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Brief Note

A note on tensile instabilities and loss of ellipticity for a fiber-reinforced nonlinearly elastic solid

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In this paper we examine the loss of ellipticity and the associated failure of fiber-reinforced compressible nonlinearly elastic solids under deformations leading to fiber extension. In particular, the analysis concerns a material model that consists of an isotropic base material augmented by a reinforcement depending on the fiber direction and referred to as a reinforcing model. We examine a reinforcement that introduces additional stiffness under simple shear deformations in the fiber direction. In previous contributions it was shown for this material that loss of ellipticity under uniaxial tensile loading in the fiber direction requires a non-convex reinforcing model. Here we generalize this result and show that loss of ellipticity under plane deformations not associated with uniaxial loading in the fiber direction but also creating fiber extension may occur for convex reinforcing models.

Key words: nonlinear elasticity, loss of ellipticity, fiber reinforcement, reinforcing models, transverse isotropy.

1. Introduction

In recent years, different materials have been analyzed in the context of anisotropic finite-strain elasticity. These include, among other, biological, composite and synthetic solids. In particular, many of these materials are modelled as fiber-reinforced nonlinearly elastic solids since these materials often exhibit nonlinear behavior during service and are anisotropic. In nonlinear elasticity the constitutive equation of the material can be given in terms of a strain-energy function that depends on independent deformation invariants. In this framework, some of the studies that can be found in the literature just focus on certain characteristics of the strain energies, for instance, polyconvexity [1, 2], ellipticity [3] or deformation invariant formulations [4] (see also the references therein). Other works focus on the mechanical response of the constitutive equations since these are used to model different materials, for instance [5]. Furthermore, a variety of phenomena related to the behavior of fiber-reinforced materials have been observed. These include, among other topics, fiber kink broadening [6] and cavitation instabilities [7].

A recent series of articles [8–13] within the framework of nonlinear elasticity has developed a continuum mechanical model to observe fiber failure or fiber instabilities in fiber-reinforced nonlinearly elastic solids under plane deformation. The onset of failure is assumed to occur at loss of ellipticity of the governing differential equations. At the breakdown of ellipticity the surfaces of discontinuity may arise inside the material. Depending on the loading regime and on the direction of the normal to these surfaces of discontinuity relative to the fiber direction, the mechanism of failure was interpreted in terms of fiber kinking, fiber splitting, fiber de-bonding or matrix failure. Furthermore, the loss of ellipticity condition is related to both the convexity of the function that gives the strain energy of the material as well as the convexity and monotonicity of the nominal stress in particular directions of that material. Fiber kinking and fiber splitting were associated with fiber compression. Fiber de-bonding and matrix failure were associated with fiber extension. The failure mechanisms were established under uniaxial loadings in the fiber direction. For plane deformations that give fiber compression the loss of ellipticity analysis and the associated failure was also carried out.

The anisotropy of homogeneous fiber-reinforced nonlinearly elastic solids is characterized by two independent deformation invariants in three dimensions. We denote these invariants by I_4 and I_5. It is well known that the isotropy of a compressible material is characterized by the three invariants I_1, I_2, I_3 of the Cauchy–Green deformation tensors. The combination of these five invariants give the more general homogeneous transversely isotropic and nonlinearly elastic solid. For fiber-reinforced nonlinearly elastic materials, the strain energy function is considered to be given by two terms: one term that reflects the isotropic character of the material, i.e. a function that depends only on the three invariants I_1, I_2, I_3; and another term that reflects the transversely isotropic character of the material, i.e. a function that depends on the two direction-dependent invariants I_4, I_5. The second anisotropic term is referred to as the reinforcing model. In our analysis and following [8] a constitutive model consisting of an isotropic base material augmented by a uniaxial reinforcement depending on only one of the two anisotropic invariants has been used.

A detailed discussion of these points can be found in [8–13]. One question remained open: whether or not for plane deformations not related to uniaxial loading in the fiber direction and obeying simultaneously $I_5 > 1$ and $I_4 > 1$ the loss of ellipticity requires non-convex reinforcing models. This is on what we focus here and, in particular, we show that for I_5-based anisotropy convex reinforcing models may lose ellipticity under those circumstances.

The paper is organized as follows. In Sec. 2, we introduce the basic notation and some necessary relations in nonlinear elasticity. The material model and the ellipticity condition are introduced in Sec. 3. In Sec. 4, the main result of the paper is established, namely, it is shown that a convex reinforcing model depending on I_5 may lose ellipticity under plane deformations not related to uniaxial tensile loading in the fiber direction but that also create fiber extension. The analysis is not restricted to plane deformations. In Sec. 5 we summarize and discuss briefly the results obtained in the previous sections.

2. Notation

For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ we let $\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|\mathbf{a}\|^2_{\mathbb{R}^3} = \langle \mathbf{a}, \mathbf{a} \rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3\times 3}$ the set of real 3×3 second order tensors, written in capital letters. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle \mathbf{X}, \mathbf{Y} \rangle_{\mathbb{M}^{3\times 3}} = \operatorname{tr} \mathbf{X}\ \mathbf{Y}^T$, and thus the Frobenius tensor norm is $\|\mathbf{X}\|^2 = \langle \mathbf{X}, \mathbf{X} \rangle_{\mathbb{M}^{3\times 3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{M}^{3\times 3}$. The identity tensor on $\mathbb{M}^{3\times 3}$ will be denoted by $\mathbf{I}$, so that $\operatorname{tr} \mathbf{X} = \langle \mathbf{X}, \mathbf{I} \rangle$. We let Sym and Psym denote the symmetric and positive-definite symmetric tensors respectively. By $\operatorname{Adj} \mathbf{X}$ we denote the tensor of transposed cofactors $\operatorname{Cof}(\mathbf{X})$ such that $\operatorname{Adj} \mathbf{X} = \det \mathbf{X}\, \mathbf{X}^{-1} = \operatorname{Cof}(\mathbf{X})^T$ if $\mathbf{X} \in \mathrm{GL}(\mathbf{3}, \mathbb{R})$. For vectors $\xi, \eta \in \mathbb{R}^n$ we have the tensor product $(\xi \otimes \eta)_{ij} = \xi_i\, \eta_j$. We write the polar decomposition in the form $\mathbf{F} = \mathbf{R}\,\mathbf{U} = \operatorname{polar}(\mathbf{F})\,\mathbf{U}$ with $\mathbf{R} = \operatorname{polar}(\mathbf{F})$ the orthogonal part of $\mathbf{F}$. In general we work in the context of nonlinear, finite elasticity. For total deformation $\varphi \in C^1(\overline{\Omega}, \mathbb{R}^3)$ we have the deformation gradient $\mathbf{F} = \nabla\varphi \in C(\overline{\Omega}, \mathbb{M}^{3\times 3})$. The first and second differential of a scalar-valued function $W(\mathbf{F})$ are written $D_{\mathbf{F}} W(\mathbf{F}).\mathbf{H}$ and $D^2_{\mathbf{F}} W(\mathbf{F}).(\mathbf{H}, \mathbf{H})$, respectively.

3. Constitutive equations and ellipticity

We consider a homogeneous transversely isotropic compressible elastic solid. The transverse isotropy is characterized by the existence of a single fiber direction defined by a unit vector field denoted by $\mathbf{a}$ in the reference configuration. The strain-energy function (defined per unit reference volume) W depends on the five independent invariants mentioned in Sec. 1, i.e. on I_1, I_2, I_3, I_4 and I_5. Hence, we write

$$W = W(I_1, I_2, I_3, I_4, I_5). \tag{3.1}$$

The invariants I_1, I_2 and I_3 are the principal invariants of the left Cauchy–Green deformation tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, or of the right Cauchy–Green deformation tensor

$\mathbf{C} = \mathbf{F}^T\mathbf{F}$, where $\mathbf{F}$ is the deformation gradient tensor relative to the natural or undeformed configuration. Therefore,

$$(3.2) \qquad I_1(\mathbf{C}) := \mathrm{tr}[\mathbf{C}], \qquad I_2(\mathbf{C}) := \frac{1}{2}\left[(\mathrm{tr}[\mathbf{C}])^2 - \mathrm{tr}[\mathbf{C}]^2\right], \qquad I_3(\mathbf{C}) := \det\mathbf{C}.$$

The invariants I_4 and I_5 are associated with the fiber reinforcement and depend on $\mathbf{a}$ as well as on $\mathbf{C}$. They are defined by

$$(3.3) \qquad \begin{aligned} I_4(\mathbf{F}) &:= \langle \mathbf{C}, \mathbf{a}\otimes\mathbf{a}\rangle = \langle \mathbf{a}, \mathbf{C}.\mathbf{a}\rangle = \langle \mathbf{F}.\mathbf{a}, \mathbf{F}.\mathbf{a}\rangle = \|\mathbf{F}.\mathbf{a}\|^2\,, \\ I_5(\mathbf{F}) &:= \langle \mathbf{C}^2, \mathbf{a}\otimes\mathbf{a}\rangle = \langle \mathbf{a}, \mathbf{C}^2.\mathbf{a}\rangle = \langle \mathbf{C}.\mathbf{a}, \mathbf{C}.\mathbf{a}\rangle = \|\mathbf{C}.\mathbf{a}\|^2\,, \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F}\,. \end{aligned}$$

We note that $\sqrt{I_4}$ has an immediate interpretation as the stretch in the direction $\mathbf{a}$, as can be seen from $(3.3)_1$. Thus, I_4 registers deformations that modify the length of the fiber. In particular, in terms of rectangular Cartesian basis vectors $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, with $\mathbf{a} = \mathbf{i}_1$, we have simply $I_4 = C_{11}$. Similarly, $I_5 = C_{11}^2 + C_{12}^2 + C_{13}^2$. Hence, in general, I_5 registers changes in the fiber reinforcement length by means of the indicator C_{11} and shear deformations via the indicators C_{12} and C_{13}. In a more general sense, the invariant I_5 is related to the fiber stretch but registers, additionally, the reaction of the reinforcement to shear deformations and to deformations of surface area elements normal to the fiber direction [9]. It follows that $I_4 > 1$ implies that $I_5 > 1$.

For fiber-reinforced materials, it is customary to simplify the general expression of the constitutive equation and consider that the strain energy is given by two terms: one associated with the isotropic base of the material and another one associated with the anisotropic character of the material. Therefore, the strain energy can be represented as

$$(3.4) \qquad \widehat{W}(\mathbf{F}) := W_{\mathrm{iso}}(I_1, I_2, I_3) + W_{\mathrm{aniso}}(I_4, I_5)\,.$$

where W_{iso} characterizes the isotropic base and W_{aniso} characterizes the anisotropic part of the material model. The latter term is referred as **reinforcing model**. It follows that W_{iso} is a function that depends at most on the three invariants I_1, I_2, I_3. Similarly, W_{aniso} is a function that depends at most on the two invariants I_4, I_5. In our ellipticity analysis we will further restrict W_{aniso} to be a function of just I_5. Furthermore, we will discard W_{iso} as it may be appropriate for strongly anisotropic materials. When we refer to convexity of W_{aniso}, it is clear that we mean the convexity w.r.t. I_5 and not the convexity of the composed function $\mathbf{F} \mapsto W_{\mathrm{aniso}}(I_5(\mathbf{F}))$. It follows that the nominal stress $\mathbf{S}_1$ corresponding to $\widehat{W}(\mathbf{F})$ is

$$(3.5) \qquad \begin{aligned} \mathbf{S}_1 = {} & 2W_{\mathrm{iso}1}\mathbf{F}^T + 2W_{\mathrm{iso}2}(I_1\mathbf{I} - \mathbf{C})\mathbf{F}^T + 2I_3W_{\mathrm{iso}3}\mathbf{F}^{-1} \\ & + 2W_{\mathrm{aniso}4}\mathbf{a}\otimes\mathbf{F}\mathbf{a} + 2W_{\mathrm{aniso}5}(\mathbf{a}\otimes\mathbf{F}\mathbf{C}\mathbf{a} + \mathbf{C}\mathbf{a}\otimes\mathbf{F}\mathbf{a}), \end{aligned}$$

where the subscripts $1, \ldots, 3$ on W_{iso} indicate differentiation with respect to $I_1, \ldots, I_3$, respectively, the subscripts $4, 5$ on W_{aniso} indicate differentiation with respect to I_4 and I_5, and $\mathbf{I}$ is again the identity tensor.

The energy function and the stress must vanish in the reference configuration (where $I_1 = I_2 = 3$ and $I_3 = I_4 = I_5 = 1$). Restrictions on $\widehat{W}$ in the reference configuration can be found in [9].

For plane deformations only two of I_1, I_2, I_3 are independent. Furthermore, if the fiber direction is assumed to be in the considered plane, I_4 and I_5 are connected through I_1 and I_3 [9]. The in-plane part of the material response depends then just only on I_1, I_2 and I_4 or on any equivalent set of three independent invariants. The ellipticity of the governing two-dimensional equations also depends on only one anisotropic invariant, either I_4 or I_5. Nevertheless, reinforcing models depending on either of the invariants I_4, I_5, or a combination of both, introduce a distinct anisotropic character to the material model due to the splitting of the strain energy. Whence, and from that point of view, the ellipticity analysis of different reinforcing models has to be considered separately.

3.1. Equilibrium and ellipticity

In the absence of body forces the equation of equilibrium can be written as

$$\operatorname{Div} \mathbf{S}_1 = \mathbf{0}\,, \qquad \mathbf{S}_1 = \frac{\partial \widehat{W}}{\partial \mathbf{F}} = D\widehat{W}(\mathbf{F})\,, \tag{3.6}$$

where $\mathbf{S}_1$ is the nominal stress tensor. We say that $\widehat{W}(\mathbf{F})$ induces a (strictly) Legendre–Hadamard elliptic system or that the equations of equilibrium $(3.6)_1$ are elliptic if

$$\forall \xi, \eta \in \mathbb{R}^3,\ \xi, \eta \neq 0: \quad D^2_{\mathbf{F}}\widehat{W}(\mathbf{F}).(\xi \otimes \eta, \xi \otimes \eta) \geq \mathbf{0} \quad (>)\,, \tag{3.7}$$

where $D^2_{\mathbf{F}}\widehat{W}(\mathbf{F}).(\mathbf{H}, \mathbf{H})$ denotes the second differential of $\widehat{W}(\mathbf{F})$ evaluated in the direction $\mathbf{H}$. In what follows we will consider a very special, simplified case in which $W_{\text{iso}} \equiv 0$. This assumption is taken since we are interested in highly anisotropic materials. The effect of the matrix in a composite material in different analysis is disregarded with respect to the effect of fiber reinforcement. For instance, it is usually the case that the elastic modulus of the fiber is much higher than the one of the matrix and the fiber is assumed to bear the load in the fiber direction. Whence, the constitutive equation is

$$\widehat{W}(\mathbf{F}) = W_{\text{aniso}}(I_5(\mathbf{F})) = W_{\text{aniso}}(\|\,\mathbf{C}.\mathbf{a}\,\|^2)\,. \tag{3.8}$$

We focus on the ellipticity analysis of this strain energy function, i.e on the ellipticity analysis of the invariant I_5. To this end we need to compute the second differential of $\widehat{W}(\mathbf{F})$, namely $D^2_{\mathbf{F}}\widehat{W}(\mathbf{F}).(\mathbf{H},\mathbf{H})$ and evaluate the second differential for rank-one tensors $\mathbf{H}=\xi\otimes\eta$. First, we compute the first differential. It is

$$D_{\mathbf{F}}\widehat{W}(\mathbf{F}).\mathbf{H} = W'_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2)\ 2\langle \mathbf{C}.\mathbf{a}, (\mathbf{F}^T\mathbf{H}+\mathbf{H}^T\mathbf{F}).\mathbf{a}\rangle\,. \tag{3.9}$$

Hence, the second differential is obtained as

$$\begin{aligned} D^2_{\mathbf{F}}\widehat{W}(\mathbf{F}).(\mathbf{H},\mathbf{H}) &= 4\,W''_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2)\,\langle \mathbf{C}.\mathbf{a}, (\mathbf{F}^T\mathbf{H}+\mathbf{H}^T\mathbf{F}).\mathbf{a}\rangle^2 \\ &\quad + 2W'_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2)\ \left[\|(\mathbf{F}^T\mathbf{H}+\mathbf{H}^T\mathbf{F}).\mathbf{a}\|^2 + 2\langle \mathbf{C}.\mathbf{a}, \mathbf{H}^T\mathbf{H}.\mathbf{a}\rangle\right]\,. \end{aligned} \tag{3.10}$$

Now we specify the direction $\mathbf{H}$ as $\mathbf{H}=\xi\otimes\eta$. Since for arbitrary $\mathbf{v}\in\mathbb{R}^3$

$$\mathbf{H}^T\mathbf{H}.\mathbf{v} = (\xi\otimes\eta)^T(\xi\otimes\eta).\mathbf{v} = (\eta\otimes\xi)\,\xi\langle\eta,\mathbf{v}\rangle = \eta\,|\xi|^2\langle\eta,\mathbf{v}\rangle\,, \tag{3.11}$$

it follows that

$$\mathbf{H}^T\mathbf{H} = |\xi|^2\ \eta\otimes\eta\,. \tag{3.12}$$

Moreover,

$$\begin{aligned} (\mathbf{F}^T\mathbf{H}+\mathbf{H}^T\mathbf{F}).\mathbf{a} &= (\mathbf{F}^T(\xi\otimes\eta)+(\xi\otimes\eta)^T\mathbf{F}).\mathbf{a} \\ &= \mathbf{F}^T(\xi\otimes\eta).\mathbf{a}+(\eta\otimes\xi)\mathbf{F}.\mathbf{a} = \mathbf{F}^T.\xi\,\langle\eta,\mathbf{a}\rangle+\eta\,\langle\xi,\mathbf{F}.\mathbf{a}\rangle\,. \end{aligned} \tag{3.13}$$

Using (3.12) and (3.13), the second differential (3.10) can be written finally as

$$\begin{aligned} &D^2_{\mathbf{F}}\widehat{W}(\mathbf{F}).(\xi\otimes\eta,\xi\otimes\eta) \\ &\quad = 4\,W''_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2)\ \big(\langle\mathbf{C}.\mathbf{a},\mathbf{F}^T.\xi\rangle\,\langle\eta,\mathbf{a}\rangle+\langle\mathbf{C}.\mathbf{a},\eta\rangle\,\langle\mathbf{F}^T.\xi,\mathbf{a}\rangle\big)^2 \\ &\quad + 2W'_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2)\ \big[\|\,\mathbf{F}^T.\xi\,\langle\eta,\mathbf{a}\rangle+\eta\,\langle\xi,\mathbf{F}\rangle.\mathbf{a}\,\|^2 + 2\,|\xi|^2\,\langle\eta,\mathbf{a}\rangle\,\langle\mathbf{F}.\mathbf{a},\mathbf{F}.\eta\rangle\big]\,. \end{aligned} \tag{3.14}$$

A simple proof of the necessary and sufficient conditions for $\widehat{W}$ to be elliptic in the reference configuration can be found in [10]. In what follows we just note that the reinforcing model that we are considering, i.e. $W_{\text{aniso}}(I_5(\mathbf{F}))$, obeys the condition

$$W'_{\text{aniso}}(I_5)>0\ (<0)\qquad\text{for}\quad I_5>1\ (<1),\qquad W'_{\text{aniso}}(1)=0, \tag{3.15}$$

and that

$$W'_{\text{aniso}}(I_5)\to-\infty\,(\infty)\qquad\text{as}\qquad I_5\to 0\,(\infty). \tag{3.16}$$

Without any loss of generality we may take $W_{\text{aniso}}(1)=0$. These conditions guarantee that $W_{\text{aniso}}(I_5(\mathbf{F}))$ is elliptic in the reference configuration. Furthermore, by continuity, it is elliptic in some neighbourhood of the reference configuration in the space of deformation gradients $\mathbf{F}$. These conditions also guarantee that the strain energy and the stress vanish in the reference configuration.

4. Ellipticity of I_5-reinforcing models

We are now concerned with the ellipticity of the energy function (3.8). More in particular, we focus on the ellipticity analysis of this strain energy function when the fiber is extended, i.e. when $I_4 > 1$ (that implies $I_5 > 1$). Since by (3.15) the strain energy function is initially convex in some neighbourhood of the reference configuration, we try to establish if the strain energy needs to lose convexity to lose ellipticity, i.e. we try to construct non-elliptic deformations that give fiber extension when $W''_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2)$ and $W'_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2)$ are positive. It has been proved in [13] that under uniaxial loading in the fiber direction, the loss of ellipticity implies that the strain energy is not convex. Here we focus on deformations that do not give uniaxial load in the fiber direction. It is shown that the strain energy may lose ellipticity and be convex with respect to I_5 simultaneously.

4.1. Simple cases: uniaxial load in the principal directions

Let us investigate first two simple cases: uniaxial load in the fiber direction and uniaxial load transverse to the fiber direction. For the former case, $\eta = \lambda\,\mathbf{a}$, $\lambda \in \mathbb{R}$. Under these circumstances, the second differential (3.14) can be written as

$$
\begin{aligned}
(4.1)\qquad & D^2_{\mathbf{F}}\widehat{W}(\mathbf{F}).(\xi\otimes\eta,\xi\otimes\eta) \\
&= 4\,W''_{\text{aniso}}(\|\,\mathbf{C}.\mathbf{a}\,\|^2)\,\left(\langle\mathbf{C}.\mathbf{a},\mathbf{F}^T.\xi\rangle\,\lambda\,\|\mathbf{a}\|^2 + \lambda\,\langle\mathbf{C}.\mathbf{a},\mathbf{a}\rangle\,\langle\mathbf{F}^T.\xi,\mathbf{a}\rangle\right)^2 \\
&\quad + 2W'_{\text{aniso}}(\|\,\mathbf{C}.\mathbf{a}\,\|^2)\,\Big[\|\,\mathbf{F}^T.\xi\,\langle\eta,\mathbf{a}\rangle + \eta\,\langle\xi,\mathbf{F}.\mathbf{a}\rangle\,\|^2 \\
&\quad + 2\,|\xi|^2\,\lambda^2\,\langle\mathbf{a},\mathbf{a}\rangle\,\langle\mathbf{F}.\mathbf{a},\mathbf{F}.\mathbf{a}\rangle\Big].
\end{aligned}
$$

It follows that if $W''_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2)$ and $W'_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2)$ are positive, then (4.1) is positive and loss of ellipticity is not possible.

Let us focus now on uniaxial load transverse to the fiber direction. In this case $\langle\eta,\mathbf{a}\rangle = 0$. Under these circumstances, the second differential (3.14) can be written as

$$
\begin{aligned}
(4.2)\qquad D^2_{\mathbf{F}}\widehat{W}(\mathbf{F}).(\xi\otimes\eta,\xi\otimes\eta) &= 4W''_{\text{aniso}}\left(\|\mathbf{C}.\mathbf{a}\|^2\right)\,\left(\langle\mathbf{C}.\mathbf{a},\eta\rangle\langle\mathbf{F}^T.\xi\mathbf{a}\rangle\right)^2 \\
&\quad + 2W'_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2)\,\|\,\eta\,\langle\xi,\mathbf{F}.\mathbf{a}\rangle\,\|^2\ .
\end{aligned}
$$

As in the previous case, if $W''_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2)$ and $W'_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2)$ are positive, then (4.2) is positive and loss of ellipticity is not possible. These results are in agreement with the results given in [9].

4.2. The case with $I_4 \geq 1$ as a side condition

Now, loss of ellipticity for (3.8) is analyzed when the condition $I_4(\mathbf{F}) = \|\mathbf{F}.\mathbf{a}\|^2 \geq 1$ is imposed. We take the second differential from (3.14) as

$$D^2_{\mathbf{F}}\widehat{W}(\mathbf{F}).(\xi\otimes\eta,\xi\otimes\eta) = 4\,W''_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2)\,\big(\langle\mathbf{C}.\mathbf{a},\mathbf{F}^T.\xi\rangle\,\langle\eta,\mathbf{a}\rangle + \langle\mathbf{C}.\mathbf{a},\eta\rangle\,\langle\mathbf{F}^T.\xi,\mathbf{a}\rangle\big)^2 + 2W'_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2)\,\Big[|\mathbf{F}^T.\xi|^2\,\langle\eta,\mathbf{a}\rangle^2 + 2\langle\mathbf{F}^T.\xi,\eta\rangle\,\langle\eta,\mathbf{a}\rangle\,\langle\mathbf{F}^T.\xi,\mathbf{a}\rangle + |\xi|^2\,\langle\mathbf{F}^T.\xi,\mathbf{a}\rangle^2 + 2\,|\xi|^2\,\langle\eta,\mathbf{a}\rangle\,\langle\mathbf{F}.\mathbf{a},\mathbf{F}.\eta\rangle\Big]\,. \tag{4.3}$$

Without any loss of generality, we assume that $|\xi| = \mathbf{1}$. It follows that the second differential (4.3) can be written as

$$\begin{aligned} &D^2_{\mathbf{F}}\widehat{W}(\mathbf{F}).(\xi\otimes\eta,\xi\otimes\eta)\\ &= 4\,W''_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2)\,\big(\langle\mathbf{C}.\mathbf{a},\mathbf{F}^T.\xi\rangle\,\langle\eta,\mathbf{a}\rangle + \langle\mathbf{C}.\mathbf{a},\eta\rangle\,\langle\mathbf{F}^T.\xi,\mathbf{a}\rangle\big)^2\\ &+ 2W'_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2)\,\Big[|\mathbf{F}^T.\xi|^2\,\langle\eta,\mathbf{a}\rangle^2 + 2\langle\mathbf{F}^T.\xi,\eta\rangle\,\langle\eta,\mathbf{a}\rangle\,\langle\mathbf{F}^T.\xi,\mathbf{a}\rangle\\ &+ \langle\mathbf{F}^T.\xi,\mathbf{a}\rangle^2 + 2\,\langle\eta,\mathbf{a}\rangle\,\langle\mathbf{F}.\mathbf{a},\mathbf{F}.\eta\rangle\Big]\,. \end{aligned} \tag{4.4}$$

We try to find $\mathbf{F}$ and directions ξ, $\eta \in \mathbb{R}^3$ such that (4.4) will be nonpositive. We additionally assume $|\eta| = 1$. The deformation gradient $\mathbf{F} \in \mathbb{M}^{3\times3}$ has to satisfy the following nonlinear two conditions:

$$\begin{aligned} \det[\mathbf{F}] > 0\,, &\quad \text{non-singular deformation}\,,\\ \|\mathbf{F}.\mathbf{a}\|^2 \geq 1\,, &\quad \text{the side condition } I_4 \geq 1\,. \end{aligned} \tag{4.5}$$

We further look for $\mathbf{F}$ that obey the condition

$$\begin{aligned} &\|\mathbf{F}^T.\xi\|^2 = \varepsilon^2\,\|\xi\|^2, \quad \text{compression, but not in the fiber direction,}\\ &\langle\eta,\mathbf{a}\rangle < 0\,, \qquad \langle\mathbf{C}.\mathbf{a},\eta\rangle = \langle\mathbf{F}.\mathbf{a},\mathbf{F}.\eta\rangle > 0 \end{aligned} \tag{4.6}$$

where ε is a small positive number. The nine components of $\mathbf{F} \in \mathbb{M}^{3\times3}$ have to obey the five conditions in (4.5)–(4.6). After a straightforward computation, the second differential (4.4) using (4.5), $(4.6)_1$ and $|\eta| = 1$ yields

$$\begin{aligned} D^2_{\mathbf{F}}\widehat{W}(\mathbf{F}).(\xi\otimes\ \eta,\xi\otimes\eta) \leq \varepsilon^2\,\big(&16\,W''_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2)\,\|\mathbf{C}.\mathbf{a}\|^2\\ &+8\,W'_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2)\big) + 4\,W'_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2)\,\langle\eta,\mathbf{a}\rangle\,\langle\mathbf{F}.\mathbf{a},\mathbf{F}.\eta\rangle\,. \end{aligned} \tag{4.7}$$

Now, let $\varepsilon \to 0$ and use $(4.6)_2$ in (4.7). It follows that, since $W'_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2) > 0$,

$$D^2_{\mathbf{F}}\widehat{W}(\mathbf{F}).(\xi \otimes \eta, \xi \otimes \eta) < \mathbf{0}\,. \tag{4.8}$$

Let us complete the proof by showing that the five conditions (4.5)–(4.6) can be met in the **planar case**. For some $\alpha > 0$ and $\delta > 0$, we take

$$\begin{gathered}
\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \eta = \frac{1}{\sqrt{1+\alpha^2}} \begin{pmatrix} -\alpha \\ 1 \end{pmatrix}, \qquad \xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
\langle \mathbf{a}, \eta \rangle = \frac{-\alpha}{\sqrt{1+\alpha^2}} < 0\,, \\
\mathbf{F} = \begin{pmatrix} \beta & \gamma \\ 0 & \sqrt{\delta - \gamma^2} \end{pmatrix}, \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \begin{pmatrix} \beta^2 & \beta\gamma \\ \beta\gamma & \delta \end{pmatrix}, \\
\det[\mathbf{F}] = \beta\,\sqrt{\delta - \gamma^2}, \\
\mathbf{F}^T = \begin{pmatrix} \beta & 0 \\ \gamma & \sqrt{\delta - \gamma^2} \end{pmatrix}, \qquad \mathbf{F}^T.\xi = (\sqrt{\delta - \gamma^2}) \cdot \xi\,, \\
\|\mathbf{F}^T.\xi\|^2 = \delta - \gamma^2\,, \\
\langle \mathbf{C}.\mathbf{a}, \eta \rangle = \frac{-\alpha\,\beta^2 + \beta\,\gamma}{\sqrt{\delta - \gamma^2}}, \qquad \langle \mathbf{C}.\mathbf{a}, \mathbf{a} \rangle = \beta^2\,.
\end{gathered} \tag{4.9}$$

Now choose

$$\beta > 1\,, \qquad -\alpha\,\beta^2 + \beta\,\gamma > 0\,, \qquad \delta - \gamma^2 = \varepsilon^2\,. \tag{4.10}$$

This example shows a non-elliptic deformation gradient obeying $I_4 \geq 1$ for positive $W''_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2)$ and $W'_{\text{aniso}}(\|\mathbf{C}.\mathbf{a}\|^2)$ and finishes our discussion.

5. Discussion

This analysis has been motivated by instability phenomena in fiber-reinforced composite solids. In particular, the materials under consideration are isotropic base materials augmented by a function that accounts for the existence of fiber reinforcement (the reinforcing model). The onset of fiber failure is established on the basis of loss of ellipticity of the governing differential equations for the considered elastic materials. A detailed analysis of the ellipticity status of the compressible I_5 reinforcing model, without restricting to plane strain deformation, has been provided under fiber extension. In particular, it has been found that loss of ellipticity (and hence fiber failure) is to be expected under fiber extension for which $I_4 > 1$ and $I_5 > 1$. Nevertheless, this is not the situation *under uniaxial tensile loading in the fiber direction*, for which loss of ellipticity does require that convexity of the reinforcing model $W_{\text{aniso}}(I_5)$ should be lost at a prior deformation. It follows then that the mechanical response curves under plane deformation of these materials will be convex at loss of ellipticity [5]. Therefore, loss of ellipticity and the associated failure is not signalled by the loss of monotonicity or convexity of the mechanical response of the material. This situation is not analogous to the one established for reinforcing models depending on I_4, for which ellipticity loss was related to the loss of convexity or monotonicity of the nominal stress in the fiber direction [11], [12].

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