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On the traction problem in mechanics

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In this paper, we show how to solve the traction problem for the Lamé and Stokes systems by means of a double layer potential. In this way we complete the results of [5], where Cialdea and Hsiao, employing a method introduced by the first author in [1], solve the Dirichlet problem for Lamé and Stokes systems by means of a simple layer potential.

Key words: Lamé equations, Stokes system, boundary integral equations.

1. Introduction

It is well-known that the classical method of solving the Dirichlet (Neumann) problem for Laplace equation

$$\begin{cases} \Delta u = 0 & & \text{in } \Omega, \\ u = f & \left(\dfrac{\partial u}{\partial \nu} = f\right) & \text{on } \Sigma, \end{cases}$$

consists in representing the solution by means of a double (simple) layer potential:

$$u(\mathbf{x}) = \int_{\Sigma} \varphi(\mathbf{y}) \frac{\partial}{\partial \nu_{\mathbf{y}}} s(\mathbf{x},\mathbf{y}) d\sigma_{\mathbf{y}}, \quad \left(u(\mathbf{x}) = \int_{\Sigma} \varphi(\mathbf{y}) s(\mathbf{x},\mathbf{y}) d\sigma_{\mathbf{y}}\right), \qquad \mathbf{x} \in \Omega,$$

where $s(\mathbf{x},\mathbf{y})$ is the fundamental solution of Laplace equation. Another approach, which is also interesting, consists in representing the solution of the Dirichlet problem by means of a simple layer potential. In this case the boundary condition leads to an integral equation of the first kind:

$$\int_{\Sigma} \varphi(\mathbf{y}) s(\mathbf{x},\mathbf{y}) d\sigma_{\mathbf{y}} = f(\mathbf{x}), \qquad \mathbf{x} \in \Sigma. \tag{1.1}$$

Following a Fichera's idea (see [9], p. 11), Cialdea [1] takes the differential of both sides of (1.1), obtaining in this way the following singular integral equation:

$$\int_{\Sigma} \varphi(\mathbf{y}) d_{\mathbf{x}}[s(\mathbf{x},\mathbf{y})] d\sigma_{\mathbf{y}} = df(\mathbf{x}), \ ^{1)} \qquad \mathbf{x} \in \Sigma, \tag{1.2}$$

in which the unknown is a scalar function φ, while $\mathbf{df}$ is a differential form of degree 1. Let us denote by $J\varphi$ the left-hand side of (1.2). It is shown in [1], Theorem I, that the singular operator J can be reduced on the left ([6, 8]), that is there exists a linear and continuous operator J' such that

$$J'J\varphi(\mathbf{x}) = -\frac{1}{4}\varphi(\mathbf{x}) + L^2\varphi(\mathbf{x}),$$

where L is a compact operator. By means of such a reduction, one can show that there exists one and only one solution $\varphi \in L^p(\Sigma)$ for any given $f \in W^{1,p}(\Sigma)$.

As remarked in [5], this method can be applied also for solving the Neumann problem for the Laplace equation by means of a double layer potential. Moreover, this approach was generalized to the biharmonic equation ([2, 3]) in any number of variables and to the Dirichlet problem for Lamé and Stokes systems in [5]. The aim of this work is to show that this method can be used in the study of the traction problem for Lamé and Stokes systems.

We remark that the approach used in this paper does not utilize the theory of pseudo-differential operators or hypersingular integrals.

Finally, we note that for a Lamé system we have to deal with singular integrals, while for a Stokes system the difficulty is the presence of some extra eigensolutions. This implies that, in this case, we have to add a particular term to the double layer potential. A similar difficulty arises if we study the simple layer potential approach to the Stokes system as it has been shown in [13].

In the sequel, Ω is a bounded domain of $\mathbb{R}^3$ such that its boundary $\partial\Omega$ is a Lyapunov surface Σ (i.e. Σ has a uniformly Hölder continuous normal field of some exponent $l \in (0,1]$) and such that $\mathbb{R}^3 - \overline{\Omega}$ is connected; $\boldsymbol{\nu}(\mathbf{y}) = (\nu_1(\mathbf{y}), \nu_2(\mathbf{y}), \nu_3(\mathbf{y}))$ denotes the outward unit normal vector at the point $\mathbf{y} = (y_1, y_2, y_3) \in \Sigma$.

By $W^{1,p}(\Sigma)$ we denote the usual Sobolev space. By $L_1^p(\Sigma)$ we mean the space of the differential forms of degree 1 whose components are L^p functions. For the theory of differential forms see [7].

1) The symbol d_x denotes exterior differentiation [7].

2. The Lamé system

We study the so-called traction problem of the elasticity theory for an isotropic homogeneous body:

$$\begin{cases} \mu\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad\,div}\,\mathbf{u} = \mathbf{0} & \text{in } \Omega, \\ \mathbf{T}\mathbf{u} = \mathbf{f} & \text{on } \Sigma, \end{cases} \tag{2.1}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the the displacement vector, λ and μ are the *Lamé constants* satisfying the conditions [11]:

$$3\lambda + 2\mu > 0, \qquad \mu \neq 0;$$

$\mathbf{T}$ is the following operator ([11], p. 57):

$$\mathbf{T}(\partial_{\mathbf{x}}, \boldsymbol{\nu}(\mathbf{x}))\mathbf{u} = \lambda\boldsymbol{\nu}(\mathbf{x})\mathrm{div}\ \mathbf{u} + 2\mu\frac{\partial\mathbf{u}}{\partial\boldsymbol{\nu}_{\mathbf{x}}} + \mu(\boldsymbol{\nu}(\mathbf{x}) \times \mathrm{curl}\ \mathbf{u});$$

the data $\mathbf{f}$ is assumed in the space $[L^p(\Sigma)]^3$, $1 < p < +\infty$.

We seek a solution of (2.1) by means of a double layer potential

$$w_j(\mathbf{x}) = \int_{\Sigma} \varphi_h(\mathbf{y}) T_{iy}(\Gamma^h(\mathbf{x}, \mathbf{y}))\, d\sigma_{\mathbf{y}}, \tag{2.2}$$

where $\Gamma^h(\mathbf{x}, \mathbf{y})$ is the column vector of the *Kelvin matrix* $\mathbf{\Gamma}(\mathbf{x}, \mathbf{y})$ ([11], p. 84) whose components are:

$$\Gamma_{kj}(\mathbf{x}) = \frac{1}{2\pi\mu}\Big(\frac{\delta_{kj}}{|\mathbf{x}|} - \frac{(\lambda+\mu)}{2(\lambda+2\mu)}\frac{\partial^2}{\partial x_k \partial x_j}|\mathbf{x}|\Big). \tag{2.3}$$

Now we introduce $\mathbf{R} : [L^p(\Sigma)]^3 \to [L_1^p(\Sigma)]^3$, the operator given by the differential of an elastic simple layer potential on Σ:

$$R_i\boldsymbol{\varphi}(\mathbf{x}) = \int_{\Sigma} \varphi_j(\mathbf{y}) d_{\mathbf{x}}[\Gamma_{ij}(\mathbf{x}, \mathbf{y})]\, d\sigma_{\mathbf{y}}, \qquad \mathbf{x} \in \Sigma,\ ^{2)} \tag{2.4}$$

where Γ_{ij} are given by (2.3). Here, as well as in (2.2), the integrals have to be understood as singular integrals.

It can be shown that $\mathbf{R}$ is a linear and continuous operator from $[L^p(\Sigma)]^3$ into $[L_1^p(\Sigma)]^3$, $1 < p < \infty$. Moreover in [5] it has been shown that $\mathbf{R}$ can be reduced on the left [3)] by

$$\mathbf{R}_{\mathbf{0}}' : [L_1^p(\Sigma)]^3 \longrightarrow [L^p(\Sigma)]^3$$

2) See note 1).

3) For the theory of *reducing operators* see [6, 8].

defined as

$$R'_{0i}\boldsymbol{\psi} = \frac{(\lambda+\mu)(\lambda+2\mu)}{(\lambda+3\mu)}\mathcal{K}_{jj}(\boldsymbol{\psi})\nu_i + 2\mu\frac{(\lambda+2\mu)}{(\lambda+3\mu)}\mathcal{K}_{ij}(\boldsymbol{\psi})\nu_j + \mu\frac{(\lambda+\mu)}{(\lambda+3\mu)}\delta^{ij}_{sp}\nu_j\mathcal{K}_{ps}(\boldsymbol{\psi}),$$

where $\mathcal{K}_{ij}$ are the following singular integral operators ([5], p. 37):

$$\mathcal{K}_{js}(\boldsymbol{\varphi}) = *_\Sigma \int_\Sigma d_{\mathbf{x}}[S_1(\mathbf{x},\mathbf{y})] \wedge \varphi_j(\mathbf{y}) \wedge dx^s - \delta^{123}_{ihp}\int_\Sigma \frac{\partial}{\partial x_s}[K_{ij}(\mathbf{x},\mathbf{y})] \wedge \varphi_h(\mathbf{y}) \wedge dy^p, \tag{2.5}$$

$*_\Sigma$ means that if $\mathbf{a} = \mathbf{a}_0 d\sigma$ for some function $\mathbf{a}_0$, then $*_\Sigma\mathbf{a} = \mathbf{a}_0$, $S_1(\mathbf{x},\mathbf{y})$ is the double 1-form introduced by Hodge [10]:

$$S_1(\mathbf{x},\mathbf{y}) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\sum_j dx^j dy^j \tag{2.6}$$

and

$$K_{ij}(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi}\Big[\mu\frac{(\lambda+\mu)}{(\lambda+3\mu)}\frac{\partial|\mathbf{x}-\mathbf{y}|}{\partial y_j}\frac{\partial|\mathbf{x}-\mathbf{y}|}{\partial y_i}\Big]\frac{1}{|\mathbf{x}-\mathbf{y}|}.$$

Further we introduce the following singular integral operator which we shall use in the sequel:

$$\widetilde{\mathbf{R}} : [L^p_1(\Sigma)]^3 \longrightarrow [L^p(\Sigma)]^3$$

defined as

$$\widetilde{R}_i\boldsymbol{\psi} = \lambda\mathcal{K}_{jj}(\boldsymbol{\psi})\nu_i + \mu\mathcal{K}_{ij}(\boldsymbol{\psi})\nu_j + \mu\delta^{ij}_{sp}\nu_j\mathcal{K}_{ps}(\boldsymbol{\psi}). \tag{2.7}$$

Lemma 1. *Let* $\mathbf{u}$ *be in* $[W^{1,p}(\Sigma)]^3$*, then*

$$\frac{\partial w_j}{\partial x_s} = \mathcal{K}_{js}(d\mathbf{u}) \qquad \text{in} \quad \Omega,$$

where w_i *is the double layer potential (2.2) and* $\mathcal{K}_{js}$ *are given by (2.5).*

This result was proved in [5], p. 37.

Lemma 2. *Let* $\boldsymbol{\psi} \in [W^{1,p}(\Sigma)]^3$ *and* z_i *be the following 1-form:*

$$z_i(\mathbf{x}) = \lambda\mathcal{K}_{jj}(\boldsymbol{\psi})dx^i + \mu\mathcal{K}_{ij}(\boldsymbol{\psi})dx^j + \mu\delta^{ij}_{sp}\mathcal{K}_{ps}(\boldsymbol{\psi})dx^j, \quad \mathbf{x} \notin \Sigma.$$

Then the restriction of $*z_i(\mathbf{x})$ *on* Σ *is* $\widetilde{R}_i\psi$*, where* $\widetilde{R}_i$ *are given by (2.7).*

P r o o f. It follows from a theorem in [4] that there exist Hölder continuous functions a_{jsh} such that

$$\lim_{\mathbf{x}\to\mathbf{x_0}^{\pm}} *z_j(\mathbf{x}) = \pm a_{jsh}(\mathbf{x_0})\psi_{sh}(\mathbf{x_0}) + \widetilde{R}_j\psi(\mathbf{x_0}) \qquad \text{a.e. } \mathbf{x_0}\in\Sigma \tag{2.8}$$

for any $\psi_s = \psi_{sh}dx^h \in L_1^p(\Sigma)$. On the other hand, if $\psi_j = du_j$ with u_j being in $C^{1,l}(\Sigma)$ $(0 < l \le 1)$, it follows from Lemma 1 that

$$\begin{aligned} z_i(\mathbf{x}) &= \lambda\mathcal{K}_{jj}(d\mathbf{u})dx^i + \mu\mathcal{K}_{ij}(d\mathbf{u})dx^j + \mu\delta^{ij}_{sp}\mathcal{K}_{ps}(d\mathbf{u})dx^j \\ &= \lambda\frac{\partial w_j}{\partial x_j}dx^i + \mu\frac{\partial w_i}{\partial x_j}dx^j + \mu\delta^{ij}_{sp}\frac{\partial w_p}{\partial x_s}dx^j, \qquad \mathbf{x}\notin\Sigma. \end{aligned}$$

This implies that

$$\lim_{\mathbf{x}\to\mathbf{x_0}^{\pm}} *z_j(\mathbf{x}) = [T_j\mathbf{w}]_{\pm}(\mathbf{x_0}). \tag{2.9}$$

Therefore from (2.8) and (2.9) it follows that

$$2a_{jsh}(\mathbf{x_0})\frac{\partial u_s}{\partial x_h} = \lim_{\mathbf{x}\to\mathbf{x_0}^{+}} *z_j(\mathbf{x}) - \lim_{\mathbf{x}\to\mathbf{x_0}^{-}} *z_j(\mathbf{x}) = [T_j\mathbf{w}]_+(\mathbf{x_0}) - [T_j\mathbf{w}]_-(\mathbf{x_0})$$

and, because of the Lyapunov–Tauberian theorem ([11], p. 408), the last expression is zero and then

$$a_{jsh}(\mathbf{x_0})\frac{\partial u_s}{\partial x_h}(\mathbf{x_0}) = 0, \qquad \forall\ \mathbf{x_0}\in\Sigma.$$

Due to the arbitrariness of $u_s\in C^{1,l}(\Sigma)$, we conclude that $a_{jsh}(\mathbf{x_0})\equiv 0$. In view of (2.8) we get the result.

Now we show the representation theorem of traction problem of the elasticity theory.

THEOREM 1. *For any* $\mathbf{f}\in[L^p(\Sigma)]^3$, $1<p<\infty$, *such that*

$$\int_\Sigma \mathbf{f}\cdot(\mathbf{a}+\mathbf{b}\wedge\mathbf{x})d\sigma = 0, \quad \forall\ \mathbf{a},\ \mathbf{b}\in\mathbb{R}^3 \tag{2.10}$$

any solution of (2.1) can be represented in the form of an elastic double layer potential (2.2) with the density $\boldsymbol{\varphi}\in[W^{1,p}(\Sigma)]^3$. *Moreover, (2.2) is a solution of (2.1) if, and only if, its density* $\boldsymbol{\varphi}$ *is given by*

$$\varphi_i(\mathbf{x}) = \int_\Sigma \psi_j(\mathbf{y})\Gamma_{ij}(\mathbf{x},\mathbf{y})\,d\sigma_{\mathbf{y}}, \qquad \mathbf{x}\in\Sigma, \tag{2.11}$$

$\boldsymbol{\psi} \in [L^p(\Sigma)]^3$ *being a solution of the singular integral equation*

$$-\boldsymbol{\psi} + \mathbf{V}^2\boldsymbol{\psi} = \mathbf{f} \tag{2.12}$$

where $\mathbf{V}$ *is given by*

$$V_j\boldsymbol{\psi}(\mathbf{x}) = \int_\Sigma \psi_h(\mathbf{y}) T_{iy}(\Gamma^h(\mathbf{x},\mathbf{y}))\, d\sigma_{\mathbf{y}}. \tag{2.13}$$

P r o o f. We want to represent a solution of (2.1) by means of a double layer potential (2.2). It follows from [5] that any $\boldsymbol{\varphi} \in [W^{1,p}(\Sigma)]^3$ can be written as a simple layer potential (2.11); we obtain from the definition of the operator $\mathbf{R}$ (2.4) that

$$R_i\boldsymbol{\psi} = d\varphi_i. \tag{2.14}$$

From (2.7), (2.14) we obtain

$$\widetilde{R}_i\mathbf{R}\boldsymbol{\psi} = \lambda\mathcal{K}_{jj}(d\boldsymbol{\varphi})\nu_i + \mu\mathcal{K}_{ij}(d\boldsymbol{\varphi})\nu_j + \mu\delta^{ij}_{sp}\mathcal{K}_{ps}(d\boldsymbol{\varphi})\nu_j.$$

Moreover, by Lemma 2, $\widetilde{R}_i\mathbf{R}\boldsymbol{\psi}$ is the restriction on Σ of $*z_i$ with

$$z_i(\mathbf{x}) = \lambda\frac{\partial w_j}{\partial x_j}dx^i + \mu\frac{\partial w_i}{\partial x_j}dx^j + \mu\delta^{ij}_{sp}\frac{\partial w_p}{\partial x_s}dx^j,$$

where $\mathbf{w}$ is given by (2.2). Hence

$$\widetilde{R}_i\mathbf{R}\boldsymbol{\psi} = T_i(\mathbf{w}), \qquad i = 1,2,3 \qquad \text{on } \Sigma.$$

Then we obtain

$$\widetilde{R}_i(d\boldsymbol{\varphi}) = f_i, \qquad i = 1,2,3 \qquad \text{on } \Sigma. \tag{2.15}$$

On the other hand, from Green's representation formula [11] for $\mathbf{x} \in \Omega$

$$\mathbf{w}(\mathbf{x}) = -2\mathbf{u}(\mathbf{x}) + \int_\Sigma \boldsymbol{\Gamma}(\mathbf{x},\mathbf{y})\mathbf{T}_{\mathbf{y}}(\partial_{\mathbf{y}},\boldsymbol{\nu})\mathbf{u}(\mathbf{y})\, d\sigma_{\mathbf{y}}$$

on Σ, we find that

$$\mathbf{T}\mathbf{w} = -\boldsymbol{\psi} + \mathbf{V}^2\boldsymbol{\psi},$$

where $\mathbf{V}$ is given by (2.13). Then there exists a solution $\boldsymbol{\varphi} \in [W^{1,p}(\Sigma)]^3$ of (2.15) if, and only if, there exists a solution $\boldsymbol{\psi} \in [L^p(\Sigma)]^3$ of (2.12). It is well known that, since the compatibility conditions (2.10) are satisfied, there exists $\boldsymbol{\Phi} \in [L^p(\Sigma)]^3$ such that

$$-\boldsymbol{\Phi} + \mathbf{V}\boldsymbol{\Phi} = \mathbf{f}; \tag{2.16}$$

and, on the other hand, there exists $\boldsymbol{\psi} \in [L^p(\Sigma)]^3$ satisfying the equation

$$\boldsymbol{\psi} + \mathbf{V}\boldsymbol{\psi} = \boldsymbol{\Phi} \tag{2.17}$$

Consequently, Eqs. (2.16) and (2.17) imply (2.12).

3. The Stokes system

In this section we consider the traction problem related to the Stokes system for the viscous fluid flow [12]:

$$\begin{cases} \mu\Delta\mathbf{u} - \operatorname{grad} p = \mathbf{0} & \text{in } \Omega, \\ \operatorname{div}\,\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{T}\mathbf{u} = \mathbf{f} & \text{on } \Sigma, \end{cases} \tag{3.1}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity vector, p is the pressure, $\mu > 0$ is the coefficient of kinematic viscosity and $\mathbf{T}$ is the vector whose components are given by:

$$T_j\mathbf{u} = \Big[-\delta_{ij}p + \mu\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)\Big]\boldsymbol{\nu}_i. \tag{3.2}$$

A fundamental solution of the Stokes system is given by the fundamental velocity tensor and the pressure vector:

$$\gamma_{ij}(\mathbf{x},\mathbf{y}) = -\frac{1}{4\pi\mu}\Big[\frac{\delta_{ij}}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}|\mathbf{x}-\mathbf{y}|\Big],$$

$$\varepsilon_j(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi}\frac{\partial}{\partial x_j}\frac{1}{|\mathbf{x}-\mathbf{y}|}.$$

By a simple layer potential with density $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ we mean the integrals:

$$\begin{aligned} v_h(\mathbf{x}) &= \int_\Sigma \varphi_j(\mathbf{y})\gamma_{hj}(\mathbf{x},\mathbf{y})d\sigma_{\mathbf{y}}, \\ r(\mathbf{x}) &= \int_\Sigma \varepsilon_h(\mathbf{x},\mathbf{y})\varphi_h(\mathbf{y})d\sigma_{\mathbf{y}}, \end{aligned} \tag{3.3}$$

and by a double layer potential with density $\boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3)$ we mean the integrals:

$$\begin{aligned} w_j(\mathbf{x}) &= \int_\Sigma \psi_h(\mathbf{y})T'_{j\mathbf{y}}[\gamma^h(\mathbf{x},\mathbf{y})]d\sigma_{\mathbf{y}}, \\ q(\mathbf{x}) &= 2\mu\int_\Sigma \frac{\partial}{\partial \nu_{\mathbf{y}}}[\varepsilon_h(\mathbf{x},\mathbf{y})]\psi_h(\mathbf{y})d\sigma_{\mathbf{y}}, \end{aligned} \tag{3.4}$$

where T'_j is the adjoint of (3.2)

$$T'_j\mathbf{u} = \Big[\delta_{ij}p + \mu\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)\Big]\nu_i$$

and $\gamma^h(\mathbf{x},\mathbf{y}) = (\gamma_{ih}(\mathbf{x},\mathbf{y}))$ is the h-th column vector of $\gamma_{ij}(\mathbf{x},\mathbf{y})$.

Now we introduce the following integral operators:

$$\Theta_j(\boldsymbol{\psi}) = *_\Sigma \int_\Sigma d_{\mathbf{x}}[S_1(\mathbf{x},\mathbf{y})] \wedge \phi(\mathbf{y}) \wedge dx^j \tag{3.5}$$

for any $\phi \in L_1^p(\Sigma)$, where S_1 is the Hodge 1-form (2.6) and

$$\mathcal{H}_{ij}\boldsymbol{\psi} = \Theta_j(\psi_i) - \delta_{shp}^{123} \int_\Sigma \frac{\partial}{\partial x_j}\Big[H_{si}(\mathbf{x},\mathbf{y})\Big] \wedge \psi_h(\mathbf{y}) \wedge dy^p \tag{3.6}$$

for any $\boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3) \in [L_1^p(\Sigma)]^3$, where

$$H_{si}(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{y}|}\frac{\partial}{\partial y_s}|\mathbf{x}-\mathbf{y}|\frac{\partial}{\partial y_i}|\mathbf{x}-\mathbf{y}|.$$

Moreover, let us define the following singular integral operator $\mathbf{F} : [L^p(\Sigma)]^3 \to [L_1^p(\Sigma)]^3$:

$$F_i\boldsymbol{\varphi}(\mathbf{x}) = \int_\Sigma \varphi_j(\mathbf{y}) d_{\mathbf{x}}[\gamma_{ij}(\mathbf{x},\mathbf{y})] d\sigma_{\mathbf{y}}, \quad \mathbf{x} \in \Sigma. \tag{3.7}$$

In [5] it has been shown that $\mathbf{F}$ can be reduced on the left by the following operator

$$\mathbf{F}' : [L_1^p(\Sigma)]^3 \longrightarrow [L^p(\Sigma)]^3$$

defined as

$$F_j'\boldsymbol{\psi} = \mu[2\delta_{ij}\Theta_h(\psi_h) + \mathcal{H}_{ij}(\boldsymbol{\psi}) + \mathcal{H}_{ji}(\boldsymbol{\psi})]\nu_i, \tag{3.8}$$

where Θ_h and $\mathcal{H}_{ij}$ are given by (3.5) and (3.6) respectively.

The following two lemmas are proved in [5].

LEMMA 3. *Let $\mathbf{u} \in [W^{1,p}(\Sigma)]^3$. Then for $\mathbf{x} \notin \Sigma$,*

$$\frac{\partial w_i}{\partial x_j} = \mathcal{H}_{ij}(d\mathbf{u}),$$

$$q_h = -2\mu\Theta_h(du_h),$$

where w_i and q_h are given by (3.4) and $d\mathbf{u} = (du_1, du_2, du_3)$.

LEMMA 4. *Let $\boldsymbol{\psi} \in [L_1^p(\Sigma)]^3$ and ϑ_j be the following 1-form:*

$$\vartheta_j(\mathbf{x}) = \mu[2\delta_{ij}\Theta_h(\psi_h) + \mathcal{H}_{ij}(\boldsymbol{\psi}) + \mathcal{H}_{ji}(\boldsymbol{\psi})]dx^i, \quad \mathbf{x} \notin \Sigma;$$

then the restriction of $*\vartheta_j(\mathbf{x})$ on Σ is $F_j'\psi$, where $F_j'\psi$ are given by (3.8).

THEOREM 2. *The following Fredholm equation*

$$-\frac{1}{4}\boldsymbol{\psi} + \mathbf{K}^2\boldsymbol{\psi} = \mathbf{f}, \tag{3.9}$$

where $\mathbf{f} \in [L^p(\Sigma)]^3$ *and* $\mathbf{K}$ *is the following integral operator*

$$K_j\boldsymbol{\psi}(\mathbf{x}) = \int_\Sigma \varphi_h(\mathbf{y}) T_{jx}(\gamma^h(\mathbf{x},\mathbf{y}))\, d\sigma_{\mathbf{y}}, \tag{3.10}$$

admits a solution $\boldsymbol{\psi} \in [L^p(\Sigma)]^3$ *if, and only if, Eqs. (2.10) and*

$$\int_\Sigma \mathbf{f}\cdot\boldsymbol{\psi}_0\, d\sigma = 0 \tag{3.11}$$

are satisfied, where $\boldsymbol{\psi}_0 \in [C^l(\Sigma)]^3$, $0 < l \leq 1$, *is an eigensolution of the following homogeneous equation:*

$$\frac{\boldsymbol{\gamma}}{2} + \mathbf{K}^*\boldsymbol{\gamma} = \mathbf{0}. \tag{3.12}$$

P r o o f. We remark that the Fredholm equation (3.12) has only one solution $\boldsymbol{\psi}_0 \in [L^q(\Sigma)]^3$ [4)], $p^{-1} + q^{-1} = 1$, since $\boldsymbol{\nu}$ is the only eigensolution of its adjoint equation, see ([12], p. 57–62). Let us investigate now the following homogeneous equation:

$$-\frac{1}{4}\boldsymbol{\psi} + \mathbf{K}^2\boldsymbol{\psi} = \left(\frac{\mathbf{I}}{2} + \mathbf{K}\right)\left(-\frac{\boldsymbol{\psi}}{2} + \mathbf{K}\boldsymbol{\psi}\right) = \mathbf{0}. \tag{3.13}$$

Since $\boldsymbol{\nu}$ is the only solution of the equation: $\left(\frac{\mathbf{I}}{2} + \mathbf{K}\right)\boldsymbol{\nu} = \mathbf{0}$, Eq. (3.13) is satisfied if, and only if, $\boldsymbol{\psi}$ is a solution of the Fredholm equation:

$$-\frac{\boldsymbol{\psi}}{2} + \mathbf{K}\boldsymbol{\psi} = c\boldsymbol{\nu}, \tag{3.14}$$

where c is an arbitrary real constant. Moreover, since the solutions of the corresponding homogeneous adjoint equation

$$-\frac{\boldsymbol{\gamma}}{2} + \mathbf{K}^*\boldsymbol{\gamma} = \mathbf{0} \tag{3.15}$$

are the rigid displacements ([12], p. 57–62), we conclude that (3.14) has a solution, because $\int_\Sigma (\mathbf{a} + \mathbf{b}\wedge\mathbf{x})\cdot\boldsymbol{\nu}\, d\sigma = \int_\Omega \operatorname{div}\, (\mathbf{a} + \mathbf{b}\wedge\mathbf{x})\, d\mathbf{x} = 0$, $\forall\ \mathbf{a},\ \mathbf{b} \in \mathbb{R}^3$.

[4)] $\boldsymbol{\psi}_0$ has to be the Hölder function because of a standard regularity results (see, e.g. [11]).

Let $\boldsymbol{\varphi}_0$ be a particular solution of (3.14) with $c = 1$. Then a solution of (3.13) is given by

$$\boldsymbol{\psi} = k\boldsymbol{\varphi}_0 + k_1\mathbf{s}_1 + \ldots + k_6\mathbf{s}_6,$$

where $\mathbf{s}_1, \ldots, \mathbf{s}_6$ are linear independent rigid displacements. Thus the dimension of the kernel of $\Lambda = -\frac{1}{4}\mathbf{I} + \mathbf{K}^2$ is less than or equal to 7. To prove that the dimension is 7, we consider the following equation:

$$-\frac{\boldsymbol{\gamma}}{4} + \mathbf{K}^{*2}\boldsymbol{\gamma} = \mathbf{0}. \tag{3.16}$$

On the one hand, we have

$$-\frac{\boldsymbol{\gamma}}{4} + \mathbf{K}^{*2}\boldsymbol{\gamma} = \left(-\frac{\mathbf{I}}{2} + \mathbf{K}^*\right)\left(\frac{\boldsymbol{\gamma}}{2} + \mathbf{K}^*\boldsymbol{\gamma}\right)$$

and then $\boldsymbol{\psi}_0$ is an eigensolution of (3.16). On the other hand, we have

$$-\frac{\boldsymbol{\gamma}}{4} + \mathbf{K}^{*2}\boldsymbol{\gamma} = \left(\frac{\mathbf{I}}{2} + \mathbf{K}^*\right)\left(-\frac{\boldsymbol{\gamma}}{2} + \mathbf{K}^*\boldsymbol{\gamma}\right)$$

and thus also $\mathbf{s}_1, \ldots, \mathbf{s}_6$ are eigensolutions of (3.16). In order to prove that the dimension of the kernel of Λ is 7, we show that $\boldsymbol{\psi}_0, \mathbf{s}_1, \ldots, \mathbf{s}_6$ are linearly independent. On the contrary, if $\boldsymbol{\psi}_0, \mathbf{s}_1, \ldots, \mathbf{s}_6$ are linearly dependent, keeping in mind that $\mathbf{s}_1, \ldots, \mathbf{s}_6$ are linearly independent, we must have: $\boldsymbol{\psi}_0 = c_1\mathbf{s}_1 + \ldots + c_6\mathbf{s}_6$. Therefore $\boldsymbol{\psi}_0$ is a rigid displacement and so it satisfies (3.15). Then

$$\boldsymbol{\psi}_0(\mathbf{x}) = \frac{1}{2}\boldsymbol{\psi}_0(\mathbf{x}) + \mathbf{K}^*\boldsymbol{\psi}_0(\mathbf{x})$$

that is $\boldsymbol{\psi}_0 = \mathbf{0}$, and this is an absurd. □

THEOREM 3. *For any $\mathbf{f} \in [L^p(\Sigma)]^3$, $1 < p < \infty$, such that (2.10) and (3.11) are satisfied, a solution of (3.1) can be represented in the form of a double layer potential (3.4) with density $\boldsymbol{\varphi} \in [W^{1,p}(\Sigma)]^3$. Moreover, (3.4) is a solution of (3.1) if, and only if, its density $\boldsymbol{\varphi}$ is given by*

$$\varphi_h(\mathbf{x}) = \int_\Sigma \psi_j(\mathbf{y})\gamma_{hj}(\mathbf{x}, \mathbf{y})\, d\sigma_\mathbf{y}, \quad \mathbf{x} \in \Sigma, \tag{3.17}$$

$\boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3) \in [L^p(\Sigma)]^3$ being a solution of the Fredholm equation (3.9).

P r o o f. We seek a solution of (3.1) in the form of a double layer potential (3.4). It follows from ([5], p. 35) that any $\boldsymbol{\varphi} \in [W^{1,p}(\Sigma)]^3$ such that $\int_\Sigma \boldsymbol{\varphi} \cdot \boldsymbol{\nu}\, d\sigma = 0$, can be written as a simple layer potential (3.17). Thus from (3.7) we obtain

$$F_h\boldsymbol{\psi} = d\varphi_h. \tag{3.18}$$

Moreover from (3.8), (3.17) we have that

$$F_i'\mathbf{F}\boldsymbol{\psi} = \mu[2\delta_{ij}\Theta_h(F_h\boldsymbol{\psi}) + \mathcal{H}_{ij}(\mathbf{F}\boldsymbol{\psi}) + \mathcal{H}_{ji}(\mathbf{F}\boldsymbol{\psi})]\nu_i$$
$$= \mu[2\delta_{ij}\Theta_h(d\varphi_h) + \mathcal{H}_{ij}(d\boldsymbol{\varphi}) + \mathcal{H}_{ji}(d\boldsymbol{\varphi})]\nu_i.$$

Set

$$\vartheta_i(\mathbf{x}) = \mu\Big[\delta_{ij}\Theta_h(d\varphi_h) + \mathcal{H}_{ij}(d\boldsymbol{\varphi}) + \mathcal{H}_{ji}(d\boldsymbol{\varphi})\Big]dx^i, \quad \mathbf{x} \in \Omega,$$

then it follows from Lemma 3. that

$$\vartheta_i(\mathbf{x}) = \mu\Big[-\delta_{ij}\frac{q(\mathbf{x})}{\mu} + \frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\Big]dx^i, \quad \mathbf{x} \in \Omega$$

and from Lemma 4 that $F_i'\,\mathbf{F}\boldsymbol{\psi}$ is the restriction of $*\vartheta_i$. Hence

$$F_i'\mathbf{F}\boldsymbol{\psi} = T_i(\mathbf{w}), \qquad i = 1, 2, 3, \qquad \text{on } \Sigma.$$

Then, keeping in mind (3.18), we have

$$F_i'(d\boldsymbol{\varphi}) = f_i, \qquad i = 1, 2, 3, \qquad \text{on } \Sigma. \tag{3.19}$$

On the other hand, from the Green's representation formula ([12], p. 54) for $\mathbf{x} \in \Omega$

$$w_j(\mathbf{x}) = \int_\Sigma \varphi_h(\mathbf{y})T_{i\mathbf{y}}'[\gamma^h(\mathbf{x},\mathbf{y})]d\sigma_{\mathbf{y}} = \varphi_j(\mathbf{x}) + \int_\Sigma \gamma_{ih}(\mathbf{x},\mathbf{y})T_h(\boldsymbol{\varphi}(\mathbf{y}))d\sigma_{\mathbf{y}},$$

and on Σ we find that [5], p. 35

$$\mathbf{T}\mathbf{w} = -\frac{1}{4}\boldsymbol{\psi} + \mathbf{K}^2\boldsymbol{\psi},$$

where $\mathbf{K}$ is the compact operator given by (3.10). Therefore

$$\mathbf{F}'(d\boldsymbol{\varphi}) = \mathbf{F}'\mathbf{F}\boldsymbol{\psi} = -\frac{1}{4}\boldsymbol{\psi} + \mathbf{K}^2\boldsymbol{\psi}.$$

This shows that the operator $\mathbf{F}' \circ d : \tilde{W} = \boldsymbol{\varphi} \in [W^{1,p}(\Sigma)]^3 / \int_\Sigma \boldsymbol{\varphi}\cdot\boldsymbol{\nu}\,d\sigma = 0\} \to [L^p(\Sigma)]^3$ can be reduced on the right. Then there exists a solution $\boldsymbol{\varphi} \in \tilde{W}$ of (3.19) if, and only if, $\mathbf{f}$ satisfies the compatibility conditions (3.11). On the other hand, $\int_\Sigma \boldsymbol{\varphi}\cdot\boldsymbol{\nu}\,d\sigma = 0$ being, it follows from ([5], p. 35) that $\boldsymbol{\varphi}$ can be represented by (3.17). Therefore, it follows from Theorem 2 that there exists a solution $\boldsymbol{\psi} \in [L^p(\Sigma)]^3$ of (3.9), if, and only if, Eqs. (2.10) and (3.11) are satisfied.

Now we show how to modify the previous result in order to solve the traction problem (3.1) when the data f satisfies only the necessary conditions (2.10).

THEOREM 4. *For any* $\mathbf{f} \in [L^p(\Sigma)]^3$, $1 < p < \infty$, *such that (2.10) are satisfied, a solution of (3.1) can be represented in the following form:*

$$\mathbf{u} = \mathbf{w} + c\mathbf{v_0} \tag{3.20}$$

where $\mathbf{w}$ *is a double layer potential (3.4) with density* $\boldsymbol{\varphi} \in [W^{1,p}(\Sigma)]^3$, $\mathbf{v}_0$ *is a simple layer potential (3.3) with density* $\boldsymbol{\psi}_0 \in [C^l(\Sigma)]^3$, $0 < l \leq 1$, $\boldsymbol{\psi}_0$ *being a fixed eigensolution of (3.12) and*

$$c = \frac{\int\limits_\Sigma \mathbf{f} \cdot \boldsymbol{\psi}_0 \, d\sigma}{\int\limits_\Sigma \boldsymbol{\psi}_0^2 \, d\sigma}.$$

P r o o f. It is clear that (3.20) satisfies the Stokes system $(3.1)_{1-2}$. Imposing the boundary condition on Σ to $\mathbf{u}$ we obtain

$$\mathbf{T}\mathbf{w} = \mathbf{f} - c\mathbf{T}\mathbf{v}_0.$$

Because of the previous theorem, we have only to show that the following compatibility conditions:

$$\int\limits_\Sigma (\mathbf{f} - \mathbf{T}\mathbf{v}_0) \cdot (\mathbf{a} + \mathbf{b} \wedge \mathbf{x}) d\sigma = 0, \quad \forall\ \mathbf{a},\, \mathbf{b} \in \mathbb{R}^3,$$
$$\int\limits_\Sigma (\mathbf{f} - c\mathbf{T}\mathbf{v}_0) \cdot \boldsymbol{\psi}_0 \, d\sigma = 0$$

hold true. The first ones are obviously satisfied, while the other one is verified because from ([12], p. 56) and (3.12) we obtain

$$\begin{aligned} \int\limits_\Sigma \mathbf{T}\mathbf{v}_0 \cdot \boldsymbol{\psi}_0 \, d\sigma &= \frac{1}{2} \int\limits_\Sigma \boldsymbol{\psi}_0^2 \, d\sigma - \int\limits_\Sigma f\mathbf{K}\boldsymbol{\psi}_0 \cdot \boldsymbol{\psi}_0 \, d\sigma \\ &= \frac{1}{2} \int\limits_\Sigma \boldsymbol{\psi}_0^2 \, d\sigma - \int\limits_\Sigma \boldsymbol{\psi}_0 \cdot \mathbf{K}^*\boldsymbol{\psi}_0 \, d\sigma = \int\limits_\Sigma \boldsymbol{\psi}_0^2 \, d\sigma. \end{aligned}$$

$$< \Phi\,(F) >= \left\{ \begin{array}{lll} 0 & & \text{for } \ t \leq t_d \\ \left.\begin{array}{lll} \Phi(F) & \text{if} & F > 0 \\ 0 & \text{if} & F \leq 0 \end{array}\right\} & & \text{for } \ t > t_d \end{array} \right. \tag{3.21}$$

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