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On stability of equilibrium in linear thermoviscoelasticity

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We present some results concerning the uniform Lyapunov stability of the generalized solution of an equilibrium problem for thermoviscoelastic materials [9]. These results are similar to those obtained in [4–7]. The results in [4] were subsequently developed for exterior domains in [8].

Key words: thermoviscoelastic material, Lyapunov stability, continuous dependents of initial data.

1. Introduction

Influenced by [1–3], the author obtained in [4–7] some results regarding the stability and continuous dependence of solution to equilibrium equations of inhomogeneous anisotropic and non-convolutive linear viscoelasticity. Precisely, denoting by $\mathbf{u}$ the displacement field and by $\boldsymbol{\sigma}$, $\mathbf{C}$, and $\mathbf{G}$, respectively, the stress, elasticity, and relaxation tensor fields, we have considered materials with the constitutive equation

$$\boldsymbol{\sigma}(\mathbf{x},t) = \mathbf{C}(\mathbf{x},t)\nabla\mathbf{u}(\mathbf{x},t) + \int\limits_0^t \mathbf{G}(\mathbf{x},t,s)\nabla\mathbf{u}(\mathbf{x},s)ds.$$

For such materials it has been proved that the solution $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$ of the viscoelastic equilibrium problem, in the case of bounded domains Ω, is *uniformly Lyapunov stable* (u.L.s.) with respect to the pairs of measures [1]

$$\mu_t(\mathbf{u}) := \frac{1}{2}\int\limits_\Omega \left[\rho(\mathbf{x})|\dot{\mathbf{u}}(\mathbf{x},t)|^2 + \nabla\mathbf{u}(\mathbf{x},t)\cdot\mathbf{C}(\mathbf{x},t)\nabla\mathbf{u}(\mathbf{x},t)\right]d\mathbf{x}, \quad \mu_0(\mathbf{u})$$

and

$$\nu_t(\mathbf{u}) := \int\limits_0^t d\tau \int\limits_\Omega \nabla\mathbf{u}(\mathbf{x},s)^2 d\mathbf{x}, \qquad \mu_0(\mathbf{u}).$$

Using this result, some other pairs of measures, with respect to which the viscoelastic equilibrium is stable, were pointed out.

Among other hypotheses assuring the existence of viscoelastic equilibrium, the obtained stability results heavily depend on the *dissipation condition*

$$\int_0^{td} d\tau \int_\Omega \boldsymbol{\sigma}^{(v)}(\mathbf{x},\tau)\cdot\nabla\dot{\mathbf{u}}(\mathbf{x},\tau)d\mathbf{x} \geq 0,$$

where $\boldsymbol{\sigma}^{(v)}(\mathbf{x},\tau) = \int_0^\tau \mathbf{C}(\mathbf{x},\tau,s)\nabla\mathbf{u}(\mathbf{x},s)ds$ is the *viscoelastic part* of the stress tensor.

The stability problems for materials of the "creep" or "relaxation" type, are subsequently studied in [8], for the exterior of a bounded region, by using the two above pairs of measures for the initial and current perturbations.

In this paper we get some results concerning the u.L.s., (uniform Lyapunov stability) similar to those obtained in [4–7], for the generalized solution of a Boundary-Initial Value Problem, called the *Problem* (P) in what follows, for linear thermoviscoelastic materials.

The constitutive equations of the linear thermoviscoelastic materials under consideration are presented in Sec. 2. The constitutive equations are those established by Navarro in [9].

The functional framework of the *Problem* (P), the definition of the generalized solution of this problem and sufficient conditions guaranteeing its existence, are briefly discussed in Sec. 3.

An energy equation, in fact a Lagrange–Brun-type identity [10] for thermoviscoelastic equilibrium solution, is derived in Sec. 4. This energy equation is actually the essential ingredient in our stability analysis of the solution to *Problem* (P) (*thermoviscoelastic equilibrium*).

In Sec. 5, the core of the paper, sufficient conditions are given for the Lyapunov stability of thermoviscoelastic equilibrium. In fact we present the pairs of measures [1] in comparison with which the stability is defined. All the results of this section rely on the *dissipation condition* $(5.1)_1$ which extends to thermoviscoelastic materials the above-mentioned condition in the case of viscoelastic materials. A special attention is paid in this section to the dimensional analysis of the material constants appearing in the pairs of measures, against which the u.L.s. of thermoviscoelastic equilibrium is defined.

2. Linear thermoviscoelastic materials

We consider an inhomogeneous anisotropic linear thermoviscoelastic material which, in a reference configuration with zero stress and absolute temperature $\theta_0 > 0$, occupies the Lipschitzian domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$ [11].

We denote by $\mathbf{u}(t) := \mathbf{u}(\mathbf{x}, t)$ the displacement vector, by $\vartheta(\mathbf{x}, t)$ the absolute temperature, and by $\theta(t) := \vartheta(\mathbf{x}, t) - \theta_0$ the temperature variation fields at position $\mathbf{x} \in \overline{\Omega}$ and time $t \in \mathbb{R}$.

The constitutive equations of the linear thermoviscoelastic material under consideration are those proposed in [9], namely

$$\mathbf{T}(t) = \mathbf{G}(0)\nabla\mathbf{u}(t) - \theta(t)\mathbf{L}(0) + \int_{-\infty}^{t} \dot{\mathbf{G}}(t-\tau)\nabla\mathbf{u}(\tau)d\tau - \int_{-\infty}^{t} \theta(\tau)\dot{\mathbf{L}}(t-\tau)d\tau, \tag{2.1}$$

$$\rho\eta(t) = \mathbf{L}(0)\cdot\nabla\mathbf{u}(t) + \rho\frac{c(0)}{\theta_0}\theta(t) + \int_{-\infty}^{t} \dot{\mathbf{L}}(t-\tau)\cdot\nabla\mathbf{u}(\tau)d\tau + \frac{\rho}{\theta_0}\int_{-\infty}^{t} \dot{c}(t-\tau)\theta(\tau)d\tau, \tag{2.2}$$

where $\rho := \rho(\mathbf{x})$, $\mathbf{T}(t) := \mathbf{T}(\mathbf{x}, t)$, and $\eta(t) := \eta(\mathbf{x}, t)$ are, respectively, the mass density, the Cauchy stress, and the specific entropy fields, while

$$\mathbf{G}(s) := \mathbf{G}(\mathbf{x}, s), \qquad \mathbf{L}(s) := \mathbf{L}(\mathbf{x}, t), \qquad c(s) := c(\mathbf{x}, s), \qquad s \geq 0,$$

are the relaxation tensor fields of the fourth, second and zero orders, respectively. Suppose that G_{ijkl} and L_{ij}, the components of $\mathbf{G}$ and $\mathbf{L}$ in the Cartesian coordonate system, satisfy the following symmetry properties on $\overline{\Omega} \times (-\infty, +\infty)$:

$$G_{ijkl} = G_{klij} = G_{jikl}, \qquad L_{ij} = L_{ji}, \qquad i, j, k, l = 1, 2, 3.$$

These assumptions do not follow from the Claudius–Duhem inequality within the framework of the theory of simple materials with fading memory.

Fourier's law for the heat flux vector $\mathbf{q}(t) := \mathbf{q}(\mathbf{x}, t)$

$$\mathbf{q}(t) = -\mathbf{K}\nabla\theta(t), \tag{2.3}$$

where $\mathbf{K} := \mathbf{K}(\mathbf{x})$ is the thermal conductivity tensor field, completes the constitutive equations of the linear thermoviscoelastic material. A sufficient condition for the Claudius–Duhem inequality to be fulfilled is the *uniform positive semi-definiteness* of the second order field $\mathbf{K}$, i.e.

$$\mathbf{v}\cdot\mathbf{K}(\mathbf{x})\mathbf{v} \geq 0, \qquad \forall\,\mathbf{x} \in \Omega, \qquad \forall\,\mathbf{v} \in \mathbb{R}^3. \tag{2.4}$$

The local equations of motion and coupled heat conduction are

$$(2.5)\qquad \begin{aligned} &\rho\ddot{\mathbf{u}}(t) = \operatorname{div}\mathbf{T}(t) + \rho\mathbf{b}(t),\\ &\theta_0\rho\dot{\eta}(t) = \operatorname{div}(K\nabla\theta(t)) + \rho r(t), \qquad \text{on } \Omega\times(0,\infty), \end{aligned}$$

where $\mathbf{b}(t) := \mathbf{b}(\mathbf{x},t)$ is the specific externally applied body force and $r(t) := r(\mathbf{x},t)$ is the specific external heat supply.

3. The boundary-initial value problem

We assume that

$$(3.1)\qquad \mathbf{b}(\mathbf{x},t) = \mathbf{0}, \qquad r(\mathbf{x},t) = 0, \qquad (\mathbf{x},t)\in\Omega\times(0,\infty),$$

and that the displacement and temperature difference fields are known up to the initial time $t = 0$, i.e.

$$(3.2)\qquad \mathbf{u}(\mathbf{x},t) = \mathbf{u}^{(1)}(\mathbf{x},t), \qquad \theta(\mathbf{x},t) = \theta^{(1)}(\mathbf{x},t), \qquad (\mathbf{x},t)\in\overline{\Omega}\times(-\infty,0],$$

where $\mathbf{u}^{(1)}$ and $\theta^{(1)}$ are given functions.

Using (2.1), (2.2), (3.1), and (3.2), the Eqs. (2.5) can be written in the form

$$(3.3)\qquad \begin{aligned} &\rho\ddot{\mathbf{u}}(t) = \operatorname{div}[\mathbf{G}(0)\nabla\mathbf{u}(t) - \theta(t)\mathbf{L}(0) + \mathbf{T}^{(v)}(t)] + \mathbf{b}_0(t),\\ &\frac{d}{dt}\left[\mathbf{L}(0)\cdot\nabla\mathbf{u}(t) + \frac{1}{\theta_0}\rho c(0)\theta(t) + \rho\eta^{(v)}(t)\right] = \frac{1}{\theta_0}\operatorname{div}(\mathbf{K}\nabla\theta(t)) + r_0(t), \end{aligned}$$

where $\mathbf{T}^{(v)}(t)$ — the *viscous part of the stress* — and $\eta^{(v)}(t)$ — the *viscous part of the entropy* — are defined as follows:

$$(3.4)\qquad \begin{aligned} &\mathbf{T}^{(v)}(t) := \int_0^t \dot{\mathbf{G}}(t-\tau)\nabla\mathbf{u}(\tau)d\tau - \int_0^t \theta(\tau)\dot{\mathbf{L}}(t-\tau)d\tau,\\ &\rho\eta^{(v)}(t) := \int_0^t \dot{\mathbf{L}}(t-\tau)\cdot\nabla\mathbf{u}(\tau)d\tau + \frac{1}{\theta_0}\rho\int_0^t \dot{c}(t-\tau)\theta(\tau)d\tau \end{aligned}$$

and

$$(3.5)\qquad \begin{aligned} &\mathbf{b}_0(t) := \operatorname{div}\left[\int_{-\infty}^0 \dot{\mathbf{G}}(t-\tau)\nabla\mathbf{u}^{(1)}(\tau)d\tau - \int_{-\infty}^0 \theta^{(1)}(\tau)\dot{\mathbf{L}}(t-\tau)d\tau\right],\\ &r_0(t) := -\frac{d}{dt}\left[\int_{-\infty}^0 \dot{\mathbf{L}}(t-\tau)\cdot\nabla\mathbf{u}^{(1)}(\tau)d\tau + \frac{1}{\theta_0}\rho\int_{-\infty}^0 \dot{c}(t-\tau)\theta^{(1)}(\tau)d\tau\right] \end{aligned}$$

are the *body force term* and *heat supply term, resulting from the history of strain and temperature.*

Consider the following Boundary-Initial Value Problem (in short Problem (P)):

Find a pair of functions

$$(\mathbf{u},\theta):\Omega\times(0,T)\to\mathrm{I\!R}^3\times\mathrm{I\!R},\qquad T\in(0,\infty)$$

satisfying (3.3) on $\Omega\times(0,T)$, the *null boundary condition*

$$(\mathbf{u}(\mathbf{x},t),\theta(\mathbf{x},t))=(\mathbf{0},0)\quad\text{on } \partial\Omega\times[0,T), \tag{3.6}$$

and the *initial condition*

$$(\mathbf{u}(\mathbf{x},0),\dot{\mathbf{u}}(\mathbf{x},0),\theta(\mathbf{x},0))=(\mathbf{u}^{(0)}(\mathbf{x}),\mathbf{v}^{(0)}(\mathbf{x}),\theta^{(0)}(\mathbf{x}))\quad\text{on } \overline{\Omega}, \tag{3.7}$$

where

$$(\mathbf{u}^{(0)}(\mathbf{x}),\mathbf{v}^{(0)}(\mathbf{x}),\theta^{(0)}(\mathbf{x}))=(\mathbf{u}^{(1)}(\mathbf{x},0),\dot{\mathbf{u}}^{(1)}(\mathbf{x},0),\theta^{(1)}(\mathbf{x},0))\qquad\text{on } \overline{\Omega}. \tag{3.8}$$

With regard to the null boundary condition (3.6) it is observed that the stability of every solution belonging to a linear problem is determined by the stability of null equilibrium solution.

Results concerning the uniqueness, existence, smoothness, and asymptotic behavior of the generalized solution of the Problem (P) are obtained in [9]. In fact, the generalized solution $(\mathbf{u},\theta)$ of the Problem (P), when formulated in a dimensionless form, is an element $(\mathbf{u},\theta)$ of the Hilbert space, obtained as the completion of the space

$$\{(\mathbf{v},\alpha):(\mathbf{v},\alpha)\in C^\infty([0,T];\mathbf{W}_0^{1,2}(\Omega))\times C^\infty([0,T];W_0^{1,2}(\Omega))\}$$

with respect to the norm induced by the inner product

$$\langle(\mathbf{v}_1,\alpha_1),(\mathbf{v}_2,\alpha_2)\rangle = \int_0^T d\tau\int_\Omega\Big[\dot{\mathbf{v}}_1(\tau)\cdot\dot{\mathbf{v}}_2(\tau)\nabla\mathbf{v}_1(\tau)\cdot\nabla\mathbf{v}_2(\tau)+\alpha_1(\tau)\alpha_2(\tau) + \int_0^t\nabla\alpha_1(s)\cdot\nabla\alpha_2(s)ds\Big]d\mathbf{x},$$

and satisfying the variational problem (4.15) of [9] (see Def. 4.4).

Here $\mathbf{W}_0^{1,2}(\Omega)$ is the space of vectorial functions $\mathbf{u}=(u_1,u_2,u_3):\Omega\to\mathrm{I\!R}^3$ with $u_k:\Omega\to\mathrm{I\!R}$, $k=1,2,3$, belonging to the Sobolev space $W_0^{1,2}(\Omega)$ [11, 12].

It is proved (see Theorem 4.2 of [9]) that if the initial data and supply terms satisfy the conditions:

$$(\mathbf{v}^{(0)}, \mathbf{v}^{(0)}, \theta^{(0)}) \in \mathbf{W}_0^{1,2}(\Omega) \times \mathbf{L}^2(\Omega) \times L^2(\Omega) \tag{3.9}$$

and

$$(\mathbf{b}_0, r_0) \in L^2([0,T); \mathbf{W}^{-1,2}(\Omega) \times L^2(\Omega)), \quad \dot{\mathbf{b}}_0 \in \mathbf{L}^2([0,T); \mathbf{W}^{-1,2}(\Omega)), \tag{3.10}$$

where $\mathbf{W}^{-1,2}(\Omega)$ is the dual space of $\mathbf{W}_0^{1,2}(\Omega)$ and $\mathbf{L}^2(\Omega)$ is the space of $\mathrm{I\!R}^3$-valued functions with components in $L^2(\Omega)$, then there exists a unique generalized solution of the Problem (P).

We suppose that all conditions, imposed in [9] upon the density and relaxation tensor fields of the thermoviscoelastic material guaranteeing (3.9) and (3.10) and therefore the existence and uniqueness of generalized solution of the Problem (P), are satisfied.

Concluding this section we remind that the application

$$\mathbf{v} \mapsto \|\nabla \mathbf{v}\|_0 := \left(\int_\Omega |\nabla \mathbf{v}(\mathbf{x})|^2 d\mathbf{x} \right)^{1/2} \in [0, \infty), \quad \mathbf{v} \in \mathbf{W}_0^{1,2}(\Omega), \tag{3.11}$$

is a norm on $\mathbf{W}_0^{1,2}(\Omega)$ and, on the basis of Poincaré's inequality [12, 14], this norm is equivalent to the norm

$$\mathbf{v} \mapsto \|\mathbf{v}\|_{1,2} := \|\mathbf{v}\|_0 + \|\nabla \mathbf{v}\|_0 \in [0, \infty), \qquad \mathbf{v} \in \mathbf{W}_0^{1,2}(\Omega), \tag{3.12}$$

where

$$\mathbf{v} \mapsto \|\mathbf{v}\|_0 := \left(\int_\Omega |\mathbf{v}(\mathbf{x})|^2 d\mathbf{x} \right)^{1/2} \in [0, \infty), \qquad \mathbf{v} \in \mathbf{L}^2(\Omega), \tag{3.13}$$

is the norm on $\mathbf{L}^2(\Omega)$.

4. A preliminary energy equation

Summing the scalar product of $(3.3)_1$ by $\dot{\mathbf{u}}(t)$ and the product of $(3.3)_2$ by $\theta(t)$, integrating the result over Ω, applying the divergence theorem and taking into account the boundary condition (3.6), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \int_\Omega \left[\rho(|\dot{\mathbf{u}}(t)|^2 + \frac{1}{\theta_0} c(0) \theta^2(t)) + \nabla \mathbf{u}(t) \cdot \mathbf{G}(0) \nabla \mathbf{u}(t) \right] d\mathbf{x} \\ = -\frac{1}{\theta_0} \int_\Omega \nabla \theta(t) \cdot \mathbf{K} \nabla \theta(t) d\mathbf{x} + \pi(t). \end{aligned} \tag{4.1}$$

Here

$$\pi(t) = P_0(t) - P^{(v)}(t), \qquad t \in [0,T), \tag{4.2}$$

where

$$P_0(t) = \int_\Omega [\mathbf{b}_0(t)\cdot\dot{\mathbf{u}}(t) + r_0(t)\theta(t)]d\mathbf{x} \tag{4.3}$$

is the *total power of the body force and heat supply resulting from the past history of the strain and temperature*, while

$$P^{(v)}(t) = \int_\Omega [\mathbf{T}^{(v)}(t)\cdot\nabla\dot{\mathbf{u}}(t) + \rho\dot{\eta}^{(v)}(t)\theta(t)]d\mathbf{x} \tag{4.4}$$

is the *total viscous power*, i.e. the sum of the *total viscous stress power*

$$\int_\Omega \mathbf{T}^{(v)}(t)\cdot\nabla\dot{\mathbf{u}}(t)d\mathbf{x} = \int_0^t d\tau\int_\Omega \nabla\dot{\mathbf{u}}(t)\cdot[\theta(t)\dot{\mathbf{L}}(t-\tau) - \dot{\mathbf{G}}(t-\tau)\nabla\mathbf{u}(\tau)]d\mathbf{x} \tag{4.5}$$

and the *total viscous entropy power*

$$\begin{aligned}\int_\Omega \rho\dot{\eta}^{(v)}(t)\theta(t)d\mathbf{x} &= \int_\Omega \theta(t)\frac{d}{dt}\left\{\int_0^t\left[\dot{\mathbf{L}}(t-\tau)\frac{1}{\theta_0}\rho\dot{c}(t-\tau)\theta(\tau)\right]d\tau\right\}d\mathbf{x}\\ &= \int_\Omega \theta(t)\left[\dot{\mathbf{L}}(0)\cdot\nabla\mathbf{u}(t) + \frac{1}{\theta_0}\rho\dot{c}(0)(t)\right]d\mathbf{x}\\ &+ \int_0^t d\tau\int_\Omega \theta(t)\left[\ddot{\mathbf{L}}(t-\tau)\cdot\nabla\mathbf{u}(\tau) + \frac{1}{\theta_0}\rho\ddot{c}(t-\tau)\theta(\tau)\right]d\mathbf{x}.\end{aligned} \tag{4.6}$$

If we integrate over $[0,t]\subset[0,T)$, we get

$$\mathcal{E}_t(\mathbf{u},\theta) = \mathcal{E}_0(\mathbf{u},\theta) - \frac{2}{\theta_0}\int_0^t d\tau\int_\Omega \nabla\theta(\tau)\cdot\mathbf{K}\nabla\theta(\tau)d\mathbf{x} + 2\int_0^t \pi(\tau)d\tau, \tag{4.7}$$

where

$$\mathcal{E}_t(\mathbf{u},\theta) := \int_\Omega \left[\rho(|\dot{\mathbf{u}}(t)|^2 + \frac{1}{\theta_0}c(0)\theta^2(t)) + \nabla\mathbf{u}(t)\cdot\mathbf{G}(0)\nabla\mathbf{u}(t)\right]d\mathbf{x}. \tag{4.8}$$

5. Stability of the thermoviscoelastic equilibrium

In this section we point out some uniform Lyapunov stability (u.L.s.) results of the thermoviscoelastic equilibrium, i.e. the solution of the Problem (P). We have in view generalized solutions of the Problem (P) corresponding to different initial conditions (3.7) which satisfy also the *dissipation condition*

$$(5.1)_1 \qquad \int_0^t \pi(\tau)d\tau \le 0 \iff \int_0^t P_0(\tau)d\tau \le \int_0^t P^{(v)}(\tau)d\tau, \qquad t \in [0,T).$$

This hypothesis is most important for obtaining all the results of this section. It means that, in any time interval $[0,t] \subset [0,T)$, the total work done by the body force and heat supply arising from the past history of strain and temperature, does not exceed the total work corresponding to the viscous parts of stress tensor and entropy.

In the particular case

$$\mathbf{u}^{(1)}(\mathbf{x},t) = \mathbf{0}, \qquad \theta^{(1)}(\mathbf{x},t) = 0, \qquad \text{on } \overline{\Omega} \times (-\infty, 0],$$

we have $P_0(\tau) = 0$, $\tau \in [0,t]$, and the dissipation condition $(5.1)_1$ becomes

$$(5.1)_2 \qquad \int_0^t P^{(v)}(\tau)d\tau \ge 0.$$

Remark 1. The analogue of this hypothesis for linear viscoelasticity is essential in deriving the sufficient condition assuring u.L.s. of the viscoelastic equilibrium [4, 7].

Theorem 1. *The solution of the Problem* (P) *satisfying the dissipation condition* $(5.1)_1$ *is u.L.s. with respect to the measures [9, Secs. 6–8].*

$$(5.2) \qquad \mathcal{E}_t(\mathbf{u},\theta) \quad \textit{and} \quad \mathcal{E}_0(\mathbf{u},\theta),$$

that is

$$(5.3) \qquad \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon)(=\varepsilon) : \mathcal{E}_0(\mathbf{u},\theta) < \delta \Longrightarrow \mathcal{E}_t(\mathbf{u},\theta) < \varepsilon, \qquad t \in [0,T).$$

P r o o f. Taking into account (2.4) and $(5.1)_1$, from (4.7) we obtain

$$(5.4) \qquad \mathcal{E}_t(\mathbf{u},\theta) \le \mathcal{E}_0(\mathbf{u},\theta), \qquad t \in [0,T),$$

and therefore we have (5.3). □

Remark 2. Lyapunov stability of the thermoviscoelastic equilibrium implies its continuous dependence on the initial data, whereas the continuous dependence of the thermoviscoelastic equilibrium on the initial data does not imply its Lyapunov stability [9] (Sec. 8). On the other hand, from (4.7) it follows that $(5.1)_1$ implies (5.4) and therefore follows the continuous dependence of thermoviscoelastic equilibrium on the initial data. Thus the dissipation condition $(5.1)_1$ of the solution of the Problem (P) is a sufficient condition for the continuous dependence of this solution on the initial data.

We shall use the following assumptions [9] belonging to the set of hypothesis assuring the existence of the generalized solution of the Problem (P):

$$0 < \rho_0 = \operatorname*{ess\,inf}_{\Omega} \rho(\mathbf{x}) \le \operatorname*{ess\,sup}_{\Omega} \rho(\mathbf{x}) = \rho_1 < \infty, \tag{5.5}$$

$$0 < c_0 = \operatorname*{ess\,inf}_{\Omega} c(\mathbf{x},0) \le \operatorname*{ess\,sup}_{\Omega} c(\mathbf{x},0) = c_1 < \infty. \tag{5.6}$$

Supposing that $\mathbf{G}(0)$ is a *positive definite tensor*, *uniformly with respect to* $\mathbf{x} \in \Omega$, [13, Section 29], on *Lin* (the space of second order tensors) [14], it results that there exists the constant $g_0 > 0$ such that

$$\int_{\Omega} \nabla \mathbf{u}(t) \cdot \mathbf{G}(0) \nabla \mathbf{u}(t) d\mathbf{x} \ge g_0 \|\nabla \mathbf{u}(t)\|_0^2, \qquad t \in [0,T), \tag{5.7}$$

and we obtain

$$g_0 \|\nabla \mathbf{u}(t)\|_0^2 \le \int_{\Omega} \nabla \mathbf{u}(t) \cdot \mathbf{G}(0) \nabla \mathbf{u}(t) d\mathbf{x} \le g_1 \|\nabla \mathbf{u}(t)\|_0^2, \qquad t \in [0,T), \tag{5.8}$$

where

$$0 < g_1 = \operatorname*{ess\,sup}_{\Omega} \left(\sum_{i,j,k,l=1}^{n} G_{ijkl}^2(0) \right)^{1/2} < \infty.$$

We notice that the material constants $\rho_0, \rho_1,\ c_0, c_1,\ g_0, g_1$, have the physical dimensions

$$\begin{gathered} \dim \rho_0 = \dim \rho_1 = ML^{-3}; \qquad \dim c_0 = \dim c_1 = L^2 T^{-2} \Theta^{-1}; \\ \dim g_0 = \dim g_1 = ML^{-1} T^{-2}. \end{gathered} \tag{5.9}$$

From (5.5), (5.6), (5.8), and (4.8) we obtain

$$\rho_0 \mu_t(\mathbf{u},\theta) \le \mathcal{E}_t(\mathbf{u},\theta) \le \mathcal{E}_0(\mathbf{u},\theta) \le \rho_1 \mu_0(\mathbf{u},\theta), \qquad t \ge 0, \tag{5.10}$$

where

$$\begin{aligned} \mu_t(\mathbf{u},\theta) &:= \|\dot{\mathbf{u}}(\cdot,t)\|_0^2 + \alpha_0\|\theta(\cdot,t)\|_0^2 + \beta_0\|\nabla\mathbf{u}(\cdot,t)\|_0^2, \\ \mu_0(\mathbf{u},\theta) &:= \|\dot{\mathbf{u}}(\cdot,0)\|_0^2 + \alpha_1\|\theta(\cdot,0)\|_0^2 + \beta_1\|\nabla\mathbf{u}(\cdot,0)\|_0^2. \end{aligned} \tag{5.11}$$

Here $\alpha_0 = c_0/\theta_0 > 0$, $\alpha_1 = c_1/\theta_0 > 0$, $\beta_0 = g_0/\rho_0 > 0$, $\beta_1 = g_1/\rho_1 > 0$ are material constants with

$$\dim\alpha_0 = \dim\alpha_1 = L^2T^{-2}\Theta^{-2}, \qquad \dim\beta_0 = \dim\beta_1 = L^2T^{-2}. \tag{5.12}$$

Thus we have the following

THEOREM 2. *The solution of the Problem* (P) *satisfying the dissipation condition* $((5.1)_1)$ *is u.L.s. with respect to the measures*

$$\mu_t(\mathbf{u},\theta) \quad \textit{and} \quad \mu_0(\mathbf{u},\theta).$$

If we write Poincaré's inequality for the displacement $\mathbf{u}(\cdot,t) \in W_0^{1,2}(\Omega)$, $t \in [0,T)$, in the dimensional form [14]

$$\gamma\delta\|\mathbf{u}(\cdot,t)\|_0 \le \|\nabla\mathbf{u}(\cdot,t)\|_0, \tag{5.13}$$

where $\gamma > 0$ is a dimensionless (genuine) constant depending only on Ω and δ is the unit having $\dim\delta = L^{-1}$; then, owing to the equivalence of norms (3.11) and (3.12) on $W_0^{1,2}(\Omega)$, we have

$$\gamma_0[\![\mathbf{u}(\cdot,t)]\!]_{1,2}^2 \le \|\nabla\mathbf{u}(\cdot,t)\|_0^2 \le \gamma_1[\![\mathbf{u}(\cdot,t)]\!]_{1,2}^2, \qquad t \in [0,T) \tag{5.14}$$

where $\gamma_0 > 0$, $\gamma_1 > 0$ are genuine constants and

$$[\![\mathbf{u}(\cdot,t)]\!]_{1,2} := \delta\|\mathbf{u}(\cdot,t)\|_0 + \|\nabla\mathbf{u}(\cdot,t)\|_0, \qquad t \in [0,T), \tag{5.15}$$

is the dimensional form of (3.12).

From (5.11) and (5.14) we obtain

$$\begin{aligned} \nu_t(\mathbf{u},\theta) &:= \|\dot{\mathbf{u}}(\cdot,t)\|_0^2 + \alpha_0\|\theta(\cdot,t)\|_0^2 + \gamma_0\beta_0[\![\mathbf{u}(\cdot,t)]\!]_{1,2}^2 \le \mu_t(\mathbf{u},\theta), \\ \mu_0(\mathbf{u},\theta) &\le \|\dot{\mathbf{u}}(\cdot,0)\|_0^2 + \alpha_1\|\theta(\cdot,0)\|_0^2 + \gamma_1\beta_1[\![\mathbf{u}(\cdot,0)]\!]_{1,2}^2 := \nu_0(\mathbf{u},\theta). \end{aligned} \tag{5.16}$$

From here it results that the thermoviscoelastic equilibrium (the solution of Problem (P)) is u.L.s. with respect to the measures

$$\nu_t(\mathbf{u},\theta) \quad \text{and} \quad \nu_0(\mathbf{u},\theta).$$

Now we point out a *partial stability result* of thermoviscoelastic equilibrium *with respect* to the displacement $\mathbf{u}(\mathbf{x},t)$.

Theorem 3. *If* $(5.1)_1$ *holds, then the displacement component of the solution* $(\mathbf{u},\theta)$ *of the Problem* (P) *is u.L.s. with respect to the measures*

$$\|\dot{\mathbf{u}}(\cdot,t)\|_0^2 + a_0\|\mathbf{u}(\cdot,t)\|_0^2 \quad \text{and} \quad \nu_0(\mathbf{u},\theta),$$

where $a_0 = \gamma_0\beta_0\delta^2 > 0$, $\dim a_0 = T^{-2}$.

The result follows from $(5.16)_1$ and (5.15).

If, instead of (2.4), we suppose the thermal conductivity tensor $\mathbf{K}$ is positive definite, uniformly with respect to $\mathbf{x} \in \Omega$, then we have

$$\int_\Omega \nabla\theta(t)\cdot\mathbf{K}\nabla\theta(t)d\mathbf{x} \geq k\|\nabla\mathbf{u}(\cdot,t)\|_0^2, \quad t\in[0,T), \tag{5.17}$$

where $k > 0$ has the dimension of the conductivity tensor $\mathbf{K}$, i.e.

$$\dim k = MLT^{-3}\Theta^{-1}. \tag{5.18}$$

From $(5.1)_1$, (4.7), and (5.17) we get

$$\lambda_t(\mathbf{u},\theta) := \mathcal{E}_t(\mathbf{u},\theta) + \frac{2k}{\theta_0}\int_0^t \|\nabla\theta(\cdot,s)\|_0^2 ds \leq \mathcal{E}_0(\mathbf{u},\theta), \quad t\in[0,T). \tag{5.19}$$

So we have the following:

Theorem 4. *Under hypothesis* (5.17)*, the solution of the Problem* (P) *satisfying* $(5.1)_1$ *is u.L.s. with respect to the measures*

$$\lambda_t(\mathbf{u},\theta) \quad \text{and} \quad \lambda_0(\mathbf{u},\theta) = \mathcal{E}_0(\mathbf{u},\theta).$$

We should remember that (5.17) is one of the hypotheses ensuring the existence of generalized solution to the Problem (P) [9].

Now we demonstrate the *partial stability result* of thermoviscoelastic equilibrium *with respect to temperature* $\theta(\mathbf{x},t)$.

Theorem 5. *If* (5.17) *and* $(5.1)_1$ *hold, then the temperature component* θ *of the solution* $(\mathbf{u},\theta)$ *of the Problem* (P) *is u.L.s. with respect to the measures*

$$\varphi_t(\theta) := \|\theta(\cdot,t)\|_0^2 + \kappa\int_0^t \|\nabla\theta(\cdot,s)\|_0^2 ds \quad \text{and} \quad \mathcal{E}_0(\mathbf{u},\theta), \tag{5.20}$$

where $\kappa = 2k/\rho_0\alpha_0\theta_0 > 0$ *and* $\dim\kappa = L^2T^{-1}$.

Indeed, from (5.10), $(5.11)_1$, and (5.19) we have

$$\|\theta(\cdot,t)\|_0^2 + \kappa \int_0^t \|\nabla\theta(\cdot,s)\|_0^2 ds \le h\mathcal{E}_0(\mathbf{u},\theta) \tag{5.21}$$

where $h = (\rho_0\alpha_0)^{-1} > 0$ and $\dim h = M^{-1}LT^2\Theta^2$.

Remark 3. To some extent this theorem reflects the fact that, in the framework of the theory simple materials with memory, the present value of the temperature gradient does not enter the constitutive equations.

Applying the Poincaré's inequality to the function $\theta(\cdot,t) \in W_0^{1,2}(\Omega)$, $t \in [0,T)$, we obtain

$$\gamma\delta\|\theta(\cdot,t)\|_0 \le \|\nabla\theta(\cdot,s)\|_0, \quad s \in [0,T). \tag{4.13$'$}$$

Introducing this result into (5.21) we obtain

$$\|\theta(\cdot,t)\|_0^2 + a \int_0^t \|\theta(\cdot,s)\|_0^2 ds \le h\mathcal{E}_0(\mathbf{u},\theta), \tag{5.22}$$

where $a = \gamma^2\kappa\delta^2 > 0$ with

$$\dim a = T^{-1}. \tag{5.23}$$

Multiplying (5.22) by e^{at} (at is dimensionless!) and integrating the result over the time interval $[0,t) \subset [0,T)$, we get

$$\int_0^t \|\theta(\cdot,s)\|_0^2 ds \le \frac{h}{a}(1-e^{-at})\mathcal{E}_0(\mathbf{u},\theta), \quad t \in [0,T),$$

whence we obtain the following *global boundedness* of temperature

$$\lim_{t\to T} \int_0^t \|\theta(\cdot,s)\|_0^2 ds := \int_0^T ds \int_\Omega \theta^2(\mathbf{x},s)d\mathbf{x} \le \frac{h}{a}\mathcal{E}_0(\mathbf{u},\theta), \tag{5.24}$$

where $h/a > 0$ and $\dim(h/a) = M^{-1}LT^3\Theta^2$.

6. Concluding remarks

The results of Sec. 5. are practically the same if, instead of the null boundary condition (3.6), we consider the null mixed boundary conditions for components $\mathbf{u}$ and θ of the solution $(\mathbf{u},\theta)$ of the system (3.3).

The solution $(\mathbf{u},\theta)$ of the Problem (P) is completely determined by the initial history $(\mathbf{u}^{(1)},\theta^{(1)})$ and therefore the fact whether a solution $(\mathbf{u},\theta)$ satisfies or not the dissipation condition $(5.1)_1$ depends only on the initial history.

A solution $(\mathbf{u},\theta)$ of the Problem (P) is called a *finite energy solution* [15] if there exists a constant $M \geq \mathcal{E}_0(\mathbf{u},\theta)$ such that

$$\mathcal{E}_t(\mathbf{u},\theta) + \frac{2}{\theta_0}\int_0^t ds \int_\Omega \nabla\theta(s)\cdot \mathbf{K}\nabla\theta(s)d\mathbf{x} \leq M, \qquad t \in [0,T).$$

Of course, a solution $(\mathbf{u},\theta)$ of the Problem (P) satisfying the dissipation condition $(5.1)_1$ is a finite energy solution with $M = \mathcal{E}_0(\mathbf{u},\theta)$.

It is my belief that the properties of finite energy solutions of the system (3.3) are essential in studying the asymptotic partition and equipartition of energies [15] in linear thermoviscoelastic materials.

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