Arch. Mech., **58**, 4–5, pp. 363–380, Warszawa 2006

A variational principle applied to the dynamics of a liquid with diffusing gas bubbles

P. GIOVINE

Dipartimento di Meccanica e Materiali - Università 'Mediterranea'
Via Graziella, 1, Località Feo di Vito, I–89060 Reggio Calabria, Italy
e-mail: `giovine@unirc.it`

In Memory of Professor Henryk Zorski

The dynamic balance equations for bubbly liquids are deduced by evaluating the variation of a spatial Hamiltonian functional for immiscible mixtures. The constraint of incompressibility for the liquid is considered by choosing suitable "paths" of variation for the functions which describe the motion of the mixture and, although this appears to be a novelty, the equations obtained are in agreement with those derived from other theories, except for an inviscid drag term due to inertia forces and depending on changes of the radius of bubbles.

Key words: Hamilton's principle, dynamical balance equations, immiscible mixtures, bubbly liquids.

Notations

$\alpha, \beta, \gamma, \ldots$	scalars,
$\mathbf{u}, \mathbf{v}, \mathbf{w}, \ldots$	vectors,
$\mathbf{R}, \mathbf{S}, \mathbf{T}, \ldots$	tensors,
$\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$	sets,
α, δ	positive functions,
β	volume fraction of the gas,
γ_i	true density of constituent i,
ϵ	positive parameter,
ζ, ζ_2	radius of spherical drop of fluid and inclusion of gas, respectively,
ϑ	quadruple of functions $(\rho_2, \beta, \mathbf{v}_1, \mathbf{v}_2)$,
κ	density per unit volume of the kinetic energy,
μ_1, μ_2, ϕ	constitutive functions of the kinetic energy,
ξ_i	radial coordinate of constituent i,
π	constant pi,
ϖ_1, ϖ_1^{in}	pressure in the liquid phase and at interface, respectively,
ρ_i	bulk mass density of constituent i,
$\varsigma\sigma$	interfacial potential energy of surface tension coefficient ς,
τ	time,

χ	positive constant coefficient,
ψ	potential energy of internal actions,
ω	potential energy of external body forces,
$\mathbf{f}$	interaction force between phases,
$\mathbf{f}_i^\epsilon$	function distorting the flow-cylinder $\mathbf{y}_i$,
$\mathbf{n}$	outer normal of $\partial\mathcal{B}$,
$\mathbf{s}_i^{\mathbf{x}}$	flow-line through $\mathbf{x}$ of constituent i,
$\mathbf{u}_i$	function of class $\mathcal{C}^1(\Omega, \mathcal{V})$,
$\mathbf{v}_i$	peculiar velocity of constituent i,
$\mathbf{x}$	place of the region $\mathcal{B}$,
$\mathbf{y}_i$	flow-cylinder of constituent i,
$\mathbf{A}$	invertible tensor,
$\mathbf{F}_i^\epsilon$	gradient of $\mathbf{f}_i^\epsilon$,
$\mathbf{I}$	identity tensor,
$\mathbf{R}_i$	Reynolds stress tensor of constituent i,
$\mathcal{A}$	class of all quadruples,
$\mathcal{B}, \mathcal{B}_*$	current and reference region, respectively,
$\mathcal{E}$	three-dimensional Euclidean space,
$\mathcal{H}$	Hamiltonian functional for bubbly liquids,
$\mathcal{P}$	sub-region,
$\mathcal{V}$	the translation space of $\mathcal{E}$,
$\partial\mathcal{B}$	boundary of the region $\mathcal{B}$,
$\delta_j\mathcal{H}$	first variation of the functional $\mathcal{H}$,
Ω	the set $\overline{\mathcal{B}} \times [\tau_0, \tau_1]$,
div	divergence operator in current configuration,
grad	gradient operator in current configuration,
$(\cdot)_1, (\cdot)_2$	index for liquid and gaseous component, respectively,
$(\cdot)_*$	value of quantity in reference configuration,
$(\cdot)'^{i}$	material time derivative following the motion of the constituent i,
$(\cdot)^\epsilon$	quantity depending on the parameter ϵ.

1. Introduction

A LIQUID CONTAINING gas bubbles that diffuses relative to the liquid is a special case of a binary immiscible mixture, in which, namely, one could distinguish from each other the parts filled by different constituents.

Thus, the general procedure illustrated in [1] to compute the first variation of a spatial Hamiltonian functional for an immiscible mixture can be generalized and applied to bubbly liquids considered as continua composed of two fluids, one compressible (the gas) and the other incompressible (the liquid), which have independent motions; this procedure has been deduced in [2] to overcome some contests on the variational principles for fluids (with regard to these disputes see the introductions of both works). In fact, this new spatial formulation avoids the criticism raised in [3] and [4] about the difficulty originating from the velocities being unknown functions in the Eulerian variational principles [5]: the Monge potentials involved in the Clebsch representation for velocities suffer from indeterminacies and redundancies in their definition and are not endowed with a clear physical meaning; on the other hand, the spatial approach seems to be imperative for fluid mixtures for which a unique reference placement is generally not available (see [6]).

When the expression of the kinetic energy for the Hamiltonian functional is chosen, the virtual mass effects due to the diffusion of the bubbles, and the contributions of the local volume changes of the gas due to expansions or contractions of the bubbles, are held in proper consideration by adding to classical terms one representing the microscopic nonuniformities in the flow of constituents, as well as another one depicting the local expansional inertial effects.

The inertia forces coming from the last contribution include an inviscid term of drag related to the variation of the volume fraction of the gas phase and usually absent from other theories for bubbly liquids. Normally, in fact, only the viscosity of the liquid is supposed to be the cause of the existence of drag terms, a hypothesis supported, apart from the paradox of d'Alembert, also by initial experiments effected on the solid bodies dipped in a liquid (see, for example, the chapter 5 of [7]); but already in [8] Drew and Lahey Jr. observed that the data of some experimental proofs on the fluids with bubbles do not confirm such close correlation between the viscosity of the liquid and the forces of drag.

The other terms which appear in the equations of motion, here obtained in the conservative case, are consistent with those proposed in [8] and [9]; it is also shown that Reynolds stresses and the mutual force between the phases fulfil the principle of material frame indifference, while the third balance equation of the theory is reduced to the generalized Rayleigh equation for the dynamics of bubbles, when the reference micro-volume contains only one small spherical bubble (see Equation (4.7) of [10]).

Finally, it needs to observe that Drumheller and Bedford in [11] have already used a variational method, but of material type, for the study of liquids with bubbles, however the mixture proposed by them does not satisfy the principle of equipresence and the request for immiscibility for the constitutive variable (see the Sec. 5A.4 of [12]), while they have considered the forces due to virtual inertia as forces of interaction between the constituents, and then equal and opposite: this is not true, in general, what is seen in our case in which such effects are inserted in the expression of the total kinetic energy.

2. Preliminaries

In this section notation and essential results which are required for the application of the variational method described in [1] and [2] are presented and a continuum model of a two-phase immiscible mixture composed of an incompressible liquids with bubbles of gas is proposed.

Precisely, the two components of the mixture are considered as interpenetrating the continua that contemporaneously fill a given region $\mathcal{B}$ of the three-dimensional Euclidean space $\mathcal{E}$, saturating it, in a certain time interval $[\tau_0,\ \tau_1]$ during which the motion is observed.

In accordance with BOWEN in [12], for whom *'a mixture is immiscible if volume fractions effect the response'*, the fields that describe the motion are the velocities $\mathbf{v}_i$ $(i = 1, 2)$ of constituents, the bulk mass densities ρ_i and the volume fraction of the gas β, where the indices 1 and 2 are used to indicate liquid and gaseous components, respectively.

For the hypotheses made, the bulk densities ρ_i are tied to true densities γ_i by relations

$$\rho_1 = \gamma_1(1-\beta) \qquad \text{and} \qquad \rho_2 = \gamma_2\beta \tag{2.1}$$

with $\beta \in]0, 1[$; besides, the following kinematic conditions of compatibility apply on the boundary $\partial\mathcal{B}$ of $\mathcal{B}$, if $\mathbf{n}$ is the outer normal of $\partial\mathcal{B}$:

$$\mathbf{v}_1 \cdot \mathbf{n} = 0 \qquad \text{and} \qquad \mathbf{v}_2 \cdot \mathbf{n} = 0. \tag{2.2}$$

Let $\Omega := \overline{\mathcal{B}} \times [\tau_0, \tau_1]$. The flow–cylinder of the i-th constituent is the function $\mathbf{y}_i : \Omega \to \mathcal{B}$ defined by

$$\mathbf{y}_i(\mathbf{x}, \tau) := \mathbf{s}_i^{\mathbf{x}}(\tau), \qquad \forall(\mathbf{x}, \tau) \in \Omega, \tag{2.3}$$

where, for each $\mathbf{x}$ in $\mathcal{B}$, $\mathbf{s}_i^{\mathbf{x}} : [\tau_0, \tau_1] \to \mathcal{B}$ represents the flow-line through $\mathbf{x}$ of the i-th constituent, that is the function that resolves the following ordinary differential equation with initial values:

$$\frac{d\mathbf{s}_i^{\mathbf{x}}}{d\tau}(\tau) = \mathbf{v}_i(\mathbf{s}_i^{\mathbf{x}}(\tau), \tau) \qquad \text{and} \qquad \mathbf{s}_i^{\mathbf{x}}(\tau_0) = \mathbf{x}; \tag{2.4}$$

we observe that, for Equations (2.2) and (2.3), it is $\mathbf{y}_i(\mathcal{B}, \tau) = \mathcal{B}$, for each $\tau \in [\tau_0, \tau_1]$.

The mass of each constituent conserves, therefore the following relation is valid

$$\frac{d}{d\tau} \int\limits_{\mathbf{y}_i(\mathcal{P},\tau)} \rho_i(\cdot, \tau) \equiv 0, \qquad \forall\mathcal{P} \subset \mathcal{B}, \tag{2.5}$$

which, in local form, is equivalent to, by the Appendix of [2],

$$\frac{\partial\rho_1}{\partial\tau} + \operatorname{div}(\rho_1\mathbf{v}_1) = 0 \qquad \text{and} \qquad \frac{\partial\rho_2}{\partial\tau} + \operatorname{div}(\rho_2\mathbf{v}_2) = 0. \tag{2.6}$$

Since the liquid is incompressible (*i.e.* $\gamma_1 =$ constant), its volume does not change during the motion of the mixture, therefore, in addition to equations of conservation of mass (2.5), it is required that the following constraint holds for the medium:

$$\text{(2.7)} \qquad \int\limits_{\mathbf{y}_1(\mathcal{P},\tau)} (1-\beta(\cdot,\tau)) \equiv \chi \quad \left(\text{or, in local form,} \quad -\frac{\partial \beta}{\partial \tau} + \operatorname{div}\left[(1-\beta)\mathbf{v}_1\right] = 0\right),$$

with $\mathbf{y}_1$ defined by $(2.3)_1$, $\mathcal{P}$ a subregion of $\mathcal{B}$ and χ a positive constant.

The size of the bubbles of dispersed gas in the liquid, however small, is such that the effects of the virtual inertia of translation, representing the local nonuniformities in the flow of constituents when they move with respect to one another, must be taken into account, besides those due to the inertia of classical translation. Additionally, local microvariations of the volume of bubbles move the ambient liquid, that is set in motion, giving rise to effects of expansional inertia. The kinetic energy of the mixture has, therefore, the following density per unit volume κ which generalizes the proposals in [1] and [11]:

$$\text{(2.8)} \qquad \kappa\left(\rho_2, \beta, \operatorname{grad}\beta, \frac{\partial \beta}{\partial \tau}, \mathbf{v}_1, \mathbf{v}_2\right) := \frac{1}{2}\rho_2\left[\mathbf{v}_2^2 + \mu_2(\rho_2,\beta)\left(\frac{\partial \beta}{\partial \tau} + \mathbf{v}_2\cdot\operatorname{grad}\beta\right)^2\right]$$
$$+ \frac{1}{2}\left[\rho_1\mathbf{v}_1^2 + \gamma_1\mu_1(\rho_2,\beta)\left(\frac{\partial \beta}{\partial \tau} + \mathbf{v}_1\cdot\operatorname{grad}\beta\right)^2 + \gamma_1\phi(\rho_2,\beta)(\mathbf{v}_2-\mathbf{v}_1)^2\right],$$

where the expression of functions μ_1, μ_2 and ϕ depends on the shape of the bubbles as well as on the kind of expansional movement of the bubbles themselves.

The potential energy of the mixture includes the energies of deformation of constituents and that due to the surface tension of the bubbles of coefficient ς, respectively inserted as an energy of internal actions of density per unit volume $\psi(\rho_2,\beta)$ and an interfacial energy of density $\varsigma\sigma(\beta,\delta)$, with $\delta := \frac{1}{2}\operatorname{grad}\beta\cdot\operatorname{grad}\beta$, which depends on the changes of the curvature at the interface.

3. Constitutive choices

Explicit expressions for the constitutive functions μ_1, μ_2, ϕ and σ, independent of ρ_2 and $\operatorname{grad}\beta$, can be obtained if one imagines an element of the continuum as made from a spherical drop of variable radius ζ of an incompressible perfect fluid containing one much smaller concentric spherical inclusion of radius ζ_2, also this variable, of a compressible gas that expands or contracts homogeneously. Consequently, it is possible to fix a reference region $\mathcal{B}_*$ and, therefore, the local radial coordinates ξ_i of constituents, with the origin in the centre of the spheres, depend on its coordinate ξ_* in $\mathcal{B}_*$; they are given by the following relations:

$$
\begin{aligned}
\xi_1 &= \zeta + (\xi_* - \zeta_*)\frac{\zeta - \zeta_2}{\zeta_* - \zeta_{2*}}, && \text{if} \qquad \xi_* > \zeta_{2*},\\
\xi_2 &= \xi_* \frac{\zeta_2}{\zeta_{2*}}, && \text{if} \qquad \xi_* < \zeta_{2*}
\end{aligned}
\tag{3.1}
$$

where, now and in the course of the paper, the asterisk $*$ means the value of the quantity in the reference configuration $\mathcal{B}_*$. In this context, the volume fraction of the inclusion and the constraint of liquid incompressibility are then:

$$
\beta = \left(\frac{\zeta_2}{\zeta}\right)^3 \qquad \text{and} \qquad \left(\frac{\zeta}{\zeta_*}\right)^3 = \frac{1-\beta_*}{1-\beta}.
\tag{3.2}
$$

Thus, the average density of kinetic energy per unit volume associated with each element, as an effect of the homogeneous expansions or contractions of the element itself with the inclusion, is equal to

$$
\frac{1}{2}\left\{\frac{1}{\frac{4}{3}\pi\zeta^3}\left[\int_{\zeta_2}^{\zeta} \gamma_1 (\xi_1^{\prime 1})^2 4\pi\xi_1^2\, d\xi_1 + \int_0^{\zeta_2} \gamma_2 (\xi_2^{\prime 2})^2 4\pi\xi_2^2\, d\xi_2\right]\right\}
\tag{3.3}
$$

(see, also, [13]), where $(\cdot)^{\prime i}\left(:= \dfrac{\partial(\cdot)}{\partial\tau} + (\operatorname{grad}(\cdot))\mathbf{v}_i\right)$ denotes the material time derivative of the quantity $(\cdot)$ following the motion of the i-th constituent. By sparing the Reader details of the algebraic manipulations, we record here the result, when relations (3.2) are taken into account:

$$
\begin{aligned}
&\frac{1}{2}\gamma_1 \frac{\zeta_{2*}^2(1-\beta_*)^{2/3}(1-\beta^{1/3})}{3\beta_*^{2/3}\beta^{1/3}(1-\beta)^{8/3}}\left(\frac{\partial(1-\beta)}{\partial\tau} + \mathbf{v}_1\cdot\operatorname{grad}(1-\beta)\right)^2\\
&\qquad + \frac{1}{2}\gamma_2 \frac{\zeta_{2*}^2(1-\beta_*)^{2/3}}{15\beta_*^{2/3}\beta^{1/3}(1-\beta)^{8/3}}\left(\frac{\partial\beta}{\partial\tau} + \mathbf{v}_2\cdot\operatorname{grad}\beta\right)^2.
\end{aligned}
\tag{3.4}
$$

By using (2.1), the constitutive expressions for the coefficients in formula (2.8) are then:

$$
\begin{aligned}
&\mu_1 = \frac{\zeta_{2*}^2(1-\beta_*)^{2/3}(1-\beta^{1/3})}{3\beta_*^{2/3}\beta^{1/3}(1-\beta)^{11/3}}, \qquad \mu_2 = \frac{\zeta_{2*}^2(1-\beta_*)^{2/3}}{15\beta_*^{2/3}\beta^{4/3}(1-\beta)^{8/3}},\\
&\phi = \frac{1}{2}\beta\frac{1+2\beta}{1-\beta},
\end{aligned}
\tag{3.5}
$$

the last one for the virtual inertia of translation being obtained in Sec. 93 of [14].

Now we observe that the density of the interfacial potential energy is equal to the product of the surface tension ς and the surface area $4\pi\zeta_2^2$ of the bubble. By multiplying by the number of bubbles per unit volume, the density becomes $\dfrac{3\varsigma\beta}{\zeta_2}$ and, by using the following relation obtained from formulae (3.2),

$$\zeta_2 = \zeta_{2*}\left(\frac{\beta(1-\beta_*)}{\beta_*(1-\beta)}\right)^{1/3}, \tag{3.6}$$

it reduces to

$$\varsigma\sigma = \frac{3\varsigma\beta_*^{1/3}\beta^{2/3}(1-\beta)^{1/3}}{\gamma_1\zeta_*(1-\beta_*)^{1/3}(1-\beta)}. \tag{3.7}$$

REMARK 1. A more general model of that presented above for an elementary drop of the bubbly liquid could easily include the effects of relative rotation of the inclusion and attribute an ellipsoidal shape to the bubbles. But then the analysis would become more complex, because the inclusion would have to be modelled through a continuum with microstructure and a non-local expression for the kinetic energy density would be required, as in formulae (16) of Sec. 98 and (7) of Sec. 99 of [14].

4. Hamilton's variational principle

Let $\vartheta := (\rho_2, \beta, \mathbf{v}_1, \mathbf{v}_2)$ be a quadruple of functions that satisfy Eqs. (2.2), (2.5) and $(2.7)_1$ and let $\mathcal{A}$ be the class of all quadruples ϑ whose elements are assigned functions at the times τ_0 and τ_1. Hamilton's variational principle states that: *in the class $\mathcal{A}$ of possible motions of the mixture, the natural one is described by the quadruple which extremises the time integral of the sum of kinetic co-energy and virtual work.*

Now, by requiring the density of kinetic co–energy to be homogeneous of second degree in the material time derivatives, it coincides with the kinetic energy κ (see [15]). Also, we suppose that the virtual work is done by conservative forces only and define the following Hamiltonian functional for bubbly liquids $\mathcal{H}$:

$$\mathcal{H}[\rho_2, \beta, \mathbf{v}_1, \mathbf{v}_2] := \int_\Omega \left\{\kappa\left(\rho_2, \beta, \mathrm{grad}\beta, \frac{\partial\beta}{\partial\tau}, \mathbf{v}_1, \mathbf{v}_2\right) - (\rho_1+\rho_2)\omega - \psi(\rho_2, \beta) - \varsigma\sigma(\beta, \delta)\right\}, \tag{4.1}$$

where $\omega = \widehat{\omega}(\mathbf{x})$ is the density per unit mass of the potential energy of external body forces. Hamilton's principle then affirms that the natural motion is the one for which the first variation of the functional $\mathcal{H}$ vanishes.

We need to compute the first variation of $\mathcal{H}$ at a given quadruple ϑ in $\mathcal{A}$ along three different 'paths' of quadruples starting from ϑ in order to evaluate the dynamics equations of bubbly liquids (see Sec. 3 of [1]). To this end, we distort the flow-cylinder $\mathbf{y}_i$ of the i-th component of the mixture by choosing a function $\mathbf{u}_i$ of class $\mathcal{C}^1(\Omega, \mathcal{V})$, with $\mathcal{V}$ the translation space of $\mathcal{E}$, such that

$$\mathbf{u}_i(\mathbf{x},\tau) = 0, \qquad \operatorname{grad}\mathbf{u}_{\mathrm{i}}(\mathbf{x},\tau) = 0 \qquad \text{and} \qquad \frac{\partial \mathbf{u}_{\mathrm{i}}}{\partial \tau}\mathbf{u}_{\mathrm{i}}(\mathbf{x},\tau) = 0, \quad \forall(\mathbf{x},\tau) \in \partial\Omega, \tag{4.2}$$

and by defining for every $\epsilon \in [0, \epsilon_0]$ the function $\mathbf{f}_i^\epsilon : \Omega \to \mathcal{E}$ by

$$\mathbf{f}_i^\epsilon(\mathbf{x},\tau) := \mathbf{x} + \epsilon\,\mathbf{u}_i(\mathbf{x},\tau), \quad \forall(\mathbf{x},\tau) \in \Omega; \tag{4.3}$$

the positive constant ϵ_0 must be such that $\mathbf{f}_i^\epsilon$ is always a $\mathcal{C}^1$-diffeomorphism of $\mathcal{B}$ onto $\mathcal{B}$, for any given $\tau \in [\tau_0, \tau_1]$. In addition, we assume that the transformation $\mathbf{f}_1^\epsilon$ preserves the volume under the motion, so that it is isochoric.

In [2] it is shown that the distorted function $\mathbf{y}_i^\epsilon : \Omega \to \mathcal{B}$,

$$\mathbf{y}_i^\epsilon(\mathbf{x},\tau) := \mathbf{f}_i^\epsilon(\mathbf{y}_i(\mathbf{x},\tau),\tau), \quad (\mathbf{x},\tau) \in \Omega, \tag{4.4}$$

is the flow-cylinder of the i-th constituent that corresponds to the following velocity $\mathbf{v}_i^\epsilon$, mass density ρ_i^ϵ and fraction volume β_i^ϵ fields belonging to the same quadruple of $\mathcal{A}$:

$$\begin{aligned} \mathbf{v}_i^\epsilon(\mathbf{f}_i^\epsilon(\mathbf{x},\tau),\tau) :&= \mathbf{F}_i^\epsilon(\mathbf{x},\tau)\mathbf{v}_i(\mathbf{x},\tau) + \epsilon\frac{\partial \mathbf{u}_i}{\partial \tau}(\mathbf{x},\tau), \\ \rho_i^\epsilon(\mathbf{f}_i^\epsilon(\mathbf{x},\tau),\tau) :&= \frac{\rho_i(\mathbf{x},\tau)}{\det\left[\mathbf{F}_i^\epsilon(\mathbf{x},\tau)\right]}, \end{aligned} \tag{4.5}$$

where $\mathbf{F}_i^\epsilon := \operatorname{grad}\mathbf{f}_{\mathrm{i}}^\epsilon = \mathbf{I} + \epsilon\operatorname{grad}\mathbf{u}_{\mathrm{i}}$, with $\mathbf{I}$ the identity tensor, and

$$\beta_i^\epsilon(\mathbf{f}_i^\epsilon(\mathbf{x},\tau),\tau) := \beta(\mathbf{x},\tau). \tag{4.6}$$

In fact, we can also note that, by the isochoricity of $\mathbf{f}_1^\epsilon$ and the Eq. $(4.6)_1$,

$$\int\limits_{\mathbf{y}_1^\epsilon(\mathcal{P},\tau)} (1 - \beta_1^\epsilon(\cdot,\tau)) = \int\limits_{\mathbf{y}_1(\mathcal{P},\tau)} (1 - \beta(\cdot,\tau)), \quad \forall\mathcal{P} \subset \mathcal{B}, \tag{4.7}$$

and thus (2.7) holds for β_1^ϵ whenever it does for β.

Now, it is necessary to specify that the variation $\epsilon\mathbf{u}_i$ to the actual placement of the i-th component leads to a variation, besides the functions ρ_i^ϵ, β_i^ϵ and $\mathbf{v}_i^\epsilon$, also of all their derivatives. In particular, for our functional $\mathcal{H}$, defined by

Eq. (4.1), we are only interested in the variations of $\operatorname{grad}\beta$ and $\partial\beta/\partial\tau$. If we point out with $\partial\beta(\mathbf{x},\tau)/\partial(\mathbf{x},\tau)$ the four-vector $\left(\operatorname{grad}\beta, \dfrac{\partial\beta}{\partial\tau}\right)^T$, where the apex T indicates transposition, it is possible to draw the required variation explicitly (see Sec. 37 of [16]): by using relations (4.3) and (4.6), it is, in fact,

$$\text{(4.8)}\qquad \frac{\partial\beta_i^\epsilon(\mathbf{f}_i^\epsilon,\tau)}{\partial(\mathbf{f}_i^\epsilon\tau)} = \left(\frac{\partial(\mathbf{f}_i^\epsilon(\mathbf{x},\tau),\tau)}{\partial(\mathbf{x},\tau)}\right)^{-T} \frac{\partial\beta_i^\epsilon(\mathbf{f}_i^\epsilon(\mathbf{x},\tau),\tau)}{\partial(\mathbf{x},\tau)}$$
$$= \left[\mathbf{I} + \epsilon\left(\left(\frac{\partial\mathbf{u}_i(\mathbf{x},\tau)}{\partial(\mathbf{x},\tau)}\right)^T, 0\right)\right]^{-1} \frac{\partial\beta(\mathbf{x},\tau)}{\partial(\mathbf{x},\tau)},$$

where the current vectors and tensors in (4.8) are understood as four-dimensional.

Finally, we need a third path to obtain the extended Rayleigh equation. Let α be a positive function of class $\mathcal{C}^1$ defined on Ω, zero in τ_0 and τ_1 and such that, for every $\epsilon \in [0,\epsilon_0]$, the function

$$\text{(4.9)}\qquad \beta_0^\epsilon(\mathbf{x},\tau) := \beta(\mathbf{x},\tau) + \epsilon\alpha(\mathbf{x},\tau), \qquad \forall(\mathbf{x},\tau)\in\Omega,$$

satisfies the relation $\beta_0^\epsilon(\Omega) \subset]0,1[$, so pertaining to a quadruple of $\mathcal{A}$ also. We observe that, in this case, it is not a change of the actual placement, then it is

$$\text{(4.10)}\qquad \frac{\partial\beta_0^\epsilon(\mathbf{x},\tau)}{\partial(\mathbf{x},\tau)} = \frac{\partial[\beta+\epsilon\alpha](\mathbf{x},\tau)}{\partial(\mathbf{x},\tau)}.$$

Therefore, the quadruple $\vartheta_0^\epsilon := (\rho_2, \beta_0^\epsilon, \mathbf{v}_1, \mathbf{v}_2)$ belongs to $\mathcal{A}$ as $\vartheta_1^\epsilon := (\rho_2, \beta_1^\epsilon, \mathbf{v}_1^\epsilon, \mathbf{v}_2)$ and $\vartheta_2^\epsilon := (\rho_2^\epsilon, \beta_2^\epsilon, \mathbf{v}_1, \mathbf{v}_2^\epsilon)$ do; when $\epsilon = 0$, by (4.5), (4.6) and (4.9), it follows that $\vartheta_j^0 = \vartheta$, for $j = 0, 1, 2$. The mappings $\epsilon \mapsto \vartheta_j^\epsilon$ ($j = 0, 1, 2$) represent the three different required paths in $\mathcal{A}$ which start from ϑ.

5. The equations of motion and the generalized Rayleigh equation

Now we are in a position to define the first variation $\delta_j\mathcal{H}$ of the functional $\mathcal{H}$ along the three previously characterised paths as follows:

$$\text{(5.1)}\qquad \delta_j\mathcal{H}(\vartheta) := \left.\frac{d}{d\epsilon}\mathcal{H}[\vartheta_j^\epsilon]\right|_{\epsilon=0}, \qquad j = 0,1,2;$$

the variations $\delta_0\mathcal{H}(\vartheta)$ and $\delta_i\mathcal{H}(\vartheta)$ are therefore linear functionals of α and $\mathbf{u}_i$, respectively.

To compute $\delta_0\mathcal{H}$, we express the value of $\mathcal{H}$ in ϑ_0^ϵ by using (2.1), (2.8), (4.1) and (4.9):

$$\mathcal{H}[\vartheta_0^\epsilon] = \int\limits_\Omega \left[\rho_1\left(\frac{1}{2}\mathbf{v}_1^2 - \omega\right) + \rho_2\left(\frac{1}{2}\mathbf{v}_2^2 - \omega\right)\right] \tag{5.2}$$
$$+ \int\limits_\Omega \left\{\frac{1}{2}\gamma_1\phi(\rho_2, \beta+\epsilon\alpha)(\mathbf{v}_2 - \mathbf{v}_1)^2\right.$$
$$+ \frac{1}{2}\,\gamma\mu_1\,(\rho_2, \beta+\epsilon\alpha)\left[\frac{\partial(\beta+\epsilon\alpha)}{\partial\tau} + \mathbf{v}_1\cdot\operatorname{grad}(\beta+\epsilon\alpha)\right]^2 - \psi(\rho_2, \beta+\epsilon\alpha)$$
$$\left.+ \frac{1}{2}\,\rho_2\mu_2(\rho_2, \beta+\epsilon\alpha)\left[\frac{\partial(\beta+\epsilon\alpha)}{\partial\tau} + \mathbf{v}_2\cdot\operatorname{grad}(\beta+\epsilon\alpha)\right]^2 - \varsigma\sigma\big(\beta+\epsilon\alpha, \delta_0^\epsilon\big)\right\},$$

where $\delta_0^\epsilon := \dfrac{1}{2}\operatorname{grad}\beta_0^\epsilon\cdot\operatorname{grad}\beta_0^\epsilon$. We observe that the first integral does not depend on ϵ and so we use definition (5.1) to obtain

$$\delta_0\mathcal{H}(\vartheta) = \int\limits_\Omega \left\{\left[\frac{1}{2}\gamma_1\frac{\partial\phi}{\partial\beta}(\mathbf{v}_2 - \mathbf{v}_1)^2 - \frac{\partial\psi}{\partial\beta} - \varsigma\frac{\partial\sigma}{\partial\delta}\operatorname{grad}\beta\cdot\operatorname{grad}\alpha\right.\right. \tag{5.3}$$
$$\left.-\varsigma\frac{\partial\sigma}{\partial\beta} + \frac{1}{2}\gamma_1\frac{\partial\mu_1}{\partial\beta}(\beta')^2 + \frac{1}{2}\rho_2\frac{\partial\mu_2}{\partial\beta}(\beta')^2\right]\alpha$$
$$\left.+ \gamma_1\mu_1\beta'^1\left(\frac{\partial\alpha}{\partial\tau} + \mathbf{v}_1\cdot\operatorname{grad}\alpha\right) + \rho_2\mu_2\beta'^2\left(\frac{\partial\alpha}{\partial\tau} + \mathbf{v}_2\cdot\operatorname{grad}\alpha\right)\right\}.$$

The continuity equation (2.6), the conditions that α is zero in τ_0 and τ_1 and the integration by parts permit us to express $\delta_0\mathcal{H}$ as a linear functional of α, thus, by requiring it to vanish for each α, we derive the following equation:

$$\rho_1\left(\frac{\mu_1(\rho_2,\beta)}{1-\beta}\beta'^1\right)'^1 - \frac{1}{2}\gamma_1\frac{\partial\mu_1}{\partial\beta}(\rho_2,\beta)(\beta'^1)^2 \tag{5.4}$$
$$+ \rho_2\left[\big(\mu_2(\rho_2,\beta)\beta'^2\big)'^2 - \frac{1}{2}\frac{\partial\mu_2}{\partial\beta}(\rho_2,\beta)(\beta'^2)^2\right] = \operatorname{div}\left(\varsigma\frac{\partial\sigma}{\partial\beta}(\beta,\delta)\operatorname{grad}\beta\right)$$
$$+ \frac{1}{2}\gamma_1\frac{\partial\phi}{\partial\beta}(\rho_2,\beta)(\mathbf{v}_2 - \mathbf{v}_1)^2 - \frac{\partial\psi}{\partial\beta}(\rho_2,\beta) - \varsigma\frac{\partial\sigma}{\partial\beta}(\beta,\delta).$$

The new first variation, necessary to find the equation of motion for incompressible liquid, is obtained by computing the value of the functional $\mathcal{H}$ at ϑ_1^ϵ:

$$\begin{aligned}(5.5)\qquad \mathcal{H}[\vartheta_1^\epsilon] = \int_\Omega \rho_2\left(\frac{1}{2}\mathbf{v}_2^2-\omega\right) + \int_\Omega \Bigg\{\rho_1^\epsilon\left(\frac{1}{2}(\mathbf{v}_1^\epsilon)^2-\omega\right)\\ + \frac{\gamma_1\mu_1(\rho_2,\beta_1^\epsilon)}{2}\left(\frac{\partial\beta_1^\epsilon(\cdot,\tau)}{\partial(\cdot,\tau)}\cdot(\mathbf{v}_1^\epsilon,1)^T\right)^2 + \frac{\tilde{\mu}_2(\rho_2,\beta_1^\epsilon)}{2}\left(\frac{\partial\beta_1^\epsilon(\cdot,\tau)}{\partial(\cdot,\tau)}\cdot(\mathbf{v}_2,1)^T\right)^2\\ +\frac{1}{2}\gamma_1\phi(\rho_2,\beta_1^\epsilon)(\mathbf{v}_2-\mathbf{v}_1^\epsilon)^2-\varsigma\sigma(\beta_1^\epsilon,\delta_1^\epsilon)-\psi(\rho_2,\beta_1^\epsilon)\Bigg\},\end{aligned}$$

where

$$\delta_i^\epsilon := \frac{1}{2}\frac{\partial\beta_i^\epsilon}{\partial\mathbf{f}_i^\epsilon}\cdot\frac{\partial\beta_i^\epsilon}{\partial\mathbf{f}_i^\epsilon},\qquad i=1,2,$$

and

$$\tilde{\mu}_2(\rho_2,\beta) := \rho_2\mu_2(\rho_2,\beta).$$

A misleading error occurred in developments after formula (3.22) of [1], so we make here the right ones explicit, obviously adapted to the more general actual context.

Also in (5.5) the first integral is independent of ϵ, therefore, if we apply relations (4.5), (4.6) and (4.8) and we make the change of variables from $(\mathbf{f}_1^\epsilon,\tau)$ to $(\mathbf{x},\tau)$, which has the determinant equal to one for its isochoricity, the second integral becomes

$$\begin{aligned}(5.6)\qquad \int_\Omega \Bigg\{\rho_1\left[\frac{1}{2}(\mathbf{v}_1+\epsilon\mathbf{u}_1'^{1})^2-\omega(\mathbf{f}_1^\epsilon)\right]\\ +\frac{1}{2}\gamma_1\Bigg\{\phi(\rho_2(\mathbf{f}_1^\epsilon,\tau),\beta)\Big[\mathbf{v}_2(\mathbf{f}_1^\epsilon,\tau)-\mathbf{v}_1-\epsilon\mathbf{u}_1'^{1}\Big]^2\\ +\mu_1(\rho_2(\mathbf{f}_1^\epsilon,\tau),\beta)\Bigg\{\Bigg[\left(\mathbf{I}+\epsilon\left(\left(\frac{\partial\mathbf{u}_1(\cdot,\tau)}{\partial(\cdot,\tau)}\right)^T,0\right)\right)^{-1}\frac{\partial\beta(\cdot,\tau)}{\partial(\cdot,\tau)}\Bigg]\cdot(\mathbf{v}_1+\epsilon\mathbf{u}_1'^{1},1)^T\Bigg\}^2\Bigg\}\\ -\psi\left(\rho_2(\mathbf{f}_1^\epsilon,\tau),\beta\right)-\varsigma\sigma\Big(\beta,\frac{1}{2}\Big[(\mathbf{I}+\epsilon\,\mathrm{grad}\,\mathbf{u}_1)^{-T}\,\mathrm{grad}\,\beta\Big]\\ \cdot\Big[(\mathbf{I}+\epsilon\,\mathrm{grad}\,\mathbf{u}_1)^{-T}\mathrm{grad}\,\beta\Big]\Big)\Bigg\}\\ +\frac{1}{2}\tilde{\mu}_2(\rho_2(\mathbf{f}_1^\epsilon,\tau),\beta)\Bigg\{\Bigg[\left(\mathbf{I}+\epsilon\left(\left(\frac{\partial\mathbf{u}_1(\cdot,\tau)}{\partial(\cdot,\tau)}\right)^T,0\right)\right)^{-1}\frac{\partial\beta(\cdot,\tau)}{\partial(\cdot,\tau)}\Bigg]\cdot(\mathbf{v}_2(\mathbf{f}_1^\epsilon,\tau),1)^T\Bigg\}^2.\end{aligned}$$

By using the formula of derivation of the inverse tensor, $\dfrac{d}{d\epsilon}\left\{\mathbf{A}^{-1}\right\} = -\mathbf{A}^{-1}\left[\dfrac{d\mathbf{A}}{d\epsilon}\right]\mathbf{A}^{-1}$ for each invertible tensor $\mathbf{A}$ dependent on ϵ, we find that

$$\frac{d}{d\epsilon}\left\{\left[\mathbf{I}+\epsilon\left(\left(\frac{\partial\mathbf{u}_1(\mathbf{x},\tau)}{\partial(\mathbf{x},\tau)}\right)^T,0\right)\right]^{-1}\frac{\partial\beta(\mathbf{x},\tau)}{\partial(\mathbf{x},\tau)}\right\}\Bigg|_{\epsilon=0} = -\left(\left(\frac{\partial\mathbf{u}_1(\mathbf{x},\tau)}{\partial(\mathbf{x},\tau)}\right)^T,0\right)\frac{\partial\beta(\mathbf{x},\tau)}{\partial(\mathbf{x},\tau)}. \tag{5.7}$$

The equation (5.7) furnishes the expression of the first variations of $\operatorname{grad}\beta$ and $\dfrac{\partial\beta}{\partial\tau}$, that are, respectively, $-(\operatorname{grad}\mathbf{u}_1)^T\operatorname{grad}\beta$ and $-\dfrac{\partial\mathbf{u}_1}{\partial\tau}\cdot\operatorname{grad}\beta$. By replacing the expression (5.6) in (5.5) and by using the relation (5.7), we have:

$$\begin{aligned}\delta_1\mathcal{H}(\vartheta) &= \int_\Omega\Big\{\rho_1(\mathbf{v}_1\cdot\mathbf{u}_1'^1 - \mathbf{u}_1\cdot\operatorname{grad}\omega) + \gamma_1\phi\,(\mathbf{v}_2-\mathbf{v}_1)\cdot[(\operatorname{grad}\mathbf{v}_2)\mathbf{u}_1 - \mathbf{u}_1'^1] \\ &\quad + \left[\tilde\mu_2\beta'^2\left((\operatorname{grad}\mathbf{v}_2)\mathbf{u}_1-\mathbf{u}_1'^2\right) + \varsigma\frac{\partial\sigma}{\partial\delta}(\operatorname{grad}\mathbf{u}_1)\operatorname{grad}\beta\right]\cdot\operatorname{grad}\beta \\ &\quad + \left\{\frac{1}{2}\frac{\partial\tilde\mu_2}{\partial\rho_2}\left(\beta'^2\right)^2 + \frac{\gamma_1}{2}\left[\frac{\partial\mu_1}{\partial\rho_2}\left(\beta'^1\right)^2 + \frac{\partial\phi}{\partial\rho_2}(\mathbf{v}_2-\mathbf{v}_1)^2\right] - \frac{\partial\psi}{\partial\rho_2}\right\}\mathbf{u}_1\cdot\operatorname{grad}\rho_2\Big\}.\end{aligned} \tag{5.8}$$

In analogous manner we obtain the last first variation, needed to find the equation of motion of the compressible gas, by computing the value of the functional $\mathcal{H}$ at ϑ_2^ϵ; thus we find that

$$\begin{aligned}\delta_2\mathcal{H}(\vartheta) &= \int_\Omega\Big\{\rho_2(\mathbf{v}_2\cdot\mathbf{u}_2'^2 - \operatorname{grad}\omega\cdot\mathbf{u}_2) + \gamma_1\phi\,(\mathbf{v}_2-\mathbf{v}_1)[\mathbf{u}_2'^2 - (\operatorname{grad}\mathbf{v}_1)\mathbf{u}_2] \\ &\quad + \left[\varsigma\frac{\partial\sigma}{\partial\delta}(\operatorname{grad}\mathbf{u}_2)\operatorname{grad}\beta + \gamma_1\mu_1\beta'^1\left((\operatorname{grad}\mathbf{v}_1)\mathbf{u}_2 - \mathbf{u}_2'^1\right)\right]\cdot\operatorname{grad}\beta \\ &\quad + \Big\{\frac{\tilde\mu_2}{2}\left(\beta'^2\right)^2 + \frac{\gamma_1}{2}\left[\mu_1\left(\beta'^1\right)^2 + \phi\,(\mathbf{v}_2-\mathbf{v}_1)^2\right] - \psi - \varsigma\sigma \\ &\quad -\rho_2\left\{\frac{1}{2}\frac{\partial\tilde\mu_2}{\partial\rho_2}\left(\beta'^2\right)^2 + \frac{\gamma_1}{2}\left[\frac{\partial\mu_1}{\partial\rho_2}\left(\beta'^1\right)^2 + \frac{\partial\phi}{\partial\rho_2}(\mathbf{v}_2-\mathbf{v}_1)^2\right] - \frac{\partial\psi}{\partial\rho_2}\right\}\Big\}\operatorname{div}\mathbf{u}_2\Big\}.\end{aligned} \tag{5.9}$$

Repeated recourses to the integration by parts, together with the use of equations of continuity (2.6) and relations (4.2) and (5.4), allow us to express the right-hand sides of relations (5.8) and (5.9) in a much more compact way as explicit linear functionals of $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively; in particular, by imposing the first variations $\delta_i \mathcal{H}(\vartheta)$ to vanish for any function $\mathbf{u}_i$, $i = 1, 2$, we obtain the following further equations of motion for a bubbly liquid:

$$
\begin{aligned}
\rho_1 \left[\frac{\partial \mathbf{v}_1}{\partial \tau} + (\operatorname{grad} \mathbf{v}_1)\mathbf{v}_1 \right] &= -\rho_1 \operatorname{grad} \omega \\
&- \operatorname{div} \left[\left(\psi - \frac{1}{2}\rho_2 \mu_2 (\beta'^{2})^2 \right) \mathbf{I} + \varsigma \frac{\partial \sigma}{\partial \delta} \operatorname{grad} \beta \otimes \operatorname{grad} \beta \right] \\
&- \left\{ \rho_1 \left(\frac{\mu_1 \beta'^{1}}{1-\beta} \right)'^{1} - \frac{\gamma_1}{2} \left[\frac{\partial \mu_1}{\partial \beta} \mu_1 (\beta'^{1})^2 + \frac{\partial \phi}{\partial \beta} (\mathbf{v}_2 - \mathbf{v}_1)^2 \right] \right. \\
&\left. - \varsigma \frac{\partial \sigma}{\partial \delta} \beta + \operatorname{div} \left(\varsigma \frac{\partial \sigma}{\partial \delta} \operatorname{grad} \beta \right) \right\} \operatorname{grad} \beta + \rho_1 \left[\frac{\phi(\mathbf{v}_2 - \mathbf{v}_1)}{1-\beta} \right]'^{1} \\
&+ (\operatorname{grad} \mathbf{v}_2)^T \left[\gamma_1 \phi (\mathbf{v}_2 - \mathbf{v}_1) \right] + \frac{\gamma_1}{2} \left[\frac{\partial \mu_1}{\partial \rho_2} \left(\beta'^{1} \right)^2 + \frac{\partial \phi}{\partial \rho_2} (\mathbf{v}_2 - \mathbf{v}_1)^2 \right] \operatorname{grad} \rho_2
\end{aligned}
\tag{5.10}
$$

and

$$
\begin{aligned}
\rho_2 \left[\frac{\partial \mathbf{v}_2}{\partial \tau} + (\operatorname{grad} \mathbf{v}_2)\mathbf{v}_2 \right] &= -\rho_2 \operatorname{grad} \omega - \operatorname{div} \left[\gamma_1 \phi (\mathbf{v}_2 - \mathbf{v}_1) \otimes (\mathbf{v}_2 - \mathbf{v}_1) \right] \\
&- \operatorname{grad} \left\{ \rho_2 \left[\frac{\partial \psi}{\partial \rho_2} - \frac{\gamma_1}{2} \left(\frac{\partial \mu_1}{\partial \rho_2} \left(\beta'^{1} \right)^2 + \frac{\partial \phi}{\partial \rho_2} (\mathbf{v}_2 - \mathbf{v}_1)^2 \right) \right] - \frac{\rho_2^2 \left(\beta'^{2} \right)^2}{2} \frac{\partial \mu_2}{\partial \rho_2} \right\} \\
&- \rho_2 \left[\left(\mu_2 \beta'^{2} \right)'^{2} - \frac{1}{2} \frac{\partial \mu_2}{\partial \beta} (\beta'^{2})^2 \right] \operatorname{grad} \beta + \frac{\partial \psi}{\partial \rho_2} \operatorname{grad} \rho_2 - \rho_1 \left[\frac{\phi(\mathbf{v}_2 - \mathbf{v}_1)}{1-\beta} \right]'^{1} \\
&- (\operatorname{grad} \mathbf{v}_2)^T \left[\gamma_1 \phi (\mathbf{v}_2 - \mathbf{v}_1) \right] - \frac{\gamma_1}{2} \left[\frac{\partial \mu_1}{\partial \rho_2} \left(\beta'^{1} \right)^2 + \frac{\partial \phi}{\partial \rho_2} (\mathbf{v}_2 - \mathbf{v}_1)^2 \right] \operatorname{grad} \rho_2 .
\end{aligned}
\tag{5.11}
$$

6. Physical analysis, interpretation and comparisons

Firstly we observe from Eqs. (5.10) and (5.11) that our immiscible mixture describing a liquid with bubbles of gas satisfies Truesdell's first and second 'metaphysical' principles for ordinary mixtures (see [17]). Moreover, developments similar to those performed in [2], permit us to demonstrate that the third one, 'The motion of the mixture is governed by the same equations as is a single body', is also fulfilled.

Now we can obtain a good understanding of the effects of virtual inertia and of variations of bubble volume from equations of balance of linear momentum (5.10) and (5.11) and from the generalized Rayleigh equation (5.4) which result in our theory and we can show how the forces acting on each constituent of the mixture are affected by them.

The relevant terms, appearing with an opposite sign in Eqs. (5.10) and (5.11), represent the interaction force $\mathbf{f}$ between the phases:

$$\mathbf{f} := \rho_1 \left[\frac{\phi\,(\mathbf{v}_2 - \mathbf{v}_1)}{1-\beta} \right]^{\prime_1} + (\operatorname{grad} \mathbf{v}_2)^T \left[\gamma_1 \phi\,(\mathbf{v}_2 - \mathbf{v}_1) \right] + \frac{\gamma_1}{2} \left[\frac{\partial \mu_1}{\partial \rho_2} \left(\beta^{\prime_1}\right)^2 + \frac{\partial \phi}{\partial \rho_2} (\mathbf{v}_2 - \mathbf{v}_1)^2 \right] \operatorname{grad}\rho_2; \tag{6.1}$$

nevertheless, the terms containing ϕ in the right-hand sides of those equations, terms which express the virtual mass forces, do not add up to zero as usually assumed by some authors (see [11] and [18]): in fact there is present a contribution $\frac{1}{2}\gamma_1 \frac{\partial \phi}{\partial \beta} (\mathbf{v}_2 - \mathbf{v}_1)^2$ to the net volumetric force in the liquid phase only and an additional Reynolds stress $\rho_2 \frac{\partial \phi}{\partial \rho_2} (\mathbf{v}_2 - \mathbf{v}_1)^2\, \mathbf{I} - \gamma_1 \phi\,(\mathbf{v}_2 - \mathbf{v}_1) \otimes (\mathbf{v}_2 - \mathbf{v}_1)$ acting on the gas phase which has not a corresponding term in the liquid one. This last term can model the situation in which gas bubbles rise uniformly through quiescent liquid since their passage generates liquid phase velocity fluctuations; the remaining part of the Reynolds stress $\mathbf{R}_2$ for the compressible constituent is isotropic and represents the bulk pressure which consists of the classical pressure $-\rho_2 \frac{\partial \psi}{\partial \rho_2}$, of the 'pseudo'-chemical potential $\frac{1}{2}\rho_2^2 \frac{\partial \mu_2}{\partial \rho_2} \left(\beta^{\prime_2}\right)^2$ and of a part that stems from volume changes of the bubbles $\frac{1}{2}\gamma_1 \rho_2 \frac{\partial \mu_1}{\partial \rho_2} \left(\beta^{\prime_1}\right)^2$.

The liquid Reynolds stress is given by $\mathbf{R}_1 := -\left(\psi - \frac{1}{2}\rho_2 \mu_2 (\beta^{\prime_2})^2\right) \mathbf{I} + \varsigma \frac{\partial \sigma}{\partial \delta} \operatorname{grad} \beta \otimes \operatorname{grad} \beta$, where the addendum besides the Archimedean buoyancy force allows a difference in the diagonal elements in the direction of the volumetric bubble fraction gradient related to the interfacial geometric changes. It is clear that $\mathbf{R}_1$ and $\mathbf{R}_2$ are objective tensors since they depend on the relative velocity of the gas with respect to the liquid phase $(\mathbf{v}_2 - \mathbf{v}_1)$, on the gradient of a scalar field $\operatorname{grad}\beta$ and on the material time derivatives $\beta^{\prime_i}$, for $i = 1, 2$, which are all independent of the coordinate frame (see the Appendix of [8]).

The force of interaction $\mathbf{f}$ is also an objective quantity; to show this, we prefer to introduce an indifferent form already known (see [8] and, moreover, Eq. (7.5) of [19] with $r = 2$ and $n = 1$). It is

$$\begin{aligned}(6.2)\qquad \mathbf{f} = &\left[\rho_1 \frac{\partial}{\partial \beta}\left(\frac{\phi}{1-\beta}\right)\beta'^{1} + \gamma_1 \frac{\partial \phi}{\partial \rho_2}\rho_2'^{1}\right](\mathbf{v}_2 - \mathbf{v}_1) \\ &+ \gamma_1 \phi\left[\operatorname{grad}\mathbf{v}_1 + (\operatorname{grad}\mathbf{v}_1)^T\right](\mathbf{v}_2 - \mathbf{v}_1) \\ &+ \gamma_1\phi\left[\mathbf{v}_2'^{1} - \mathbf{v}_1'^{2} + \frac{1}{2}\operatorname{grad}(\mathbf{v}_2 - \mathbf{v}_1)^2\right] \\ &+ \frac{\gamma_1}{2}\left[\frac{\partial \mu_1}{\partial \rho_2}\left(\beta'^{1}\right)^2 + \frac{\partial \phi}{\partial \rho_2}(\mathbf{v}_2 - \mathbf{v}_1)^2\right]\operatorname{grad}\rho_2.\end{aligned}$$

The term $\gamma_1\phi(\operatorname{grad}\mathbf{v}_1)^T(\mathbf{v}_2 - \mathbf{v}_1)$ of Eq. (6.2) is an inertial lift force; $\gamma_1\phi(\operatorname{grad}\mathbf{v}_1)(\mathbf{v}_2 - \mathbf{v}_1)$ can be viewed as a convective acceleration, while $\gamma_1\phi\left[\mathbf{v}_2'^{1} - \mathbf{v}_1'^{2} + (\operatorname{grad}(\mathbf{v}_2 - \mathbf{v}_1))^T(\mathbf{v}_2 - \mathbf{v}_1)\right]$ represents the volumetric force of virtual mass; the last one measures the forces related to variations in the density of the gas and so it is very small with respect to others.

The first two terms on the right-hand side of the relation (6.2) have clearly the form of a force of drag, but they have not, just as clearly, the characteristic aspect of a classical force of resistance of Stokes' type for viscous fluids: here we study perfect fluids only; effectively it is a force of inertial drag related to convective derivatives, with respect to the liquid velocity, of the radius of bubbles and of the bulk mass density ρ_2. As observed in the introduction, terms of this type are not usually present in the known theories, but in [20] the authors showed that some models for bubbly liquids, which consider viscous drag force only, conflict evidently with experimental observations of the motion of these fluids in a circular pipe. Thus they asked 'whether this is due to an inadequate assumption about the variables of importance in the viscous drag coefficient, or whether an important term is missing'.

Here we can only observe that in bubbly liquids there could be fairly significant fluctuations of the radius of dispersed bubbles, and thus of the volume fraction β; moreover, similar terms could be recognized in recent theories (see, for example, Eq. (100) of [9]).

Finally, we want to recover the classical Rayleigh–van Wijngaarden equation (4.7) of [10] for the growth of a single bubble in an infinite perfect liquid from our balance equation (5.4). Thus firstly we have to reduce our constitutive functions to expressions of Sec. 3; moreover, we suppose that the dispersed phase occupies relatively small regions within which no large pressure gradients can be supported, therefore the gas pressure at the interface is equal to that inside the bubble and the potential energy ψ is thus related to the pressure difference in the liquid phase: $\varpi - \varpi_1^{in} = \dfrac{\partial\psi}{\partial\beta}$.

Since the density of the gas is typically small compared to the density of the liquid, the only local expansional kinetic energy is the kinetic energy of the liquid that is displaced as the bubble expands or contracts, thus Eq. (5.4) reduces to

$$\varpi_1^{in} - \varpi_1 = \varsigma\frac{\partial\sigma}{\partial\delta}\beta + \rho_1\left(\frac{\mu_1}{1-\beta}\beta'^1\right)'^1 - \frac{1}{2}\gamma_1\frac{\partial\mu_1}{\partial\beta}(\beta'^1)^2 - \frac{1}{2}\gamma_1\frac{\partial\phi}{\partial\beta}(\mathbf{v}_2-\mathbf{v}_1)^2. \tag{6.3}$$

Next we approximate the expression of the functions which appear in the Eq. (6.3), when $\beta \ll 1$, and obtain, by (3.7) and (3.5),

$$\frac{\partial\sigma}{\partial\beta} \propto \frac{2}{\beta^{1/3}}, \qquad \frac{\mu_1}{1-\beta} \propto \frac{1}{3\beta^{1/3}}, \qquad \frac{1}{2}\frac{\partial\mu_1}{\partial\beta} \propto -\frac{1}{18\beta^{4/3}} \quad \text{and} \quad \frac{1}{2}\frac{\partial\phi}{\partial\beta} \propto \frac{1}{4}, \tag{6.4}$$

for suitable constants. Now, it is sufficient to observe that $\zeta \approx \zeta_*$, for $(3.2)_2$, and therefore it is $\beta \propto \zeta_2^3$, for $(3.2)_1$, thus we have the required Eq. (4.7) of [10]:

$$\varpi_1^{in} = \varpi_1 + \frac{2\varsigma}{\zeta_2} + \gamma_1\left[\frac{3}{2}(\zeta_2'^1)^2 + \zeta_2(\zeta_2'^1)'^1 - \frac{1}{4}(\mathbf{v}_2-\mathbf{v}_1)^2\right]. \tag{6.5}$$

Obviously, if the bubble moves with the liquid in absence of surface tension, we also find the classical equation of Rayleigh for the dynamics of bubbles.

7. Conclusions

In this work, bubbly liquids are considered as a two-phase immiscible mixture of a compressible and an incompressible fluid; the dynamic equations of balance in the conservative case are obtained from a Hamiltonian variational principle deduced from a more general procedure by imposing the constraint of incompressibility on the liquid in the choice of appropriate paths of variation for functions describing the motion.

The proposed model is perfectly consistent with previous known theories derived from averaged space equations or from variational methods of material type, even if it satisfies all Truesdell's metaphysical principles in addition to the principles of material frame indifference and of equipresence, while those theories fail in some respects.

Furthermore, in the force of interaction between phases, a new term of inviscid drag due to inertia forces and related to convective derivatives, following the liquid velocity, of the bubbles radius and of the bulk mass density of the gas appears: this term was already hypothesized by Lahey Jr., Sim and Drew in their experimental observations of the motion of bubbly fluids in a circular pipe [20].

Acknowledgments

The author acknowledges the financial support by the University 'Medi–terranea' of Reggio Calabria through the program of scientific research "RdB 2003 – Constitutive Models for Materials with Microstructure". The author thanks Luigi M. Scarfò for his help in the revision.

References

1. P. Giovine, *Sulla dinamica di una miscela di due fluidi comprimibili e non miscibili*, Rend. Mat. Acc. Lincei, **1**, 9, 377–385, 1990.
2. E.G. Virga, *On the variation of the Hamiltonian functional for an immiscible mixture*, Arch. Rat. Mech. Analysis, **103**, 51–59, 1989.
3. F. Bampi, A. Morro, *The inverse problem of the calculus of variations applied to continuum physics*, J. Math. Physics, **23**–II, 2312–2321, 1982.
4. F. Bampi, A. Morro, *The connection between variational principles in Eulerian and Lagrangian descriptions*, J. Math. Physics, **25**, 2418–2421, 1984.
5. G. Capriz, *Spatial variational principles in continuum mechanics*, Arch. Rat. Mech. Analysis, **85**, 2, 99–109, 1984.
6. G. Capriz, P. Giovine, *On effects of virtual inertia during diffusion of a dispersed medium in a suspension*, Arch. Rat. Mech. Analysis, **98**, 2, 115–122, 1987.
7. L. Prandtl, O.G. Tietjens, Applied Hydro– and Aeromechanics, Dover Publications Inc., New York 1934.
8. D.A. Drew, R.T. Lahey Jr., *Application of general constitutive principles to the derivation of multi-dimensional two-phase flow equations*, Int. J. Multi. Flow, **5**, 243–264, 1979.
9. D.Z. Zhang, A. Prosperetti, *Ensemble phase-averaged equations for bubbly flows*, Phys. Fluids, **6**, 2956–2970, 1994.
10. A. Biesheuvel, L. van Wijngaarden, *Two-phase flow equations for a dilute dispersion of gas bubbles in liquid*, J. Fluid Mech., **148**, 301–318, 1984.
11. D.S. Drumheller, A. Bedford, *A theory of bubbly liquids*, J. Acoust. Soc. Am., **66**, 197–208, 1979.
12. R.M. Bowen, *Diffusion models implied by the theory of mixtures* [in:] C. Truesdell, Rational Thermodynamics (2$^{\mathrm{nd}}$ edition), Springer-Verlag, 237–263, New York 1984.
13. P. Giovine, *The theory of continua with voids as a basis for the study of aerated lubrication*, Proc. of 4-th AIMETA Congress on Tribology, 161–168, Pisa 1996.
14. H. Lamb, *Hydrodynamics* (6-th edition), Cambridge University Press, New York 1953.
15. G. Capriz, P. Giovine, *On microstructural inertia,* Math. Model Meth. Appl. Sc., **7**, 211–216, 1997.
16. I.M. Gelfand, S.V. Fomin, *Calculus of variations*, Prentice-Hall Inc., Englewood Cliffs, New Jersey 1963.

17. C. Truesdell, *Thermodynamics of diffusion*, [in:] Rational Thermodynamics (2-nd edition), Springer-Verlag, 218–236, New York 1984.

18. D.Y. Hsieh, *On dynamics of bubbly liquids*, Adv. in Appl. Mech., **26**, 63–133, 1988.

19. J.E. Adkins, *Nonlinear diffusion II. Constitutive equations for mixtures of isotropic fluids*, Phil. Trans. R. Soc. London, **A255**, 635–648, 1963.

20. R.T. Lahey Jr., S. Sim, D.A. Drew, *An evaluation of interfacial drag models for bubbly two-phase flows*, ASME Symposium on Nonequilibrium Transfer Processes, 1979.

Received November 29, 2005; revised version May 30, 2006.