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Thermodynamical and self-consistent approach to inelastic ferromagnetic polycrystals

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In Memory of Professor Henryk Zorski

GEOMETRIC AND KINEMATIC aspects of intragranular as well as intergranular plastic deformation of ferromagnetic polycrystals are discussed. Elastic strain is covered by the effective field homogenization method inside a representative volume element (RVE). By applying this method, an effective magnetostriction 4-tensor is determined. The evolution equation formed by tensor representation having incremental form is postulated to model inelastic metals. The rate-dependence takes place by means of stress rate-dependent value of the initial yield stress. Concept of the M. Zorawski deformation geometry is extended on the basis of constrained micro- and free macrorotations in intermediate reference configuration. This has as a consequence that the evolution equation for plastic spin of RVE is an outcome of the evolution equation for plastic stretching. The macroscopic evolution equation is based on Vakulenko's concept of thermodynamic time. A tensor representation for magnetomechanical interaction is proposed and susceptibility coefficients for iron are calibrated.

Key words: anisotropic Eshelby tensor, tensor representation, Vakulenko's thermodynamic time, susceptibility coefficients.

1. Introduction

THE PRINCIPAL OBJECTIVE of this paper is to give a simplified approach to inelasticity of (inherently polycrystalline) ferromagnetic materials aimed to serve primarily for nondestructive magnetic examination of inelastic behavior of reactor steels (cf. [30, 47, 56]). The subject is complicated and requires a careful examination of geometry as well as thermodynamics of deformation process.

As a starting point the geometry of the deformation with multiplicative deformation gradient must be extended in order to include magneto-mechanical interaction. This is done in the easiest way following the Kröner's "*incompatibility method*" which in the paper [15] was applied to magnetostriction. The authors have assumed that total incompatibility composed of purely elastic and magnetostrictive strain is zero while its constituents are not. They assumed isoclinicity of magnetization vectors in the natural state elements (intermediate local refer-

ence configuration) and their inhomogeneity in instantly deformed configuration which is responsible for magnetostrictive strains. Such an assumption in accord with our geometrical approach (cf. [32]) and will be accepted henceforth. An extension of their reasoning to the more general case of thermo-elasto-magneto-plastic strain is allowed if Kröner's formula

$$\mathbf{F} = \mathbf{F}_E \mathbf{F}_P \tag{1.1}$$

is understood as follows. The incompatible tensor $\mathbf{F}_E$ is obtained by cutting the body into infinitesimal pieces which are free of neighbors, brought to the reference temperature in the absence of external magnetic field. The other constituent on the right-hand side of (1.1), namely $\mathbf{F}_P$, contains the strain caused by irreversible magnetization as well as by pure plastic strain. This will be further elaborated in the next section.

Our approach to evolution equations does not take as granted *associativity of flow rule*, i.e. the normality of the plastic stretching tensor onto a yield surface [28, 34, 35], even if such an approach is accepted in the majority of the papers dealing with the subject. Such a normality is seriously questioned not only by the theoretical but by experimental results as well[1)]. For these reasons the normality is at first abandoned and instead of such an assumption evolution equations (exposed in the second subsection of this section) are based on the appropriate geometry of deformation and tensor representation. Each reasonable constitutive theory must have a tight relationship to thermodynamics. Here we mention some possible approaches.

- A very attractive approach to the extended thermodynamics has been proposed in [42] with a rational analysis of thermodynamic processes leading to the desired thermodynamic restrictions of general constitutive equations. This approach with the LIU's theorem [25] was applied by this author to viscoplastic materials in [34] and to inelastic composite materials in [35]. In spite of its beauty, an inherent coldness function (which is not quite clear from the experimental point of view) is inevitable.
- An alternative approach to extended thermodynamics following [8] was applied to thermoplasticity of irradiated materials in [28] and viscoplasticity of single ferromagnetic crystals in [37].
- The approach of endochronic thermodynamics with properly chosen thermodynamic time is the succeeding choice. It will be presented here in the version of VAKULENKO [53].

[1)] In the paper [16] a comparison between tension and torsion was one of the first signs of such a discrepancy. This subject has been discussed in detail in [41] where also experiments dealing with cruciform specimens [1] are included. In so-called J_2-theories with corners where a lot of "normals" to yield surface exist, this normality is in fact abandoned.

- Statistical thermodynamics is a mighty tool in treating choice of internal variables. It will be partly used here in defining magnetization distributions. In this field, the developments of ZORSKI [57] and KRÖNER [22] are very important.

The analysis in this paper is aimed to a description of fast multiaxial experiments on austenitic steels like AISI 316H having face-centered cubic lattice (compare [36]), as well other steels with body-centered cubic lattices. For this sake it is essential to reduce the number of material constants to be found from the available experiments. In other words, the general desire is always to make evolution equations with a minimal number of material constants even if these equations originate from very general functionals. The evolution equations usually comprise plastic stretching (often named by experimentalists as plastic strain-rate tensor) as well as plastic spin. The latter is understood by some authors as a trigger in localization behavior while some others require independence of these two evolution equations what greatly complicates the identification problem. This issue will be discussed as well.

The means applied in the paper to realize the stated goal are listed below. A micro-evolution equation having incremental form is postulated. The rate dependence takes place by means of stress rate-dependent value of the initial yield stress. Thus, such materials are quasi-rate dependent. Such an approach follows the results of dynamic experiments on AISI-steels performed at JRC-Ispra, Italy, at strain rates in the interval [0.001, 1000] 1/s (described in [36]). The still controversial plastic spin issue is treated by extending the concept of M. Zorawski deformation geometry, postulating constrained micro- and free macro-rotations in intermediate reference configuration. The self-consistent method (effective field variant) resulting from paper [24] provides effective stiffness tensors for RVE from individual grains and leads to a simplification of tedious calculations of their effective fourth-rank stiffness tensors of individual grains.

Throughout the paper thermal, elastic and magnetostrictive strains are assumed to be small and being approximately additive. This does not hold for plastic strains which are finite in general. However, in the fourth section devoted to the self-consistent method, the plastic strains are also considered as small due to assumptions in the "*effective field theory*" developed in the papers [18, 19, 24].

At present, only the first deformation gradients are considered. Thus, higher order effects like mechanical couple stresses (requiring a gradient viscoplasticity treated in [39]) as well as phase transitions, ferroelectric and ferrimagnetic effects, intrinsic spin, exchange forces and gyromagnetic effects are ignored (with negligible precessional velocity of magnetization). The considered process is electromagnetically slow enough such that ratio of particle velocity and speed of light is negligible. Due to size of paper, boundary conditions are not analyzed.

The notation used in this paper might be briefly shown by summation over repeated Cartesian indices: $\overrightarrow{H}\overrightarrow{M} = H_a M_a$, $(\overrightarrow{H} \otimes \overrightarrow{M})_{ab} = H_a M_b$, $(\mathbf{AB})_{ab} = A_{ac} B_{cb}$, $\mathrm{tr}(\mathbf{AB}) = A_{ac} B_{ca}$, $(\mathbb{D}\mathbf{E})_{ab} = D_{abcd} E_{dc}$. The usual notation: $2\,\mathrm{sym}\mathbf{A} = \mathbf{A} + \mathbf{A}^T$ and $2\,\mathrm{skw}\mathbf{A} = \mathbf{A} - \mathbf{A}^T$ is also applied here. Due to the limited space, some second order effects (treated in detail in [39]) have to be dropped from the consideration. For the same reason, many important references cannot be included into the already long list of references.

2. Micro and macro-geometry

2.1. Magneto-thermo-mechanical distortions

As a prerequisite, a correct geometric description of the inelastic deformation process analyzed is necessary. Consider a polycrystalline body in a real configuration $(k) \equiv (k(t))$ with dislocations, magnetization $\overrightarrow{M}(X,t)$ and an inhomogeneous temperature field $T(X,t)$ (where t stands for time and X for the considered particle of the body) subject, to external stress (i.e. surface tractions) and external magnetic field $\overrightarrow{H}$. Corresponding to (k) there exists, usually, an initial reference configuration $(K) = (k\,(t_0))$ with dislocations at a homogeneous temperature T_0 without surface tractions. Due to these defects, such a configuration is not stress-free but contains an equilibrated residual stress (so-called "back-stress"). It is generally accepted that linear mapping function $\mathbf{F}(.,t) : (K) \to (k)$ is a compatible second rank total deformation gradient tensor. In the papers, the dealing with continuum representations of dislocation distributions, the configuration (k) is imagined to be cut into small elements denoted by $(n) \equiv (n(t))$, these being subsequently brought to the temperature of (K) free of neighbors. The deformation "gradient" tensor $\mathbf{F}_E(.,t) : (n) \to (k)$ obtained in such a way is incompatible and should be called the thermoelastic distortion tensor whereas (n)-elements are commonly named as natural state local reference configurations (cf. for instance [15, 21, 32]). Of course, the corresponding plastic distortion tensor is not compatible. Here $\mathbf{F}$ is found by comparison of material fibres in (K) and (k) while $\mathbf{F}_E$ is determined by crystallographic vectors in (n) and (k). Multiplying (1.1) from the right-hand side by $\mathbf{F}_E\,(.,t_0)^{-1}$ we reach at slightly modified Kröner's decomposition rule[2)]. The formula (1.1) is often wrongly named the Lee's decomposition formula. It is worthy of note that $\mathrm{curl}\,\mathbf{F}_E\,(.,t)^{-1} \neq 0$ and this incompatibility is commonly attributed to an asymmetric second-order tensor of dislocation density[3)].

[2)] This modification, representing mapping $(k(t_0)) \to (k(t))$ and introduced by TEODOSIU [51] is necessary to account for DASHNER replacement invariance [12].

[3)] This definition covers only geometrically necessary dislocations (GND) without statistically stored dislocations (SSD) appearing at dislocation loops and dipoles. For this reason in

Taking into account the above discussion, it is reasonable to decompose irreversible *magneto-plastic* distortion as follows:

$$\mathbf{F}_P = \mathbf{F}^R_\mu \mathbf{F}_p. \tag{2.1}$$

It contains irreversible residual magnetic distortion $\mathbf{F}^R_\mu$ and pure plastic distortion $\mathbf{F}_p$. On the other hand, *magneto-thermo-elastic distortion* may be decomposed by means of:

$$\mathbf{F}_E = \mathbf{F}_e \mathbf{F}_\theta \mathbf{F}^r_\mu \tag{2.2}$$

with pure elastic distortion $\mathbf{F}_e$, thermal distortion $\mathbf{F}_\theta$ as well as reversible magnetic distortion $\mathbf{F}^r_\mu$. Now it is reasonable to define the *thermo-magnetic quasi-plastic distortion tensor* as the product of thermal and magnetic distortion:

$$\mathbf{F}_\omega = \mathbf{F}_\theta \mathbf{F}^r_\mu \mathbf{F}^R_\mu. \tag{2.3}$$

The name "quasi-plastic" was introduced by ANTHONY in [2]. Suppose now that thermal as well as magnetostrictive strains are much smaller than plastic strain. This is confirmed by experiments even for plastic strains of the order of 1%. Then for thermo-magnetic quasi-plastic strain, a linear decomposition approximately holds:

$$\mathbf{E}_\omega \approx \mathbf{E}_\theta + \mathbf{E}^r_\mu + \mathbf{E}^R_\mu \equiv \mathbf{E}_\theta + \mathbf{E}_\mu. \tag{2.4}$$

Here $\mathbf{E}_\alpha = (\mathbf{F}^T_\alpha \mathbf{F}_\alpha - \mathbf{1})/2,\ \alpha \in \{\omega, \theta, \mu\}$ are Lagrangian strains and $\mathbf{E}_\mu$ is called *magnetostrictive strain*. Due to magnetic symmetry it is bilinear function of magnetization vector [31]. Its constituents are *reversible magnetostrictive strain* and *irreversible magnetostrictive strain*. Their explicit forms are given in the next section.

2.2. Polycrystal strains

Let us imagine now that a typical (n)-element (called in the sequel *representative volume element* and denoted by RVE) is composed of N_g single crystal grains such that each Λ-th grain has N_s slip systems $\mathbf{A}_{\alpha\Lambda} = \overrightarrow{s}_{\alpha\Lambda} \otimes \overrightarrow{n}_{\alpha\Lambda}$. For instance, for fcc-crystals $N_s = 24$. Here $\overrightarrow{s}_{\alpha\Lambda}$ is the unit slip vector and $\overrightarrow{n}_{\alpha\Lambda}$ is the unit vector normal to the slip plane. Comparing a RVE in $(n(t))$ and

the paper [22], a more precise definition is given by infinite number of correlation functions composed of fundamental dyadic of Burgers vector and dislocation line tangent vector. On the other hand, $\mathbf{F}_P$ and its first gradients allow diverse non-Euclidean interpretations covering not only GND but implanting the Eshelbian strains as well (cf. details in [32, 40]).

$(n(t_0))$ we may write a formula similar to Kröner's formula (1.1) holding for the microplastic distortion tensor

$$\mathbf{\Pi}_{\Lambda} := \mathbf{\Pi}_{\Lambda E}\mathbf{\Pi}_{\Lambda P}, \tag{2.5}$$

whose components are the residual microelastic distortion tensor $\mathbf{\Pi}_{\Lambda E}$ and microplastic distortion tensor $\mathbf{\Pi}_{\Lambda P}$. Then the polar decomposition gives

$$\mathbf{\Pi}_{\Lambda E} = \mathbf{R}_{\Lambda}\mathbf{U}_{\Lambda E}.$$

Here microrotation satisfies the relations $\mathbf{R}_{\Lambda}^T\mathbf{R}_{\Lambda} = \mathbf{1}$ and its time-rate equals to $D_t\mathbf{R}_{\Lambda} = \mathbf{\Omega}_{\Lambda}\mathbf{R}_{\Lambda}$. Therefore, slip systems dyadics evolve according to: $\mathbf{A}_{\alpha\Lambda}(t) = \mathbf{R}_{\Lambda}(t)\mathbf{A}_{\alpha\Lambda}(t_0)\mathbf{R}_{\Lambda}^T(t)$ since microrotations must be constrained inside each RVE. By making use of these dyadics as well as microrotations we may write

$$\mathbf{U}_{\Lambda E} = \mathrm{diag}(1 + \lambda_{k\Lambda}),\ k \in \{1, 2, 3\}$$

as well as

$$\mathbf{\Pi}_{\Lambda P} := 1 + \sum_{\alpha} \gamma_{\alpha\Lambda}\mathbf{A}_{\alpha\Lambda},$$

where $\gamma_{\alpha\Lambda}\,(\alpha \in \{1, N_s\}, \Lambda \in \{1, N_g\})$ are plastic microshears inside the Λ–th grain. If a RVE has the volume $\Delta V = \sum_{\Lambda} \Delta V_{\Lambda}$ and the microplastic deformation tensors for individual grains are

$$\mathbf{C}_{\Pi\Lambda} = \mathbf{\Pi}_{\Lambda P}^T\mathbf{U}_{\Lambda E}^2\mathbf{\Pi}_{\Lambda P} \equiv \left[\mathbf{1} + \sum_{\alpha} \gamma_{\alpha\Lambda}\mathbf{A}_{\alpha\Lambda}^T\right]\ \mathbf{U}_{\Lambda E}^2\ \left[\mathbf{1} + \sum_{\alpha} \gamma_{\alpha\Lambda}\mathbf{A}_{\alpha\Lambda}\right],$$

then their volume average named *macroplastic deformation tensor* $\mathbf{C}_P := \mathbf{F}_P^T\mathbf{F}_P$ has the following form:

$$\mathbf{C}_P = \langle \mathbf{C}_{\Pi\Lambda} \rangle = \left\langle \mathbf{\Pi}_{\Lambda}^T\mathbf{\Pi}_{\Lambda} \right\rangle \equiv \frac{1}{\Delta V} \sum_{\Lambda} \mathbf{\Pi}_{\Lambda}^T\mathbf{\Pi}_{\Lambda}\Delta V_{\Lambda}. \tag{2.6}$$

Moreover, in the corresponding polar macro-decomposition $\mathbf{F}_P = \mathbf{R}_P\mathbf{U}_P$ the macroplastic rotation tensor $\mathbf{R}_P$ can be taken arbitrary (according to ZORAWSKI [58]), and might be fixed either to be a unit tensor or to have Mandel's isoclinicity property (details are given in [41]). For a definition of isoclinicity we should find average crystal directions in RVE(t) and RVE(t_0) and make them coincident. Accepting henceforth the first choice we obtain the relationship

$$\mathbf{R}_P = \mathbf{1} \quad \Rightarrow \quad \mathbf{F}_P = \mathbf{U}_P = \mathbf{C}_P^{1/2}, \tag{2.7}$$

which greatly simplifies the macroplastic spin issue (cf. [41]).

The above introduced microrotations of grains permit to formulate the exact relationship for material time-rate of microplastic distortion tensor (cf. Fig. 1)

$$D_t\mathbf{\Pi}_{\Lambda P} = \sum_{\alpha} \mathbf{A}_{\alpha\Lambda} D_t\gamma_{\alpha\Lambda} + \gamma_{\alpha\Lambda} D_t\mathbf{A}_{\alpha\Lambda}.$$

Here the aforementioned constrained microrotations must fulfil the relationship

$$D_t\mathbf{A}_{\alpha\Lambda} = \mathbf{\Omega}_\Lambda \mathbf{A}_{\alpha\Lambda} + \mathbf{A}_{\alpha\Lambda}\mathbf{\Omega}_\Lambda^T,$$

such that the microplastic stretching and microplastic spin tensors read:

$$\mathbf{D}_{\Lambda\Pi} = \mathbf{R}_\Lambda^T \left(\mathbf{D}_{\Lambda P} + D_t \log \mathbf{U}_{\Lambda E}\right) \mathbf{R}_\Lambda, \qquad \mathbf{W}_{\Lambda\Pi} = \mathbf{R}_\Lambda^T \mathbf{W}_{\Lambda P} \mathbf{R}_\Lambda + \mathbf{W}_{\Lambda E},$$

with

$$\mathbf{W}_{\Lambda E} = D_t\mathbf{R}_\Lambda^T \mathbf{R}_\Lambda$$

and

$$D_t \log \mathbf{U}_{\Lambda E} = \operatorname{diag}\left\{ D_t\lambda_{k\Lambda} \left(1 + \lambda_{k\Lambda}\right)^{-1} \right\}, \quad k \in \{1, 2, 3\}$$

as well as the notations

$$2\mathbf{D}_{\Lambda P} = D_t\mathbf{\Pi}_{\Lambda P}\mathbf{\Pi}_{\Lambda P}^{-1} + \mathbf{\Pi}_{\Lambda P}^{-T} D_t\mathbf{\Pi}_{\Lambda P}^T, \qquad 2\mathbf{W}_{\Lambda P} = D_t\mathbf{\Pi}_{\Lambda P}\mathbf{\Pi}_{\Lambda P}^{-1} - \mathbf{\Pi}_{\Lambda P}^{-T} D_t\mathbf{\Pi}_{\Lambda P}^T.$$

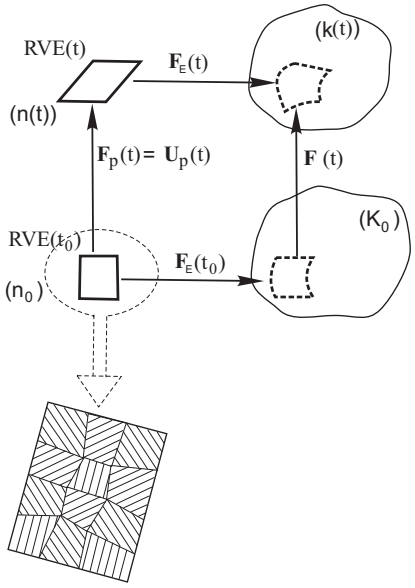

FIG. 1. Principal configurations of a polycrystalline body with illustration of free macro- and constrained micro-rotation.

The corresponding macroplastic stretching and macroplastic spin tensors follow now directly from (2.7) in the form:

$$2\mathbf{D}_P = D_t\mathbf{U}_P\mathbf{U}_P^{-1} + \mathbf{U}_P^{-1}D_t\mathbf{U}_P, \qquad 2\mathbf{W}_P = D_t\mathbf{U}_P\mathbf{U}_P^{-1} - \mathbf{U}_P^{-1}D_t\mathbf{U}_P. \tag{2.8}$$

It is worthy of note that such a representation considerably reduces the number of necessary material constants if some evolution equations for macro-quantities $\mathbf{D}_P$ and $\mathbf{W}_P$ are chosen in such a way to follow from the tensor representation. Connection of the macroplastic stretching with (2.6) by means of $2\mathbf{D}_P = \mathbf{U}_P^{-1}D_t\mathbf{C}_P\mathbf{U}_P^{-1}$ is then straightforward and is obtained from:

$$D_t\mathbf{C}_{\Pi\Lambda} = \mathbf{\Pi}_{\Lambda P}^T D_t\mathbf{U}_{\Lambda E}^2\mathbf{\Pi}_{\Lambda P} + \left(\sum_\alpha \mathbf{A}_{\alpha\Lambda}D_t\gamma_{\alpha\Lambda} + \gamma_{\alpha\Lambda}D_t\mathbf{A}_{\alpha\Lambda}\right)\mathbf{U}_{\Lambda E}^2\mathbf{\Pi}_{\Lambda P} + \mathbf{\Pi}_{\Lambda P}^T D_t\mathbf{U}_{\Lambda E}^2\left(\sum_\alpha \mathbf{A}_{\alpha\Lambda}^T D_t\gamma_{\alpha\Lambda} + \gamma_{\alpha\Lambda}D_t\mathbf{A}_{\alpha\Lambda}^T\right)$$

by means of the spatial averaging throughout a RVE i.e. $D_t\mathbf{C}_P = \langle D_t\mathbf{C}_{\Pi\Lambda}\rangle$.

In the paper [38] an initial attempt has been made to model transition of plastic strain from a grain to its neighbors. Yet an application of such an idea to computer simulation of inelastic behavior of RVE would require too long computing time. Instead of that, here a self-consistent method is applied. It will be explained in more detail in the subsequent section.

3. Macroscopic evolution and constitutive equations

3.1. Evolution equations by extended thermodynamics

Let us consider only the magnetomechanical terms in the balance laws and constitutive equations. More precisely the scope of this subsection is limited by the following assumption:

ASSUMPTION 1. Ferroelectric and ferrimagnetic effects, intrinsic spin, exchange forces and gyromagnetic effects are ignored (with negligible precessional velocity of magnetization). The considered process is slow enough so that the ratio of particle velocity and speed of light are negligible[4)].

Then, the next reduced set of objective and Galileo-invariant state variables [30, 46] should be introduced in general

$$\Gamma := \{\mathbf{E}, \mathbf{E}_P, A, T, \mathrm{GRAD}\,T, \vec{\mathcal{Q}}, \vec{\mathcal{M}}, \vec{\mathcal{M}}_R\}, \quad \Gamma \in \mathcal{G}. \tag{3.1}$$

[4)] Details of such a situation are explained in [31, 37]

The tensorial quantities used here should be connected with the convective material X-coordinates[5] in the deformed instant (k)-configuration (cf. Fig. 1). Herein

$2\,\mathbf{E} = \mathbf{F}^T\mathbf{F} - \mathbf{1} \equiv \mathbf{C} - \mathbf{1}$ is the Lagrangean total strain tensor,

$2\mathbf{E}_P = \mathbf{F}_P^T\mathbf{F}_P - \mathbf{1} \equiv \mathbf{C}_P - \mathbf{1}$ – Lagrangean plastic strain tensor,

$\mathrm{GRAD}\,T \equiv \mathbf{F}^{-1}\mathrm{grad}\,T$ – temperature gradient,

$\vec{\mathcal{Q}} \equiv J\mathbf{F}^{-1}\vec{q}$ – the heat flux vector,

$\vec{\mathcal{M}} = J\mathbf{F}^{-1}\vec{m}$ – the magnetization vector,

$\vec{\mathcal{M}}_R$ – the corresponding irreversible (residual) magnetization vector,

A – the volume-defined dislocation density (number of dislocation lines per unit volume).

Capital letters are reserved for such a convective representation. Differential operator $\mathrm{GRAD} \equiv \overrightarrow{C}^K\partial_K\otimes$ is referred to such coordinate frame, whereas $\mathrm{grad} \equiv \overrightarrow{g}^a\partial_a\otimes$ is used to indicate the same operator in spatial (possibly Cartesian) coordinate frame of (k)-configuration. Accordingly, $\mathrm{GRAD}\,T, \vec{\mathcal{Q}}$, $\vec{\mathcal{M}}$,$\vec{\mathcal{M}}_R$ have convective material either covariant or contravariant components. The above set Γ may be otherwise understood as a point belonging to the *extended configuration (deformation-temperature-magnetic) space* $\mathcal{G}$. Its subset $\{\mathbf{E}_P, A, \vec{\mathcal{M}}_R\}$ collects internal variables responsible for irreversible behavior.

To this configuration point corresponds a reaction point represented by the set

$$\Delta_1 := \{\mathbf{T}_K, u, s, \vec{S}, \vec{\mathcal{E}}, \vec{\mathcal{B}}\}, \qquad \Delta_1 \in \mathcal{D}, \tag{3.2}$$

where[6]

$\mathbf{T}_K = J\mathbf{F}^{-1}\mathbf{T}_k\mathbf{F}^{-T}$ – the symmetric Piola-Kirchhoff stress tensor of the second kind related to the material convective coordinates of (k)-configuration wherein $\mathbf{T}_k$ is the Cauchy stress,

u and s – the internal energy and the entropy densities,

$\vec{S} \equiv J\mathbf{F}^{-1}\vec{s}$ – the entropy flux vector,

$\vec{\mathcal{B}} = \vec{b}\mathbf{F}$ – the magnetic induction vector.

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[5] Another choice is to accept convective structural coordinates in one of the local natural state configurations depicted in Fig. 1. However, this is more difficult here since it is necessary to account for non-Euclidean expressions of the differential operators (cf. [41] for details).

[6] Our convective magnetic vectors coincide with those in the comprehensive reference ([31, p. 169]) with the exception of magnetic induction and magnetization, where $\vec{\mathcal{M}} = J\,\mathbf{C}^{-1}\vec{\mathcal{M}}_{\mathrm{Maugin}}$ and $\vec{\mathcal{B}} = J^{-1}\,\mathbf{C}\vec{\mathcal{B}}_{\mathrm{Maugin}}$ holds.

By means of $\mathcal{D}$ the *extended stress space* is indicated, whose objective and Galileio-invariant elements are listed in (3.2). At this place the constitutive equations are simply formulated by the bijective mapping:

$$\Delta_1 = \mathcal{R}(\Gamma) \equiv \Delta_1(\Gamma) \qquad \text{or} \qquad \mathcal{R} : \mathcal{G} \to \mathcal{D}, \tag{3.3}$$

which is too general, so that the thermodynamic analysis presented henceforth is aimed to supply the restrictions concordant with the second law of thermodynamics.

The evolution functions are proposed here in such a way that they should be compatible with (3.1)–(3.3) and are collected into the set

$$\Delta_2 := \{\vec{\mathcal{Q}}^*, \mathbf{E}^*, \vec{M}^*, A^*\}, \quad \Delta_2 \in \mathcal{D}, \tag{3.4}$$

so that objective evolution equations simply read:

$$D_t\vec{\mathcal{Q}} = \vec{\mathcal{Q}}^*(\Gamma), \tag{3.5}$$

$$D_t\mathbf{E}_P = \mathbf{E}^*(\Gamma), \tag{3.6}$$

$$D_t\vec{\mathcal{M}}_R = \vec{M}^*(\Gamma), \tag{3.7}$$

$$D_t\, A = A^*(\Gamma), \tag{3.8}$$

where the material time derivative is denoted by D_t. The simplicity of the left-hand sides of (3.5)–(3.8), owing to the absence of corotational time derivatives, has the origin in the accepted material convective description of constitutive functions and variables listed in (3.1)–(3.2)[7)].

A thermodynamic process occurring in the considered body is described by the evolution equations and by the following balance laws which are equivalent but slightly modified with respect to those of [31]:

$$\rho D_t u - \Big(\mathbf{T}_K + \big(\mathbf{1}\,(\vec{B}\vec{\mathcal{M}}) - \vec{B}\otimes\vec{\mathcal{M}}\big)\mathbf{C}^{-1}\Big) : D_t\mathbf{E} - \vec{B}D_t\vec{\mathcal{M}} + \mathrm{DIV}\,\vec{\mathcal{Q}} - \rho_0 h = 0, \tag{3.9}$$

$$\rho_0 - \rho J = 0, \tag{3.10}$$

[7)] It is noted here that right-hand sides of (3.5–3.8) do not include material time derivatives of internal variables – elements of the set Δ_1 in (3.2). In other words, quasi-rate independent ferromagnetics (cf. [40]) are not covered by the above evolution equations (3.5)–(3.8). In the Subsecs. 3.2 and 3.4 this approach is extended to include such materials as well.

$$\rho D_t\vec{v} - \vec{f} - \vec{f}^{em} - \frac{1}{J}\mathrm{DIV}\left(\mathbf{T}_K\mathbf{F}^T\right) = \vec{0}, \tag{3.11}$$

$$\mathrm{skw}\mathbf{T}_K = \mathrm{skw}\Big(\mathbf{C}^{-1}(\vec{B}\otimes\vec{\mathcal{M}} - \vec{P}\otimes\vec{\mathcal{E}})\Big), \tag{3.12}$$

wherein ρ_0 and ρ are mass densities in (K) and (k), $\vec{v}$ is the velocity of the particle, while the conventional notation:

$$J\vec{f}^{em}\mathbf{F} \equiv J\,\mathrm{GRAD}\left(\frac{1}{J}\vec{B}\mathbf{F}^T\right)\mathbf{F}^{-T}\vec{\mathcal{M}}$$

is used. The above Eqs. (3.9)–(3.12) are, respectively, the equation of energy balance, the mass conservation law, the equation of balance of momentum and the equation of balance of angular momentum.

Let us remind that electric quantities are not considered in this paper. Then one of the Maxwell equations becomes identity whilst the others read:

$$D_t\vec{B} = 0, \tag{3.13}$$

$$DIV\,\vec{B} = 0, \tag{3.14}$$

$$\mathrm{CURL}(\vec{\mathcal{B}} - J^{-1}\mathbf{C}\vec{\mathcal{M}}) = \vec{0}. \tag{3.15}$$

Further consequence of the Assumption 1 is a simplification of the set of internal variables by loosing from it the gradient of the magnetization vector and assuming in such a way that balance law for magnetization [45] (i.e. balance of angular momentum of spin continuum in wording of [31]) is identically satisfied.

The balance laws listed above imply constraints on the elements of the set $\{\Gamma\}\cup\{D_t\Gamma\}$ causing breaking of their independence, what is the essence of the Liu's theorem (given in [25]).

Nonetheless, there is still another constraint on these elements in the case of inelastic deformation process: the essential notion of yield surface which divides sharply two regions of material behavior. In other words, the elastic and plastic strain ranges are separated by the yield surface[8)]. The dynamic and static scalar yield functions are here defined by means of

$$f = f(\mathbf{T}_K, T, \mathbf{E}_P, \vec{\mathcal{M}}_R) \equiv h(\Gamma), \tag{3.16}$$

[8)] An essentially important question to be posed here reads: whether such a frontier between reversible and irreversible magnetization exists or not. If the answer is affirmative, then it leads to next questions: which irreversible process (mechanical or magnetic) advances and what is the shape of magnetic "yield" surface? It is tacitly assumed in this subsection that both processes are simultaneously triggered. In Subsec. 3.4 some other possibilities are discussed.

$$f^{\#} = f(\mathbf{T}_K^{\#}, T, \mathbf{E}_P, \vec{\mathcal{M}}_R) \equiv h_0(\Gamma), \tag{3.17}$$

where $\mathbf{T}_K^{\#}$ is the static stress corresponding to the dynamic viscoplastic stress $\mathbf{T}_K$. Their difference is usually termed as the *overstress tensor* and, in the simplest case, it may be represented by a linear function of $D_t\mathbf{E}_P$ as follows:

$$\Delta\mathbf{T}_K := \mathbf{T}_K - \mathbf{T}_K^{\#} = \mathbb{P}(\Gamma) : D_t\mathbf{E}_P, \tag{3.18}$$

with $\mathbb{P}(\Gamma)$ being the fourth-rank *tensor of plastic viscosity coefficients*. Introducing the *plastic strain rate intensity* by

$$D_t p := (D_t\mathbf{E}_P : D_t\mathbf{E}_P)^{1/2} \equiv \|D_t\mathbf{E}_P\| \geq 0, \tag{3.19}$$

the classification:

$f > 0, f^{\#} = 0, Dp > 0$ – viscoplastic behavior;

$f = f^{\#} = 0, Dp = 0$ – elastoplastic frontier;

$f = f^{\#} < 0, Dp = 0$ – elastic behavior;

and the kinematic constraint[9)]:

$$< f > \; D_t f^{\#} = 0, \tag{3.20}$$

will be useful in the sequel.

All thermodynamic processes must obey the master law of nature i.e. the second law of thermodynamics which in material convective coordinate frame of (k)-configuration reads:

$$\rho D_t s + \mathrm{DIV}\,\vec{S} - \rho\frac{r}{T} = 0, \tag{3.21}$$

where r/T is the entropy source. The precisely defined thermodynamic process is a solution of evolution and balance equations which obeys (3.21). The analysis of the above entropy inequality (3.21) by the Liu's theorem may be described as follows. Introducing $s^*(\Gamma)$ and $\vec{S}^*(\Gamma)$ into (3.21), this becomes a differential inequality linear with respect to the elements of the set $\{D_t\Gamma\} \cup \{\mathrm{GRAD}\,\Gamma\}$

[9)] Here $< f > = 1$ for $f > 0$ and $f = 0$ otherwise. This notation should not be confused with averaging sign $< \bullet >$ used in the previous section.

namely:

$$\rho_0 D_t s + \mathrm{DIV}\vec{S} - \rho\frac{r}{T} - \Lambda^u\Big[\rho_0 D_t u - \vec{B}D_t\vec{\mathcal{M}} + \mathrm{DIV}\vec{\mathcal{Q}} - \rho_0 h - D_t\mathbf{E} : \Big(\mathbf{T}_K + \big(\mathbf{1}\,(\vec{B}\vec{\mathcal{M}}) - \vec{B}\otimes\vec{\mathcal{M}}\big)\mathbf{C}^{-1}\Big)\Big] - \Lambda^\ell : \mathrm{skw}\Big(\mathbf{T}_K - \mathbf{C}^{-1}\vec{B}\otimes\vec{\mathcal{M}}\Big) - \Lambda^f < f > D_t f^{\#} - \vec{\Lambda}^v\Big(\rho D_t\vec{v} - \vec{f} - \vec{f}^{em} - J^{-1}\mathrm{DIV}(\mathbf{T}_K\mathbf{F}^T)\Big) - \Lambda^A\big(D_t A - A^*(\Gamma)\big) - \Lambda^E : \Big(D_t\mathbf{E}_P - \mathbf{E}^*(\Gamma)\Big) - \vec{\Lambda}^M\big(D_t\vec{\mathcal{M}}_R - \vec{M}^*(\Gamma)\big) - \vec{\Lambda}^2\,D_t\vec{B} - \Lambda^3\,\mathrm{DIV}\vec{B} - \vec{\Lambda}^4\,\mathrm{CURL}(\vec{\mathcal{B}} - J^{-1}\mathbf{C}\vec{\mathcal{M}}) \geq 0. \tag{3.22}$$

By introducing Lagrange multipliers into (3.21) all the elements of the set $\{D_t\Gamma\}\cup\{\mathrm{GRAD}\,\Gamma\}$ except solely $\mathrm{GRAD}\,T$ (which is already included in Γ) become mutually independent. Hence, in such an extended inequality all the coefficients with the elements of the set $\{D_t\Gamma\}\cup\{\mathrm{GRAD}\,\Gamma\}$ must vanish. This gives rise to the consecutive constitutive restrictions (cf. [28]):

$$\vec{S} = \Lambda^u(T)\vec{Q} \equiv \vec{Q}/T, \tag{3.23}$$

$$\mathbf{T}_K = \mathbf{C}^{-1}\big(-\vec{\mathcal{M}}\otimes\vec{B} + (\vec{B}\vec{\mathcal{M}})\mathbf{1}\big) + \rho_0\partial_{\mathbf{E}}g + < f > T\Lambda^f\partial_{\mathbf{E}}f^{\#}\big), \tag{3.24}$$

$$s = \partial_T g + < f > T\rho^{-1}\Lambda^f\partial_T f^{\#}, \tag{3.25}$$

$$\vec{B} = -\rho_0\partial_{\vec{\mathcal{M}}}g - < f > T\Lambda^f\partial_{\vec{\mathcal{M}}}f^{\#}, \tag{3.26}$$

$$\vec{0} = \partial_{\mathrm{GRAD}\,T}g + < f > T\rho^{-1}\Lambda^f\partial_{\mathrm{GRAD}\,T}f^{\#}, \tag{3.27}$$

as well as to the residual dissipation inequality

$$\vec{\Lambda}^M\vec{M}^*(\Gamma) + \Lambda^E : \mathbf{E}^*(\Gamma) + \Lambda^A A^*(\Gamma) - T^{-2}\vec{Q}\,\mathrm{GRAD}\,T \geq 0, \tag{3.28}$$

where $g := u - s(\Lambda^u)^{-1} \equiv u - Ts$ is the free energy density. If the thermodynamic process is very close to equilibrium (cf. [28]) then the above residual inequality permits adirect application of the Onsager–Casimir reciprocity relations. Some of the above Lagrange multipliers are explicitly given by:

$$\vec{\Lambda}^M = -\rho_0 T^{-1}\partial_{\vec{M}}g - < f > \Lambda^f\partial_{\vec{M}}f^{\#},$$

$$\Lambda^E = -\rho_0 T^{-1}\partial_{\mathbf{E}} g - < f > \Lambda^f \partial_{\mathbf{E}} f^{\#},$$
$$\Lambda^A = -\rho_0 T^{-1}\partial_A g - < f > \Lambda^f \partial_A f^{\#},$$

whereas the others vanish

$$\vec{\Lambda}^v = 0, \qquad \vec{\Lambda}^2 = 0, \qquad \Lambda^3 = 0, \qquad \vec{\Lambda}^4 = 0, \qquad \Lambda^\ell = 0.$$

The details of the procedure given above are presented in the reference [28] where thermoplasticity of the neutron-irradiated steels was considered.

3.2. Small magnetoelastic strains of isotropic plastically deformed insulators and generalized normality

In order to illustrate the above derived constitutive and evolution equations we accept in this subsection the following very simplifying assumptions for an isotropic body:

ASSUMPTION 2. Elastic and thermal strain as well as strain induced by reversible and irreversible magnetization are small of the same order but plastic strain itself is finite (cf. also [28])

Such an assumption corresponds to the so-called piezomagnetism processes when magnetization is generated by straining processes (illustrated in [13, 26]).

Let us take into account the fact that by its very nature, the mechanical stress disappears when pure elastic strain vanishes and, similarly, the local magnetic field equals zero if the reversible magnetization vanishes. Then, according to [2, 15] and the above relationship (2.4), it is reasonable to introduce *magnetostrictive quasi-plastic strain* by means of

$$\mathbf{E}_\mu = \mathbb{L}\mathbf{g}. \tag{3.29}$$

Here $\mathbb{L}$ is the fourth rank tensor of magnetostriction constants, symmetric only in the indices of the first as well as the second pair, whereas the notation: $\mathbf{g} \equiv \overrightarrow{\gamma}\otimes\overrightarrow{\gamma}$ stands for diadics of the unit vector of the magnetization vector ($\operatorname{tr}\mathbf{g} = 1$).

For convenience the elastic strain tensor expressed in material convective coordinates, accepted in previous subsection, then reads:

$$\begin{aligned}\widetilde{\mathbf{E}}_e &:= \frac{1}{2}\mathbf{F}_p^T\mathbf{F}_\mu^T\mathbf{F}_\theta^T(\mathbf{F}_e^T\mathbf{F}_e - \mathbf{1})\mathbf{F}_\theta\mathbf{F}_\mu\mathbf{F}_p \\ &\equiv \mathbf{E} - \mathbf{E}_p - \mathbf{F}_p^T(\mathbf{E}_\theta + \mathbf{E}_\mu)\mathbf{F}_p.\end{aligned} \tag{3.30}$$

As it has been already discussed in detail in the first section, the constituents of the *Lagrangian strain tensor*, namely, $\widetilde{\mathbf{E}}_e, \mathbf{E}_\theta, \mathbf{E}_\mu$ as well as $\mathbf{E}_p$, are incompatible.

With these facts taken into account and the above introduced Assumption 2, the mechanical and magnetic constitutive equations are presented henceforth.

First, mechanical part of the stress tensor must be linear in elastic strain (3.30). In the case of elastic and thermal isotropy case we would have the relationship:

$$\mathbf{T}_K = \widetilde{\mathbb{D}}(\mathbf{E}_p)\widetilde{\mathbf{E}}_e \equiv (c_1\mathbf{1} + c_2\mathbf{E}_p + c_3\mathbf{E}_p^2)\ \mathrm{tr}\,\widetilde{\mathbf{E}}_e + 2c_4\widetilde{\mathbf{E}}_e + c_5\ (\mathbf{E}_p\widetilde{\mathbf{E}}_e + \widetilde{\mathbf{E}}_e\mathbf{E}_p) + c_6(\mathbf{E}_p^2\widetilde{\mathbf{E}}_e + \widetilde{\mathbf{E}}_e\mathbf{E}_p^2), \tag{3.31}$$

which clearly exhibits a plastic strain-induced anisotropy.

Before formulating the magnetic constitutive equation, let us transform the vectors of magnetization and magnetic field into the corresponding second rank skew-symmetric tensors. This is done by means of the product with material convective Ricci third-rank permutation tensor as follows:

$$\mathbf{H} \equiv \mathcal{E}\vec{H} = -\mathbf{H}^T, \qquad \mathbf{M}_a \equiv \mathcal{E}\vec{\mathcal{M}}_a = -\mathbf{M}_a^T, \qquad a \in \{r, R\}. \tag{3.32}$$

In the above replacements

$$\vec{\mathcal{M}}_r := \vec{\mathcal{M}} - \vec{\mathcal{M}}_R \tag{3.33}$$

is the *reversible magnetization vector* while the skew-symmetric second-rank tensors $\mathbf{H}, \mathbf{M}_r$ and $\mathbf{M}_R$ should replace corresponding vectors.

As for magnetic constitutive equation we first remark that we have accepted the Heavisisde–Lorentz form of Maxwell equations (cf. table on page 59 of [31]). Thus from the last of these equations (3.15) (i.e. CURL $\vec{\mathcal{H}} = \vec{0}$) we see that the relationship between the magnetic induction field vector $\vec{\mathcal{B}}$ and the *internal magnetic field vector* $\vec{\mathcal{H}}$ (opposing the local magnetic field vector) reads:

$$\vec{\mathcal{H}} := \vec{\mathcal{B}} - J^{-1}\mathbf{C}\vec{\mathcal{M}}. \tag{3.34}$$

Due to the Assumption 2, the magnetic constitutive equation for an isotropic ferromagnetic must have the form:

$$\mathbf{H} = c_7\mathbf{M}_r + c_8(\mathbf{E}_p\mathbf{M}_r + \mathbf{M}_r\mathbf{E}_p) + c_9(\mathbf{E}_p^2\mathbf{M}_r + \mathbf{M}_r\mathbf{E}_p^2), \tag{3.35}$$

which is linear in reversible magnetization. The hereinabove constitutive expression for $\mathbf{H}$ has been derived from (3.26) and (3.34) under Assumption 1 by means of tensorial representations for the proper orthogonal group [49][10)].

Equation (3.31) is the generalized Hooke's law accounting for plastic strain-induced mechanical anisotropy. It is noteworthy that the constitutive equation for internal magnetic field predicts *magnetic anisotropy induced by the same cause.*

[10)] The skew-symmetric tensors are favored instead of the corresponding vectors for convenience and more compact representation. Of course, an equivalent formulation using products of vectors $\vec{\mathcal{M}}_r$ and $\vec{\mathcal{M}}_R$ with symmetric second rank tensor $\mathbf{E}_p$ is also possible.

According to the Assumption 2, the free energy function generating (3.31) and (3.35) must be quadratic in elastic strain and reversible magnetization i.e.:

$$g = \frac{1}{2}c_1\ i_1^2 + \frac{1}{2}c_2\ i_2^2 + \frac{1}{2}c_3\ i_3^2 + c_4\ i_4 + c_5\ i_5 + c_6\ i_6 + \frac{1}{2}c_7\ i_7 + \frac{1}{2}c_8\ i_8 + \frac{1}{2}c_9\ i_9, \tag{3.36}$$

where the consecutive proper and mixed invariants appearing in the above scalar function must be introduced (cf. [49]):

$$\begin{aligned} &i_1 = \mathrm{tr}\widetilde{\mathbf{E}}_e, && i_2 = \mathrm{tr}\{\mathbf{E}_p\widetilde{\mathbf{E}}_e\}, && i_3 = \mathrm{tr}\{\mathbf{E}_p^2\widetilde{\mathbf{E}}_e\},\\ &i_4 = \mathrm{tr}\{\widetilde{\mathbf{E}}_e^2\}, && i_5 = \mathrm{tr}\{\mathbf{E}_p\widetilde{\mathbf{E}}_e^2\}, && i_6 = \mathrm{tr}\{\mathbf{E}_p^2\widetilde{\mathbf{E}}_e^2\},\\ &i_7 = \mathrm{tr}\{\mathbf{M}_r^2\}, && i_8 = \mathrm{tr}\{\mathbf{E}_p\mathbf{M}_r^2\}, && i_9 = \mathrm{tr}\{\mathbf{E}_p^2\mathbf{M}_r^2\}. \end{aligned} \tag{3.37}$$

In the sequel the inverse forms of (3.31) and (3.35) will be useful. They can be written in the following way:

$$\widetilde{\mathbf{E}}_e = (\gamma_1\mathbf{1} + \gamma_2\mathbf{E}_p + \gamma_3\mathbf{E}_p^2)\ \mathrm{tr}\,\mathbf{T}_K + 2\gamma_4\ \mathbf{T}_K + \gamma_5\ (\mathbf{E}_p\mathbf{T}_K + \mathbf{T}_K\mathbf{E}_p) + \gamma_6\ (\mathbf{E}_p^2\mathbf{T}_K + \mathbf{T}_K\mathbf{E}_p^2), \tag{3.38}$$

and

$$\mathbf{M}_r = \gamma_7\mathbf{H} + \gamma_8(\mathbf{E}_p\mathbf{H} + \mathbf{H}\mathbf{E}_p) + \gamma_9(\mathbf{E}_p^2\mathbf{H} + \mathbf{H}\mathbf{E}_p^2). \tag{3.39}$$

The relationships between sets $\{c_1, ..., c_9\}$ and $\{\gamma_1, ..., \gamma_9\}$ can be found as follows. Let us multiply (3.31) as well as (3.38) by the tensors $\mathbf{1}$, $\mathbf{E}_p$ and $\mathbf{E}_p^2$ finding traces of both sides. If we introduce notations:

$$\begin{aligned} &s_1 = \mathrm{tr}\mathbf{T}_K, && s_2 = \mathrm{tr}\{\mathbf{E}_p\mathbf{T}_K\}, && s_3 = \mathrm{tr}\{\mathbf{E}_p^2\mathbf{T}_K\},\\ &s_4 = \mathrm{tr}\{\mathbf{T}^2\}, && s_5 = \mathrm{tr}\{\mathbf{E}_p\mathbf{T}_K^2\}, && s_6 = \mathrm{tr}\{\mathbf{E}_p^2\mathbf{T}_K^2\},\\ &s_7 = \mathrm{tr}\{\mathbf{H}^2\}, && s_8 = \mathrm{tr}\{\mathbf{E}_p\mathbf{H}^2\}, && s_9 = \mathrm{tr}\{\mathbf{E}_p^2\mathbf{H}^2\}, \end{aligned} \tag{3.40}$$

then such a procedure will produce relationships between $\{c_1, ..., c_6\}$ and $\{\gamma_1, ..., \gamma_6\}$. Of course, the same procedure applied to (3.35) as well as (3.39) would connect sets $\{c_7, ..., c_9\}$ and $\{\gamma_7, ..., \gamma_9\}$.

Similarly, the evolution equations for plastic strain rate and residual magnetization rate are explicitly stated by the following formulae:

$$
\begin{aligned}
(3.41)\qquad D_t\mathbf{E}_p =< f > \Big(& d_1\mathbf{1} + d_2\widetilde{\mathbf{E}}_e + d_3(\widetilde{\mathbf{E}}_e\mathbf{E}_p + \mathbf{E}_p\widetilde{\mathbf{E}}_e) \\
&+ d_4(\widetilde{\mathbf{E}}_e\mathbf{E}_p^2 + \mathbf{E}_p^2\widetilde{\mathbf{E}}_e) + d_5\mathbf{E}_p + d_6\mathbf{E}_p^2 \\
&+ d_7(\mathbf{M}_r\mathbf{E}_p - \mathbf{E}_p\mathbf{M}_r) + d_8(\mathbf{M}_r\mathbf{E}_p^2 - \mathbf{E}_p^2\mathbf{M}_r) \\
&+ d_9(\mathbf{M}_R\mathbf{E}_p - \mathbf{E}_p\mathbf{M}_R) + d_{10}(\mathbf{M}_R\mathbf{E}_p^2 - \mathbf{E}_p^2\mathbf{M}_R) \\
&+ d_{11}(\mathbf{E}_p\mathbf{M}_r\mathbf{E}_p^2 - \mathbf{E}_p^2\mathbf{M}_r\mathbf{E}_p) \\
&+ d_{12}(\mathbf{E}_p\mathbf{M}_R\mathbf{E}_p^2 - \mathbf{E}_p^2\mathbf{M}_R\mathbf{E}_p)\Big),
\end{aligned}
$$

$$
\begin{aligned}
(3.42)\qquad D_t\mathbf{M}_R = e_1\mathbf{M}_r + e_2(\mathbf{M}_r\mathbf{E}_p + \mathbf{E}_p\mathbf{M}_r) + e_3\big(&\mathbf{M}_r\mathbf{E}_p^2 \\
&+ \mathbf{E}_p^2\mathbf{M}_r\big) + e_4\mathbf{M} =_R + e_5(\mathbf{M}_R\mathbf{E}_p + \mathbf{E}_p\mathbf{M}_R) \\
&+ e_6(\mathbf{M}_R\mathbf{E}_p^2 + \mathbf{E}_p^2\mathbf{M}_R) + e_7(\mathbf{E}_p\widetilde{\mathbf{E}}_e - \widetilde{\mathbf{E}}_e\mathbf{E}_p) \\
&+ e_8(\mathbf{E}_p^2\widetilde{\mathbf{E}}_e - \widetilde{\mathbf{E}}_e\mathbf{E}_p^2) + e_9(\mathbf{E}_p^2\widetilde{\mathbf{E}}_e\mathbf{E}_p - \mathbf{E}_p\widetilde{\mathbf{E}}_e\mathbf{E}_p^2).
\end{aligned}
$$

It should be noted here that all the scalar coefficients in above relationships (3.31)–(3.35) and (3.41)–(3.42) are functions of the principal invariants of the plastic strain tensor $\mathbf{E}_p$. Of course, if plastic strain itself is small, then the corresponding complete linearization of constitutive and evolution equations is straightforward what might be of interest especially if dynamic effects are considered, i.e. wave equations of the linearized problem are written (cf. [28]). Evolution equations would then be reduced to Onsager–Casimir reciprocity relations.

The above evolution equations (3.41) and (3.42) become very much simplified if a generalized loading function Ω is assumed. Such a function would have the consecutive orthogonality properties (cf. [30, 37])

$$
(3.43)\qquad D_t\mathbf{E}_p = D_t\Lambda\frac{\partial\Omega}{\partial\mathbf{T}_K} \quad \text{and} \quad D_t\mathbf{M}_R = D_t\Lambda\frac{\partial\Omega}{\partial\mathbf{H}},
$$

where the material time rate of a scalar function Λ vanishes if the yield functions f as well as $f^{\#}$ are either negative or zero (cf. (3.20)). These constitutive equations (3.43) include magneto-mechanical reversible and irreversible interactions. To underline that the evolution equations are of quasi rate-independent type we use here $D_t\Lambda$ instead of Λ [40, 41]. A development of Ω into a power series whose terms are products of invariants (3.40), would give rise to different magnetomechanical irreversible couplings. Such a procedure has been applied in [41]

for thermomechanical processes. It is worthy of note that even in the simplest case when loading function is a quadratic function of $\mathbf{H}$ and $\mathbf{T}$:

$$\Omega = \frac{1}{2}\omega_1\ s_1^2 + \frac{1}{2}\omega_2\ s_4 + \frac{1}{2}\omega_3\ s_7, \tag{3.44}$$

such a coupling may appear through $D_t\Lambda$. This function leads by means of (3.43) into the consecutive two evolution equations

$$D_t\mathbf{E}_p = D_t\Lambda\,(\omega_1\mathbf{1}\ \mathrm{tr}\,\mathbf{T}_K + \omega_2\ \mathbf{T}_K), \tag{3.45}$$

$$D_t\mathbf{M}_R = D_t\Lambda\,\omega_3\mathbf{H}, \tag{3.46}$$

whose simplicity follows from the above very special loading scalar function Ω. They are oversimplified what will be seen from the following considerations.

3.3. Magnetoelasticity of cubic crystals by tensor representation

In order to show the simplest magnetomechanical interaction let us consider in this subsection magnetoelasticity of iron crystals. If we neglect the exchange energy, then the potential energy for a magnetic domain $\alpha \in \{1,\ldots,6\}$ inside a single crystal grain Γ reads (cf. [31], p. 378):

$$w_\Gamma^\alpha = \frac{1}{2}\mathbf{e}_E\mathbb{D}\mathbf{e}_E + M_s^2\mathbf{e}_E\mathbb{B}\mathbf{g}^\alpha - \mu_0 M_s\overrightarrow{H}\,\overrightarrow{\gamma}^\alpha + \frac{1}{4}M_s^4\mathbf{g}^\alpha\mathbb{K}_1\mathbf{g}^\alpha + \frac{1}{4}M_s^6\mathbf{g}^\alpha(\mathbf{g}^\alpha\mathfrak{K}_2\mathbf{g}^\alpha) \tag{3.47}$$

with the notation: $\mathbf{g}^\alpha \equiv \overrightarrow{\gamma}^\alpha\otimes\overrightarrow{\gamma}^\alpha$ (cf. (3.29)). Following [11] it has been assumed here that strain $\mathbf{e}_E$ and local magnetic field $\overrightarrow{H}$ are uniform for the considered grain. Strictly speaking, all the terms on the right-hand side of (3.47) except μ_0, M_s and $\overrightarrow{H}$ should have subscript Γ to show-grain dependence. For simplicity, all these subscripts are not written. In the above equation the first term on the right-hand side is the magnetoelastic energy w_Γ^σ, the second term shows the magnetostrictive energy $w_\Gamma^{\sigma\mu}$, magneto-static energy w_Γ^{mag} is the next term, whereas the last two terms are responsible for the magneto-crystalline energy w_Γ^{an}. If we subtract the magnetostrictive strain from magnetoelastic strain by means of $\mathbf{e}_e^\alpha \equiv\ \mathbf{e}_E - \mathbf{e}_\mu^\alpha$ (cf. also (3.29)) and take into account that square of the magnetostrictive strain is negligible (cf. also Appendix in [11]), then we get:

$$w_\Gamma^{\sigma\mu} + w_\Gamma^{\mathrm{mag}} \approx -\mathrm{tr}(\sigma^\alpha\mathbf{e}_\mu^\alpha) - \mu_0 M_s\overrightarrow{H}\,\overrightarrow{\gamma}^\alpha. \tag{3.48}$$

For cubic crystals, the magnetocrystalline 4-tensor $\mathbb{K}_1$ and 6-tensor $\mathfrak{K}_2$ in the crystallographic frame (CF) have the special forms [31]:

$$
\begin{aligned}
(\mathbb{K}_1)_{\kappa\lambda\mu\nu} &= K_1 \sum_{\alpha} \delta_{\alpha\kappa}\delta_{\alpha\lambda}\delta_{\alpha\mu}\delta_{\alpha\nu}, \\
(\mathfrak{K}_2)_{\alpha\beta\kappa\lambda\mu\nu} &= K_2 \delta_{1\alpha}\delta_{1\beta}\delta_{2\kappa}\delta_{2\lambda}\delta_{3\mu}\delta_{3\nu}.
\end{aligned}
\tag{3.49}
$$

The base of (CF) is determined by the easy directions [100], [010] and [001] while unit vectors $\overrightarrow{\gamma}^{\alpha}, \alpha \in \{1, \ldots, 6\}$ are either parallel or antiparallel to this base.

For the sake of illustration the Fig. 2 shows the energy (3.47) for iron as a function of direction in the mentioned CF (cf. also [10]).

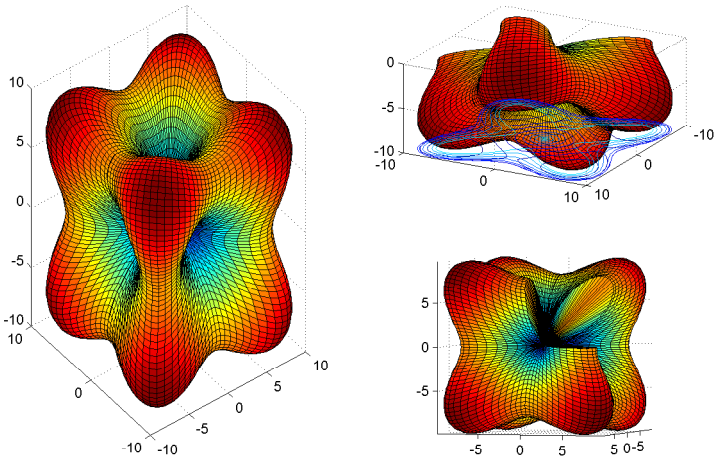

FIG. 2. Magneto-crystalline energy of iron.

An issue of high importance for polycrystals is concerned with volume fractions of magnetisations inside the magnetic sub-domains. Throughout a typical cubic Λ-grain it is assumed according to [6, 10] that they fulfil the Boltzmann distribution:

$$
f_{\Lambda}^{\alpha} = \frac{\exp(-A_D) w_{\Lambda}^{\alpha}}{\sum_{\beta} \exp(-A_D) w_{\Lambda}^{\beta}}, \qquad \alpha, \beta \in \{1, \ldots, 6\}.
\tag{3.50}
$$

It is important to note here that the first and two last terms in (3.47) are constant throughout the Λ-grain. Therefore, only energies (3.48) changing from domain to domain actually appear in (3.50).

The coefficient A_D was calibrated for iron by DANIEL in his PhD-thesis [10] on the basis of measurements performed by Webster in 1925. Such an identification

means that the coefficient A_D depends either on the experiment or the material or both[11)] Having found these volume fractions, the magnetisation vector inside Λ-grain reads:

$$\overrightarrow{M}_{\Lambda} := \sum_{\alpha=1}^{6} f_{\Lambda}^{\alpha} \overrightarrow{M}_{\Lambda}^{\alpha}, \tag{3.51}$$

where constituents $\overrightarrow{M}_{\Lambda}^{\alpha}$ are directed along the "easy" magnetic directions $\overrightarrow{\gamma}^{\alpha}$, namely [100], [010], [001], $[\bar{1}00]$, $[0\bar{1}0]$ and $[00\bar{1}]$.

In order to see how multiaxial stress influences magnetic susceptibility tensor, let us apply (3.50) and (3.51) to a grain of iron cubic crystal acted upon by arbitrarily oriented external magnetic field inside the CF-plane [100] and [010]

$$\overrightarrow{H} = H_0 \Big(\cos(\theta_H) \quad \sin(\theta_H) \quad 0 \Big)$$

with $\theta_H \in [0, \pi/4]$ while $H_0 \in [20, 2000]$A/m.

To choose typical and representative stress histories let us remind that only deviatoric stress influences magnetostriction. Then the following two cases are worthwhile to be considered.

CASE 1. Let the stress tensor acting on the grain be

$$\boldsymbol{\sigma} = \sigma_0 \begin{pmatrix} \cos^2(\theta_\sigma) & \sin(\theta_\sigma)\cos(\theta_\sigma) & 0 \\ \sin(\theta_\sigma)\cos(\theta_\sigma) & \sin^2(\theta_\sigma) & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with $\theta_\sigma \in [0, \pi/4]$ while stress magnitudes are varied in the interval $\sigma_0 \in [0, 160]$ MPa. Thus its compressive component acts along the CF-[001], the direction while the tension component acts arbitrarily oriented inside the plane [100] and [010].

CASE 2. Let the stress tensor be situated in the same CF-plane as $\overrightarrow{H}$, i.e.

$$\boldsymbol{\sigma} = \sigma_0 \begin{pmatrix} \cos(2\theta_\sigma) & \sin(2\theta_\sigma) & 0 \\ \sin(2\theta_\sigma) & -\cos(2\theta_\sigma) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

[11)] As explained by SOMMERFELD in [48], for a small body immersed in a huge closed thermodynamic system, the coefficient $A_D = 1/kT$ where $k = 1.3806505 \cdot 10^{-23}$[J/K] is the Boltzmann *universal* constant and T is the absolute temperature. It is important to underline that this coefficient does not depend on the process and the energies. Such an approach was applied in the paper [33] where field equations of ZORSKI's statistical theory of dislocations [57] were closed by Boltzmann's partition function in a very special case of 2D screw dislocations. Since a magnetic domain is much smaller than RVE, a good preliminary choice would be to accept Sommerfeld's suggestion. The essential difference between these two approaches is the question of universality of A_D. Final judgement must be drawn from experiments.

with $\theta_\sigma \in [0, \pi/4]$ and $\sigma_0 \in [0, 160]$ MPa. Since, in general, $\theta_\sigma \neq \theta_H$ directions, of external magnetic field vector and major principal stress differ in general. In this case both principal stresses (tension and compression) are also arbitrarily oriented inside plane [100] and [010].

From the numerical results of the above two programs of magnetomechanical histories with calculated values for $\overrightarrow{M}$, at this point we propose an identification of stress and magnetic field-dependent susceptibilities. Such a formula could be useful for fast finding of the influence of stress on susceptibilities in each grain when magnetic constitutive equation for magnetization of a polycrystal is analyzed.

The fact that reversible magnetization depends on stress and magnetic field and it disappears with external magnetic field, suggests the tensor generators in the tensor representation formula (cf. [49]):

$$\overrightarrow{M} = \overrightarrow{H} \sum_{\alpha} \chi_\alpha \boldsymbol{\Psi}_\alpha. \tag{3.52}$$

Thus, symmetric and skew-symmetric tensor generators are formed by products of the tensors $\mathbf{1}, \boldsymbol{\sigma}$ and $\overrightarrow{H} \otimes \overrightarrow{H}$. As a result we obtain

$$\overrightarrow{M} = \Big(\mathbf{1}\big(\chi_1 + \chi_2 H^2 + \chi_3 \overrightarrow{H}\boldsymbol{\sigma}\overrightarrow{H}\big) + \boldsymbol{\sigma}\big(\chi_4 + \chi_5 H^2\big)\Big)\overrightarrow{H} \equiv \boldsymbol{\chi}\overrightarrow{H}. \tag{3.53}$$

Such a reduced representation is chosen to have the smallest number of material "constants" $\chi_1, \ldots \chi_5$. It has odd powers of magnetic field in order to maintain magnetic symmetry requirement [31, 46]. Its linearity in stress is the simplest approach taking into account magnetomechanical interaction. When the stress disappears, it fulfils the symmetry requirement for a cubic crystal.

Input data for the numerical simulation are taken here for iron from [9, 11] as follows: $\lambda_{[100]} = 21 \cdot 10^{-6}$ – magnetostriction constant for the easy direction, $\mu_0 = 4\pi \cdot 10^{-7}$ – vacuum magnetic permeability, $\chi_0 = 2000$ – initial magnetic susceptibility determined in Webster's experiments (cf. [11]) and $A_D = 1.6 \cdot 10^{-3}[\mathrm{m^3 J^{-1}}]$ is the Buiron–Daniel constant appearing in distribution (3.50). From these data the corresponding saturated magnetization equals to: $M_s = (3\chi_0/\mu_0 A_D)^{1/2}[\mathrm{A/m}]$. Calibration results for both above cases are shown in the following two figures. Each $(H_{\max}, \sigma_0)$ point encompasses fitting for all χ-coefficients including all the points inside the 4-domain: $H \in [0, H_{\max}]$, $\theta_H \in [0, \pi/4]$, $\sigma \in [-\sigma_0, \sigma_0]$, $\theta_\sigma \in [0, \pi/4]$. Obviously, the proposed approximation (3.53) is satisfactory in domains where χ-coefficients are only slightly variable.

In order to present the results for χ-coefficients in a more explicit way let us introduce the scaling coefficients $\kappa_H \equiv H/H_{\max}$ and $\kappa_\sigma \equiv \sigma/\sigma_0$. For simplicity, consider special case $\kappa_H = \kappa_\sigma \equiv \kappa$. Then the dependence of correlation coefficient

of magnetization calculated by (3.50) and (3.51) as well as by (3.53) on scaling coefficient κ can be determined. As an example let us take $\kappa = 0.5$ corresponding to $H = 1000$ A/m and $\sigma_0 = 80$ MPa. For in-plane case the calibrated constants read $\chi = \{1726, -5.57 \times 10^{-4}, 6.97 \times 10^{-4}, 803, -1.21 \times 10^{-3}\}$, $\eta = 0.944$ whereas for out-of-plane stress we have $\chi = \{1661, -7.43 \times 10^{-4}, 1.12 \times 10^{-4}, 2250, -1.55 \times 10^{-3}\}$, $\eta = 0.896$. The later correlation is not acceptable whereas the first one is much better. From these two cases it may be concluded that the proposed tensor representation (3.53) is more suitable for in-plane case[12]. The special stress-free case is easily observed from the Figs. 3 and 4.

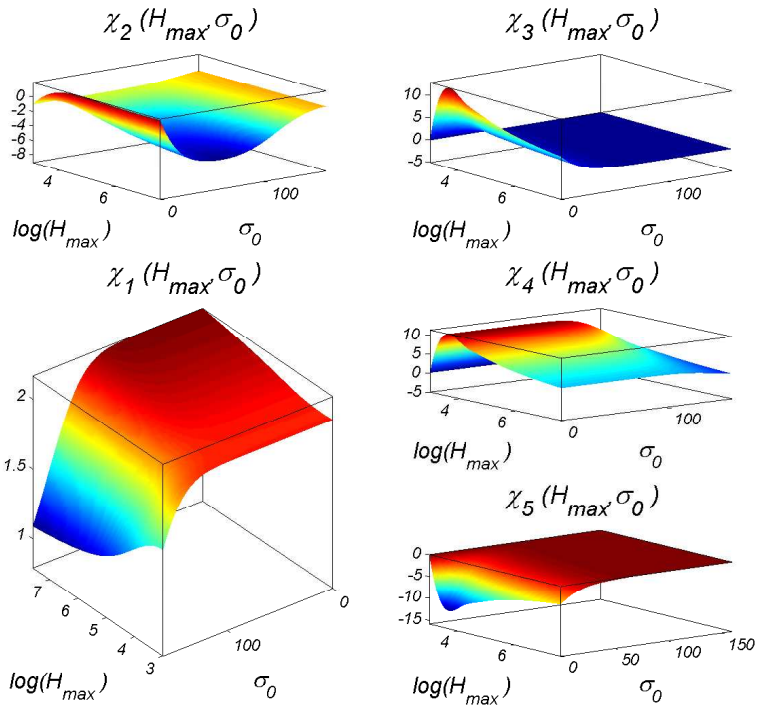

FIG. 3. Principal magnetic susceptibilities $\chi_1/1000$, $1000 \cdot \chi_2$, χ_3, $1000 \cdot \chi_4$ and χ_5 of mono-iron for diverse magnetic fields H [A/m] and biaxial stress magnitude σ_0 [MPa]. Out-of-plane case.

At the end of this subsection let us remark that the formula (3.53) might be further improved the introducing Boehler's structural tensors for cubic crystals [4] at the expense of additional susceptibility coefficients[13]. Of course, such a result is physically more justified due to correct material symmetries.

[12] A further improvement of the results for the susceptibilities could be obtained if rotations of domain magnetisations from $\vec{H}$ towards easy directions are calculated according to [7].

[13] Even the isotropic representation (3.53) predicts induced magnetic anisotropy i.e. different orientations of $\vec{H}$ and $\vec{M}$ (cf. [9] Sec. 13) unless stress disappears.

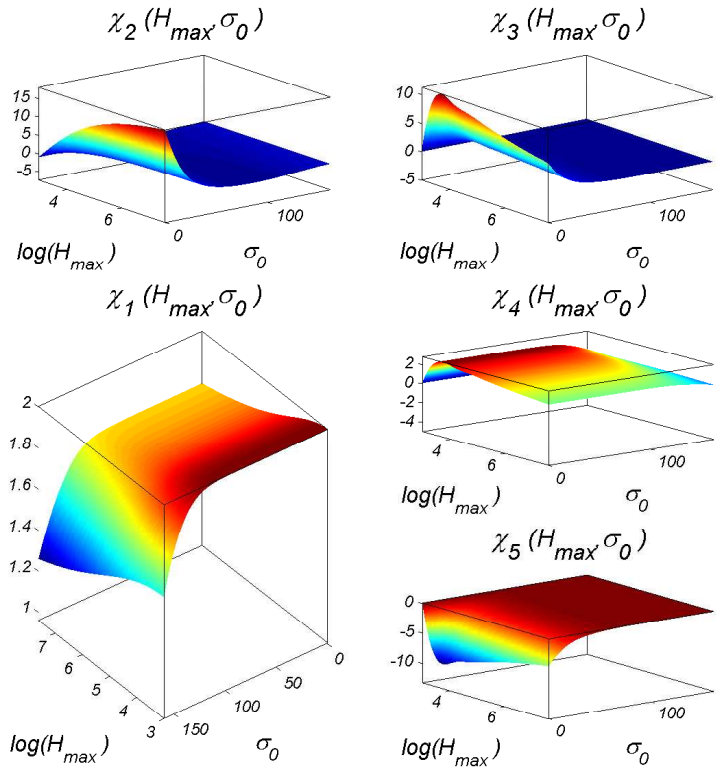

FIG. 4. Principal magnetic susceptibilities $\chi_1/1000$, $1000 \cdot \chi_2$, χ_3, $1000 \cdot \chi_4$ and χ_5 of mono-iron for diverse magnetic field H [A/m] and biaxial stress magnitude σ_0 [MPa]. Out-of-plane case.

Moreover in a polycrystal each grain orientation is either random or depends on orientation function originating from texture. A comparison of simplified representation (3.53) with the corresponding cubic crystal representation for full randomness is a worthwhile task.

It should be mentioned that MOTOGI and MAUGIN in [44] considered the subdomain distribution from convexity property of "stocked" energy g_p (cf. (3.54)) without introducing microdistribution (3.50).

3.4. Approach by endochronic thermodynamics

1. Let us first briefly discuss *purely mechanical inelastic irreversible behavior of steels* given in [38]. The specific free energy of the considered body is taken to be of the form

$$g = g_E\left(\mathbf{E}_E, T\right) + g_p\left(\lambda, T\right), \tag{3.54}$$

where λ is the *isotropic hardening parameter*. Its time rate is given by

$$D_t\lambda := \mathbf{T}_K : D_t\mathbf{E}_p, \tag{3.55}$$

having the meaning of plastic power. Since the free energy is assumed in the form (3.54), we have the plastic part of dissipation

$$\aleph^p = (1 - \rho\partial_\lambda g)\, D_t\lambda.$$

The total thermoplastic dissipation appearing in the second law of thermodynamics is denoted by $\aleph$, namely $\aleph \equiv T\big(\rho D_t s + \operatorname{div}(\mathbf{q}/T)\big) \geq 0$, where ρ is the mass density, T is the absolute temperature, $\mathbf{q}$ is the heat flux vector and s is the specific entropy.

The plastic dissipation served Vakulenko to introduce his *thermodynamic time* [53] by the hereditary function

$$\zeta(t) := \int_0^t \psi\big(\aleph^p(t')\big)\, dt'. \tag{3.56}$$

The function $\zeta(t)$ is piecewise continuous and nondecreasing in the way that $D_t\zeta(t) = 0$ within elastic ranges and $D_t\zeta(t) > 0$ when plastic deformation takes place. Splitting the whole time history into a sequence of infinitesimal segments, Vakulenko represented the plastic strain tensor as a functional of stress and stress rate history.

Moreover, in the paper [38] the accumulated plastic strain

$$\varepsilon^p_{eq}(\zeta) \equiv \int_0^\zeta \|D_t\mathbf{E}_p(\xi)\|\, d\xi, \tag{3.57}$$

as the important inelastic history parameter, was included into the memory kernel, extending in such a way the formerly mentioned Vakulenko's arguments. Another important generalization of his model in [38] was an extension of the function ψ to have the nonlinear power form:

$$\psi(\aleph^p) = (\aleph^p)^a\,. \tag{3.58}$$

The exponent a is of a great importance since it shows the speed of ageing. For example, $a < 1$ may be named *decelarated ageing* whereas $a > 1$ would define *accelarated ageing*. By such a classification the Vakulenko's value $a = 1$ might be termed *steady ageing*.

Now, according to Vakulenko's postulate we have:

$$\mathbf{E}_p(\zeta) = \int_0^\zeta \mathbf{\Psi}\left[\zeta-\xi, \mathbf{T}_K(\xi), D_\xi \mathbf{T}_K(\xi), \varepsilon^p_{eq}(\xi)\right]\, d\xi. \tag{3.59}$$

Of course, this integral equation is adopted to our case of finite plastic strains and absence of plastic rotation. Differentiation of (3.59) with respect to the thermodynamic time gives

$$\partial_\zeta \mathbf{E}_p = \mathbf{\Psi}\left[0, \mathbf{T}_K(\zeta), D_\zeta \mathbf{T}_K(\zeta), \varepsilon^p_{eq}(\zeta)\right] + \int_0^\zeta \partial_\zeta \mathbf{\Psi}\left[\zeta-\xi, \mathbf{T}_K(\xi), D_\xi \mathbf{T}_K(\xi), \varepsilon^p_{eq}(\xi)\right]\, d\xi. \tag{3.60}$$

Further analysis of the above integral equation is given in the next subsection.

2. Let us apply now the above explained concept to evolution of irreversible magnetization. Again we have non-steady ageing speed defined by the exponent a by means of:

$$D_t\zeta = (\aleph^{PM})^a \equiv \left(\vec{H} D_t \vec{M}_R + tr(\mathbf{T}_K D_t \mathbf{E}_p)\right)^a, \tag{3.61}$$

but now irreversible power induced by magnetisation must be taken into account. It is included in the magnetoplastic dissipation $\aleph^{PM}$. Suppose now that magnetomechanical interaction occurs only through equivalent plastic strain history. Then the magnetic evolution equation in its integral form may be taken as

$$\vec{M}_R(\zeta) := \int_0^\zeta \vec{\Psi}(\varepsilon^p_{eq}(z), \zeta - z, \vec{H}(z))\, dz, \tag{3.62}$$

where the corresponding endochronic memory is characterized by the thermodynamic time (3.61). Choosing a special form of the integral kernel as follows[14)]

$$\vec{\Psi} = \vec{H}(z)\, \omega(\varepsilon^p_{eq})\, \exp\left\{-\beta\,(\zeta - z)\right\}, \tag{3.63}$$

we would arrive at the following simple explicit evolution equation for residual magnetization vector:

$$D_\zeta \vec{M}_R = \omega \vec{H} - \beta \vec{M}_R. \tag{3.64}$$

[14)] Such an exponential kernel is typical for endochronic theories (cf. [54]).

However, in this equation the derivative is taken not with respect to real but to thermodynamic time. In order to transform it to real time, let us first introduce the *irreversible magnetic power* by means of

$$D_t\lambda_\mu := \vec{H} D_t \vec{M}_R, \tag{3.65}$$

following the same notation in (3.55). Now, when we introduce (3.65) into (3.61) and multiply this by the magnetic field vector and time derivative of thermodynamic time, we get a nonlinear algebraic equation:

$$\left(\frac{D_t\lambda_\mu}{\omega|\vec{H}|^2 - \vec{H}\vec{M}_R}\right)^{a/(1-a)} - D_t\lambda_\mu := D_t\lambda. \tag{3.66}$$

This equation explicitly characterizes the magnetoplastic interaction. Its validity should be checked by experiments where simultaneously stress, plastic strain, magnetic field and residual magnetization are measured. The two interesting special cases may be drawn from this equation:

- If plastic power is approximately equal to zero, then we would have a *thermoelastic irreversible magnetization*. Since $D_t\lambda \approx 0$, the Eq. (3.66) gives a simplified time rate of the thermodynamic time:

$$D_t\zeta = (D_t\lambda_\mu)^a = \left(\omega|\vec{H}|^2 - \vec{H}\vec{M}_R\right)^{a/(1-a)}. \tag{3.67}$$

- Another special case of interest would be choice $a = 1$ which might be called *Vakulenko's coupled magneto-viscoplasticity*. For such a choice, the time rate of ζ reads:

$$D_t\zeta = D_t\lambda_\mu + D_t\lambda = \frac{D_t\lambda}{1 - \omega|\vec{H}|^2 + \vec{H}\vec{M}_R}. \tag{3.68}$$

In both cases the evolution equation for residual magnetization in real time domain has the form:

$$D_t\vec{M}_R = \left(\omega\vec{H} - \beta\vec{M}_R\right) D_t\zeta. \tag{3.69}$$

Suppose for simplicity that constitutive equation (3.53) holds not only for a grain but for RVE as well. Then $\vec{M}_r = \boldsymbol{\chi}\vec{H}$ and using (3.34) we arrive at the

integral evolution equation connecting magnetic induction and magnetic field vectors:

$$\vec{B}(\zeta) = \vec{H}(\zeta) + J(\zeta)^{-1}\mathbf{C}(\zeta)\Big(\boldsymbol{\chi}(\zeta)\vec{H}(\zeta) + \int_0^{\zeta} \vec{\Psi}(\varepsilon^p_{eq}(z), \zeta - z, \vec{H}(z))\, dz\Big), \tag{3.70}$$

where the explicit form of the right Cauchy–Green total deformation tensor[15] $\mathbf{C} = \mathbf{1} + 2\mathbf{E}$ is found from the relationship (3.30). Here the magnetomechanical interaction appears through total as well as plastic strain history and the fact that susceptibilities depend on stress. Obviously, the oversimplified equation (3.46) might hold only in the case of negligible β. We believe that the above integral equation could be used for some nondestructive experimental checking of the order of magnitude of magnetomechanical interactions at low cycle fatigue or at some other experiments designed to establish the characteristic points of inelastic behavior of steels or some other ferromagnetic materials.

3.5. Low-cycle fatigue of ferromagnetics

The constitutive and evolution equations described in previous subsections might be used to describe piezomagnetic behavior induced by low-cycle fatigue of ferromagnetics.

Such a process has been investigated in the paper [13]. A cylindrical specimen of AISI 1018 was uniaxially treated by push-pull tests on a MTS-810 servo-hydraulic testing machine such that total strain was periodic and triangularly shaped $\|\mathbf{E}\| \in \{0,\ 0.009\}$ with cycle duration of 2 s. Magnetic induction due to piezomagnetic effect was also almost periodic with very slight changes of periodicity with increase of the relative number of cycles N/N_f, and cumulation of phase delay with respect to strain with growth of the accumulated plastic strain. Maxima and minima of total Lagrangean strain $\mathbf{E}$ are displaced with respect to minima and maxima of the magnetic induction vector[16]. According to (3.57) we calculate the *accumulated plastic strain* as a function of time by means of

$$\epsilon^p_{eq}(t) := \int_0^t \|D_t\mathbf{E}_p(\tau)\|\ d\tau. \tag{3.57$'$}$$

[15] Terminology is taken from [52] Sec. 24.

[16] Here for convenience again magnetic induction is represented by the vector $\vec{B}$.

Now, if uniaxial components of tensors $\mathbf{E}, \vec{B}, \vec{M}_r, \vec{M}_R$ are denoted by means of $E_{11}, B_1, M_{r1}, M_{R1}$, then the following memory-type equation emerging from (3.46) and (3.62)

$$B_1(t) := \int_0^t J(\epsilon^p_{\text{eq}}, t-\tau)\, D_\tau\lambda(\tau)\, H_1(\tau)\; d\tau, \tag{3.71}$$

could cover the delay between the measured functions $E_{11}(t)$ and $B_1(t)$. Time differentiation of the above relationship gives rise to the expression:

$$D_tB_1(t) := J(\epsilon^p_{\text{eq}}, 0)\, D_t\lambda(t)H_1(t) + \int_0^t \frac{\partial}{\partial t} J(\epsilon^p_{\text{eq}}, t-\tau)\, D_\tau\lambda(\tau)\, H_1(\tau)\; d\tau. \tag{3.72}$$

In the above integro-differential equation, the second term on the right-hand side is responsible for the above mentioned change of time delay and the deflection of pure periodicity of $B_1(t)$. Therefore, it is much smaller than the first part. On the other hand, if the constitutive equation $\overrightarrow{B} = \mu_0\big((\mathbf{1}+\boldsymbol{\chi})\overrightarrow{H} + \overrightarrow{M}_R\big)$ (by means of (3.53)) is used, then we would have

$$\begin{aligned} D_tB_1 &= \mu_0 D_tM_{R1} + \mu_0 D_t\big((1+\chi_{11})H_1\big), \\ D_tE_{11} &= D_tE_{e11} + D_tE_{\mu 11} + D_tE_{p11}. \end{aligned} \tag{3.73}$$

Now, the relationships (3.68) and (3.55) in our case lead to:

$$D_tM_{R1} = \big(\omega H_1 - \beta M_{R1}\big)\frac{\sigma_{11} D_tE_{p11}}{1-\omega H_1^2 + H_1 M_{R1}}. \tag{3.74}$$

The above three equations show a clear magnetomechanical interaction. They may serve for identification of material functions from the LCF uniaxial tension experiments (like those in [13]). Simultaneous zeros of $D_t\mathbf{E}_p$ and $D_t\overrightarrow{M}_R$ (ensuing from (3.45) and (3.46)) are a consequence of the model where it is assumed that irreversible magnetisation and plastic strain are triggered at the same time instant.

Instead of using (3.73) and (3.74), it is possible to find a more explicit endochronic kernel in (3.72). Let us remind that the Langevin function is suggested in many references as the best approximant for anhysteretic curve[17] (cf. for instance [43]). Then, using the data in (18.137) and figure 22.1 from [9] for such

[17] Let us recall that this function has the form $\mathfrak{L}(H) = \coth(H) - 1/H$.

cubic crystals[18)] it is possible to depict the consecutive figure by translating the anhysteretic curve by *coercive* magnetic field H_c either to the right or to the left depending on sign of time rate of magnetic field. Dropping indices for simplicity this may be represented by the following kernel:

$$\Psi(\zeta,z) = M_0\Big[\delta\big(H(z)-H(\zeta)+H_c(\lambda)\mathrm{sgn}(D_tH)\big) - \delta\big(H(z)-H(\zeta)\big)\Big]\mathfrak{L}\big(H(z)\big)D_zH, \tag{3.75}$$

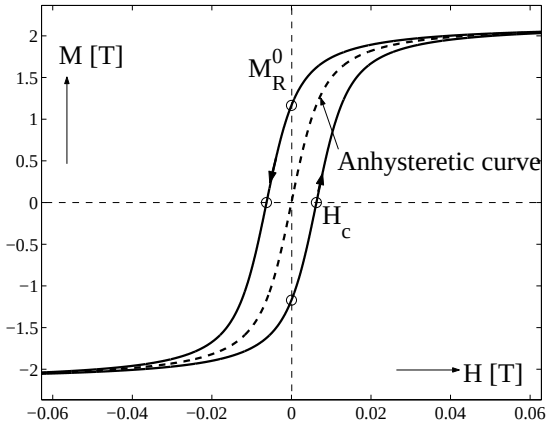

FIG. 5. Soft ferromagnetic steel behavior approximated by Langevin function according to [9].

where $\mathrm{sgn}(x) = 1$ for $x > 0$ and $\mathrm{sgn}(x) = -1$ for $x < 0$. If this kernel is inserted into the integral equation (3.62) then it gives rise to the following explicit expression for residual magnetization:

$$M_R = M - M_r = M_0\,\mathfrak{L}\big(H - H_c\,\mathrm{sgn}(D_tH)\big) - M_0\,\mathfrak{L}(H). \tag{3.76}$$

It is worthy of noting that herein the magneto-mechanical interaction is taken into account by *dependence of the coercive field on plastic power* λ whose time rate is given by (3.55).

Turning again to the paper [13] it may be concluded that the dependence $H_c(\lambda)$ and Eq. (3.76) enable to take into account complex disturbances of shape from simple periodicity of $E_{11}(t)$ towards more complicated shape of $B_1(t)$, as well as their relative delays of minima and maxima. It may be thus concluded that such an equation could be fruitful for description of magneto-viscoplastic phenomena occurring at low cycle fatigue.

[18)] In the case of a soft ferromagnetic steel this author suggests the following data: $(M)|_{H=0} = 0.832M_{\mathrm{sat}}$, $(dM/dH)|_{\max} = 0.6M_{\mathrm{sat}}/H_c$ where $H_c = 0.0063T$ is the *coercive* magnetic field and $M_{\mathrm{sat}} = 2.15T$ is the *saturated magnetization value*.

4. Effective field method for ferromagnetic polycrystals

Let us now assume that in the relationship (3.30) all the strains are small, so that an approximate additivity holds. Then the Eulerian strains fulfil the following relation:

$$\mathbf{e} = \mathbf{e}_e + \mathbf{e}_\theta + \mathbf{e}_\mu + \mathbf{e}_p \equiv \mathbf{e}_e + \mathbf{e}^{\omega p}, \tag{4.1}$$

with: $2\mathbf{e}_\alpha = \mathbf{1} - \mathbf{F}_\alpha^{-T}\mathbf{F}_\alpha^{-1}$, $\alpha \in \{e, \theta, \mu, p\}$. Here the term $\mathbf{e}^{\omega p}$ is the inelastic "eigen-strain" (with thermal, magnetostrictive and plastic parts) representing a source of internal stresses (cf. [18]). Suppose that a RVE is composed of many grains which may be modeled by randomly oriented ellipsoids. If such a RVE has a homogeneous elastic strain, magnetization and temperature then it is customary for the effective field method (cf. [18]) to consider such a grain as an inclusion implanted (by "*eigen-strains*") into a hypothetical matrix with average properties over grain orientations (with notation $\frac{1}{N_g}\sum_{\Gamma} \mathcal{A}_\Gamma \equiv \langle \mathcal{A}_\Gamma \rangle$):

$$\mathbb{D}_0 = \langle \mathbb{D}_\Gamma \rangle, \qquad \mathbb{L}_0 = \langle \mathbb{L}_\Gamma \rangle, \qquad \alpha_0 = \langle \alpha_\Gamma \rangle. \tag{4.2}$$

Two main hypotheses of effective field method (according to [23, 18]) are:

HYPOTHESIS 1. Inside each ellipsoidal inclusion the strain is homogeneous.

HYPOTHESIS 2. Ergodic property holds, i.e. properties of the inclusions are statistically independent of their spatial distribution.

The first hypothesis in [27] is named "step-constant approximation". If these two hypotheses are fulfilled and the RVE is acted upon by some "*external*" stress $\boldsymbol{\sigma}_0$, then performing the procedure applied in [23, 24] the stress and strain inside a Γ-grain become:

$$\begin{aligned} \boldsymbol{\sigma}_\Gamma &= \boldsymbol{\sigma}_0 + \textstyle\sum_\Delta \mathbb{S}_0 * \big([\![\mathbb{D}_\Delta^{-1}]\!]\boldsymbol{\sigma}_\Delta + [\![\mathbf{e}_\Delta^{\omega p}]\!]\big)\delta_\Delta, \\ \mathbf{e}_\Gamma &= \mathbf{e}_0 - \textstyle\sum_\Delta \mathbb{K}_0 * \big([\![\mathbb{D}_\Delta]\!]\mathbf{e}_{e\Delta} - \mathbb{D}_0[\![\mathbf{e}_\Delta^{\omega p}]\!]\big)\delta_\Delta, \end{aligned} \tag{4.3}$$

where $(\mathbb{A} * \mathbf{a})(x) := \int \mathbb{A}(x - x')\mathbf{a}(x')dx'$, $\mathbb{A} \in \{\mathbb{K}_0, \mathbb{S}_0\}$, $[\![\mathbf{a}_\Delta]\!] \equiv \mathbf{a}_\Delta - \mathbf{a}_0$ is the jump of $\mathbf{a}$ across Δ-grain boundary and δ_Δ is the *characteristic function* of Δ-grain being unity when position vector points to Δ-grain and zero otherwise. The corresponding characteristic function for whole RVE is obtained by summation of characteristic functions of all the grains belonging to the considered RVE. The kernel $\mathbb{S}_0$ was determined in [18]:

$$\mathbb{S}_0(x) = \mathbb{D}_0\mathbb{K}_0(x)\mathbb{D}_0 - \mathbb{D}_0\delta(x) \tag{4.4}$$

by means of the kernel (in KUNIN's notation [23]):

$$\mathbb{K}_0 = -\mathrm{def}\mathbf{G}_0\mathrm{def} \tag{4.5}$$

built by the Green function $\mathbf{G}_0$ corresponding to stiffness $\mathbb{D}_0$ (cf. (4.2)) and $\mathrm{def}\,\overrightarrow{a} \equiv \mathrm{sym}(\nabla \otimes \overrightarrow{a})$.

For simplicity, let us concentrate our attention on the special case of magnetoelastic strain, leaving out plastic and thermal strains. Then effective stiffness and effective magnetostriction tensor are defined by means of

$$\langle \boldsymbol{\sigma}_\Gamma \rangle = \mathbb{D}_{\mathrm{eff}} \langle \mathbf{e}_{e\Gamma} \rangle, \qquad \langle \mathbf{e}_{\mu\Gamma} \rangle = \mathbb{L}_{\mathrm{eff}} \langle \mathbf{g}_\Gamma \rangle. \tag{4.6}$$

Here the volume averages give the macroscopic elastic strain, macroscopic stress, macroscopic magnetostrictive strain and magnetization dyadic:

$$\mathbf{e}_e = \langle \mathbf{e}_{e\Lambda} \rangle\,, \qquad \mathbf{T} = \langle \boldsymbol{\sigma}_\Lambda \rangle\,, \qquad \mathbf{e}_\mu = \langle \mathbf{e}_{\mu\Lambda} \rangle\,, \qquad \mathbf{g} = \langle \mathbf{g}_\Lambda \rangle\,,$$

respectively.

Let us note that in the simplest linear case, without thermal and plastic strains taken into account, for ellipsoids randomly oriented with crystallographic frames deflecting from their semi-axes, getting a solution of the set of coupled integral equations (4.3) towards the homogenization formulae (4.6) is tremendous. For this reason some reasonable simplifications are inevitable.

In homogenization theories for composites with particulate inclusions there are two distinct self-consistent approaches:

- effective medium approach, where it is assumed that each inclusion behaves as isolated and immersed in a medium having effective constants $\mathbb{D}^{\mathrm{eff}}$ and
- effective field approach with an assumption that again each inclusion behaves approximately as isolated and situated in the matrix with elasticity constants $\mathbb{D}_{\mathrm{M}}$, while the influence of neighboring inclusions is taken into account by means of the *effective strain field* $\mathbf{e}_{\mathrm{eff}}$ acting on the considered inclusion [27].

In this paper the second approach is employed. The mentioned effective strain in the case of pure elastic strain covers in the second formula of (4.3) all the terms in the sum with $\Delta \neq \Gamma$ under the assumption that the correlation field induced by all other inclusions on Γ-grain is of the same shape as the Γ-grain itself but with larger dimensions. This assumption is simply illustrated by Fig. 6. Under such assumption the authors in [19] found ($\mathbb{I}$ is unit 4-tensor):

$$\mathbf{e}_{\mathrm{eff}}(x) = \left(\mathbb{I} - p_\Gamma \mathbb{A}_\Phi \langle [\![\mathbb{D}_\Gamma]\!] \mathbb{M}_\Gamma \rangle\right)^{-1} \mathbf{e}_0, \tag{4.7}$$

with

$$\mathbb{M}_\Gamma = (\mathbb{I} + \mathbb{A}(x_\Gamma)[\![\mathbb{D}_\Gamma]\!])^{-1},\ \mathbb{A}(x) = \int \mathbb{K}_0(x - x')dx' \ \text{ and } \ \mathbb{A}_\Phi = \int \mathbb{K}_0(x)\Phi(x)dx.$$

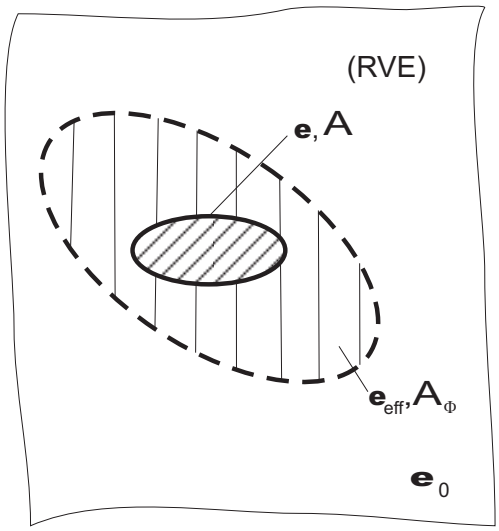

FIG. 6. Effective self consistent field homogenization.

The function $\mathbf{\Phi}(x)$ describes correlation distribution inside RVE. Having found $\mathbf{e}_{\text{eff}}$ the average strain equals to

$$\langle \mathbf{e}_{\Gamma} \rangle = \langle \mathbb{M}_{\Gamma} \mathbf{e}_{\text{eff}} \rangle. \tag{4.8}$$

Substituting this expression into (4.3) leads to the effective stiffness 4-tensor:

$$\mathbb{D}_{\text{eff}} = \mathbb{D}_0 + \sum_{\Gamma} p_{\Gamma} [\![\mathbb{D}_{\Gamma}]\!] \Big(\mathbb{I} + \big(\mathbb{A}(x_{\Gamma}) - p_{\Gamma} \mathbb{A}_{\Phi} \big) [\![\mathbb{D}_{\Gamma}]\!] \Big)^{-1}. \tag{4.9}$$

In the special case when all the inclusions have parallel semi-axes coinciding with their crystallographic frames, we have $\mathbb{A}(x_{\Gamma}) = \mathbb{A}_{\Phi}$ $(\forall \Gamma \in \{1, N_g\})$ and the above relationship takes the form of the Mori–Tanaka effective stiffness: $\mathbb{D}_{\text{eff}} = \mathbb{D}_0 + \sum_{\Gamma} p_{\Gamma} [\![\mathbb{D}_{\Gamma}]\!] \left(\mathbb{I} + (1 - p_{\Gamma}) \mathbb{A} [\![\mathbb{D}_{\Gamma}]\!] \right)^{-1}$.

Suppose now that throughout a RVE the magnetostrictive strains are balanced so that average stress-induced magnetostrictive grain strains equal zero. Then by making use of the ergodic hypothesis 4. we obtain

$$\mathbb{L}^T_{\text{eff}} \boldsymbol{\sigma}_0 - \mathbb{L}^T_0 \boldsymbol{\sigma}_0 - \langle \mathbb{L}^T_{\Gamma} \boldsymbol{\sigma}_{\Gamma} \rangle = \mathbf{0}, \tag{4.10}$$

where $(\mathbb{L}^T)_{cdab} = (\mathbb{L})_{abcd}$. Due to linearity of (4.3) we may introduce the stress concentration 4-tensor by means of the substitution $\boldsymbol{\sigma}_{\Gamma} = \mathbb{N}^{\sigma}_{\Gamma} \boldsymbol{\sigma}_0$. Knowing this stress concentration tensor we find the effective magnetostriction 4-tensor from:

$$\mathbb{L}^T_{\text{eff}} = \mathbb{L}^T_0 + \langle [\![\mathbb{L}^T_{\Gamma}]\!] \mathbb{N}^{\sigma}_{\Gamma} \rangle. \tag{4.11}$$

This concentration tensor is found from the reduced version of the first of Eq. (4.3) when inelastic terms are dropped, i.e.

$$\mathbb{N}^{\sigma}_{\Gamma} = \mathbb{I} + \sum_{\Delta} \mathbb{S}_0 * [\![\mathbb{D}^{-1}_{\Delta}]\!]\mathbb{N}^{\sigma}_{\Delta}. \tag{4.12}$$

Omitting details of the derivation we just present the result for the tensor $\mathbb{L}^{T}_{\text{eff}}$ derived in the same way as the effective thermal expansion tensor in [24]:

$$\mathbb{L}^{T}_{\text{eff}} = \mathbb{L}^{T}_{0} + \langle\, \mathbb{B}_{\Gamma} \rangle^{-1} \sum_{\Delta} \langle\, \mathbb{B}_{\Delta}\, [\![\mathbb{L}^{T}_{\Delta}]\!] \,\rangle \tag{4.13}$$

with $\mathbb{B}_{\Delta} = \Big(\mathbb{I} + \mathbb{D}_0\big(\mathbb{I} - \mathbb{A}(x_{\Delta})\mathbb{D}_0\big)[\![\mathbb{D}^{-1}_{\Delta}]\!]\Big)^{-1}$. Let us finally quote the effective thermal expansion tensor derived by LEVIN in [24]:

$$\alpha_{\text{eff}} = \alpha_0 + \langle\, \mathbb{B}_{\Gamma} \rangle^{-1} \sum_{\Delta} \langle\, \mathbb{B}_{\Delta}\, [\![\alpha_{\Delta}]\!] \,\rangle. \tag{4.14}$$

In this way we have completed all the necessary effective constitutive tensors entering Hooke's law for the RVE based on microstructural thermo-magneto-mechanical properties of individual grains. It is noteworthy that effective magnetostriction as well as effective thermal expansion tensors are derived from pure elasticity consideration. According to Taylor's assumption, the plastic strain is assumed to be homogeneous throughout RVE being equal for all the grains [19)].

5. Some concluding comments

The subject has been treated by tensor representation applying either the non-associativity of flow rule with extended thermodynamics or generalized normality which includes orthogonality of residual magnetization rate on the generalized loading surface which includes mechanical as well as magnetic state variables. While plastic strain is finite, the thermoelasto-magnetostrictive strain is assumed to be small. Small magneto-elasto-viscoplastic strains are then considered in detail in order to analyze magnetomechanical interaction at low-cycle fatigue. Furthermore, endochronic thermodynamics with Vakulenko's thermodynamic time made possible an account of the (experimentally observed) time

[19)] Let us note that the effective field approach to effective susceptibilities by means of grain-based constitutive equation is more complicated than the above analysis due to essential non-linearity of (3.53). It seems that a variational approach like that used in [55] is more suitable. Anyway, if we suppose that $\overrightarrow{H}_{\Gamma} = \langle \overrightarrow{H}_{\Gamma} \rangle$, then the Wiener upper bound depending on the RVE-average of stress could be obtained (cf. also [11]).

delay between stress and magnetic field histories. Such a result could be useful in inelastic testing with magnetic fields either induced or applied.

Geometrical approach based on early papers in the field of continuum theory of dislocations leads to the essential difference between micro and macro-spin, having as the origin constrained micro-rotations of grains inside a representative volume element. Here an Eshelbian approach is applied assuming that quasi-plastic (thermomagnetoplastic) strain is unconstrained whereas elastic strain is constrained. Since a RVE, having volume of an infinitesimal volume element, cannot be disintegrated any more, micro-spin does not follow from plastic micro-stretching. In previous papers of the author (reviewed shortly in [41]) treating purely thermomechanical strain histories of viscoplastic polycrystals, such an idea has been proved to be very successful. Namely, purely elastic micro-strains have been assumed to be covered by a self-consistent method (effective medium or effective field approach by Levin) whereas for plastic stretching as well as residual magnetization rate, quasi rate-independent incremental macro-evolution equations are postulated. The rate-dependence takes place by means of stress rate dependent value of the initial yield stress. The macroscopic magneto-inelastic evolution equations obey the Vakulenko's concept of thermodynamic time. The macro-evolution equation for plastic spin of RVE results from the corresponding evolution equation for plastic stretching. The same does not hold true for plastic micro-spin.

This paper has been dealt with viscoplasticity of ferromagnetic materials. The evolution equations have been derived either from inelastic materials of differential type or from the loading function generalized normality. In both cases tensor representation is applied to such a set of evolution equations. Restrictions concerning the set of field equations are established by means of the extended irreversible thermodynamics (the version which follows exposition in [8]). Small magnetoelastic strains of isotropic insulators are considered in detail in two special cases of finite as well as small plastic strain. As one example, a low-cycle fatigue of ferromagnetics is considered with special account of the time delay between stress and magnetic field histories. To describe such an experimental evidence, an integro-differential equation is proposed, whose equivalent plastic strain-dependent kernel covers the observed delay.

Concluding this section it is inevitable to compare the foregoing results with the existing achievements in the field. The major contributions to viscoplasticity of ferromagnetic materials have been given by MAUGIN and his collaborators in [30, 31]. The principal assumptions assumed in this section are closer to the scope of the first of these references, where

- small strain case together with absence of exchange forces and gyromagnetic effects has been assumed;

- the assumed on hysteresis effects have been analysed and
- evolution equations have been derived by normality of plastic strain rate and residual magnetization rate onto loading surface.

On the other hand, we have presented here the following results:

- in the case of finite plastic strains, the magnetic anisotropy induced by plastic strain is predicted by (3.42), where development of residual magnetization by mechanical terms is also evident;
- the influence of magnetization on plastic strain rate is obtained even in the case of isotropic ferromagnetic materials;
- the extended thermodynamics procedure allows for more general history effects, with inhomogeneities of magnetization being taken into account;
- the obtained relationships with couplings allow for magnetic measurements of inelastic phenomena, but the measurements will show their order of magnitude and practical measurability of these phenomena;
- in general, the developed theory is of non-associated type for plastic strain rate and residual magnetization rate are not perpendicular to the yield surface;
- albeit a generalized normality is much simpler with smaller number of material constants, a careful examination of the experiments on piezomagnetism and magnetostriction processes would give the final judgement concerning the question which theory should be applied;
- endochronic thermodynamical approach is less general than the approach based on extended thermodynamics, but it is much more suitable for explicit description and calibration of inelastic magneto-mechanical experiments such as low-cycle fatigue stress-strain-induction histories.

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