fig. 7b) .

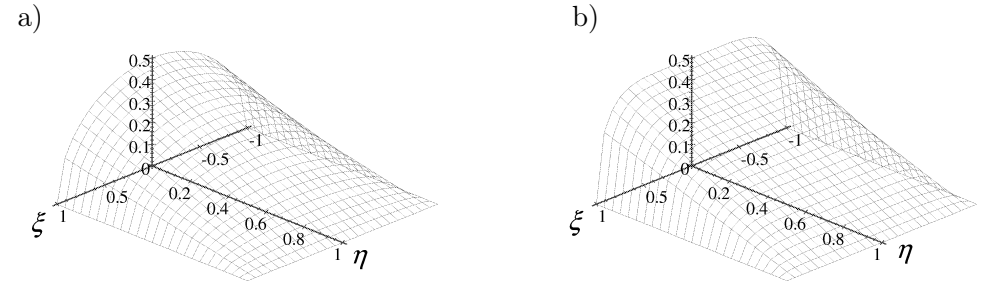

FIG. 8. Diagrams of the dimensionless intrinsic microvibration amplitude for $\Omega = 1$, $\Gamma = 20$ and $f(\xi) = 0.5\sqrt{1-\xi^2}$ (Fig. 8a), $f(\xi) = 0.5\sqrt{1-\xi^6}$ (Fig. 8b) .

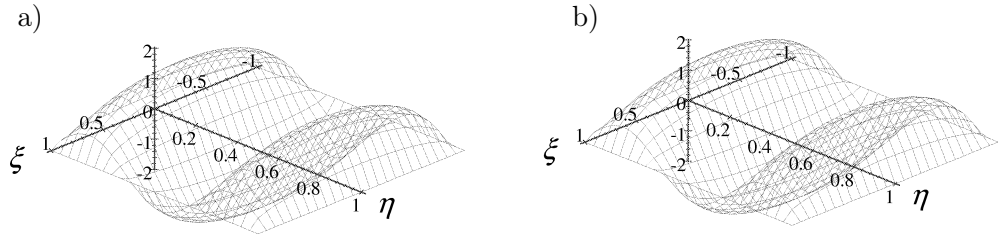

FIG. 9. Diagrams of the dimensionless intrinsic microvibration amplitude for $\Omega = 1.1$, $\Gamma = 20$ and $f(\xi) = 0.5\sqrt{1-\xi^2}$ (Fig. 9a), $f(\xi) = 0.5\sqrt{1-\xi^6}$ (Fig. 9b).

6. Summary of new results

The following new results and information on elastodynamics of the FGL solids have been obtained in this contribution.

1. A mathematical model for investigations of a linear-elastic behavior of FGL was derived on the basis of a certain heuristic assumption. The kinematics

of FGL in the framework of this model is described by two vector fields **u** and **v** which satisfy general model equations (2.5), (2.6). Fields **u** and **v** describe the averaged and fluctuational parts of the displacement field, respectively. In contrast to the homogenized model equations (2.9) the model equations (2.5), (2.6) make it possible to analyze the effect of microstructure size of FGL on its overall behavior. Moreover, the accuracy of the obtained solution to model equations (2.5), (2.6) can be estimated *a posteriori* by conditions (2.2).

2. A simplified form (3.6), (3.3) of model equations for averaged displacement **u** and intrinsic fluctuation **r** was formulated. Equation (3.3) for **r** is independent of **u** and describe certain boundary- and initial-layer type phenomena. In the framework of a homogenized model field **r** is identically equal to zero. This result is a generalization of the results obtained for periodic laminates in [7].

3. The aforementioned model was applied to the analysis of intrinsic microvibrations in a certain functionally graded laminate. It was shown that the character of intrinsic microvibrations depends on the vibration frequency. For lower frequencies we observe certain boundary-layer phenomena in which intrinsic fluctuations are strongly decaying. For higher frequencies the behavior of FGL is different and leads to the existence of resonance microvibrations.

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