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Yield criteria in anisotropic finite elasto-plasticity

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The paper deals with several descriptions of the yield criteria, within the constitutive framework of anisotropic finite multiplicative elasto-plasticity. We put into evidence appropriate yield criteria, defined either in stress or strain spaces and we analyse the Eulerian setting attached to $\mathbf{\Sigma}$-models, first described in relaxed configuration, when $\mathbf{\Sigma}$ represents the Mandel's stress measure or the quasistatic Eshelby stress tensor. We compare our anisotropic model with the elasto-plastic model, usually adopted in computational finite plasticity.

1. Introduction

The paper deals with several descriptions of the yield criteria, within the constitutive framework of anisotropic elasto-plastic materials with local current relaxed configurations (lcrc), denoted by K_t, proposed by Cleja-Ţigoiu and Soós [3] (see also Cleja-Ţigoiu [1]) and based on the multiplicative decomposition of the deformation gradient $\mathbf{F}$, see Mandel [19], Teodosiu [26], Rice [24], into its elastic and plastic components

$$\mathbf{F} = \mathbf{E}\mathbf{P}, \tag{1.1}$$

denoted $\mathbf{E}$ and $\mathbf{P}$.

• The behaviour of the material is elastic with respect to a moving set of plastically deformed configurations (lcrc), while the irreversible variables, i.e. the plastic component and the internal variables, are defined by the rate-independent evolution equations, related to the appropriate yield surfaces. The mathematical description of the (icrc) has been introduced simultaneously with the law of materials, taking into account the physical origin and mechanical significance, attributed by Teodosiu [26], Kratochvil [12], Mandel [19] to the elastic and plastic parts of deformation. Based on the material symmetry concept introduced and developed by Cleja-Ţigoiu and Soós [2, 3], it is proved that the material symmetry group with respect to K_t is fixed $g_{K_t} = g_k$. Here g_k characterizes the structural preexisting anisotropy (a) in the undeformed body.

The evolving anisotropy (b) is induced by the development of the plastic deformation, via the internal variables, $\boldsymbol{\alpha}$. Both anisotropies (a) and (b) are involved in the constitutive models.

The different descriptions of the yield surfaces were introduced and analyzed in MANDEL [19], MIEHE [21, 25], CLEJA-ŢIGOIU and MAUGIN, [6], motivated by the work conjugate pair of variables. The presence of $\boldsymbol{\Sigma}$, non-symmetric Mandel's stress tensor (also called quasi-static Eshelby's tensor, see MAUGIN [22]), naturally appears as a conjugate variable to the rate of plastic deformation in relaxed configuration (called also intermediate configuration by certain authors), see thermodynamic considerations in [26, 19, 16, 9, 10]. The yield criteria in $\boldsymbol{\Sigma}$-space, as an extension of the Huber–Mises condition, were derived for compressible as well as for incompressible elasto-plastic material. In our approach to anisotropic body we developed models for two material symmetry groups g_1 and g_4, in the case of the transverse isotropy, and for g_6 – orthotropic material, respectively, see CLEJA-ŢIGOIU [7, 8]. The symmetry groups g_k are just those defined by I-SHIH LIU in [14].

• The inelastic part of deformation $\mathbf{P}$ and the internal variables $\boldsymbol{\alpha}$ are defined by the evolution equation written either in strain or in stress formulation. The different stress measures are involved in the evolution equations: $\boldsymbol{\Pi}$ – symmetric Piola–Kirchhoff stress tensor or $\boldsymbol{\Sigma}$, generally non-symmetric stress tensor for anisotropic materials, $\mathbf{T}$ – Cauchy stress tensor. They are related by the formulae

$$\begin{aligned} &\frac{\boldsymbol{\Pi}}{\tilde{\rho}} = (\det\mathbf{E})\mathbf{E}^{-1}\frac{\mathbf{T}}{\rho}\mathbf{E}^{-T}, \qquad \boldsymbol{\Sigma} = \mathbf{G}\frac{\boldsymbol{\Pi}}{\tilde{\rho}} \\ &\text{with} \qquad \mathbf{G} = \mathbf{E}^T\mathbf{E}, \qquad \tilde{\rho} = |\det\mathbf{E}|\rho, \end{aligned} \tag{1.2}$$

$\boldsymbol{\Pi}$, $\boldsymbol{\Sigma}$ are related to the unstressed configuration, and $\mathbf{G}$ denotes the elastic (right) strain tensor.

• In finite multiplicative elasto-plasticity many attempts have been made in order to obtain physically motivated restrictions, based on the appropriate generalizations of the classical dissipation postulates. We mention several forms of the Il'yushin postulate, in connection with Drucker postulate, say for instance in MARIGO [20], in LUCCHESI and PODIO–GUIDUGLI [17], in LUCCHESI and ŠILHAVÝ [18] within the framework of *isotropic* materials with *elastic range* for isothermal and respectively, non- isothermic processes, in KRAWIETZ [13] for finite plasticity, basically formulated for stress cycles in relaxed configurations. Ily'iushin-type dissipation postulate formulated by CLEJA-ŢIGOIU in [4] (see also [5] for other consequences of the dissipative restrictions), extends for anisotropic multiplicative elasto-plasticity the previous results.

We also mention Mandel's nine-dimensional flow rule, [19], LUBLINER's consequences, in [15, 16], the models in CLEJA-ŢIGOIU and MAUGIN [6], based on the conjugate rates of deformation with stress measures, as well as the models of associative plasticity developed in SIMO [25], all of them derived from the maximum dissipation postulate. In manipulating these postulates the constitutive framework is essential.

In the present paper, we deal in Sec. 2 with different descriptions of the yield functions, within the constitutive framework of elasto-plastic models with relaxed configurations for generally anisotropic material, anisotropy being characterized by the invariance properties listed in Sec. 4. In Sec. 3 we perform the analyse for Eulerian setting attached to $\mathbf{\Sigma}$-models, first described in relaxed configuration but without any specification of special anisotropy. In the Sec. 4 we emphasize the peculiar feature of anisotropic elasto-plastic models versus the isotropic ones.

We propose also a new rate-type constitutive equations, compatible with the dissipation postulate proposed in [4], in Sec. 5. We put into evidence appropriate descriptions of the yield criteria, as image and preimage of the elastic ranges, defined either in stress or strain spaces, following the above descriptions, and we give the reasons for adopting a certain model.

Further we shall use the following notations:
Lin – the set of all second order tensors;
Sym – all symmetric elements of Lin, Sym^+ all positive definite tensors of Sym;
Ort^+– all proper rotation of the orthogonal group Ort;
$\mathbf{a}\cdot\mathbf{b}$ – the scalar product of the vectors $\mathbf{a},\mathbf{b}$;
$\mathbf{A}\cdot\mathbf{B} := \mathrm{tr}\,\mathbf{A}\mathbf{B}^T$– the scalar product of $\mathbf{A},\mathbf{B}\in$ Lin; $\mathbf{A}^T$ the transpose of the tensor $\mathbf{A}$; $\mathbf{A}^s = \dfrac{1}{2}\left(\mathbf{A}+\mathbf{A}^T\right)$ and $\mathbf{A}^a = \dfrac{1}{2}\left(\mathbf{A}-\mathbf{A}^T\right)$ – the symmetrical and respectively skew-symmetrical parts of $\mathbf{A}\in$ Lin; $\mathbf{I}$ is the identity tensor;
$\mathcal{E}^T$ – the transpose of $\mathcal{E}$ – fourth order tensor, defined for all $\mathbf{A},\mathbf{B}\in$ Lin by $\mathcal{E}^T\mathbf{A}\cdot\mathbf{B} := \mathbf{A}\cdot\mathcal{E}\mathbf{B}$;
$\partial_{\mathbf{G}}\,\varphi(\mathbf{G},\boldsymbol{\alpha})$ – the partial derivative of the function $\varphi(\mathbf{G},\boldsymbol{\alpha})$ with respect to $\mathbf{G}$;
H is Heaviside's function; $< z >= 1/2\,(z+ \mid z \mid)$, $\forall z\in R$– the set of all real numbers;
$\hat{\mathbf{C}}^t$ denotes the history of the strain tensor up to time t; superposed dot denotes material time derivative, replaced sometimes by d/dt in order to avoid ambiguities, see (3.16);
$\rho_0, \tilde{\rho}, \rho$ are mass densities in initial, relaxed and actual configurations.

2. Different description of the yield surfaces

Within the constitutive framework of elasto-plastic materials with local current relaxed configurations (lcrc), denoted K_t (see [3, 1]), the yield conditions have been formulated in the appropriate stress spaces, relative to K_t. These conditions involve either $\mathbf{\Pi}$ or $\mathbf{\Sigma}$ and $\boldsymbol{\alpha}$ – internal variables. It can be proved, see [1, 4], that:

1. Any **stress formulation**, for instance in the case of $\mathbf{\Sigma}$ models, leads to a model in **elastic strain formulation**, with the relationship between the appropriate yield functions, and constitutive functions, respectively, in the form:

$$\begin{aligned}&\widetilde{\mathcal{F}}(\mathbf{G},\boldsymbol{\alpha}) = \hat{\mathcal{F}}(\hat{\mathbf{\Sigma}}(\mathbf{G},\boldsymbol{\alpha}),\boldsymbol{\alpha}), \qquad \widetilde{\mathcal{B}}(\mathbf{G},\boldsymbol{\alpha}) = \hat{\mathcal{B}}(\hat{\mathbf{\Sigma}}(\mathbf{G},\boldsymbol{\alpha}),\boldsymbol{\alpha}), \quad \text{where}\\ &\mathbf{\Sigma} = \hat{\mathbf{\Sigma}}(\mathbf{G},\boldsymbol{\alpha}) \quad \text{the elastic-type constitutive equation,}\end{aligned} \tag{2.1}$$

in terms of Mandel's stress measure $\mathbf{\Sigma}$.

Note the similar relationships for $\mathbf{\Pi}$ models:

$$\begin{aligned}&\widetilde{\mathcal{F}}(\mathbf{G},\boldsymbol{\alpha}) = \mathcal{F}(h(\mathbf{G},\boldsymbol{\alpha}),\boldsymbol{\alpha}), \qquad \widetilde{\mathcal{B}}(\mathbf{G},\boldsymbol{\alpha}) = \mathcal{B}(h(\mathbf{G},\boldsymbol{\alpha}),\boldsymbol{\alpha}), \quad \text{where}\\ &\frac{\mathbf{\Pi}}{\tilde{\rho}} = h(\mathbf{G},\boldsymbol{\alpha}) \quad \text{the elastic-type constitutive equation,}\end{aligned} \tag{2.2}$$

this time in terms of the Piola–Kirchoff stress tensor $\mathbf{\Pi}$.

2. Any elasto-plastic models in elastic strain formulation lead to models in strain formulation, i.e. in a Lagrangian form, with the yield function

$$\mathcal{F}(\mathbf{C},\mathbf{Y}) := \widetilde{\mathcal{F}}(\mathbf{P}^{-T}\mathbf{C}\mathbf{P}^{-1},\boldsymbol{\alpha}) \equiv \widetilde{\mathcal{F}}(\mathbf{G},\boldsymbol{\alpha}) \quad \text{with} \quad \mathbf{Y} \equiv (\mathbf{P}^{-1},\boldsymbol{\alpha}). \tag{2.3}$$

In (2.3) $\mathbf{Y}$ denotes the irreversible variables, while the elastic strain $\mathbf{G}$ introduced in (1.2) and the total strain $\mathbf{C}$ are related via the formula

$$\mathbf{G} = \mathbf{P}^{-T}\mathbf{C}\mathbf{P}^{-1}, \quad \text{with} \quad \mathbf{C} = \mathbf{F}^T\mathbf{F}, \tag{2.4}$$

provided as a consequence of the multiplicative decomposition of the deformation gradient (1.1).

The yield conditions are formulated within the **elasto-plastic models** for materials with relaxed configurations.

• Let us briefly describe the $\mathbf{\Sigma}$-models with respect to relaxed configurations, K_t.

The elastic type constitutive equation, relative to the relaxed configurations, in terms of Mandel's stress measure is characterized by

$$\begin{aligned}&\mathbf{\Sigma} = \hat{\mathbf{\Sigma}}(\mathbf{G},\boldsymbol{\alpha}), \qquad \text{with the constraints}\\ &\mathbf{G}^{-1}\hat{\mathbf{\Sigma}}(\mathbf{G},\boldsymbol{\alpha}) = \hat{\mathbf{\Sigma}}^T(\mathbf{G},\boldsymbol{\alpha})\mathbf{G}^{-1},\\ &\hat{\mathbf{\Sigma}}(\mathbf{I},\boldsymbol{\alpha}) = 0 \qquad \text{and conversely}\\ &\hat{\mathbf{\Sigma}}(\mathbf{S},\boldsymbol{\alpha}) = 0 \quad \forall \mathbf{S} \in \mathrm{Sym}^+ \quad \Longleftrightarrow \quad \mathbf{S} = \mathbf{I}.\end{aligned} \tag{2.5}$$

The last statements in (2.5) formalize the relaxation assumption, which confers to the configurations K_t the attribute to be relaxed.

REMARK 1. We recall here that the plastic part of deformation is defined as $\mathbf{P}(t) \equiv K_t(K_0)^{-1}$, while $\mathbf{E}(t) \equiv \nabla\chi(t)K_t^{-1}$, for any χ – motion of the body. The local configurations K_t for all $t \in R$ are defined in NOLL's sense, [23], i.e. they can be viewed as an invertible linear transformation.

The **irreversible behaviour** is described by rate-independent evolution equations given in the form:

$$\dot{\mathbf{P}}\mathbf{P}^{-1} = \hat{\mu}\,\hat{\mathcal{B}}(\boldsymbol{\Sigma}, \boldsymbol{\alpha}), \qquad \dot{\boldsymbol{\alpha}} = \hat{\mu}\,\hat{\mathbf{l}}(\boldsymbol{\Sigma}, \boldsymbol{\alpha}), \tag{2.6}$$

attached to the yield function defined in $\boldsymbol{\Sigma}$-space, dependent on $\boldsymbol{\alpha}$, i.e. to $\hat{\mathcal{F}}(\boldsymbol{\Sigma}, \boldsymbol{\alpha})$, via the Khun–Tucker and the consistency conditions

$$\begin{aligned} &\hat{\mathcal{F}}(\cdot, \boldsymbol{\alpha}) : \mathcal{D}_{\hat{\mathcal{F}}} \subset \mathrm{Lin} \longrightarrow R, \quad \hat{\mathcal{F}}(0, \boldsymbol{\alpha}) < 0 \qquad \text{and} \\ &\hat{\mu} \geq 0, \quad \hat{\mu}\,\hat{\mathcal{F}} = 0, \quad \hat{\mathcal{F}} \leq 0, \quad \text{and} \quad \hat{\mu}\,\dot{\hat{\mathcal{F}}} = 0 \quad \text{consistency condition.} \end{aligned} \tag{2.7}$$

We add the initial condition

$$\mathbf{P}(t_0) = \mathbf{I}, \qquad \boldsymbol{\alpha}(t_0) = 0. \tag{2.8}$$

• A larger class of elasto-plastic materials with relaxed configurations can be described in **strain formulation**, with respect to time-dependent K_t – relaxed configurations. The elastic type constitutive equation delivers the current value of the Piola–Kirchhoff stress tensor

$$\begin{aligned} &\frac{\boldsymbol{\Pi}}{\tilde{\rho}} = \mathbf{h}(\mathbf{G}, \boldsymbol{\alpha}), \quad \text{with} \quad \mathbf{G} = \mathbf{E}^T\mathbf{E}, \\ &\mathbf{h}(\mathbf{S}, \boldsymbol{\alpha}) = 0 \quad \text{for} \quad \mathbf{S} \in \mathrm{Sym}^+ \quad \Longleftrightarrow \quad \mathbf{S} = \mathbf{I} \quad \text{relaxation property.} \end{aligned} \tag{2.9}$$

The evolution equations for the plastic part of deformation $\mathbf{P}$ and internal variable $\boldsymbol{\alpha}$ with respect to the current relaxed configuration K_t are written in the form

$$\begin{aligned} &\dot{\mathbf{P}}\mathbf{P}^{-1} = \lambda\,\widetilde{\mathcal{B}}(\mathbf{G}, \boldsymbol{\alpha}), \qquad \dot{\boldsymbol{\alpha}} = \lambda\,\widetilde{\mathbf{l}}(\mathbf{G}, \boldsymbol{\alpha}), \\ &\lambda \geq 0, \qquad \lambda\widetilde{\mathcal{F}} = 0, \qquad \widetilde{\mathcal{F}} \leq 0, \qquad \lambda\dot{\widetilde{\mathcal{F}}} = 0 \qquad \text{consistency condition,} \end{aligned} \tag{2.10}$$

at which we add the initial condition (2.8).

The yield function $\widetilde{\mathcal{F}}$ is dependent on the elastic strain tensor $\mathbf{G}$ and internal variables $\boldsymbol{\alpha}$.

REMARK 2. The evolution in time for $\mathbf{Y} \equiv (\mathbf{P}^{-1}, \boldsymbol{\alpha})$ is governed by the solutions of Cauchy problem, for a given total strain history, $\mathbf{C}(\mathbf{X}, \cdot) : [t_0, \infty) \rightarrow \mathrm{Sym}$. The differential system pulled back to the initial configuration through the procedure mentioned in (2.3), is written in an invariant form with respect to a change of frame in the actual configuration. In [1] the mathematical aspects concerning the solutions of the appropriate Cauchy problem are analysed. Just the Lagrangian formulation of the differential system describing irreversible behaviour allows to prove the coherent evolution of the solutions and the consistency of the model.

Consequently, via the solution of the appropriate differential system formulated for $\mathbf{Y}$, the following definitions for the elastic range (or the current elastic domain), in elastic strain space and in total strain space respectively, become possible:

Definition 1. *The* **elastic range** *(in the total strain space) and the reduced elastic range (in the elastic strain space) at any time t, for a given history of the total strain up to time t,* $\hat{\mathbf{C}}^t$*, denoted by* $\mathcal{U}(\hat{\mathbf{C}}^t)$, $\mathcal{U}_{\mathcal{R}}(\hat{\mathbf{C}}^t)$*, are defined by*

$$(2.11)\qquad \begin{aligned} \mathcal{U}(\hat{\mathbf{C}}^t) &: = \{\mathbf{B} \in \mathrm{Sym}^+ \mid \mathcal{F}(\mathbf{B}, \mathbf{Y}(t)) \leq 0\}\ , \\ \mathcal{U}_{\mathcal{R}}(\hat{\mathbf{C}}^t) &: = \{\mathbf{A} \in \mathrm{Sym}^+ \mid \widetilde{\mathcal{F}}(\mathbf{A}, \boldsymbol{\alpha}(t)) \leq 0\}, \end{aligned}$$

$\mathcal{F}$ denotes the yield function in the initial configuration, while $\widetilde{\mathcal{F}}$ is the yield function in the relaxed configuration, in a strain description.

The yield surfaces, or the yield conditions, can be characterized as the boundaries of the elastic ranges (2.11), in the total strain space and in the elastic strain space respectively, by

$$(2.12)\qquad \begin{aligned} \partial\mathcal{U}(\hat{\mathbf{C}}^t) &: = \{\mathbf{B} \in \mathrm{Sym}^+ \mid \mathcal{F}(\mathbf{B}, \mathbf{Y}(t)) = 0\}, \\ \partial\mathcal{U}_{\mathcal{R}}(\hat{\mathbf{C}}^t) &= \{\mathbf{A} \in \mathrm{Sym}^+ \mid \widetilde{\mathcal{F}}(\mathbf{A}, \boldsymbol{\alpha}(t)) = 0\}. \end{aligned}$$

The elastic ranges (2.4) have the same topological properties, due to the formula (2.4), which defines an homomorphism between the elastic strain space and the total strain space for any fixed $\mathbf{P}$. $\mathcal{U}_{\mathcal{R}}(\hat{\mathbf{C}}^t)$ is considered to be the closure of a connected open set, with identity strain tensor inside.

3. Elasto-plastic models in Eulerian setting

Starting from elasto-plastic models in the intermediate configuration we associate the description in the actual configuration for generally anisotropic materials, by the push-forward procedure.

Taking into account the constitutive description for $\boldsymbol{\Sigma}$-models, (2.5), (2.6), and the kinematic relation between the velocity gradient $\mathbf{L} \equiv \mathrm{grad}\mathbf{v} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ and the rates of elastic $\mathbf{L}^e \equiv \dot{\mathbf{E}}\mathbf{E}^{-1}$ and plastic $\mathbf{L}^p \equiv \dot{\mathbf{P}}\mathbf{P}^{-1}$ parts of deformation, derived as a straightforward consequence of the multiplicative decomposition (1.1)

$$(3.1)\qquad \mathbf{L} \equiv \dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\mathbf{E}}\mathbf{E}^{-1} + \mathbf{E}\dot{\mathbf{P}}\mathbf{P}^{-1}\mathbf{E}^{-1}, \qquad \mathbf{L}^e = \dot{\mathbf{E}}\mathbf{E}^{-1}, \qquad \mathbf{L}^p = \dot{\mathbf{P}}\mathbf{P}^{-1},$$

we pass to the following equivalent formulation (3.3) together with (3.4), under the hypothesis of the **existence** of a smooth **stress potential** $\varphi(\cdot, \boldsymbol{\alpha}(t))$ such that

$$\frac{\boldsymbol{\Pi}(t)}{\tilde{\rho}} = 2\, \partial_{\mathbf{G}} \varphi(\mathbf{G}, \boldsymbol{\alpha}) = \mathbf{h}(\mathbf{G}, \boldsymbol{\alpha}). \tag{3.2}$$

The irreversible behaviour is described by rate-independent evolution equations given in the form:

• evolution equations for $(\mathbf{E}, \boldsymbol{\alpha})$

$$\dot{\mathbf{E}}\mathbf{E}^{-1} = \mathbf{L} - \hat{\mu}\, \mathbf{E}\hat{\mathcal{B}}(\boldsymbol{\Sigma}, \boldsymbol{\alpha})\mathbf{E}^{-1}, \qquad \dot{\boldsymbol{\alpha}} = \hat{\mu}\, \hat{\mathbf{l}}(\boldsymbol{\Sigma}, \boldsymbol{\alpha}), \tag{3.3}$$

with the conditions do define the plastic multiplier written in (2.7).

The elastic type behaviour is given in terms of $\boldsymbol{\Sigma}$, from (1.2) and (3.2), in the form

$$\boldsymbol{\Sigma} = \hat{\boldsymbol{\Sigma}}(\mathbf{G}, \boldsymbol{\alpha}) \equiv 2\, \mathbf{G}\, \partial_{\mathbf{G}} \varphi(\mathbf{G}(t), \boldsymbol{\alpha}(t)). \tag{3.4}$$

In order to obtain the consistency of the model in the sense that:

• for any smooth history $\mathbf{F}(\mathbf{X}, \cdot) : [t_0, \infty) \longrightarrow \mathrm{Lin}$ given at a fixed material point $\mathbf{X}$, there exists at every time a well-defined elastic part of deformation $\mathbf{E}$ and the set of internal variable $\boldsymbol{\alpha}$, we introduce the causality assumption (similar to those stipulated in [1]): the plastic multiplier $\hat{\mu}$ is uniquely determined, if the current value of the appropriate stress is situated on the current yield surface.

In the framework of $\boldsymbol{\Sigma}$-model, the current yield surface (or the yield condition) in Mandel's stress space can be regarded as the boundary of the current **elastic stress range** (see [5]) defined for a given value of internal variables, $\boldsymbol{\alpha}$, by

$$\mathcal{K}(\boldsymbol{\alpha}) = \{\boldsymbol{\Sigma} \in \mathrm{Lin} \mid \quad \hat{\mathcal{F}}(\boldsymbol{\Sigma}, \boldsymbol{\alpha}) \leq 0\}. \tag{3.5}$$

The current elastic stress range, (3.5), is considered to be the closure of a connected open set, with zero stress inside. Moreover, the yield surface represented by $\partial\mathcal{K}(\boldsymbol{\alpha})$ is considered to be a differential manifold of the class C^1. The pre-image under the elastic stress function $\hat{\boldsymbol{\Sigma}}$, via formula (3.4),

$$\mathcal{U}(\boldsymbol{\alpha}) = \{\mathbf{G} \in \mathrm{Sym} \mid \quad \hat{\mathcal{F}}(\hat{\boldsymbol{\Sigma}}(\mathbf{G}, \boldsymbol{\alpha}), \boldsymbol{\alpha}) \leq 0\} \tag{3.6}$$

plays a basical role in dissipative postulate, formulated for anisotropic finite elasto-plastic materials in [4].

Remark 3. The image of the elastic (reduced) range $\partial\mathcal{U}_{\mathcal{R}}(\hat{\mathbf{C}}^t)$ under the elastic stress function is generally accepted as definition of the elastic stress range in the appropriate stress space, [20, 17]. The topological boundary of the elastic stress range (or domain) defines the yield condition in stress space. In our approach to finite elasto-plasticity, the transformation map $\hat{\mathbf{\Sigma}}$ (i.e. the elastic stress function in $\mathbf{\Sigma}$ from (3.4)) is not a one-to-one function, even for structurally isotropic material. $\hat{\mathbf{\Sigma}}$ does not map the boundary of certain elastic reduced range into the boundary of its image, i.e. the associated yield condition in stress space is not well-definite. In order to avoid the issue produced by the lack of the injectivity, a converse procedure to define the admissible stresses has been adopted in [4, 5]. The pre-image of the elastic stress range under the elastic map $\hat{\mathbf{\Sigma}}$ gives rise to an elastic domain with all the properties imposed in the former dissipation postulate.

In order to avoid the objection formulated by LUBLINER [15, 16], relative to the nine-dimensional flow rule, in the analysis of the dissipative nature of the plastic flow [4], the elastic range has been defined as the pre-image in the elastic strain space, of the elastic stress range, via the elastic function, say here (3.4).

PROPOSITION 1. For a given smooth history of the deformation gradient $\mathbf{F}(\mathbf{X}, \cdot) : [t_0, \infty) \longrightarrow \mathrm{Lin}$, the elastic part of deformation and internal variable are defined to be a solution of the Cauchy problem, generated by the evolution equations (3.3) together with the initial condition (2.8)

$$\begin{aligned}
&\dot{\mathbf{E}}\mathbf{E}^{-1} = \mathbf{L} - \frac{<\hat{\beta}>}{\hat{\gamma}} H(\hat{\mathcal{F}})\, \mathbf{E}\, (\hat{\mathcal{B}})\, \mathbf{E}^{-1}, \quad \text{with} \quad \mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1},\\
&\dot{\boldsymbol{\alpha}} = \frac{<\hat{\beta}>}{\hat{\gamma}} H(\hat{\mathcal{F}})\hat{\mathbf{l}}(\mathbf{\Sigma}, \boldsymbol{\alpha}), \qquad \hat{\mu} = \frac{<\hat{\beta}>}{\hat{\gamma}} H(\hat{\mathcal{F}}),\\
&\hat{\beta} = \mathbf{E}((\partial_{\mathbf{G}}\, \hat{\mathbf{\Sigma}})^T[\partial_{\mathbf{\Sigma}}\, \hat{\mathcal{F}}])\mathbf{E}^T \cdot \mathbf{D}, \qquad \text{and} \quad \mathbf{D} = \{\mathbf{L}\}^s,\\
&\hat{\gamma} = (\partial_{\mathbf{G}}\, \hat{\mathbf{\Sigma}})^T[\partial_{\mathbf{\Sigma}}\, \hat{\mathcal{F}}]) \cdot \{\mathbf{G}\hat{\mathcal{B}}\}^s - \frac{1}{2}\frac{d\hat{\mathcal{F}}}{d\boldsymbol{\alpha}} \cdot \hat{\mathbf{l}},\\
&\text{here} \qquad \frac{d\hat{\mathcal{F}}}{d\boldsymbol{\alpha}} \equiv (\partial\, \hat{\mathbf{\Sigma}}_{\boldsymbol{\alpha}})^T[\partial_{\mathbf{\Sigma}}\, \hat{\mathcal{F}}] + \partial_{\boldsymbol{\alpha}}\, \hat{\mathcal{F}},
\end{aligned} \tag{3.7}$$

with $\hat{\gamma}$ – the hardening parameter, supposed to be positive. The evolution functions are calculated by composing the "hat functions" with the elastic-type constitutive function (3.4), which means by passing to the elastic strain description via the formulae (2.1).

In order to derive the expression of the plastic multiplier (3.7) we write the consistency condition, on the yield surface when $\hat{\mu} > 0$,

$$\partial_{\mathbf{\Sigma}}\hat{\mathcal{F}}(\mathbf{\Sigma}, \boldsymbol{\alpha}) \cdot \dot{\mathbf{\Sigma}} + \partial_{\boldsymbol{\alpha}}\hat{\mathcal{F}}(\mathbf{\Sigma}, \boldsymbol{\alpha}) \cdot \dot{\boldsymbol{\alpha}} = 0. \tag{3.8}$$

We calculate the rate of elastic strain using kinematic relations (3.1) together with the first evolution equation (3.3), in the form

$$\dot{\mathbf{G}} = 2\, \mathbf{E}^T\mathbf{D}\mathbf{E} - 2\hat{\mu}\, \{\mathbf{G}\hat{\mathcal{B}}\}^s \tag{3.9}$$

as well as the partial derivative of the elastic-type constitutive functions (3.4)

$$(3.10)\qquad \partial_{\mathbf{G}} \hat{\mathbf{\Sigma}}[\mathbf{A}] = 2\,\mathbf{A} \partial_{\mathbf{G}}\,\varphi(\mathbf{G},\boldsymbol{\alpha}) + 2\,\mathbf{G}\,\partial^2_{\mathbf{GG}}\,\varphi(\mathbf{G},\boldsymbol{\alpha})[\mathbf{A}], \quad \forall \quad \mathbf{A} \in \mathrm{Sym}.$$

REMARK 4. In the actual configuration the behaviour of the elasto-plastic material is described, for a given history of deformation $\mathbf{F} : [t_0, \infty) \longrightarrow \mathrm{Lin}$, by the elastic-type constitutive equation, which delivers the current value of Cauchy stress tensor,

$$(3.11)\qquad \frac{\mathbf{T}}{\rho} = 2\,\mathbf{E}\,(\partial_{\mathbf{G}} \varphi(\mathbf{G},\boldsymbol{\alpha}))\,\mathbf{E}^T$$

with $(\mathbf{E},\boldsymbol{\alpha})$ solution of the Cauchy problem (3.7), while the plastic part of deformation is calculated from (1.1), $\mathbf{P} = \mathbf{E}^{-1}\mathbf{F}$.

Let us define the tensorial internal variable $\mathbf{a}$ in actual configuration, similarly to the relationship between the Mandel stress measure and the Cauchy stress derived from (1.2), by pushing forward to the actual configuration

$$(3.12)\qquad \mathbf{a} := \mathbf{E}^{-T}\boldsymbol{\alpha}\mathbf{E}^T, \qquad \frac{\mathbf{T}}{\rho} = \mathbf{E}^{-T}\mathbf{\Sigma}\mathbf{E}^T.$$

When we push forward the constitutive representation to the actual configuration, the appropriate constitutive functions written in the intermediate configuration generate new functions dependent on the set of variables $\left(\frac{\mathbf{T}}{\rho}, \mathbf{a}, \mathbf{E}\right)$, as it follows:

$$(3.13)\qquad \begin{aligned}
\overline{\mathcal{F}}\left(\frac{\mathbf{T}}{\rho}, \mathbf{a}, \mathbf{E}\right) &: = \hat{\mathcal{F}}(\mathbf{\Sigma},\boldsymbol{\alpha}) \equiv \hat{\mathcal{F}}\left(\mathbf{E}^T \frac{\mathbf{T}}{\rho}\mathbf{E}^{-T}, \mathbf{E}^T\mathbf{a}\mathbf{E}^{-T}\right),\\
\overline{\mathcal{B}}\left(\frac{\mathbf{T}}{\rho}, \mathbf{a}, \mathbf{E}\right) &: = \mathbf{E}\,(\hat{\mathcal{B}}(\mathbf{\Sigma},\boldsymbol{\alpha}))\,\mathbf{E}^{-1},\\
\bar{\mathbf{l}}\left(\frac{\mathbf{T}}{\rho}, \mathbf{a}, \mathbf{E}\right) &: = \mathbf{E}^{-T}\,(\hat{\mathbf{l}}(\mathbf{\Sigma},\boldsymbol{\alpha}))\,\mathbf{E}^T,\\
\mathbf{n}\left(\frac{\mathbf{T}}{\rho}, \mathbf{a}, \mathbf{E}\right) &: = \mathbf{E}\,(\partial_{\mathbf{\Sigma}}\hat{\mathcal{F}}\,(\mathbf{\Sigma},\boldsymbol{\alpha}))\;\mathbf{E}^{-1},\\
\mathbf{q}\left(\frac{\mathbf{T}}{\rho}, \mathbf{a}, \mathbf{E}\right) &: = \mathbf{E}\,(\partial_{\boldsymbol{\alpha}}\hat{\mathcal{F}}(\mathbf{\Sigma},\boldsymbol{\alpha}))\,\mathbf{E}^{-1}
\end{aligned}$$

taking into account the formulae (3.12).

PROPOSITION 2. In the actual configuration the evolution equations (3.3) for $(\mathbf{E}, \mathbf{a})$ become

$$
\mathbf{L}^e \equiv \dot{\mathbf{E}}\mathbf{E}^{-1} = \mathbf{L} - \overline{\mu}\, \overline{\mathcal{B}}\left(\frac{\mathbf{T}}{\rho}, \mathbf{a}, \mathbf{E}\right),
\qquad
\frac{\bar{D}^e\, \mathbf{a}}{\bar{D}t} = \overline{\mu}\, \bar{l}\left(\frac{\mathbf{T}}{\rho}, \mathbf{a}, \mathbf{E}\right), \tag{3.14}
$$

with the objective time derivative of the field $\mathbf{a}$ introduced in (3.14) by the formulae

$$
\frac{\bar{D}^e\, \mathbf{a}}{\bar{D}t} := \dot{\mathbf{a}} + (\mathbf{L}^e)^T\, \mathbf{a} - \mathbf{a}\, (\mathbf{L}^e)^T \equiv \mathbf{E}^{-T} \dot{\boldsymbol{\alpha}} \mathbf{E}^T. \tag{3.15}
$$

In order to obtain the **rate form of the constitutive equation** in terms of Cauchy stress tensor we take the derivative with respect to time in (3.11), and replace the derivative of the elastic strain tensor $\dot{\mathbf{G}}$ from (3.9). It follows

$$
\frac{d}{dt}\left(\frac{\mathbf{T}}{\rho}\right) - \mathbf{L}^e \frac{\mathbf{T}}{\rho} - \frac{\mathbf{T}}{\rho}(\mathbf{L}^e)^T = \widetilde{\mathcal{E}}[\mathbf{D}]
- \hat{\mu}\, \widetilde{\mathcal{E}}[\mathbf{E}^{-T}\{\mathbf{G}\hat{\mathcal{B}}\}^s \mathbf{E}^{-1}] + 2\mathbf{E}\, (\partial^2_{\boldsymbol{\alpha}\mathbf{G}}\, \varphi[\dot{\boldsymbol{\alpha}}])\mathbf{E}^T, \tag{3.16}
$$

with fourth order elastic tensor $\widetilde{\mathcal{E}}$ and the measure of the influence of the hardening on the elastic-type constitutive function $\mathcal{N}$, defined below:

$$
\widetilde{\mathcal{E}}[\mathbf{A}] = 4\, \mathbf{E}(\partial^2_{\mathbf{G}\mathbf{G}}\, \varphi[\mathbf{E}^T \mathbf{A}\mathbf{E}])\mathbf{E}^T, \quad \forall \quad \mathbf{A} \in \mathrm{Sym},
\qquad
\mathcal{N}[\bar{\mathbf{l}}] \equiv 2\, \mathbf{E}(\partial^2_{\boldsymbol{\alpha}\,\mathbf{G}}\, \varphi[\mathbf{E}^T\, \bar{\mathbf{l}}\, \mathbf{E}^{-T}])\mathbf{E}^T. \tag{3.17}
$$

Further we use the notation from (3.13), as well as the equations (3.3), and the identity

$$
\mathbf{E}^{-T}\{\mathbf{G}\hat{\mathcal{B}}\}^s \mathbf{E}^{-1} = \{\mathbf{E}\hat{\mathcal{B}}\mathbf{E}^{-1}\}^s = \{\overline{\mathcal{B}}\}^s \tag{3.18}
$$

in order to transform (3.16).

THEOREM 1. *The rate-type constitutive equations for the anisotropic* $\boldsymbol{\Sigma}$-*models, pushed forward to the actual configuration, are described in the form of the differential system, for the unknowns* $\left(\frac{\mathbf{T}}{\rho}, \mathbf{a}, \mathbf{E}\right)$

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\mathbf{T}}{\rho}\right) - \mathbf{L}\frac{\mathbf{T}}{\rho} - \frac{\mathbf{T}}{\rho}\mathbf{L}^T = \widetilde{\mathcal{E}}[\mathbf{D}] - \overline{\mu}\left(2\left\{\overline{\mathcal{B}}\frac{\mathbf{T}}{\rho}\right\}^s + \widetilde{\mathcal{E}}[\{\overline{\mathcal{B}}\}^s] - \mathcal{N}[\bar{\mathbf{l}}]\right),\\
&\dot{\mathbf{E}}\mathbf{E}^{-1} = \mathbf{L} - \overline{\mu}\, \overline{\mathcal{B}},\\
&\dot{\mathbf{a}} + \mathbf{L}^T \mathbf{a} - \mathbf{a}\mathbf{L}^T = \overline{\mu}\, (\overline{\mathcal{B}}^T\, \mathbf{a} - \mathbf{a}\overline{\mathcal{B}}^T + \bar{\mathbf{l}}),
\end{aligned} \tag{3.19}
$$

with the plastic multiplier $\overline{\mu}$ *and hardening parameter* $\bar{h}_c$ *defined by*

$$(3.20)\qquad \begin{aligned} \overline{\mu} &= \frac{1}{\bar{h}_c} < \left(\mathbf{n}\frac{\mathbf{T}}{\rho} + \frac{1}{2}\,\widetilde{\mathcal{E}}[\mathbf{n}]\right)\cdot \mathbf{D} > \quad \textit{with} \\ \bar{h}_c &= \mathbf{n}\cdot\left(\{\overline{\mathcal{B}}\}^s\frac{\mathbf{T}}{\rho} + \frac{1}{2}\,\widetilde{\mathcal{E}}\,[\{\overline{\mathcal{B}}\}^s]\right) - \frac{1}{2}\mathbf{n}\cdot\mathcal{N}[\bar{\mathbf{l}}] - \frac{1}{2}\,\mathbf{q}\cdot\,\bar{\mathbf{l}}. \end{aligned}$$

We suppose that $\bar{h}_c > 0$.

A useful relationship can be obtained from (3.10) calculated for the rate of plastic part of deformation, via the formula (2.6) together with (3.14):

$$(3.21)\qquad (\partial_{\mathbf{G}}\hat{\boldsymbol{\Sigma}})^T[\dot{\mathbf{P}}\mathbf{P}^{-1}] = \hat{\mu}\,\mathbf{E}^{-1}\left(\left\{\overline{\mathcal{B}}\frac{\mathbf{T}}{\rho}\right\}^s + \frac{1}{2}\,\widetilde{\mathcal{E}}[\{\overline{\mathcal{B}}\}^s]\right)\mathbf{E}^{-T}.$$

4. Anisotropic elasto-plastic models in multiplicative plasticity

The Eulerian setting of elasto-plastic models has been derived in (3.19) without imposing any specific anisotropy. The constitutive functions which enter (3.19) are dependent on the set of variables $\left(\frac{\mathbf{T}}{\rho}, \mathbf{a}, \mathbf{E}\right)$, as it follows from (3.13).

Based on the material symmetry concept developed in [2], [3] and following the adopted point of view in [7], within the constitutive framework of $\boldsymbol{\Sigma}$-model we introduce here:

The symmetry requirement. There exists a symmetry group $g_k \subset \mathrm{Ort}^+-$ such that the constitutive functions satisfy the following restrictions:

$$(4.1)\qquad \begin{aligned} \varphi(\mathbf{Q}\mathbf{G}\mathbf{Q}^T, \mathbf{Q}\boldsymbol{\alpha}\mathbf{Q}^T) &= \varphi(\mathbf{G}, \boldsymbol{\alpha}), \\ \hat{\mathcal{F}}(\mathbf{Q}\boldsymbol{\Sigma}\mathbf{Q}^T, \mathbf{Q}\boldsymbol{\alpha}\mathbf{Q}^T) &= \hat{\mathcal{F}}(\boldsymbol{\Sigma}, \boldsymbol{\alpha}), \\ \hat{\mathcal{B}}(\mathbf{Q}\boldsymbol{\Sigma}\mathbf{Q}^T, \mathbf{Q}\boldsymbol{\alpha}\mathbf{Q}^T) &= \mathbf{Q}\hat{\mathcal{B}}(\boldsymbol{\Sigma}, \boldsymbol{\alpha})\mathbf{Q}^T, \\ \hat{\mathbf{l}}(\mathbf{Q}\boldsymbol{\Sigma}\mathbf{Q}^T, \mathbf{Q}\boldsymbol{\alpha}\mathbf{Q}^T) &= \mathbf{Q}\hat{\mathbf{l}}(\boldsymbol{\Sigma}, \boldsymbol{\alpha})\mathbf{Q}^T \end{aligned}$$

for all $\mathbf{Q} \in g_k$.

DEFINITION 2. *The elasto-plastic material is anisotropic if there exists a material symmetry group* $g_k \subset \mathrm{Ort}^+$, $g_k \neq \mathrm{Ort}^+$, *and the material is isotropic if* $g_k = \mathrm{Ort}^+$(*but it follows that* g_k *becomes equal to* Ort).

REMARK 5. In formulating the material symmetry concept, in [2, 3], we have stipulated that for the same motion χ, two sets of (clrc) $\{K_t\}$, $\{\bar{K}_t\}$ can be equivalently used to characterize mechanical response, if and only if $\bar{K}_t K_t^{-1} = \mathbf{Q} \in \mathrm{Ort}$ is a fixed orthogonal map, i.e. the condition $\bar{K}_t K_t^{-1} \in \mathrm{Ort}$, that follows from the relaxation assumption, has been reinforced. Thus, see also Remark 1, we have taken into account that $\mathbf{E}, \mathbf{P}, \mathbf{\Pi}$, hence $\mathbf{\Sigma}$ are changing into $\mathbf{E}\mathbf{Q}^T$, $\mathbf{Q}\mathbf{P}\mathbf{Q}^T$, $\mathbf{Q}\mathbf{\Pi}\mathbf{Q}^T$, $\mathbf{Q}\mathbf{\Sigma}\mathbf{Q}^T$, when the local initial configuration K_0 is rotated with an orthogonal transformation. We denoted by $\mathbf{Q}K_0$ the transformed local initial configuration, viewed as an invertible linear transformation. Here $\mathbf{Q} \in g_k$.

The tensorial internal variable must satisfy the same constraint as $\mathbf{\Sigma}$, see (3.12). In general case the set of internal variables can be split into tensorial internal variables and scalar internal variables, all of them are considered to be frame-independent. We appreciate that a well-elaborated model must specify, in addition to concrete nature on the internal variables, their transformation laws with respect to a change of frame in the actual configuration, as well as to a change of the reference configurations [3]. Only the scalar set remains unchanged by passing to the actual configuration, being also invariant with respect to symmetry group, like $(4.1)_1$.

Remark 6. Relative to the fact that g_k is viewed as a subset Ort^+, we remark that if $\mathbf{E}, \mathbf{\Pi}$, are changed into $\mathbf{E}\mathbf{Q}^T, \mathbf{Q}\mathbf{\Sigma}\mathbf{Q}^T$, then from the relationship between the Cauchy stress and Piola–Kirchoff (1.2) it follows that $\mathbf{T}$ is changed into $\det(\mathbf{Q})\mathbf{T}$, i.e. $\mathbf{T}$ remains invariant under an orthogonal transformation, if and only if $\det(\mathbf{Q}) = 1$.

Objectivity axiom has been formulated in [3]: If $(K_t, \boldsymbol{\alpha}_{K_t})$ is a set of local current configuration and internal variables associated to the motion χ, then K_t is a current local configuration for the motion χ^*, which differs from χ by a change of frame, as well $\boldsymbol{\alpha}^*_{K_t}$ – the set of internal variables corresponding to χ^*, but related to K_t, is equal to $\boldsymbol{\alpha}_{K_t}$, i.e. $\boldsymbol{\alpha}^*_{K_t} = \boldsymbol{\alpha}_{K_t}$. Here in order to simplify the writing formulae, we have omitted the dependence on the local configuration of the constitutive function and of the variables.

Remark 7. Due to the relationship (1.2), the statement: "$\mathbf{T}$ is objective if and only if $\mathbf{\Pi}$ is invaraint with respect to a change of frame in the actual configuration" is true, only under the hypothesis that $\mathbf{Q} \in \mathrm{Ort}$ is restricted to have $\det(\mathbf{Q}) = 1$, i.e. $\mathbf{Q} \in \mathrm{Ort}^+$.

Consequently, for any $\mathbf{Q}$ a rigid rotation, which defines the change of frame from the motion χ to χ^*, it follows that:

$$\begin{aligned} &\mathbf{P}^* = \mathbf{P}, \qquad \mathbf{E}^* = \mathbf{Q}\mathbf{E}, \qquad \boldsymbol{\alpha}^* = \boldsymbol{\alpha}, \qquad \mathbf{\Pi}^* = \mathbf{\Pi}, \qquad \mathbf{\Sigma}^* = \mathbf{\Sigma}, \\ &\mathbf{F}^* = \mathbf{Q}\mathbf{F}, \qquad \mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T. \end{aligned} \tag{4.2}$$

The evolution in time of the right elastic strain tensor $\mathbf{G}$ can be prescribed as a consequence of (3.9) together with (3.18) by

$$\dot{\mathbf{G}} = 2\,\mathbf{E}^T(\mathbf{D} - \hat{\mu}\,\{\bar{\mathcal{B}}\}^s)\mathbf{E}. \tag{4.3}$$

Concerning the time derivative of the tensorial internal variable $\mathbf{a}$, we remark that $(3.19)_3$ can be also written in terms of the objective derivative $D\mathbf{a}/Dt$

$$\frac{D\mathbf{a}}{Dt} = \hat{\mu}\,\mathbf{E}^{-T}\,(\hat{\mathcal{B}}^T\boldsymbol{\alpha} - \boldsymbol{\alpha}\hat{\mathcal{B}}^T + \hat{\mathbf{l}})\,\mathbf{E}^T, \qquad \frac{D\mathbf{a}}{Dt} := \frac{d}{dt}\mathbf{a} + \mathbf{L}^T\mathbf{a} - \mathbf{a}\mathbf{L}^T. \tag{4.4}$$

The equation (4.4) can be written formally in a simple form

$$\frac{D\mathbf{a}}{Dt} = \hat{\mu}\,\mathbf{E}^{-T}\,(\hat{\mathbf{m}})\,\mathbf{E}^T, \quad \text{with} \quad \hat{\mathbf{m}}(\boldsymbol{\Sigma},\boldsymbol{\alpha}) := \hat{\mathcal{B}}^T\boldsymbol{\alpha} - \boldsymbol{\alpha}\hat{\mathcal{B}}^T + \hat{\mathbf{l}}. \tag{4.5}$$

REMARK 8. For a given history of the deformation gradient, $\hat{\mathbf{F}}^t$, the set of evolution equations (4.3) and (4.5) for the fields $(\mathbf{G},\mathbf{a})$ depends on the elastic rotation $\mathbf{R}^e$, via the polar decomposition theorem for $\mathbf{E}$

$$\mathbf{E} = \mathbf{R}^e\,\mathbf{U}^e \quad \text{with} \quad \mathbf{G} = (\mathbf{U}^e)^2, \quad \text{and} \quad \mathbf{b}^e := \mathbf{E}\mathbf{E}^T \equiv \mathbf{R}^e\,\mathbf{G}\,(\mathbf{R}^e)^T. \tag{4.6}$$

Let us introduce the time derivative of the elastic left strain tensor, following [25]. From (4.6) we get the identity

$$\dot{\mathbf{b}}^e = \mathbf{L}^e\mathbf{b}^e + \mathbf{b}^e\,(\mathbf{L}^e)^T. \tag{4.7}$$

We replace the rate of the elastic part of the deformation by $(3.19)_2$ and we arrive at the equation containing the appropriate objective derivative $\mathcal{L}_{\mathbf{v}}\,\mathbf{b}^e$, expressed by

$$\mathcal{L}_{\mathbf{v}}\,\mathbf{b}^e = -2\,\hat{\mu}\,\{\bar{\mathcal{B}}\mathbf{b}^e\}^s, \quad \text{where} \quad \mathcal{L}_{\mathbf{v}}\,\mathbf{b}^e := \frac{d}{dt}\,\mathbf{b}^e - \mathbf{L}\mathbf{b}^e - \mathbf{b}^e\,\mathbf{L}^T. \tag{4.8}$$

Finally we have proved:

• The set of the constitutive equations defining the model in terms of $\left(\mathbf{b}^e, \mathbf{a}, \dfrac{\mathbf{T}}{\rho}\right)$ can be expressed in the form

$$\begin{aligned} &\mathcal{L}_{\mathbf{v}}\,\mathbf{b}^e = -2\,\hat{\mu}\,\mathbf{E}\{\hat{\mathcal{B}}(\boldsymbol{\Sigma},\boldsymbol{\alpha})\}^s\mathbf{E}^T, \qquad \frac{D\mathbf{a}}{Dt} = \hat{\mu}\,\mathbf{E}^{-T}\,(\hat{\mathbf{m}}(\boldsymbol{\Sigma},\boldsymbol{\alpha}))\,\mathbf{E}^T,\\ &\frac{\mathbf{T}}{\rho} = 2\,\mathbf{E}\,(\partial_{\mathbf{G}}\varphi(\mathbf{G},\boldsymbol{\alpha}))\,\mathbf{E}^T. \end{aligned} \tag{4.9}$$

Due to the definitions of the Mandel's stress measure and of the internal variable, $\boldsymbol{\Sigma}$ and $\boldsymbol{\alpha}$ are related to $\mathbf{T}$ and $\mathbf{a}$, and the right-hand side of the equations written in (4.9) depends on the elastic strain $\mathbf{G}$ and on the elastic rotation $\mathbf{R}^e$.

We rewrite the system (4.9) using the polar decomposition theorem (4.6)

$$
\begin{aligned}
&\mathcal{L}_{\mathbf{v}}\, \mathbf{b}^e = -2\,\hat{\mu}\, \mathbf{R}^e \mathbf{U}^e \{\hat{\mathcal{B}}(\boldsymbol{\Sigma}, \boldsymbol{\alpha})\}^s \mathbf{U}^e (\mathbf{R}^e)^T,\\
&\frac{D\mathbf{a}}{Dt} = \hat{\mu}\, \mathbf{R}^e (\mathbf{U}^e)^{-1}\, (\hat{\mathbf{m}}(\boldsymbol{\Sigma}, \boldsymbol{\alpha}))\, \mathbf{U}^e (\mathbf{R}^e)^T,\\
&\frac{\mathbf{T}}{\rho} = 2\, \mathbf{R}^e \mathbf{U}^e\, (\partial_{\mathbf{G}} \varphi(\mathbf{G}, \boldsymbol{\alpha}))\, \mathbf{U}^e (\mathbf{R}^e)^T.
\end{aligned}
\tag{4.10}
$$

We exemplify the push-forward procedure to generate the functions provided in (3.13) for g_4 – transversal isotropy, when the elastic behaviour is characterized in [7] by:

$$
\begin{aligned}
\frac{\boldsymbol{\Pi}}{\tilde{\rho}} &= \widehat{\mathcal{E}}(\mathbf{n}_1 \otimes \mathbf{n}_1)[\boldsymbol{\Delta}] \equiv [a\boldsymbol{\Delta}\mathbf{n}_1 \cdot \mathbf{n}_1 + b\,\mathrm{tr}\boldsymbol{\Delta}](\mathbf{n}_1 \otimes \mathbf{n}_1)\\
&\quad + (c\boldsymbol{\Delta}\mathbf{n}_1 \cdot \mathbf{n}_1 + d\,\mathrm{tr}\boldsymbol{\Delta})\mathbf{I} + e[(\mathbf{n}_1 \otimes \mathbf{n}_1)\boldsymbol{\Delta} + \boldsymbol{\Delta}(\mathbf{n}_1 \otimes \mathbf{n}_1)] + f\boldsymbol{\Delta},\\
\boldsymbol{\Delta} &= \frac{1}{2}(\mathbf{G} - \mathbf{I}).
\end{aligned}
\tag{4.11}
$$

The existence of the stress potential holds if and only if $b = c$.

Within the constitutive framework of $\boldsymbol{\Sigma}$-models, see [7], g_4 – invariant yield criterion, quadratical with respect to $\overline{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma} - \boldsymbol{\alpha}$, is characterized only by five material parameters (if *elastic strains are small* ($\boldsymbol{\Sigma} \simeq \boldsymbol{\Pi}/\tilde{\rho}$), $\boldsymbol{\alpha} \in \mathrm{Sym}$ and $\kappa = \mathrm{constant}$)

$$
\begin{aligned}
\mathcal{F}(\boldsymbol{\Sigma}, \boldsymbol{\alpha}) &\equiv A_1[\overline{\boldsymbol{\Sigma}} \cdot (\mathbf{n}_1 \otimes \mathbf{n}_1)]^2 + A_2[\overline{\boldsymbol{\Sigma}}^2 \cdot (\mathbf{n}_1 \otimes \mathbf{n}_1)] + A_3 \mathrm{tr}\,(\overline{\boldsymbol{\Sigma}}^2)\\
&\quad + A_4(\mathrm{tr}\,\overline{\boldsymbol{\Sigma}})[\overline{\boldsymbol{\Sigma}} \cdot (\mathbf{n}_1 \otimes \mathbf{n}_1)] + A_5(\mathrm{tr}\,\overline{\boldsymbol{\Sigma}})^2 - \kappa = 0\,,\\
&\quad \text{for} \quad \overline{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma} - \boldsymbol{\alpha}.
\end{aligned}
\tag{4.12}
$$

REMARK 9. In this case the elastic stress range $\mathcal{K}(\boldsymbol{\alpha})$ is defined using (4.12) in (3.5), while its pre-image $\mathcal{U}(\boldsymbol{\alpha})$ is derived from (3.6) with the stress function $\hat{\boldsymbol{\Sigma}} = \mathbf{G}\dfrac{\boldsymbol{\Pi}}{\tilde{\rho}}$ expressed from (4.11).

PROPOSITION 3. Let us denote by $\mathbf{S} \equiv \dfrac{\mathbf{T}}{\rho} - \mathbf{a} = \mathbf{E}^{-T}(\overline{\boldsymbol{\Sigma}})\mathbf{E}^T$, and by $\mathbf{m}_1 = \mathbf{E}\mathbf{n}_1$ – the anisotropy direction pushed forward to the actual configuration. By direct calculus the yield function supposed to be (4.12) is written in the form

$$\overline{\mathcal{F}}\left(\frac{\mathbf{T}}{\rho},\mathbf{a},\mathbf{E}\right) = A_1[\mathbf{S}(\mathbf{b}^e)^{-1}\cdot(\mathbf{m}_1\otimes\mathbf{m}_1)]^2 + A_2\mathbf{S}^2(\mathbf{b}^e)^{-1}\cdot(\mathbf{m}_1\otimes\mathbf{m}_1) + A_3\mathbf{S}\cdot\mathbf{b}^e\mathbf{S}(\mathbf{b}^e)^{-1} + A_4(\mathrm{tr}\,\mathbf{S})\mathbf{S}(\mathbf{b}^e)^{-1}\cdot(\mathbf{m}_1\otimes\mathbf{m}_1) + A_5(\mathrm{tr}\,\mathbf{S})^2-\kappa\,, \tag{4.13}$$

with the evolution of the structural anisotropy characterized by $\dot{\mathbf{m}}_1 = \dot{\mathbf{E}}\mathbf{E}^{-1}\mathbf{m}_1$, via an appropriate evolution equation $(3.19)_2$, while the elastic-type constitutive equation becomes a nonlinear one, given by

$$\frac{\mathbf{T}}{\rho} = \frac{1}{2}\Big[a(\mathbf{I}-(\mathbf{b}^e)^{-1})\mathbf{m}_1\cdot\mathbf{m}_1 + b\mathrm{tr}(\mathbf{b}^e-\mathbf{I})\Big](\mathbf{m}_1\otimes\mathbf{m}_1 + \frac{1}{2}\Big[c(\mathbf{I}-(\mathbf{b}^e)^{-1})\mathbf{m}_1\cdot\mathbf{m}_1 + d\mathrm{tr}(\mathbf{b}^e-\mathbf{I})\Big]\mathbf{b}^e + e\frac{1}{2}\Big[(\mathbf{m}_1\otimes(\mathbf{b}^e-\mathbf{I})\mathbf{m}_1 + (\mathbf{b}^e-\mathbf{I})\mathbf{m}_1\otimes\mathbf{m}_1\Big] + f\mathbf{b}^e\frac{1}{2}(\mathbf{b}^e-\mathbf{I}). \tag{4.14}$$

We shall impose restrictions (4.1) to the constitutive functions which enter the equations (4.10), for g_k = Ort. When we replace in (4.1) $\mathbf{Q}$ by the elastic rotation $\mathbf{R}^e$ from (4.6), it follows $\varphi(\mathbf{G},\boldsymbol{\alpha}) = \varphi(\mathbf{b}^e,\overline{\boldsymbol{\alpha}})$, $\hat{\mathcal{F}}(\boldsymbol{\Sigma},\boldsymbol{\alpha}) = \hat{\mathcal{F}}(\overline{\boldsymbol{\Sigma}},\overline{\boldsymbol{\alpha}})$, and so on. If we apply again (4.1) for $\mathbf{Q}\mathbf{R}^e$, for all $\mathbf{Q}\in$ Ort, the isotropy of the constitutive functions with respect to their tensorial arguments follows at once. We get the constitutive equations in the form written below:

THEOREM 2. *1. The behaviour of isotropic materials is characterized in the actual configuration by the following set of equations attached to the former* $\boldsymbol{\Sigma}$*-model:*

$$\begin{aligned} \mathcal{L}_{\mathbf{v}}\,\mathbf{b}^e &= -2\,\hat{\mu}\,\mathbf{V}^e\{\hat{\mathcal{B}}(\overline{\boldsymbol{\Sigma}},\overline{\boldsymbol{\alpha}})\}^s\mathbf{V}^e,\\ \frac{D\mathbf{a}}{Dt} &= \hat{\mu}\,(\mathbf{V}^e)^{-1}\,\hat{\mathbf{m}}(\overline{\boldsymbol{\Sigma}},\overline{\boldsymbol{\alpha}})\,\mathbf{V}^e,\\ \frac{\mathbf{T}}{\rho} &= 2\,\mathbf{V}^e\,(\partial_{\mathbf{b}^e}\varphi(\mathbf{b}^e,\overline{\boldsymbol{\alpha}}))\,\mathbf{V}^e,\\ \text{where}\quad \overline{\boldsymbol{\Sigma}} &:= \mathbf{R}^e\,\boldsymbol{\Sigma}(\mathbf{R}^e)^T,\qquad \overline{\boldsymbol{\alpha}} := \mathbf{R}^e\,\boldsymbol{\alpha}(\mathbf{R}^e)^T \end{aligned} \tag{4.15}$$

Here the objective derivatives $\mathcal{L}_{\mathbf{v}}$ *and* $\frac{D}{Dt}$ *are calculated through the formulae (4.4) and* $(4.8)_2$*, and the constitutive functions are isotropic relative to their tensorial arguments.*

2. For a given history of the deformation gradient, (4.15) becomes a differential system for the unknowns $\left(\mathbf{b}^e,\frac{\mathbf{T}}{\rho},\mathbf{a}\right)$ *due to the following relationships between the appropriate fields involved herein:*

$$(4.16)\qquad \mathbf{b}^e = (\mathbf{V}^e)^2, \qquad \frac{\mathbf{T}}{\rho} = (\mathbf{V}^e)^{-1}\overline{\mathbf{\Sigma}}\mathbf{V}^e, \qquad \mathbf{a} := (\mathbf{V}^e)^{-1}\overline{\boldsymbol{\alpha}}\,\mathbf{V}^e.$$

3. The constitutive equations (4.15) and (4.16) are objective.

Theorem 3. *Under the hypothesis that there is no influence of the hardening on the elastic properties* $\partial_{\boldsymbol{\alpha}}(\partial_{\mathbf{G}}\varphi(\mathbf{G},\boldsymbol{\alpha})) = 0$, *the elasto-plastic material allows a simple constitutive description in the form*

$$(4.17)\qquad \begin{aligned} &\frac{\mathbf{T}}{\rho} = 2\,\mathbf{b}^e\,\partial_{\mathbf{b}^e}\varphi(\mathbf{b}^e), \qquad \mathcal{L}_{\mathbf{v}}\,\mathbf{b}^e = -2\,\hat{\mu}\,\mathbf{V}^e\left\{\hat{\mathcal{B}}\left(\frac{\mathbf{T}}{\rho},\overline{\boldsymbol{\alpha}}\right)\right\}^s\mathbf{V}^e,\\ &\frac{D\mathbf{a}}{Dt} = \hat{\mu}\,(\mathbf{V}^e)^{-1}\,\hat{\mathbf{m}}\left(\frac{\mathbf{T}}{\rho},\overline{\boldsymbol{\alpha}}\right)\mathbf{V}^e, \qquad \textit{with} \qquad \mathbf{a} = (\mathbf{V}^e)^{-1}\overline{\boldsymbol{\alpha}}\,\mathbf{V}^e, \end{aligned}$$

with the yield function $\hat{\mathcal{F}}\left(\frac{\mathbf{T}}{\rho},\overline{\boldsymbol{\alpha}}\right)$, *and the evolution functions* $\hat{\mathcal{B}}$ *and* $\hat{\mathbf{m}}$ *isotropic with respect to the set of variables* $\left(\frac{\mathbf{T}}{\rho},\overline{\boldsymbol{\alpha}}\right)$.

The result is a direct consequence of the Theorem 2, taking into account that $\frac{\mathbf{T}}{\rho} = \overline{\mathbf{\Sigma}}$ follows from the permutability of $\mathbf{G}$ and $\partial_{\mathbf{G}}\varphi(\mathbf{G})$, for isotropic potential $\varphi(\mathbf{G}) = \varphi(\mathbf{b}^e)$.

5. Constitutive restrictions imposed by dissipative restrictions

We emphasize certain restrictions imposed by Ily'iushin-type dissipation postulate on the elasto-plastic models with relaxed configuration, and we derive the rate-type constitutive equation in Eulerian setting compatible with the dissipation inequality, via the modified flow rule. On the other hand, we put into evidence that certain models, compatible with the maximum dissipation postulate in Eulerian descriptions, can be reduced to isotropic elasto-plastic models with respect to intermediate configurations.

Ily'iushin-type dissipation postulate adopted by Cleja-Ţigoiu [4] requires that the work done by internal forces in the initial configuration should be positive on small cycles of strain only, i.e.:

• For all strain histories $\hat{\mathbf{C}}$ from an appropriate admissible set, with the $\hat{\mathbf{C}}(s) := \mathbf{C}(s) = \mathbf{F}^T(s)\mathbf{F}(s)$, such that

$$(5.1)\qquad \forall\, t_1, t_2 \in [0,1], \qquad 0 \le t_1 < t_2 \le 1, \qquad \hat{\mathbf{C}}(t_1) = \hat{\mathbf{C}}(t_2) \in \bigcap_{\tau\in[t_1,t_2]} \mathcal{U}(\hat{\mathbf{C}}^\tau)$$

$$\Longrightarrow \frac{1}{2}\int_{t_1}^{t_2} \frac{\mathbf{\Pi}_0(\tau)}{\rho_0}\cdot\dot{\mathbf{C}}(\tau)d\tau \ge 0.$$

Here $\mathbf{\Pi}_0$ denotes the current value of the Piola–Kirchhoff stress tensor in the initial configuration. The time interval is reduced to $[0,1]$, as the time-independent evolution equations are involved in the model [17].

$\mathcal{U}(\hat{\mathbf{C}}^\tau)$ is the elastic range in total strain space associated to the restriction of the strain history $\hat{\mathbf{C}}$ up to time τ. It is defined in (2.11), and it is the closure of connected open set, with identity strain tensor inside it.

The dissipation postulate becomes efficient only within the precise constitutive framework. Within the largest class of elasto-plastic models formulated here in elastic strain formulation (2.9), (2.10), the following results have been proved in [4].

THEOREM 4. *The dissipation postulate can be equivalently expressed by i), ii):*

i) For all $\hat{\mathbf{C}}$ sufficiently smooth, and for all $t \in [0,1)$ there exists a smooth stress potential $\varphi(\cdot, \boldsymbol{\alpha}(t)) : \mathcal{U}_{\mathcal{R}}(\hat{\mathbf{C}}^t) \longrightarrow R$, i.e a function of variables $(\mathbf{G}, \boldsymbol{\alpha})$, such that

$$\frac{\mathbf{\Pi}(t)}{\tilde{\rho}} = 2\,\partial_{\mathbf{G}}\varphi(\mathbf{G}(t), \boldsymbol{\alpha}(t)) = \mathbf{h}(\mathbf{G}, \boldsymbol{\alpha}). \tag{5.2}$$

ii) For all $t \in (0,1)$ such that $\mathbf{C}(t) \in \partial\mathcal{U}(\hat{\mathbf{C}}^t)$, the following dissipation inequality holds for all $\mathbf{A} \in \mathcal{U}(\hat{\mathbf{C}}_t)$, i.e.

$$\begin{aligned} 2\,[\mathbf{G}(t)\partial_{\mathbf{G}}\varphi(\mathbf{G}(t), \boldsymbol{\alpha}(t)) - \mathbf{G}^*(t)\partial_{\mathbf{G}}\varphi(\mathbf{G}^*(t), \boldsymbol{\alpha}(t))] \cdot \dot{\mathbf{P}}(t)\mathbf{P}^{-1}(t) \\ + [\partial_{\boldsymbol{\alpha}}\varphi(\mathbf{G}^*(t), \boldsymbol{\alpha}(t)) - \partial_{\boldsymbol{\alpha}}\varphi(\mathbf{G}(t), \boldsymbol{\alpha}(t))] \cdot \dot{\boldsymbol{\alpha}}(t) \geq 0\,, \end{aligned} \tag{5.3}$$

$$\text{where} \qquad \mathbf{G}(t) = \mathbf{P}^{-T}(t)\mathbf{C}(t)\mathbf{P}^{-1}(t), \qquad \mathbf{G}^*(t) = \mathbf{P}^{-T}(t)\mathbf{A}\mathbf{P}^{-1}(t).$$

The inequality (5.3) can be equivalently reformulated within the class of $\mathbf{\Sigma}$-models, taking into account constitutive equation (3.4). The dissipation inequality (5.3) is satisfied at any time t, if and only if, for every **accessible stress state**

$$\begin{aligned} \mathbf{\Sigma}^* \in \mathrm{Im}\hat{\mathbf{\Sigma}} := \{\mathbf{\Sigma} \mid \ \exists\ \mathbf{G} \quad \text{such that} \quad \mathbf{\Sigma} = \hat{\mathbf{\Sigma}}(\mathbf{G}, \boldsymbol{\alpha})\} \\ \text{and} \quad \hat{\mathcal{F}}(\mathbf{\Sigma}^*, \boldsymbol{\alpha}(t)) \leq 0, \\ (\mathbf{\Sigma}(t) - \mathbf{\Sigma}^*) \cdot \dot{\mathbf{P}}(t)\mathbf{P}^{-1}(t) + (\boldsymbol{\beta}(t) - \boldsymbol{\beta}^*) \cdot \dot{\boldsymbol{\alpha}}(t) \geq 0 \end{aligned} \tag{5.4}$$

hold for $\mathbf{\Sigma}(t)$ having the properties $\hat{\mathcal{F}}(\mathbf{\Sigma}(t), \boldsymbol{\alpha}(t)) = 0\,, \mathbf{\Sigma}(t) \in \mathrm{Im}(\hat{\mathbf{\Sigma}})$.

Here the *forces conjugated* to internal variables (see [10]) are considered

$$\boldsymbol{\beta}(t) := -\partial_{\boldsymbol{\alpha}}\varphi(\mathbf{G}(t), \boldsymbol{\alpha}(t)), \quad \boldsymbol{\beta}^* = -\partial_{\boldsymbol{\alpha}}\varphi(\mathbf{G}^*, \boldsymbol{\alpha}(t)) \tag{5.5}$$

with the strains $\mathbf{G}(t)$ and $\mathbf{G}^*$ giving rise to the stresses $\mathbf{\Sigma}(t)$ and $\mathbf{\Sigma}^*$:

$$\mathbf{\Sigma}(t) = \hat{\mathbf{\Sigma}}(\mathbf{G}(t), \boldsymbol{\alpha}(t))\,, \quad \mathbf{\Sigma}^* = \hat{\mathbf{\Sigma}}(\mathbf{G}^*, \boldsymbol{\alpha}(t)). \tag{5.6}$$

REMARK 10. The inequality (5.4) formalizes the maximum dissipation postulate, in the intermediate configuration.

• Further investigation of the consequences of the dissipation postulate within the class of $\mathbf{\Sigma}$-models leads to the modified flow rule, expressed in [5] by

$$(\partial_{\mathbf{G}}\hat{\mathbf{\Sigma}}(\mathbf{G},\boldsymbol{\alpha}))^T[\dot{\mathbf{P}}\mathbf{P}^{-1}] = \lambda\partial_{\mathbf{G}}\,\widetilde{\mathcal{F}}(\mathbf{G},\boldsymbol{\alpha}) + \partial^2_{\boldsymbol{\alpha}\,\mathbf{G}}\varphi(\mathbf{G},\boldsymbol{\alpha})[\dot{\boldsymbol{\alpha}}] \tag{5.7}$$

for $\mathbf{G}$ such that $\mathbf{\Sigma} = \hat{\mathbf{\Sigma}}(\mathbf{G},\boldsymbol{\alpha})$ and $\hat{\mathcal{F}}(\hat{\mathbf{\Sigma}}(\mathbf{G},\boldsymbol{\alpha}),\boldsymbol{\alpha}) = 0$, the time t will be omitted in writing the above formulae. Here in (5.7) $\lambda \geq 0$ is an arbitrary function, proportional to plastic multiplier.

Moreover if at the time t we have $\hat{\mathcal{F}}(\hat{\mathbf{\Sigma}}(\mathbf{G},\boldsymbol{\alpha}),\boldsymbol{\alpha}) < 0$, then there is no variation of the irreversible variables $\dot{\mathbf{P}}\mathbf{P}^{-1} = 0$ and $\dot{\boldsymbol{\alpha}} = 0$.

We can prove the following theorem:

THEOREM 5. *The rate form of the model compatible with the dissipation postulate is given by*

$$\frac{d}{dt}\left(\frac{\mathbf{T}}{\rho}\right) - \mathbf{L}\frac{\mathbf{T}}{\rho} - \frac{\mathbf{T}}{\rho}\mathbf{L}^T = \widetilde{\mathcal{E}}[\mathbf{D}] - 2\,\lambda\mathbf{E}\partial_{\mathbf{E}}\widetilde{\mathcal{F}}(\mathbf{G},\boldsymbol{\alpha})\mathbf{E}^T, \tag{5.8}$$

with fourth order elastic tensor $\widetilde{\mathcal{E}}$ defined in (3.17). The scalar function is proportional to the plastic multiplier, which enters the evolution equations, via the modified flow rule (5.7).

In order to prove (5.8), first we replace the terms containing the plastic multiplier in (3.19) from (3.21) and we take into account the expression of $\mathcal{N}$ introduced in (3.17). The rate-type constitutive equation in actual configuration results in the form

$$\begin{aligned}\frac{d}{dt}\left(\frac{\mathbf{T}}{\rho}\right) - \mathbf{L}\frac{\mathbf{T}}{\rho} - \frac{\mathbf{T}}{\rho}\mathbf{L}^T = \widetilde{\mathcal{E}}[\mathbf{D}] - 2\,\mathbf{E}\,(\partial_{\mathbf{G}}\hat{\mathbf{\Sigma}})^T[\dot{\mathbf{P}}\mathbf{P}^{-1}]\mathbf{E}^T \\ + 2\,\mathbf{E}(\partial^2_{\boldsymbol{\alpha}\,\mathbf{G}}\,\varphi[\dot{\boldsymbol{\alpha}}\,])\mathbf{E}^T,\end{aligned} \tag{5.9}$$

or with the objective derivative for $\dfrac{\mathbf{T}}{\rho} \equiv \boldsymbol{\tau}$ defined in the left-hand side of (5.9)

$$\begin{aligned}&\frac{D}{Dt}\left(\frac{\mathbf{T}}{\rho}\right) = \widetilde{\mathcal{E}}[\mathbf{D}] - 2\,\mathbf{E}\,((\partial_{\mathbf{G}}\hat{\mathbf{\Sigma}})^T[\dot{\mathbf{P}}\mathbf{P}^{-1}] - \partial^2_{\boldsymbol{\alpha}\,\mathbf{G}}\,\varphi[\dot{\boldsymbol{\alpha}}\,])\mathbf{E}^T, \text{with}\\ &\frac{D}{Dt}\left(\frac{\mathbf{T}}{\rho}\right) := \frac{d}{dt}\left(\frac{\mathbf{T}}{\rho}\right) - \mathbf{L}\frac{\mathbf{T}}{\rho} - \frac{\mathbf{T}}{\rho}\mathbf{L}^T.\end{aligned} \tag{5.10}$$

From the modified flow rule (5.7) we evaluate the difference

$$\mathbf{E}\,((\partial_{\mathbf{G}}\hat{\mathbf{\Sigma}})^T[\dot{\mathbf{P}}\mathbf{P}^{-1}] - \partial^2_{\boldsymbol{\alpha}\,\mathbf{G}}\,\varphi[\dot{\boldsymbol{\alpha}}\,])\mathbf{E}^T = \lambda\mathbf{E}\,\partial_{\mathbf{G}}\,\widetilde{\mathcal{F}}(\mathbf{G},\boldsymbol{\alpha})\,\mathbf{E}^T. \tag{5.11}$$

Finally we consider the model of plasticity widely used in numerical simulations, see [25, 21], typically formulated directly in the actual configuration.

We are starting from the model relative to the actual configuration, defined by:

• Uncoupled isotropic hyperelastic behaviour is described by

$$\boldsymbol{\tau} = 2\,(\partial_{\mathbf{b}^e} W^e)\,\mathbf{b}^e, \qquad \boldsymbol{\beta} := -\partial_{\boldsymbol{\alpha}} W^p, \qquad \text{when} \quad \varphi(\mathbf{b}^e, \boldsymbol{\alpha}) = W^e(\mathbf{b}^e) + W^p(\boldsymbol{\alpha}), \tag{5.12}$$

W^e and W^p are the elastic and plastic parts of the free energy function $\varphi(\mathbf{b}^e, \boldsymbol{\alpha})$.

• The evolution equations are represented by the flow rule

$$\begin{aligned} \mathcal{L}_{\mathbf{v}}\,\mathbf{b}^e &= -2\,\lambda\,\mathbf{m}_{\boldsymbol{\tau}}(\boldsymbol{\tau}, \boldsymbol{\beta})\,\mathbf{b}^e, \quad &&\text{with} \quad \mathbf{m}_{\boldsymbol{\tau}} := \partial_{\boldsymbol{\tau}} f(\boldsymbol{\tau}, \boldsymbol{\beta}), \\ \dot{\boldsymbol{\alpha}} &= \lambda\,\mathbf{m}_{\boldsymbol{\beta}}(\boldsymbol{\tau}, \boldsymbol{\beta}), &&\text{with} \quad \mathbf{m}_{\boldsymbol{\beta}} := \partial_{\boldsymbol{\beta}} f(\boldsymbol{\tau}, \boldsymbol{\beta})), \end{aligned} \tag{5.13}$$

being attached to the generic yield function $f(\boldsymbol{\tau}, \boldsymbol{\beta})$. The Lie derivative $\mathcal{L}_{\mathbf{v}}$ is defined in (4.8), $\mathbf{m}_{\boldsymbol{\tau}}, \mathbf{m}_{\boldsymbol{\beta}}$ correspond to the flow directions, while λ is the plastic multiplier, via the appropriate Khun–Tucker conditions $\lambda \geq 0, f \leq 0, \;\; \lambda f = 0$.

Here is the particular case of the equations (4.17), with only scalar internal variables, supposed to be invariant under a change of frame in the actual configuration. From the objectivity principle it follows that W^e and f are isotropic with respect to $\mathbf{b}^e$ and $\boldsymbol{\tau}$, respectively, i.e. for instance $f(\boldsymbol{\tau}, \boldsymbol{\beta}) = f(\mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^T, \boldsymbol{\beta}), \;\; \forall \;\; \mathbf{Q} \in \mathrm{Ort}$.

The principle of maximum dissipation requires the inequality

$$[\boldsymbol{\tau} - \boldsymbol{\tau}^*] \cdot \left[-\frac{1}{2}\,(\mathcal{L}_{\mathbf{v}}\,\mathbf{b}^e)(\mathbf{b}^e)^{-1} \right] + [\boldsymbol{\beta} - \boldsymbol{\beta}^*] \cdot \dot{\boldsymbol{\alpha}} \geq 0 \tag{5.14}$$

for all admissible pairs $(\boldsymbol{\tau}^*, \boldsymbol{\beta}^*) \in \mathcal{D}_{El}$.

The elastic range, denoted $\mathcal{D}_{El}$, is defined by $\mathcal{D}_{El} = \{(\boldsymbol{\tau}^*, \boldsymbol{\beta}^*) \,|\, f(\boldsymbol{\tau}^*, \boldsymbol{\beta}^*) \leq 0\}$.

We derive the model pulled back to the intermediate configuration.

Combining the kinematic relationships (3.1) that follows from the multiplicative decomposition (1.1) one obtains

$$\mathcal{L}_{\mathbf{v}}\,\mathbf{b}^e = -2\,\{\mathbf{E}(\dot{\mathbf{P}}\mathbf{P}^{-1})\,\mathbf{E}^T\}^s. \tag{5.15}$$

Using the evolution equation $(5.13)_1$, (5.15), as well as the permutability of the tensors $\boldsymbol{\tau}$ and $\mathbf{b}^e$ related through $(5.12)_1$, with W^e being an isotropic function relative to $\mathbf{b}^e$, we get

$$\mathbf{R}^e\{\dot{\mathbf{P}}\mathbf{P}^{-1}\}^s(\mathbf{R}^e)^T = \mathbf{m}_{\boldsymbol{\tau}}(\boldsymbol{\tau}, \boldsymbol{\beta}) \qquad \text{with} \qquad \mathbf{m}_{\boldsymbol{\tau}} := \partial_{\boldsymbol{\tau}} f(\boldsymbol{\tau}, \boldsymbol{\beta}). \tag{5.16}$$

Taking into account the isotropy of the yield function f we proved the following result:

Proposition 4. The model described in the actual configuration by (5.12) and (5.13) is equivalent to an isotropic elasto-plastic model, which is hyperelastic with the associated flow rule

$$\mathbf{L}^p = \lambda\, \partial_{\boldsymbol{\Sigma}} f(\boldsymbol{\Sigma}, \boldsymbol{\beta}), \qquad \dot{\boldsymbol{\alpha}} = \lambda\, \partial_{\boldsymbol{\beta}} f(\boldsymbol{\Sigma}, \boldsymbol{\beta}), \qquad \text{here} \quad \mathbf{L}^p = \dot{\mathbf{P}}\mathbf{P}^{-1},$$

(5.17)

$$\text{yield function} \qquad f(\boldsymbol{\tau}, \boldsymbol{\beta}) = f(\boldsymbol{\Sigma}, \boldsymbol{\beta}) \qquad \text{since} \quad \boldsymbol{\tau} := \frac{\mathbf{T}}{\rho} = \mathbf{R}^e \boldsymbol{\Sigma} (\mathbf{R}^e)^T.$$

We remark that the plastic spin $\{\mathbf{L}^p\}^a = 0$ is vanishing.

6. Conclusions

1. In the paper, different models within the constitutive framework of the elasto-plastic materials with relaxed configuration have been presented. In adopting a certain description of the model, the arguments of the mathematics and/or physics nature have been dominant.

2. For instance, in order to reflect at the macroscopic level, the fact that the plastic (inelastic) deformation begins to develop only if the reduced shear stress reaches a critical value, different stress measures have been involved in the yield criteria.

3. When we deal with the dissipation postulate, the strain description of the yield conditions have been imposed by the mathematical issues. As a consequence of the dissipative restrictions, it has been proved in [4] that the convexity and normality properties of the elastic range in stress space can be treated only within the constitutive framework of $\boldsymbol{\Sigma}$-model, but the normality does not mean the associated plastic flow rule.

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