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Modelling effective properties of composite materials using the inclusion concept. General considerations

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THIS PAPER IS DEVOTED to some general theoretical considerations concerning the modelling of the effective properties of composite materials based on the inclusion concept. Starting from the kinematical integral equation for inhomogeneous materials, all principal homogenisation methods are reviewed and analysed. Special attention is focused on three approaches, namely the self-consistent scheme, the Mori–Tanaka method and incremental procedure derived from the differential scheme. Mono-site and multi-site versions of these approximate solutions are considered. Limitations of the traditional self-consistent scheme are recognized. Improvements are proposed such as composite or coated inclusions and the incremental method mentioned above. Direct and iterative procedures allowing the determination of strain concentration tensors derived from the integral equation are established. The numerical implementation of all the schemes presented in this article will be considered in the next paper. Extreme configurations will be analysed such as composites with voids or very stiff inclusions with respect to matrix properties.

Key words: composite materials, inclusion concept, homogenisation methods, self-consistent, coated inclusion, incremental scheme.

1. Introduction

SINCE THE FUNDAMENTAL WORK of ESHELBY [1], the concept of ellipsoidal inclusion embedded in a surrounding matrix has been applied to model many physically important problems [2, 3, 4, 5]. The idea of using this concept for predicting the effective properties of an inhomogeneous continuum such as composite materials, has been introduced by ESHELBY himself [1]. Numerous models have been developed during the last three decades. To date, the Self-Consistent Scheme (SCS) appears as a powerful tool to predict the effective properties of polycrystalline materials, both in elasticity as well as in elasto-plasticity [6–8]. Some trials of its direct application to the natural or manufactured multi-phase

composites have demonstrated the inadequacy of such an approach [8–10]. In fact, an aberrant behaviour is predicted when the reinforcement volume or mass fraction and/or the degree of inhomogeneity between this reinforcement and the matrix become important [8, 9]. This is due to the poor description of interactions between the reinforcement and the surrounding material introduced by the self-consistent assumption. In fact, in this approach the inclusion modelling an inhomogeneity is embedded in the effective material whose properties can be very different from those of the medium surrounding the inclusion.

At least three improvements have been proposed to circumvent this fault. In the Mori–Tanaka model (MTM) [10], the inclusion directly interacts with a uniform matrix, which is submitted to an overall load or constraint. We can cite for example the works by TAKO *et al.* [11], BREBAN and BAPTISTE [12], EL MOUDEN [13], or more recently MA *et al.* [14] and XUN *et al.* [15].

To describe more accurately the interactions of the heterogeneity with the surrounding material, the idea of a "composite" inclusion has been developed. In such an approach, the reinforcement is coated with a shell of the matrix material and the resulting composite is embedded in an infinite body having the unknown properties of the homogenous equivalent medium. This model, initially introduced by FRÖHLICH and SACK [16], has been accurately solved by CHRISTENSEN and LO [17] for elastic materials and extended to the non-linear materials by HERVÉ and ZAOUI [18]. Recently, CHERKAOUI *et al.* [19] have proposed an alternative but approximate solution of the coated inclusion.

Another approximate method, called Differential Scheme (DS), was proposed by BRUGGEMAN [20] to predict the effective conductivity of inhomogeneous media. The DS is based on a progressive construction of the composite material. The effective properties of the composite are obtained by a gradual addition of infinitesimal quantities of reinforcements. The actual effective properties are determined using the small concentration relationship. To find the material global properties, a differential tensorial equation has to be solved. This method has been used and improved by many authors. ROSCOE [21] proposed an interpretation of this method based on the assumption that there exist reinforcements of different sizes. The construction of the composite begins with the insertion of the smallest inclusions and ends with the largest ones. In this way, at each step, the application of the small concentration procedure is justified. BOUCHER [22] has used the DS to predict the behaviour of porous materials. He obtained a good agreement of the predicted properties with experimental results. Works by MCLAUGHLIN [23], SALGANIK [24], LAWS and DVORAK [25], HASHIN [26] can be cited to illustrate the application of the DS in the field of viscous and cracked media. It is important to stress that all these applications are limited to the case of isotropic or transversally isotropic behaviour of the constituents.

Their geometry is frequently limited to spherical, cylindrical or penny-shaped forms.

Another type of approach, called asymptotic homogenisation method, has been developed by BENSOUSSAN *et al.* [27], DUVAUT [28], SANCHEZ–PALENCIA [29]. This method is used to calculate the effective properties of heterogeneous materials with periodic structure. The principal hypotheses are the periodic character of the phase distribution and small dimension of the period against the one of the studied domain. A unit cell can thus be defined allowing the homogenisation process. An analytic solution can be derived if the unit cell has a simple geometry. In the general case corresponding to the majority of situations, the finite element method is used to give a solution. This method has been used by BEGIS *et al.* [30] for the determination of the equivalent properties of composite materials or by BATEL [31] to find the equivalent properties of wood materials. One of the problems involved in this method is the assumption of the microstructure periodicity. Many solids, such as porous ceramics, have a random microstructure architecture and therefore this assumption may cause important errors, see TAKANO *et al.* [32]. MOULINEC and SUQUET [33–35] have proposed a numerical approach to compute the effective properties of linear and nonlinear composites with complex and periodic microstructures. Their approach allows to describe non-linear elasticity at infinitesimal strain as well as the elastic-plastic (J_2-flow theory) behaviour at small strains. This development is based on the solution of integral equation in Fourier space. The microstructure is obtained by digitalisation of any micrograph. The method is practically limited to 2D situations because of very time-consuming calculations.

The theoretical developments carried out in this paper are these of mean field theories leading to the kinematical integral equation of the Lippman-Schwinger type with piecewise uniform properties and strain tensor inside the constituents. The general tendency in this field theory is to get as narrow as possible bounds for the global elastic properties of composites, taking into account the material microstructure details. The systematic treatment initially proposed by KRÖNER and KOCH [36] and KRÖNER [37] is based on the use of point-correlation functions. More recently, PONTE CASTAÑEDA and WILLIS [38] have developed the Hashine–Shtrikman type bounds for overall moduli of a composite containing inclusions with their spatial locations having ellipsoidal forms. STOLZ and ZAOUI [39], and BORNERT *et al.* [40] have proposed the new concept of a "morphologically representative pattern" to derive bounds of Voigt–Reuss type in the case of any patterns distribution, or Hashine–Shtrikman type for isotropic pattern distributions.

The main purpose of this paper is to establish a general systematic approach for the numerical prediction of the global elastic properties of composite mate-

rials using the Eshelby's inclusion concept. The theoretical results obtained are implemented in a general homogenisation software. This implementation and some comparative results will be presented in our next paper. The kinematical integral equation of Lippman–Schwinger type is introduced in Sec. 2. This equation appears as a formal solution of the description problem of inhomogeneous materials' behaviour.

Section 3 is connected with the basic approach leading to the generalised Mori–Tanaka method, incremental scheme and self-consistent model. Following FASSI–FEHRI [41], FASSI-FEHRI *et al.* [42] and ZATTARIN and LIPINSKI [43], tensors describing the interactions between inclusions through a reference medium are introduced. These operators are used to construct one-site and multi-site approximations of the integral equation. Section 4 is concerned with the developments using coated inclusions. The new corrected version of interaction tensors, introduced first by CHERKAOUI [44] and CHERKAOUI *et al.* [19], is used to develop two classes of one-site and multi-site models based respectively on the Mori–Tanaka and self-consistent approximations. In the last paragraph, we treat the general case of multi-site incremental scheme. The developments proposed by VIÉVILLE [45, 46] VIÉVILLE and LIPINSKI [47], BROOHM *et al.* [48] are generalised in this section. The presented developments can be extended to the case of nonlinear elastic or elasto-plastic composites using stress and strain rates instead of stress and strain measures.

2. Kinematical integral equation and strain concentration tensor

Let us consider an elementary representative volume V of a macro-homogeneous and micro inhomogeneous body in the sense introduced by HILL [2]. Our goal is to determine the overall behaviour of this body. We restrict our considerations to the case of a linear elastic constitutive law and the small transformation approximation. The usual Hooke's law is supposed to be valid on the global and local levels. Let $\mathbf{E}$ and $\boldsymbol{\Sigma}$ be respectively the global strain and stress tensors such that:

$$\Sigma_{ij} = C^{\text{eff}}_{ijkl}\, E_{kl}. \tag{2.1}$$

$\mathbf{C}^{\text{eff}}$ is the global fourth-order elastic constant tensor having the usual symmetries. A similar relationship can be locally written for each point $\mathbf{r}$ of the elementary representative volume (ERV)

$$\sigma_{ij}(\mathbf{r}) = c_{ijkl}(\mathbf{r})\varepsilon_{kl}(\mathbf{r}), \tag{2.2}$$

where $\boldsymbol{\sigma}(\mathbf{r})$, $\boldsymbol{\varepsilon}(\mathbf{r})$ and $\mathbf{c}(\mathbf{r})$ are respectively the local stress, strain and elastic constant tensors. We suppose that all constituents of the ERV are perfectly bonded

together. Under these conditions, one can demonstrate the following relationships between the macroscopic and microscopic quantities:

$$\Sigma_{ij} = \frac{1}{V}\int_V \sigma_{ij}(\mathbf{r})\, dV, \tag{2.3}$$

$$E_{ij} = \frac{1}{V}\int_V \varepsilon_{ij}(\mathbf{r})\, dV. \tag{2.4}$$

The expressions (2.3) and (2.4) are obtained supposing the homogeneous boundary conditions of Dirichlet and Neumann type, see for instance WILLIS [49].

Following Hill [2], we now introduce a fourth-order strain concentration tensor **A** such that

$$\varepsilon_{ij}(\mathbf{r}) = A_{ijkl}(\mathbf{r})\, E_{kl}. \tag{2.5}$$

Combining (2.2), (2.3), and (2.5) and comparing the resulting expression with (2.1), one can determine the effective properties of the ERV

$$C^{\text{eff}}_{ijkl} = \frac{1}{V}\int_V c_{ijmn}(\mathbf{r})\, A_{mnkl}(\mathbf{r})\, dV. \tag{2.6}$$

This expression constitutes the solution of our problem. It clearly appears that the main difficulty in determining the effective properties of any inhomogeneous material is to calculate the strain concentration tensor **A**. To establish a general approach allowing the determination of this concentration tensor, we remind the formal solution describing the deformation of the representative volume of an inhomogeneous material under homogeneous boundary conditions of Dirichlet type. This solution is called kinematical integral equation and has been proposed in elasticity initially by Elmer and then generalised by DEDERICHS and ZELLER [50].

To obtain this integral equation, the Green tensor technique is used for the infinite body with the homogeneous reference properties C^r_{ijkl}. All details of such an approach are given for example in the works [49, 50, 62]. The final solution takes the form of integral equations expressing the displacement and strain fields.

$$u_m(\mathbf{r}) = U^r_m(\mathbf{r}) + \int_V G_{im,j}(\mathbf{r}-\mathbf{r}')\delta c_{ijkl}(\mathbf{r}')\varepsilon_{kl}(\mathbf{r}')dV', \tag{2.7}$$

$$\boldsymbol{\varepsilon}_{mn}(\mathbf{r}) = E^r_{mn} - \int_V \Gamma_{mnij}(\mathbf{r}-\mathbf{r}')\delta c_{ijkl}(\mathbf{r}')\varepsilon_{kl}(\mathbf{r}')dV'. \tag{2.8}$$

The following new quantities have appeared in these expressions:

- $G_{im}(\mathbf{r}-\mathbf{r}')$ – the Green tensor of the elasticity problem describing the displacement at point $\mathbf{r}$ in direction i, due to the unit force applied at point $\mathbf{r}'$ in direction m.
- $\Gamma_{mnij}(\mathbf{r}-\mathbf{r}') = -\frac{1}{2}\big(G_{im,jn}(\mathbf{r}-\mathbf{r}') + G_{in,jm}(\mathbf{r}-\mathbf{r}')\big)$ – the so-called modified Green tensor,
- $U^r_m(\mathbf{r})$ – displacement field of the reference medium,
- $E^r_{mn} = \frac{1}{2}\big(U^r_{m,n} + U^r_{n,m}\big)$ – homogeneous strain field of the reference medium which is generally different from the imposed strain $\mathbf{E}$,
- $\delta c_{ijkl}(\mathbf{r}) = c_{ijkl}(\mathbf{r}) - C^r_{ijkl}(\mathbf{r})$ – represents the deviation part of local properties from the properties of reference medium $\mathbf{C^r}$.

Expressions (2.7) and (2.8) constitute the formal solution of the problem and allow us to deduce a form of the concentration tensor $\mathbf{A}(\mathbf{r})$. This tensor is obtained via an iterative procedure, see for instance [4, 36]. To explain this procedure, let's suppose the existence of another concentration tensor such that:

$$\boldsymbol{\varepsilon}(\mathbf{r}) = \mathbf{a}(\mathbf{r}) : \mathbf{E}^r. \tag{2.9}$$

The initial or zero approximation of this concentration tensor is:

$$(\mathbf{a})_0(\mathbf{r}) = \mathbf{I} \tag{2.10}$$

leading to

$$\boldsymbol{\varepsilon}(\mathbf{r}) \approx (\boldsymbol{\varepsilon})_0 = \mathbf{E}^r, \tag{2.11}$$

where $\mathbf{I}$ is a fourth-rank unit tensor.

Introducing (2.11) into the second term of (2.8), one can deduce the first order approximations of strain field and concentration tensor.

$$(\boldsymbol{\varepsilon})_1(\mathbf{r}) = \left(\mathbf{I} - \int_V \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}') : \delta\mathbf{c}(\mathbf{r}')dV'\right) : \mathbf{E}^r$$

and

$$(\mathbf{a})_1(\mathbf{r}) = \left(\mathbf{I} - \int_V \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}') : \delta\mathbf{c}(\mathbf{r}')dV'\right).$$

This process has to be repeated in order to obtain the desired solution. KRÖNER [36, 37] has proved that this procedure is monotonically convergent to the exact values of $\boldsymbol{\varepsilon}(\mathbf{r})$ and $\mathbf{a}(\mathbf{r})$. Each supplementary step permits to take into account

a supplementary higher order correlation between the material constituents expressed by the terms $\mathbf{\Gamma}(\mathbf{r}-\mathbf{r}')$ and similar. After n steps, the concentration tensor can be written as follows:

$$
\begin{aligned}
(2.12)\qquad (\mathbf{a})_n(\mathbf{r}) = \mathbf{I} &- \int_V \mathbf{\Gamma}(\mathbf{r}-\mathbf{r}') : \delta\mathbf{c}(\mathbf{r}')\mathbf{dV}' \\
&+ \int_V \mathbf{\Gamma}(\mathbf{r}-\mathbf{r}') : \delta\mathbf{c}(\mathbf{r}') : \int_V \mathbf{\Gamma}(\mathbf{r}'-\mathbf{r}'') : \delta\mathbf{c}(\mathbf{r}'')dV''dV' \\
&...(-1)^n \int_V \mathbf{\Gamma}(\mathbf{r}-\mathbf{r}') : \delta\mathbf{c}(\mathbf{r}') : \int_V \mathbf{\Gamma}(\mathbf{r}'-\mathbf{r}'') : \delta\mathbf{c}(\mathbf{r}'') \\
&: .. : \int_V \mathbf{\Gamma}(r^{(n-1)} - r^n) : \delta\mathbf{c}(\mathbf{r}^n)dV^n..dV'
\end{aligned}
$$

In the general case, and for an arbitrary choice of the reference medium, the convergence of this procedure is rather slow.

To derive the concentration tensor $\mathbf{A}(\mathbf{r})$, which is necessary to calculate the effective material properties (see relationship (2.6)), one has to average expression (2.9) to deduce the link between $\mathbf{E}^r$ and $\mathbf{E}$. Relationship (2.4) combined with (2.9) yields:

$$
E_{ij} = \frac{1}{V}\int_V \varepsilon_{ij}(\mathbf{r})dV = \left(\frac{1}{V}\int_V a_{ijkl}(\mathbf{r})dV\right) E^r_{kl}
$$

or

$$
(2.13)\qquad E^r_{ij} = \left(\frac{1}{V}\int_V a_{ijkl}(\mathbf{r})dV\right)^{-1} E_{kl} = \langle a_{ijkl}(\mathbf{r})\rangle^{-1} E_{kl}.
$$

The brackets $\langle\bullet\rangle$ define volume averaging of the bracketed quantity. Expression (2.13) combined with relationships (2.5) and (2.9) allows us to derive the strain concentration tensor

$$
(2.14)\qquad \mathbf{A}(\mathbf{r}) = \mathbf{a}(\mathbf{r}) : \langle\mathbf{a}(\mathbf{r})\rangle^{-1},
$$

verifying the obvious relation

$$
(2.15)\qquad \langle\mathbf{A}(\mathbf{r})\rangle = \mathbf{I}.
$$

In practice, this formal solution is too complex to obtain any relevant result. For instance, the analytical expression of Green tensor exists only for isotropic or transversally isotropic materials. In order to derive such a result, we have to construct a simplified but sufficiently detailed model of the material microstructure.

3. Modelling based on the Eshelby's inclusion concept

3.1. Description of the microstructure

It is easy to understand that the quality of the predicted effective properties of an inhomogeneous material, such as composites, depends on how fine the description of its microstructure is. To define precisely the principal concepts we intend to introduce in this work, let us consider Fig. 1 representing a cut of a typical elementary volume of composite.

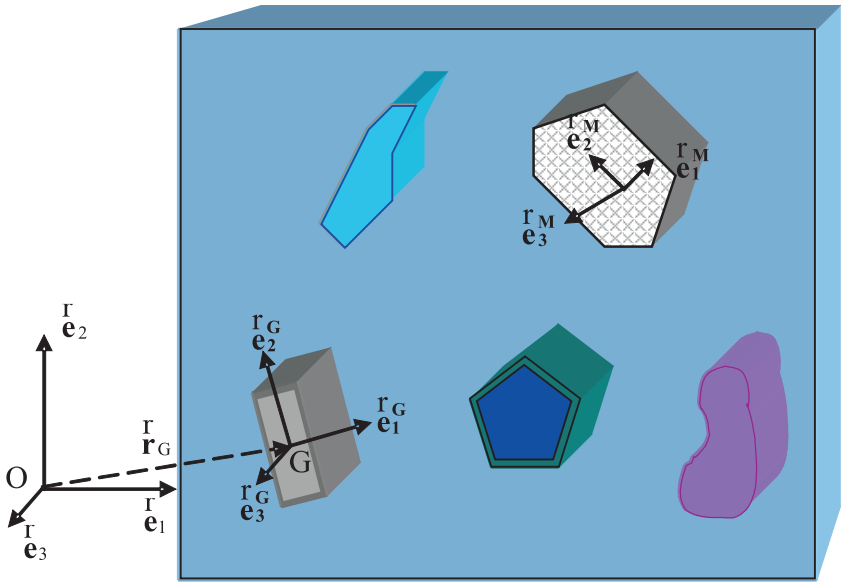

FIG. 1. Schematic representation of composite materials.

This composite is constituted by a matrix surrounding some number of inhomogeneities or reinforcements. Every reinforcement occupies a specific volume and is made of a particular material we shall call a phase. It is important to emphasize that the number of phases is usually different from the number of constituents (matrix and inhomogeneities). The number of constituents is frequently greater than the number of phases. Typically one can introduce two geometrically different inclusions of the same phase.

To define the microstructure of the composite, three various coordinate systems are brought into play. The global system $\{\vec{\mathbf{e}}_i\}$ is connected with the representative volume element. The homogenisation process will be performed with respect to this system. The second system $\left\{\vec{\mathbf{e}}_i^M\right\}$ is used to describe the orientation of the material principal frame. This notion of material frame is very important when the properties of the constitutive phases are anisotropic. The third frame

$\{\vec{\mathbf{e}}_i^G\}$ is called the geometrical system and is attached to a specific constituent. The origin of this system is associated with a particular point of reinforcement, say its centre G. The geometry of the constituent is described with respect to this system. The position of this system, and by consequence that of the constituent, is defined by the vector $\vec{\mathbf{r}}^G$. From the mathematical point of view, inclusions can be in contact or partially (as well as totally) overlaid. The contact between inclusions, in the usual mechanical sense, is not considered in this approach. The interactions between reinforcements are always transmitted via the surrounding reference medium.

The geometry of any reinforcement is always ellipsoidal. It can be simple or coated (composite inclusion). The existence of an interface between matrix and reinforcement can play a very important role in the description of the effective properties of composites. In this case, thickness of the coating and its phase (properties) have to be defined as well.

All the parameters cited above have some influence on the material effective properties. In particular circumstances, some of these parameters can be neglected without altering the resultant properties of the composite.

In the simplest description of a heterogeneous material, only the constituents' local properties and their volume fractions are taken into account. It leads to the models introduced by VOIGT [51] and REUSS [52]. The predictions of these models provide respectively the lower and upper bounds for the effective properties of the material, since no correlations between the constituents are incorporated. If one can take into consideration the shape and orientations of these constituents defined via the geometrical system, one is able to describe the material anisotropy due to the reinforcement's morphology (morphological texture). If the constituents' relative position is introduced into the model, the topological texture influence on the global anisotropic properties is taken into account. In the first approximation, only the relative position between neighbours can be considered. The greater is the distance of interactions considered in the model, the more precise is the description of the microstructure.

The necessity to be able to model precisely enough the constituents' geometry clearly appears from this brief recall.

As it has been emphasized in the Introduction, the ellipsoidal inclusion treated at first by Eshelby appears as a good compromise between the possibility to describe approximately various elementary geometries and the mathematical difficulties encountered during the concentration tensor calculation.

3.2. Concentration tensors for reinforcements of ellipsoidal shape

We now suppose that the ERV is composed of N reinforcements and a surrounding matrix which is considered as a constituent number 0. We admit that

the geometry of reinforcements can be approached by ellipsoidal inclusions. They are enumerated from 1 to N. As a consequence, the composite is made up with $(N+1)$ constituents. The volume of a given inclusion I, or family of the same type of inclusions, is denoted V^I. If the volume of ERV is supposed to be V, the volume fraction of this inclusion is equal to:

$$f^I = \frac{V^I}{V}.$$

It is admitted that the elastic properties of each constituent are homogeneous inside the constituent. In agreement with the proposed labelling of constituents, the tensor of the matrix elastic properties is denoted $\mathbf{c}^0$ and that of inclusion $\mathbf{c}^I$. In consequence, the field of the elastic properties of the material can be expressed as below:

$$\mathbf{c}(\mathbf{r}) = \mathbf{c}^0\theta^0(\mathbf{r}) + \mathbf{c}^1\theta^1(\mathbf{r}) + \cdots + \mathbf{c}^I\theta^I(r) + \cdots + \mathbf{c}^N\theta^N(\mathbf{r}) = \mathbf{c}^0\theta^0(\mathbf{r}) + \sum_{I=1}^{N} \mathbf{c}^I\theta^I(\mathbf{r}) \tag{3.1}$$

where θ^I is a characteristic function satisfying

$$\theta^I(\mathbf{r}) = \begin{cases} 1 & \forall\ \mathbf{r} \in V^I \\ 0 & \forall\ \mathbf{r} \notin V^I. \end{cases} \tag{3.2}$$

The deviation part of the elastic properties field, with respect to the reference medium, can now be expressed as:

$$\delta\mathbf{c}(\mathbf{r}) = \left(\mathbf{c}^0 - \mathbf{C}^r\right)\theta^0(\mathbf{r}) + \sum_{I=1}^{N} \left(\mathbf{c}^I - \mathbf{C}^r\right)\theta^I(\mathbf{r}) = \Delta\mathbf{c}^0\theta^0(\mathbf{r}) + \sum_{I=1}^{N} \Delta\mathbf{c}^I\theta^I(\mathbf{r}). \tag{3.3}$$

Substituting (3.3) in the integral equation (2.8) and using property (3.2) of the characteristic function, one gets:

$$\boldsymbol{\varepsilon}(\mathbf{r}) = \mathbf{E}^r - \int_{V^0} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}'){:}\Delta\mathbf{c}^0{:}\boldsymbol{\varepsilon}(\mathbf{r}')dV' - \sum_{I=1}^{N} \int_{V^I} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}'){:}\Delta\mathbf{c}^I{:}\boldsymbol{\varepsilon}(\mathbf{r}')dV'. \tag{3.4}$$

In spite of this important simplification, the above equation still remains very difficult to treat. The difficulty of this problem is related to the potential complexity of the strain field. With reference to the Eshelby's solution revealing that

the strain field is homogenous inside the ellipsoidal inclusion embedded within an infinite matrix, let us approximate the real strain field by a function

$$\boldsymbol{\varepsilon}(\mathbf{r}) = \boldsymbol{\varepsilon}^0\theta^0(\mathbf{r}) + \sum_{I=1}^{N} \boldsymbol{\varepsilon}^I\theta^I(\mathbf{r}). \tag{3.5}$$

Substituting (3.5) into the integral equation (3.4) one obtains

$$\boldsymbol{\varepsilon}(\mathbf{r}) = \mathbf{E}^r - \left(\int_{V^0} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}')dV' \right) : \Delta\mathbf{c}^0 : \boldsymbol{\varepsilon}^0 - \sum_{I=1}^{N} \left(\int_{V^I} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}')dV' \right) : \Delta\mathbf{c}^I : \boldsymbol{\varepsilon}^I. \tag{3.6}$$

One may now calculate the mean strain for each constituent in turn, using the obvious relation

$$\boldsymbol{\varepsilon}^I = \frac{1}{V^I} \int_{V^I} \boldsymbol{\varepsilon}(\mathbf{r})\, dV.$$

This leads to a system of $N+1$ equations for $N+1$ unknown strain tensors $\boldsymbol{\varepsilon}^I$ involving $(N+1)^2$ new quantities called interaction tensors. They are defined by the following expression:

$$\mathbf{T}^{IJ} = \frac{1}{V^I} \int_{V^I} \int_{V^J} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}')\, dV dV. \tag{3.7}$$

The interaction tensors have been studied among others by FASSI–FEHRI [41] and FASSI–FEHRI *et al.* [42]. It can be proved, see [41], that the following symmetry is valid:

$$V^I\mathbf{T}^{IJ} = \int_{V^I}\int_{V^J} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}')dV'dV = \int_{V^J}\int_{V^I} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}')dV'dV = V^J\mathbf{T}^{JI}. \tag{3.8}$$

A numerical calculation of these tensors has been developed by LIPINSKI [4] and improved by ZATTARIN *et al.* [53].

The system obtained above takes the form:

$$\boldsymbol{\varepsilon}^I = \mathbf{E}^{\mathbf{r}} - \sum_{J=0}^{N} \mathbf{T}^{IJ}{:}\Delta\mathbf{c}^J : \boldsymbol{\varepsilon}^J \qquad I = 0, 1, ..., N. \tag{3.9}$$

The direct solution of this system with respect to $\boldsymbol{\varepsilon}^I$ is rather difficult to obtain. However an iterative method can be used. The convergence of such an approach was already proved by KRÖNER [37]. Substituting (2.9) into (3.9) we obtain:

$$\mathbf{a}^I : \mathbf{E}^r = \mathbf{E}^r - \sum_{J=0}^{N} \mathbf{T}^{IJ} : \Delta\mathbf{c}^J : \mathbf{a}^J : \mathbf{E}^r \qquad I = 0, 1, ..., N.$$

Since the above system has to remain valid for all $\mathbf{E}^r$, one has:

$$\mathbf{a}^I = \mathbf{I} - \sum_{J=0}^{N} \mathbf{T}^{IJ} : \Delta\mathbf{c}^J : \mathbf{a}^J \qquad I = 0, 1, ..., N. \tag{3.10}$$

Solving each of these equations with respect to $\mathbf{a}^I$, I being the number of the current equation, the final expressions are obtained:

$$\left(\mathbf{a}^I\right)_{i+1} = \left(\mathbf{I} + \mathbf{T}^{II} : \Delta\mathbf{c}^I\right)^{-1} : \left(\mathbf{I} - \sum_{\substack{J=0 \\ J \neq I}}^{N} \mathbf{T}^{IJ} : \Delta\mathbf{c}^J : \left(\mathbf{a}^J\right)_i \right) \tag{3.11}$$

$$I = 0, 1, ..., N,$$

where $\left(\mathbf{a}^I\right)_i$ is an approximation of the I-th concentration tensor at iteration i. As before, the initial approximation of all concentration tensors can be taken equal to the unit tensor $\left(\mathbf{a}^I\right)_0 = \mathbf{I}$.

The principal difficulty of this approach is that the tensors $\mathbf{T}^{0I}$ cannot be estimated in the general case because the matrix is not an ellipsoidal inclusion. To answer this problem, at least two approaches, both based on particular choices of the reference medium, are possible. The first one is established adopting the Mori–Tanaka assumption claiming the matrix as the reference medium, i.e. $\mathbf{C}^r = \mathbf{c}^0$. EL MOUDEN [13] has used this approximation to generalise the Mori–Tanaka model to the multi-site situation. In his approach, the matrix has been supposed to be isotropic. The usual Mori–Tanaka model is obtained in the limit case when all the interactions between constituents are neglected, except those between the constituents and the surrounding matrix. The second one corresponds to the assumption $\mathbf{C}^r = \mathbf{C}^{\text{eff}}$. This choice of the reference medium is usual for the self-consistent type of modelling. In our case it will lead to the generalized multi-site self-consistent scheme.

3.3. Mori–Tanaka approach using matrix as reference medium

As mentioned above, this type of modelling is obtained by choosing the matrix itself as the reference homogenous material. In this case one gets:

$$\mathbf{C}^r = \mathbf{c}^0 \quad \rightarrow \quad \Delta\mathbf{c}^0 = \mathbf{c}^0 - \mathbf{C}^r = 0.$$

So, system (3.9) can be rewritten for N inclusions as follows:

$$\boldsymbol{\varepsilon}^I = \mathbf{E}^0 - \sum_{J=1}^{N} \mathbf{T}^{IJ} : \Delta \mathbf{c}^J : \boldsymbol{\varepsilon}^J \qquad I = 1, 2, ..., N \tag{3.12}$$

where $\mathbf{E}^0$ is the strain tensor of the homogeneous ERV with matrix properties. These equations do not depend on the unknown $\mathbf{T}^{0I}$ tensors. It means that the concentration tensors $\mathbf{a}^I$ can now be estimated for all $I = 1, 2, .., N$ using procedure (3.11) i.e.:

$$\left(\mathbf{a}^I\right)_{i+1} = \left(\mathbf{I} + \mathbf{T}^{II} : \Delta \mathbf{c}^I\right)^{-1} : \left(\mathbf{I} - \sum_{\substack{J=1 \\ J \neq I}}^{N} \mathbf{T}^{IJ} : \Delta \mathbf{c}^J : \left(\mathbf{a}^J\right)_i \right) \tag{3.13}$$

$$I = 1, 2, ..., N.$$

These results have to be completed by a hypothesis concerning the strain concentration tensor for the matrix. Because the geometry of this constituent can be very complex, only a rough approximation of this tensor may be envisaged.

The best choice seems to be that proposed by MORI and TANAKA [10] leading here to the multi-site generalisation of their model. Let's suppose that

$$\mathbf{E}^0 \cong \boldsymbol{\varepsilon}^0. \tag{3.14}$$

As a consequence one gets:

$$\mathbf{a}^0 = \mathbf{I}. \tag{3.15}$$

To apply expression (2.14) allowing the determination of $\mathbf{A}^I$, one has to calculate the inverse of the average value of all $\mathbf{a}^I$ tensors. Under the above approximation, one gets:

$$\left\langle \mathbf{a}^I \right\rangle^{-1} = \left(f^0 \mathbf{I} + \sum_{I=1}^{N} f^I \, \mathbf{a}^I \right)^{-1}. \tag{3.16}$$

Substituting this expression into (2.14), one obtains the following approximation of the concentration tensors:

$$\mathbf{A}^0 = \mathbf{a}^0 : \left\langle \mathbf{a}^I \right\rangle^{-1} = \left\langle \mathbf{a}^I \right\rangle^{-1} \tag{3.17}$$

and

$$\mathbf{A}^I = \mathbf{a}^I : \left\langle \mathbf{a}^I \right\rangle^{-1} = \mathbf{a}^I : \mathbf{A}^0 \qquad I = 1, 2, ..., N. \tag{3.18}$$

Expressions (2.6), (3.17) and (3.18) can now be combined to calculate the effective properties of the material; according to the generalized Mori–Tanaka model we denote $\mathbf{C}^{\mathrm{GMT}}$:

$$\mathbf{C}^{\mathrm{GMT}} = \left\langle \mathbf{c}^I : \mathbf{A}^I \right\rangle = f^0 \mathbf{c}^0 : \mathbf{A}^0 + \sum_{I=1}^{N} f^I \mathbf{c}^I : \mathbf{A}^I$$

or

$$\mathbf{C}^{\mathrm{GMT}} = \left(f^0 \mathbf{c}^0 + \sum_{I=1}^{N} f^I \mathbf{c}^I : \mathbf{a}^I \right) : \left\langle \mathbf{a}^J \right\rangle^{-1} = \left(f^0 \mathbf{c}^0 + \sum_{I=1}^{N} f^I \mathbf{c}^I : \mathbf{a}^I \right) : \mathbf{A}^0. \tag{3.19}$$

To close the reflections concerning the modelling with matrix as a reference medium, let us consider the usual one-site approximation. Such a model is obtained when all interaction tensors $\mathbf{T}^{IJ}\,(I \neq J)$ present in system (3.13) are neglected. Only the tensors describing the interactions between individual inclusions and the surrounding matrix are taken into account. In such an approach, the interactions between inclusions are described only on average via the appropriate choice of the reference medium.

In this case the procedure described by (3.13) ceases to be iterative and one gets:

$$\mathbf{a}^I = \left(\mathbf{I} + \mathbf{T}^{II} : \Delta \mathbf{c}^I \right)^{-1} \qquad I = 1, 2, ..., N. \tag{3.20}$$

Expressions (3.14) to (3.19) remain valid and allow the calculation of the composite effective properties. The usual Mori–Tanaka model is obtained in this case.

3.4. Equivalent material as reference medium

When the unknown effective properties $\mathbf{C}^{\mathrm{eff}}$ are chosen as homogeneous reference medium $\mathbf{C}^r$, the self-consistent multi-site approximation can be derived from the equations obtained. In this case $E^r = E$ and by consequence $\mathbf{A}^I = \mathbf{a}^I$ for $I = 0, 1, \ldots, N$. Such an approach has been applied by FASSI–FEHRI *et al.* [42] for the self-consistent modelling of anisotropic materials with cubic symmetries. ZATTARIN and LIPINSKI [43] have developed a general multi-site self-consistent scheme (MSCS) for arbitrary arrangements of inclusions. System (3.10) can thus

be rewritten as:

$$\mathbf{A}^I = \left(\mathbf{I} + \mathbf{T}^{II} : \Delta\mathbf{c}^I\right)^{-1} : \left(\mathbf{I} - \sum_{\substack{J=0 \\ J\neq I}}^{N} \mathbf{T}^{IJ} : \Delta\mathbf{c}^J : \mathbf{A}^J\right) \tag{3.21}$$

$$I = 0, 1, .., N.$$

Because the concentration tensors have to verify equality (2.15) and as in general, all these tensors are determined numerically, it seems to be interesting to replace one of the Eq. (3.21) by relationship (2.15). In this manner, the fundamental property of concentration tensors is always verified. In this work the concentration tensor of the matrix is determined using the relationship (3.22) deduced from (2.15):

$$\mathbf{A}^0 = \frac{1}{f^0}\left(\mathbf{I} - \sum_{I=1}^{N} f^I \mathbf{A}^I\right) \tag{3.22}$$

and the resulting system takes the form

$$\mathbf{A}^I = \left(\mathbf{I} + \mathbf{T}^{II} : \Delta\mathbf{c}^I\right)^{-1} : \left(\mathbf{I} - \sum_{\substack{J=0 \\ J\neq I}}^{N} \mathbf{T}^{IJ} : \Delta\mathbf{c}^J : \mathbf{A}^J\right) \tag{3.23}$$

$$I = 1, 2, ..., N.$$

Equation (3.22) completes the system (3.23).

The above system is not completely determined because it depends on the interaction tensors $\mathbf{T}^{I0}$ between the matrix and I-th inclusion. In accordance with the main idea of the self-consistent model, the term $\left(\mathbf{I} + \mathbf{T}^{II} : \Delta\mathbf{c}^I\right)^{-1}$ describes "on average" interactions between an inclusion I and its surrounding medium. So, it clearly appears that terms $\mathbf{T}^{I0} : \Delta\mathbf{c}^0 : \mathbf{A}^0$ play only a secondary role in the calculations of the concentration tensors. In this sense, the geometry of the matrix has only a limited influence on the concentration tensors of any other constituent. According to the Hill interpretation of the self-consistent method [2], the best results are obtained when the matrix is represented by the same geometry as that of the main inclusion, i.e. having the greatest volume fraction. If there is no main inclusion, the spherical shape should be supposed for the matrix.

The situation becomes even simpler when one intends to use the one-site self-consistent approximation. In this case, the matrix geometry is not involved in

the concentration tensors' computation. System (3.24) can be simply rewritten as follows:

$$\mathbf{A}^0 = \frac{1}{f^0}\left(\mathbf{I} - \sum_{I=1}^{N} f^I \mathbf{A}^I\right), \tag{3.24}$$
$$\mathbf{A}^I = \left(\mathbf{I} + \mathbf{T}^{II} : \Delta\mathbf{c}^I\right)^{-1} \qquad I = 1, 2, ..., N.$$

It is obvious that the calculation of the concentration tensors has to be performed iteratively because the effective properties of the material appearing in $\Delta\mathbf{c}^I$ are unknown. Moreover, every $\mathbf{A}^I$ tensor depends on all unknown $\mathbf{A}^J$ operators ($J \neq I$) in case of a multi-site approach. Acceptable starting values for these operators are for example $\left(\mathbf{C}^{\text{eff}}\right)_0 \approx \langle \mathbf{c}^I \rangle$ and $(\mathbf{A}^J)_0 = \mathbf{I}$.

After each iteration, the self-consistent approximation of the effective properties of the composite is calculated using the expression:

$$\left(\mathbf{C}^{\text{SC}}\right)_{i+1} = \sum_{I=0}^{N} f^I \mathbf{c}^I : \left(\mathbf{A}^I\right)_{i+1}. \tag{3.25}$$

This new estimation of the effective elastic constants is used as the reference material in the following iteration. The iteration process is stopped when a convergence criterion concerning the evolution of $\mathbf{C}^{\text{SC}}$ is achieved.

3.5. Incremental Scheme (IS)

As it has been mentioned in introduction, BRUGGEMAN [20] first proposed the Differential Scheme to predict the effective conductivity of inhomogeneous media. The main idea of this approximation is to use the exact solution of the Eshelby problem for small concentrations of inhomogeneities. The effective properties of the composite are obtained by a gradual addition of infinitesimal quantities of reinforcements. This method has been used and improved by many authors. We can cite BOUCHER [22], MCLAUGHLIN [23], SALGANIK [24], LAWS and DVORAK [25], HASHIN [26], NORRIS [58] and ROSCOE [59] in the case of viscoelastic materials.

The approach developed in this paper is based on the DS idea and was first proposed by VIÉVILLE [45] and VIÉVILLE and LIPINSKI [47] for one-site approximations. It consists of the finite increment injection of reinforcements. Recently, BROOHM *et al.* [55] have generalised this model to the case of a multi-site situation. The model takes into account, as it is shown below, the anisotropy of the constituent, the morphological texture of the composite as well as the spatial repartition of the inhomogeneities (topological texture). In this paper, it is extended to the case of coated inclusions. All methods proposed in the previous

section are well recognised due to an important number of papers devoted to them. One can cite the status of the one-site self-consistent model as the exact solution of the problem with a random distribution of reinforcements. In spite of the important advantage of the Incremental Scheme, derived from the Differential Scheme, this last one does not cause a similar interest.

Let us remind the results obtained by BROOM *et al.* [55] for a two-phase material whose general idea is illustrated in Fig. 2. The number of steps used to build the composite is denoted S. The general expression of the volume fraction Δf_s to be injected at step s is given by:

$$\Delta f_s = \frac{\Delta f}{1-(s-1)\Delta f} = \frac{f}{S-(s-1)f}, \tag{3.26}$$

where f is the final volume fraction of reinforcement and $\Delta f = \dfrac{f}{S}$.

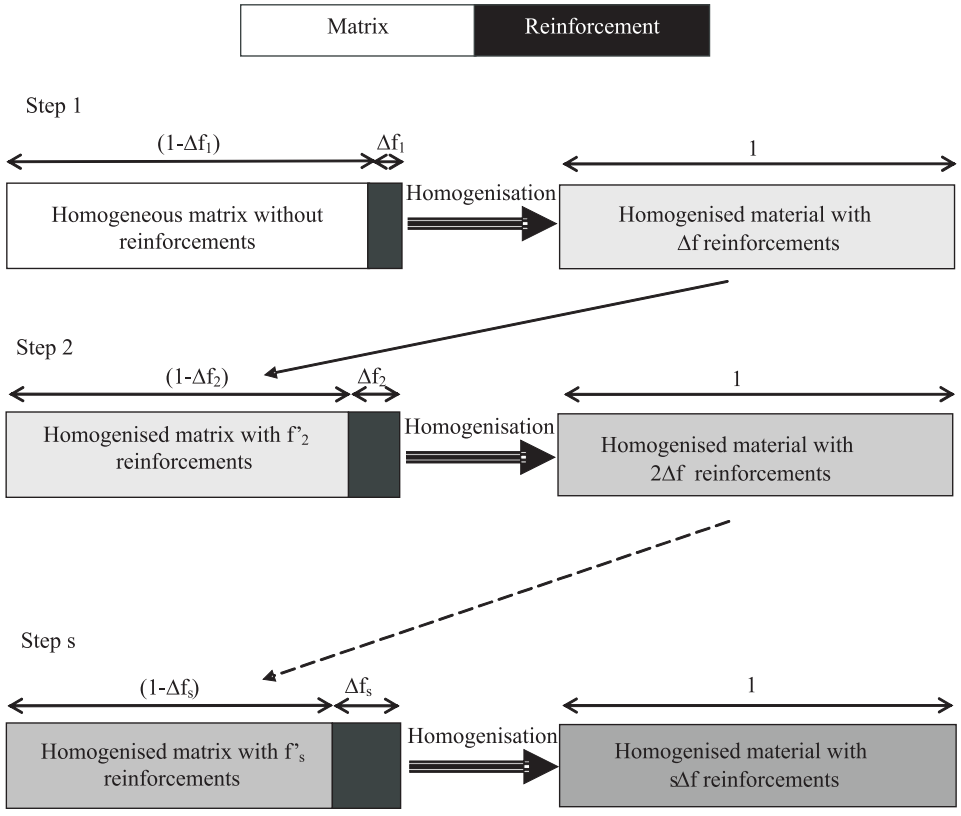

FIG. 2. Construction of a homogeneous material using the Incremental Scheme idea.

Expression (3.26) shows that the incremental volume fraction of reinforcements Δf_s continuously increases as a function of the step number s. This cor-

responds to the ROSCOE'S [21] interpretation of the DS where the filling process begins with the smallest inclusions to end up with the biggest ones. However, as it has been emphasised by HASHIN [26], this interpretation is contradictory with the requirement of infinitesimality of volume fraction increment in case of DS.

It is important to point out that the overall properties of the equivalent homogeneous material obtained in this manner depend, for a too small number of steps S, on the value of this number. However for very heterogeneous materials, when S is greater than 20, the asymptotic properties correspond to the usual DS predictions.

Let us consider now the case of a multiphase material (more than two phases) composed of a matrix with N families of reinforcements. To deduce quickly the relevant results, we bring considerations to the case of a two-phase situation mentioned above and presented in [55].

Let f^J be the volume fraction of inclusions of the family J. We denote:

$$f = \sum_{J=1}^{N} f^J \tag{3.27}$$

the volume fraction of all heterogeneities. We assume the injection procedure of reinforcements described by expression (3.26). In this homothetic approach, every family of reinforcement follows an evolution of its fraction, due to the step-by-step procedure, which has to be the same as the global evolution given by the expression (3.26). During each step, the proportions between the reinforcements of different families have to be preserved. This condition leads to the following relationship for the incremental fraction of any reinforcement family:

$$\Delta f_s^J = \frac{\Delta f}{1-(s-1)\Delta f}\frac{f^J}{f} = r_s f^J \tag{3.28}$$

which is the generalisation of expression (3.26). The proportionality factor introduced in

$$r_s = \frac{\Delta f}{[1-(s-1)\Delta f]\, f} = \frac{1}{S-(s-1)f} = \frac{1}{S-(s-1)(1-f^0)} \tag{3.29}$$

describes the radial or homothetic introduction of the reinforcements chosen. In these last expressions, f^0 represents the volume fraction of a matrix and Δf_s^J is the incremental volume fraction of the family J that must be introduced during the step s. As before, the number of steps of the composite construction is denoted S. As expected, the ratio r_i is the same for all families of reinforcements. It is equal to $\dfrac{1}{S}$ for the first step and increases to $\dfrac{1}{1+f^0(S-1)}$ during the last step.

It is important to emphasise that the above solution corresponds to the radial or proportional path in the N-dimensional space of volume fractions of reinforcements. As it has been discussed by NORRIS [58], this is not a unique possibility to fill the composite. More detailed discussion of this path-dependence can also be found in [55].

In the case of a one-site approach, this scheme takes into account the fraction of reinforcements but also the inclusions' morphology. In its multi-site version, the spatial distribution of inclusions (topological texture) is also reproduced.

At a typical step s of this approach one can apply for instance the Eshelby dilute procedure leading to the following approximation for composite effective properties:

$$\left(\mathbf{C}^{\mathrm{IS}}\right)_{s+1} = \left(\mathbf{C}^{\mathrm{IS}}\right)_{s} + r_s \sum_{I=1}^{N} f^I \left(\Delta\mathbf{c}^I\right)_s : \left(\mathbf{A}^I\right)_s, \tag{3.30}$$

where

$$\left(\Delta\mathbf{c}^I\right)_s = \mathbf{c}^I - \left(\mathbf{C}^{\mathrm{IS}}\right)_s \tag{3.31}$$

and such that for $s = 0$

$$\left(\mathbf{C}^{\mathrm{IS}}\right)_0 = \mathbf{c}^0. \tag{3.32}$$

Concentration tensors $\left(\mathbf{A}^I\right)_s$ are obtained using expression (3.23) and taking $\mathbf{C}^r = (\mathbf{C}^{\mathrm{IS}})_s$. Because the number of steps and Δf_s^J are finite, the Eshelby dilute scheme can be replaced by the Mori–Tanaka or self-consistent method, leading to more rapid convergence of the above procedure.

4. Modelling based on the concept of a coated inclusion

Let us consider a composite ellipsoidal inclusion as presented in Fig. 3. The following notation is adopted to describe such an inclusion. As before, the letter I is used to identify the inclusion (composite inclusion in this case). The core or kernel of the inclusion denoted by index K is supposed to be ellipsoidal and its elastic properties are specified by $\mathbf{c}^{I_K}$. To simplify our considerations, only the case with one coating is analysed in this paper. The discussion of the more general case of multi-coating can be found in [63]. The coating or robe is identified by the letter C. The elastic properties of the robe of the composite inclusion are defined by $\mathbf{c}^{I_c}$. Let's denote the semi-axes of the kernel a, b, and c and the thicknesses of the coating in the three principal directions respectively $\Delta a, \Delta b$, and Δc as illustrated by Fig. 3.

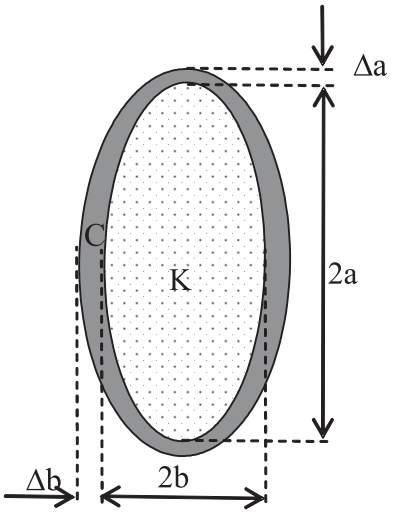

FIG. 3. Composite inclusion.

The deviation part of the field of elastic properties (3.3) with respect to a reference medium can now be specified:

(4.1)
$$\delta\mathbf{c}(\mathbf{r}) = \left(\mathbf{c}^0 - \mathbf{C}^r\right)\theta^0(\mathbf{r}) + \sum_{I=1}^{N}\left[\left(\mathbf{c}^{I_K} - \mathbf{C}^r\right)\theta^{I_K}(\mathbf{r}) + \left(\mathbf{c}^{I_C} - \mathbf{C}^r\right)\theta^{I_C}(\mathbf{r})\right]$$
$$= \Delta\mathbf{c}^0\theta^0(\mathbf{r}) + \sum_{I=1}^{N}\left(\Delta\mathbf{c}^{I_K}\theta^{I_K}(\mathbf{r}) + \Delta\mathbf{c}^{I_C}\theta^{I_C}(\mathbf{r})\right)$$

where two new characteristic functions have been introduced:

(4.2)
$$\theta^{I_K}(\mathbf{r}) = \begin{cases} 1 & \mathbf{r} \in V^{I_K} \\ 0 & \mathbf{r} \notin V^{I_K} \end{cases} \quad \text{and} \quad \theta^{I_C}(\mathbf{r}) = \begin{cases} 1 & \mathbf{r} \in V^{I_C} \\ 0 & \mathbf{r} \notin V^{I_C}. \end{cases}$$

Obviously, the following relationship is valid:

(4.3)
$$\theta^I = \theta^{I_K} + \theta^{I_C}.$$

Substituting (4.1) into integral Eq. (2.8) one gets:

$$\boldsymbol{\varepsilon}(\mathbf{r}) = \mathbf{E}^r - \int\limits_{V^0} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}'):\Delta\mathbf{c}^0:\boldsymbol{\varepsilon}(\mathbf{r}')dV - \sum_{J=1}^{N}\left(\int\limits_{V^{J_K}} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}'):\Delta\mathbf{c}^{J_K}:\boldsymbol{\varepsilon}(\mathbf{r}')dV + \int\limits_{V^{J_C}} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}'):\Delta\mathbf{c}^{J_C}:\boldsymbol{\varepsilon}(\mathbf{r}')dV\right). \tag{4.4}$$

The above expression can now be used to calculate the average value of deformation for the matrix and a composite inclusion I.

$$\boldsymbol{\varepsilon}^0 = \mathbf{E}^r - \frac{1}{V^0}\int\limits_{V^0}\int\limits_{V^0} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}') : \Delta\mathbf{c}^0 : \boldsymbol{\varepsilon}(\mathbf{r}')dVdV - \sum_{J=1}^{N}\frac{1}{V^0}\int\limits_{V^0}\int\limits_{V^{J_K}} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}') : \Delta\mathbf{c}^{J_K} : \boldsymbol{\varepsilon}(\mathbf{r}')dVdV - \sum_{J=1}^{N}\frac{1}{V^0}\int\limits_{V^0}\int\limits_{V^{J_C}} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}') : \Delta\mathbf{c}^{J_C} : \boldsymbol{\varepsilon}(\mathbf{r}')dVdV,$$

$$\boldsymbol{\varepsilon}^I = \mathbf{E}^r - \frac{1}{V^I}\int\limits_{V^I}\int\limits_{V^0} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}') : \Delta\mathbf{c}^0 : \boldsymbol{\varepsilon}(\mathbf{r}')dVdV - \sum_{J=1}^{N}\frac{1}{V^I}\left(\int\limits_{V^I}\int\limits_{V^{J_K}} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}') : \Delta\mathbf{c}^{J_K} : \boldsymbol{\varepsilon}(\mathbf{r}')dVdV + \int\limits_{V^I}\int\limits_{V^{J_C}} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}') : \Delta\mathbf{c}^{J_C} : \boldsymbol{\varepsilon}(\mathbf{r}')dVdV\right) \quad I = 1,2,...,N. \tag{4.5}$$

As before in the case of non-coated inclusions, we admit that the strain field $\boldsymbol{\varepsilon}(\mathbf{r})$ can be represented by the piecewise function given below

$$\boldsymbol{\varepsilon}(\mathbf{r}) = \boldsymbol{\varepsilon}^0\theta^0(\mathbf{r}) + \sum_{I=1}^{N}\boldsymbol{\varepsilon}^{I_K}\theta^{I_K}(\mathbf{r}) + \sum_{I=1}^{N}\boldsymbol{\varepsilon}^{I_C}\theta^{I_C}(\mathbf{r}) \tag{4.6}$$

where the average values of the strain tensor of the composite inclusion, kernel and coating have been introduced:

$$\boldsymbol{\varepsilon}^{I} = \frac{1}{V^{I}} \int\limits_{V^{I}} \boldsymbol{\varepsilon}(\mathbf{r}) dV,$$

$$\boldsymbol{\varepsilon}^{I_K} = \frac{1}{V^{I_K}} \int\limits_{V^{I_K}} \boldsymbol{\varepsilon}(\mathbf{r}) dV, \tag{4.7}$$

$$\boldsymbol{\varepsilon}^{I_C} = \frac{1}{V^{I_C}} \int\limits_{V^{I_C}} \boldsymbol{\varepsilon}(\mathbf{r}) dV.$$

Under such approximation, expressions (4.3) become:

$$\begin{aligned} \boldsymbol{\varepsilon}^{0} = \mathbf{E}^{r} &- \frac{1}{V^{0}} \int\limits_{V^{0}} \int\limits_{V^{0}} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}') dV dV : \Delta\mathbf{c}^{0} : \boldsymbol{\varepsilon}^{0} \\ &- \sum_{J=1}^{N} \frac{1}{V^{0}} \int\limits_{V^{0}} \int\limits_{V^{J_K}} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}') dV dV : \Delta\mathbf{c}^{J_K} : \boldsymbol{\varepsilon}^{J_K} \\ &- \sum_{J=1}^{N} \frac{1}{V^{0}} \int\limits_{V^{0}} \int\limits_{V^{J_C}} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}') dV dV : \Delta\mathbf{c}^{J_C} : \boldsymbol{\varepsilon}^{J_C}, \end{aligned} \tag{4.8}$$

$$\begin{aligned} \boldsymbol{\varepsilon}^{I} = \mathbf{E}^{r} &- \frac{1}{V^{I}} \int\limits_{V^{I}} \int\limits_{V^{0}} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}') dV dV : \Delta\mathbf{c}^{0} : \boldsymbol{\varepsilon}^{0} \\ &- \sum_{J=1}^{N} \frac{1}{V^{I}} \left(\int\limits_{V^{I}} \int\limits_{V^{J_K}} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}') dV dV : \Delta\mathbf{c}^{J_K} : \boldsymbol{\varepsilon}^{J_K} \right. \\ &\left. + \int\limits_{V^{I}} \int\limits_{V^{J_C}} \boldsymbol{\Gamma}(\mathbf{r}-\mathbf{r}') dV dV : \Delta\mathbf{c}^{J_C} : \boldsymbol{\varepsilon}^{J_C} \right) \qquad I = 1, 2, ..., N. \end{aligned}$$

Let us introduce two new tensors, $\mathbf{T}^{II_K}$ and $\mathbf{T}^{IJ_K}$ describing respectively the interactions, through the reference medium, between the composite inclusion I

and its kernel I_K and between the composite inclusion I and the kernel of inclusion J_K:

$$\begin{aligned}\mathbf{T}^{II_K} &= \frac{1}{V^I}\int\limits_{V^I}\int\limits_{V^{I_K}} \mathbf{\Gamma}(\mathbf{r}-\mathbf{r}')dVdV,\\ \mathbf{T}^{IJ_K} &= \frac{1}{V^I}\int\limits_{V^I}\int\limits_{V^{J_K}} \mathbf{\Gamma}(\mathbf{r}-\mathbf{r}')dVdV.\end{aligned} \tag{4.9}$$

On the other hand, since $V^J = V^{J_K} + V^{J_C}$, one has:

$$\begin{aligned}\frac{1}{V^I}\int\limits_{V^I}\int\limits_{V^{J_C}} \mathbf{\Gamma}(\mathbf{r}-\mathbf{r}')dVdV &= \frac{1}{V^I}\int\limits_{V^I}\int\limits_{V^J-V^{J_K}} \mathbf{\Gamma}(\mathbf{r}-\mathbf{r}')dVdV\\ = \frac{1}{V^I}\int\limits_{V^I}\int\limits_{V^J} \mathbf{\Gamma}(\mathbf{r}-\mathbf{r}')dVdV &- \frac{1}{V^I}\int\limits_{V^I}\int\limits_{V^{J_K}} \mathbf{\Gamma}(\mathbf{r}-\mathbf{r}')dVdV = \mathbf{T}^{IJ} - \mathbf{T}^{IJ_K}\end{aligned} \tag{4.10}$$

where the usual interaction tensor between inclusions I and J is considered. Now, expressions (4.8) can be rewritten as follows:

$$\begin{aligned}\boldsymbol{\varepsilon}^0 &= \mathbf{E}^r - \mathbf{T}^{00} : \Delta\mathbf{c}^0{:}\boldsymbol{\varepsilon}^0\\ &\quad - \sum_{J=1}^{N} \mathbf{T}^{0J_K} : \left(\Delta\mathbf{c}^{J_K}{:}\boldsymbol{\varepsilon}^{J_K} - \Delta\mathbf{c}^{J_C}{:}\boldsymbol{\varepsilon}^{J_C}\right) - \sum_{J=1}^{N} \mathbf{T}^{0J} : \Delta\mathbf{c}^{J_C}{:}\boldsymbol{\varepsilon}^{J_C},\\ \boldsymbol{\varepsilon}^I &= \mathbf{E}^r - \mathbf{T}^{I0} : \Delta\mathbf{c}^0{:}\boldsymbol{\varepsilon}^0\\ &\quad - \sum_{J=1}^{N} \mathbf{T}^{IJ_K} : \left(\Delta\mathbf{c}^{J_K}{:}\boldsymbol{\varepsilon}^{J_K} - \Delta\mathbf{c}^{J_C}{:}\boldsymbol{\varepsilon}^{J_C}\right) - \sum_{J=1}^{N} \mathbf{T}^{IJ} : \Delta\mathbf{c}^{J_C}{:}\boldsymbol{\varepsilon}^{J_C}\\ &\qquad\qquad I = 1, 2, ..., N.\end{aligned} \tag{4.11}$$

The above expressions constitute a system of $N+1$ tensorial equations with $3N+1$ unknown strain tensors, namely $\boldsymbol{\varepsilon}^0, \boldsymbol{\varepsilon}^I$, $\boldsymbol{\varepsilon}^{I_K}$, and $\boldsymbol{\varepsilon}^{I_C} (I = 1, 2, ..., N)$. To solve this system and derive the concentration tensors for all composite inclusions, one has to construct 2N supplementary equations. Such equations can be obtained by solving the problem of a composite inclusion. CHRISTENSEN and LO [17] have treated the problem of the three-phase spherical and cylindrical composite inclusions. CHERKAOUI [44], and CHERKAOUI *et al.* [19] have proposed an approximate solution for the general case of an arbitrary composite

inclusion embedded in an anisotropic matrix. HUANG *et al.* [57] have generalised the self-consistent concept to the case of multiphase inclusions. The model presented in this paper exploits the result concerning the interfacial operator obtained by CHERKAOUI [44] but is based on the system of equations presented in Appendix. This system is different from that proposed by CHERKAOUI *et al.* [19]. The results presented in [19] allow, for very thin coatings, to reproduce only approximately the exact mean strains determined by HERVÉ and ZAOUI [60] in the case of a spherical isotropic coated inclusion embedded in an isotropic matrix. The approach developed in Appendix and [63] is based on the calculation of jumps between mean strains in the kernel and coating and leads to the exact Hervé–Zaoui solution for any coating thickness. The details of this approach are presented in [63] and [64]. From this solution one can deduce, see Appendix, the existence of two concentration tensors for each composite inclusion, one for the kernel $\boldsymbol{\alpha}^{I_K}$ and a second one for the coating $\boldsymbol{\alpha}^{I_C}$, such that:

$$\begin{aligned} \boldsymbol{\varepsilon}^{I_K} &= \boldsymbol{\alpha}^{I_K} : \boldsymbol{\varepsilon}^I, \\ \boldsymbol{\varepsilon}^{I_C} &= \boldsymbol{\alpha}^{I_C} : \boldsymbol{\varepsilon}^I. \end{aligned} \tag{4.12}$$

Introducing these expressions in (4.11) one gets:

$$\begin{aligned} \boldsymbol{\varepsilon}^0 &= \mathbf{E}^r - \mathbf{T}^{00} : \Delta\mathbf{c}^0{:}\boldsymbol{\varepsilon}^0 \\ &- \sum_{J=1}^{N} \left[\mathbf{T}^{0J_K} : \left(\Delta\mathbf{c}^{J_K} : \boldsymbol{\alpha}^{J_K} - \Delta\mathbf{c}^{J_C} : \boldsymbol{\alpha}^{J_C} \right) - \mathbf{T}^{0J} : \Delta\mathbf{c}^{J_C} : \boldsymbol{\alpha}^{J_C} \right] : \boldsymbol{\varepsilon}^J, \\ \boldsymbol{\varepsilon}^I &= \mathbf{E}^r - \mathbf{T}^{I0} : \Delta\mathbf{c}^0{:}\boldsymbol{\varepsilon}^0 \\ &- \sum_{J=1}^{N} \left[\mathbf{T}^{IJ_K} : \left(\Delta\mathbf{c}^{J_K}{:}\boldsymbol{\alpha}^{J_K} - \Delta\mathbf{c}^{J_C} : \boldsymbol{\alpha}^{J_C} \right) - \mathbf{T}^{IJ} : \Delta\mathbf{c}^{J_C}{:}\boldsymbol{\alpha}^{J_C} \right] : \boldsymbol{\varepsilon}^J \\ & \qquad I = 1, 2, ..., N. \end{aligned} \tag{4.13}$$

This last result constitutes the system of $N+1$ tensorial equations for $N+1$ strain tensors $\boldsymbol{\varepsilon}^0$, and $\boldsymbol{\varepsilon}^I$ $(I = 1, 2, ..., N)$. An iterative procedure can be derived to obtain the concentration tensors for all composite inclusions. Indeed, as in the case of ordinary inclusions without coating, one can introduce into (4.13) the concentration relationships for all unknown strain tensors to obtain:

$$\begin{aligned} \mathbf{a}^0 &= \mathbf{I} - \mathbf{T}^{00} : \Delta\mathbf{c}^0{:}\mathbf{a}^0 \\ &- \sum_{J=1}^{N} \left[\mathbf{T}^{0J_K} : \ \left(\Delta\mathbf{c}^{J_K}{:}\boldsymbol{\alpha}^{J_K} - \Delta\mathbf{c}^{J_C}{:}\boldsymbol{\alpha}^{J_C} \right) - \mathbf{T}^{0J} : \Delta\mathbf{c}^{J_C}{:}\boldsymbol{\alpha}^{J_C} \right] :\mathbf{a}^J, \end{aligned} \tag{4.14}$$

(4.14) [cont.]
$$\mathbf{a}^I = \mathbf{I} - \mathbf{T}^{I0} : \Delta\mathbf{c}^0{:}\mathbf{a}^0 - \sum_{J=1}^{N} \Big[\mathbf{T}^{IJ_K} : \big(\Delta\mathbf{c}^{J_K}{:}\boldsymbol{\alpha}^{J_K} - \Delta\mathbf{c}^{J_C}{:}\boldsymbol{\alpha}^{J_C}\big) - \mathbf{T}^{IJ} : \Delta\mathbf{c}^{J_C}{:}\boldsymbol{\alpha}^{J_C} \Big] : \mathbf{a}^J \qquad I = 1, 2, ..., N.$$

Now, one can solve each of these equations with respect to $\mathbf{a}^I$ to derive an iterative procedure for the unknown concentration tensors. One gets:

$$(\mathbf{a})^0_{i+1} = \big(\mathbf{I} + \mathbf{T}^{00} : \Delta\mathbf{c}^0\big)^{-1} : \Big\{ \mathbf{I} - \sum_{J=1}^{N} \Big[\mathbf{T}^{0J_K} : \big(\Delta\mathbf{c}^{J_K}{:}\boldsymbol{\alpha}^{J_K} - \Delta\mathbf{c}^{J_C}{:}\boldsymbol{\alpha}^{J_C}\big) - \mathbf{T}^{0J} : \Delta\mathbf{c}^J_C{:}\boldsymbol{\alpha}^J_C \Big] : \big(\mathbf{a}^J\big)_i \Big\},$$

(4.15)
$$(\mathbf{a})^I_{i+1} = \Big[\mathbf{I} + \mathbf{T}^{II_K} : \big(\Delta\mathbf{c}^{I_K}{:}\boldsymbol{\alpha}^{I_K} - \Delta\mathbf{c}^{I_C}{:}\boldsymbol{\alpha}^{I_C}\big) - \mathbf{T}^{II} : \Delta\mathbf{c}^{I_C} : \boldsymbol{\alpha}^{I_C} \Big]^{-1} : \Big\{ I - \mathbf{T}^{I0} : \Delta\mathbf{c}^0{:}\big(\mathbf{a}^0\big)_i - \sum_{\substack{J=1 \\ J\neq I}}^{N} \Big[\mathbf{T}^{1J_K} : \big(\Delta\mathbf{c}^{J_K}{:}\boldsymbol{\alpha}^{J_K} - \Delta\mathbf{c}^{J_C}{:}\boldsymbol{\alpha}^{J_C}\big) - \mathbf{T}^{1J} : \Delta\mathbf{c}^J_C{:}\boldsymbol{\alpha}^J_C \Big] : \big(\mathbf{a}^J\big)_i \Big\} \qquad I = 1, 2, ..., N.$$

Similar expressions can be obtained in case of multi-coated inclusions, see [63] and [64]. This result generalises the method proposed by EL MOUDEN *et al.* [65] in case of simply coated inclusions arranged in a periodic array and based on CHERKAOUI solution [19], valid only for very thin coatings.

Two following observations can be formulated:

1. When the coating thickness tends to zero, a usual simple inclusion is obtained. In this case $\boldsymbol{\alpha}^I_K = \mathbf{I}$, $\boldsymbol{\alpha}^I_C = \mathbf{0}$ and $\mathbf{T}^{IJ_K} = \mathbf{T}^{IJ}$ and system (4.15) becomes identical to system (3.11).
2. When all interactions defined by tensors $\mathbf{T}^{IJ_K}$ and $\mathbf{T}^{IJ}$ are neglected, i.e. when $\mathbf{T}^{IJ_K} = \mathbf{T}^{IJ} = 0$, the one-site approach with composite inclusions is obtained.

It clearly appears that all the methods developed in the previous paragraph for the simple inclusion case remain valid for the composite inclusion model. Namely, one can apply the Mori–Tanaka, self-consistent or incremental approximation to determine the effective properties of composite materials. Indeed,

knowing the concentration tensor for composite inclusions given by (4.15) one can directly apply the results of Sec. 3; expression (3.19) for Mori–Tanaka model, expression (3.25) in case of self-consistent approximation or expression (3.30) when dealing with the incremental scheme.

5. Conclusions

In the literature, there is a great number of methods devoted to find the equivalent properties of heterogeneous materials. Among all, the method derived from the Eshelby solution (MTM, SCS) presents some well-identified malfunctions. Starting from the kinematical integral equation, being a formal solution of the problem of micro-homogenous and macro-heterogeneous materials' behaviour, various models based on the concept of Eshelby's inclusion have been reviewed or established. A special attention has been given to three families of approaches, namely the self-consistent, Mori–Tanaka and incremental methods. Mono-site and multi-site versions of these schemes have been analysed. To describe more precisely the interactions between the heterogeneities and the surrounding matrix, the coated inclusion concept has been exploited, both for one-site and multi-site situations. New expressions for the strain concentration tensor have been derived for such composite inclusions. They can be applied in the framework of the self-consistent, Mori–Tanaka and incremental concepts. The numerical implementation and comparison of the predicted properties with experimental results from the literature will be presented in our next paper. The domains of validity of the various models established in this paper will be identified.

Appendix. Concentration tensors for kernel and coating of a composite inclusion

Let us consider the expression defining the average value of the strain tensor of a composite inclusion I. Using the obvious relation, one gets:

$$\boldsymbol{\varepsilon}^I = \frac{1}{V^I}\int\limits_{V^I} \boldsymbol{\varepsilon}(\mathbf{r})dV = \frac{1}{V^I}\int\limits_{V^{I_K}+V^{I_C}} \boldsymbol{\varepsilon}(\mathbf{r})dV$$

$$= \frac{1}{V^I}\left(\int\limits_{V^{I_K}} \boldsymbol{\varepsilon}(\mathbf{r})dV + \int\limits_{V^{I_C}} \boldsymbol{\varepsilon}(\mathbf{r})dV\right). \tag{A.1}$$

Introducing the average strain measures for kernel and coating such that:

$$\begin{aligned} \boldsymbol{\varepsilon}^{I_K} &= \frac{1}{V^{I_K}} \int\limits_{V^{I_K}} \boldsymbol{\varepsilon}(\mathbf{r}) dV, \\ \boldsymbol{\varepsilon}^{I_C} &= \frac{1}{V^{I_C}} \int\limits_{V^{I_C}} \boldsymbol{\varepsilon}(\mathbf{r}) dV. \end{aligned} \tag{A.2}$$

relationship (A.1) becomes:

$$\boldsymbol{\varepsilon}^{I} = f^{I_K} \boldsymbol{\varepsilon}^{I_K} + f^{I_C} \boldsymbol{\varepsilon}^{I_C} \tag{A.3}$$

where:

$$\begin{aligned} f^{I_K} &= \frac{V^{I_K}}{V^I}, \\ f^{I_C} &= \frac{V^{I_C}}{V^I}. \end{aligned} \tag{A.4}$$

On the other hand, the stress and strain fields inside the coating are generally very complicated. Through the interface separating the kernel and the coating, these two tensors undergo a jump. When the thickness of the coating is thin enough, one can admit that the fields $\boldsymbol{\sigma}(\mathbf{r})$ and $\boldsymbol{\varepsilon}(\mathbf{r})$ are quite uniform through this thickness. In such a case, one can link the strain inside the kernel and coating using the interfacial operators [61, 62]. It is possible to write, for any point of the interface with the normal vector $\mathbf{n}$:

$$\boldsymbol{\varepsilon}^{C} - \boldsymbol{\varepsilon}^{K} = \mathbf{P}^{C} : (\mathbf{c}^{I_C} - \mathbf{c}^{I_K}) : \boldsymbol{\varepsilon}^{K} \tag{A.5}$$

where $\boldsymbol{\varepsilon}^{C} - \boldsymbol{\varepsilon}^{K}$ represents the strain jump for any point of the interface. Moreover, one has

$$\mathbf{P}^{C}_{ijkl} = \frac{1}{4} \left(h_{ik} n_j n_l + h_{jk} n_i n_l + h_{il} n_j n_k + h_{jl} n_i n_k \right) \tag{A.6}$$

and

$$h^{-1}_{ik} = c^{I_C}_{imkn} n_m n_n \tag{A.7}$$

is the usual Christoffel matrix.

Solving (A.5) with respect to $\boldsymbol{\varepsilon}^{C}$ one has:

$$\boldsymbol{\varepsilon}^{C} = \left[\mathbf{I} + \mathbf{P}^{C} : (\mathbf{c}^{I_K} - \mathbf{c}^{I_C}) \right] : \boldsymbol{\varepsilon}^{K}. \tag{A.8}$$

As the first approximation one can substitute $\boldsymbol{\varepsilon}^K$ in (A.8) by its average value $\boldsymbol{\varepsilon}^{I_K}$ to obtain:

$$\boldsymbol{\varepsilon}^C = \left[\mathbf{I} + \mathbf{P}^C : (\mathbf{c}^{I_K} - \mathbf{c}^{I_C})\right] : \boldsymbol{\varepsilon}^{I_K}.$$

The integration over the volume of coating yields:

$$\boldsymbol{\varepsilon}^{I_C} = \left[\mathbf{I} + \frac{1}{V^{I_C}} \int\limits_{V^{I_C}} \mathbf{P}^C dV : (\mathbf{c}^{I_K} - \mathbf{c}^{I_C})\right] : \boldsymbol{\varepsilon}^{I_K}. \tag{A.9}$$

Let us introduce a new interaction tensor $\boldsymbol{\tau}^C$ such that:

$$\boldsymbol{\tau}^C = \frac{1}{V^{I_C}} \int\limits_{V^{I_C}} \mathbf{P}^C dV. \tag{A.10}$$

CHERKAOUI [44] and CHERKAOUI *et al.* [19] have demonstrated that the tensor $\boldsymbol{\tau}^C$ is given by the following expression:

$$\boldsymbol{\tau}^C = \boldsymbol{\tau}^{I_K} - \frac{V^{I_K}}{V^{I_C}} \left(\boldsymbol{\tau}^I - \boldsymbol{\tau}^{I_K}\right) = \boldsymbol{\tau}^{I_K} - \frac{f^{I_K}}{f^{I_C}} \left(\boldsymbol{\tau}^I - \boldsymbol{\tau}^{I_K}\right) \tag{A.11}$$

where

$$\begin{aligned} \boldsymbol{\tau}^I &= \frac{1}{V^I} \int\limits_{V^I} \boldsymbol{\Gamma}\left(\mathbf{c}^{I_C}\right) dV, \\ \boldsymbol{\tau}^{I_K} &= \frac{1}{V^{I_K}} \int\limits_{V^{I_K}} \boldsymbol{\Gamma}\left(\mathbf{c}^{I_C}\right) dV \end{aligned} \tag{A.12}$$

are two new interaction tensors of ellipsoidal inclusions embedded in the surrounding material with elastic properties of coating. The modified Green tensor $\boldsymbol{\Gamma}(\mathbf{c}^{Ic})$ is calculated using the same elastic properties of coating. Because these tensors are independent of the size of inclusion, it is clear that in the case of a homothetic composite inclusion (the same shape for kernel and composite inclusion), the following equality is valid

$$\boldsymbol{\tau}^I = \boldsymbol{\tau}^{I_K}$$

and as consequence one has:

$$\boldsymbol{\tau}^C = \boldsymbol{\tau}^I. \tag{A.13}$$

In the general case, expression (A.9) takes the form:

$$\boldsymbol{\varepsilon}^{I_C} = \left[\mathbf{I} + \left(\boldsymbol{\tau}^{I_K} - \frac{f^{I_K}}{f^{I_C}} \left(\boldsymbol{\tau}^I - \boldsymbol{\tau}^{I_K}\right)\right) : (\mathbf{c}^{I_K} - \mathbf{c}^{I_C})\right] : \boldsymbol{\varepsilon}^{I_K}. \tag{A.14}$$

Substituting (A.14) into (A.3) one gets:

$$\boldsymbol{\varepsilon}^{I} = f^{I_K}\boldsymbol{\varepsilon}^{I_K} + f^{I_C}\left[\mathbf{I} + \left(\boldsymbol{\tau}^{I_K} - \frac{f^{I_K}}{f^{I_C}}\left(\boldsymbol{\tau}^{I} - \boldsymbol{\tau}^{I_K}\right)\right) : \left(\mathbf{c}^{I_K} - \mathbf{c}^{I_C}\right)\right] : \boldsymbol{\varepsilon}^{I_K}$$

or

$$\boldsymbol{\varepsilon}^{I} = \boldsymbol{\varepsilon}^{I_K} + \left(\boldsymbol{\tau}^{I_K} - f^{I_K}\boldsymbol{\tau}^{I}\right) : \left(\mathbf{c}^{I_K} - \mathbf{c}^{I_C}\right) : \boldsymbol{\varepsilon}^{I_K}. \tag{A.15}$$

This last result can be rewritten in the form of the strain concentration operation:

$$\boldsymbol{\varepsilon}^{I_K} = \left[\mathbf{I} + \left(\boldsymbol{\tau}^{I_K} - f^{I_K}\boldsymbol{\tau}^{I}\right) : \left(\mathbf{c}^{I_K} - \mathbf{c}^{I_C}\right)\right]^{-1} : \boldsymbol{\varepsilon}^{I} \tag{A.16}$$

defining the concentration tensor for the kernel of composite inclusion:

$$\boldsymbol{\alpha}^{I_K} = \left[\mathbf{I} + \left(\boldsymbol{\tau}^{I_K} - f^{I_K}\boldsymbol{\tau}^{I}\right) : \left(\mathbf{c}^{I_K} - \mathbf{c}^{I_C}\right)\right]^{-1}. \tag{A.17}$$

Since

$$f^{I_K}\boldsymbol{\alpha}^{I_K} + f^{I_C}\boldsymbol{\alpha}^{I_C} = \mathbf{I}$$

the concentration tensor of coating is equal to:

$$\boldsymbol{\alpha}^{I_C} = \frac{1}{f^{I_C}}\left(\mathbf{I} - f^{I_K}\boldsymbol{\alpha}^{I_K}\right). \tag{A.18}$$

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