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Some results on the spatial behaviour in linear porous elasticity

M. CIARLETTA[1], S. CHIRIŢĂ[2], F. PASSARELLA[1]

[(1)] *Department of Information Engineering and Applied Mathematics (DIIMA), University of Salerno, 84084 Fisciano (Sa), Italy*

[(2)] *Faculty of Mathematics, University of Iaşi, 6600–Iaşi, Romania*

In this paper we study the spatial behaviour for an extended class of isotropic and homogeneous porous materials for which the constitutive coefficients are supposed to satisfy some relaxed positive definiteness conditions. By using some appropriate measures, we are able to establish results describing the spatial behaviour of transient and steady-state solutions in these enlarged classes of porous materials.

Key words: spatial behaviour, porous materials, steady-state and transient solutions, energy not definite.

1. Introduction

The theory of elastic porous materials has been studied by Goodman and Cowin [1], Nunziato and Cowin [2] and Cowin and Nunziato [3] for describing the deformation of a continuum with voids in which the matrix material is elastic and the interstices are void of material. The nonlinear theory of porous materials was developed by Nunziato and Cowin [2] by assuming that the bulk density is the product of two fields: the matrix material density field and the volume fraction field. This representation introduces an additional degree of kinematic freedom and it is compatible with the theory of granular materials developed by Goodman and Cowin [1]. The linear theory of elastic materials with voids has been developed by Cowin and Nunziato [3]. The intended applications of the theory of elastic materials with voids are to geological materials such as rocks and soils and to artificial porous materials. Porous materials have also been studied in micromechanics (see Gibson and Ashby [4]) and in homogenization (see, for example, Jikov, Kozlov and Oleinik [5], Galka, Telega and Tokarzewski [6] and Cioranescu and Saint-Jean Paulin [7]).

The spatial behaviour of solutions in elastostatics of cylinders made of a porous elastic material was studied by Chiriţă [8, 9] and Ieşan and Quintanilla [10]. The spatial behaviour of the steady-state and transient solutions have been studied by Ieşan and Quintanilla [10] and Scalia [11], under the assumption concerning the positive definiteness of the constitutive coefficients.

Recently, some novel foam structures with negative Poisson's ratios were prepared and their mechanical behaviour and structure have been analysed (see, e.g. Lakes [12] and Caddock and Evans [13]). Such auxetic or anti-rubber materials expand laterally when stretched, in contrast to ordinary materials. Some anisotropic polymer foams have been prepared which exhibit the Poisson's ratio exceeding 1 (see Lee and Lakes [14]). Materials of this kind are expected to have interesting mechanical properties, such as high energy absorption and fracture resistance, which may be useful in applications. Possible applications of such materials in prevention of pressure sores or ulcers are outlined by Wang and Lakes [15]. Saint–Venant end effects for materials with negative Poisson's ratio are studied by Lakes [16].

In the present paper we describe two methods, relevant to the study of spatial behaviour for the steady-state and transient solutions in a linear isotropic homogeneous elastic body with voids, under relaxed conditions concerning the positive definiteness of the constitutive coefficients, including the class of auxetic materials.

In the study of the transient solutions we introduce two time-weighted surface measures and derive and discuss a second-order differential inequality which is valid for a bounded as well as an unbounded body of arbitrary form. Thus, we obtain results of the type described by Scalia [11], but for some enlarged classes of materials with voids.

For the study of the time-harmonic vibrations, we consider a right cylinder made of an isotropic and homogeneous elastic material with voids and introduce two appropriate cross-sectional measures. The treatment of this problem leads to a first-order differential inequality furnishing the information concerning the spatial behaviour of the amplitude of vibration, provided the frequency of the harmonic vibration is lower than critical frequency. Such results are established for the same enlarged classes of materials with voids as for the transient solutions.

2. Formulation of the problem

Let B be a bounded or unbounded regular region of the physical space R^3 with the piecewise smooth boundary surface ∂B. We denote by $\mathbf{n}$ the outward unit normal vector to the boundary. The region B is filled with an isotropic and homogeneous elastic porous material. We select a rectangular system of coordinates and note that vectors and tensors will have components denoted by Latin subscripts ranging over $1, 2, 3$. Summation over repeated subscripts and other conventions typical for differential operations used and superposed dot or a comma followed by a subscript are used to denote partial derivative with respect to time or the corresponding Cartesian coordinate, respectively. Further, we suppress the dependence upon the spatial variable when no confusion may occur.

In the context of linear theory for a porous elastic solid as described by COWIN and NUNZIATO [3], the equations of motion are

$$
\begin{aligned}
&t_{ji,j} + b_i = \rho \ddot{u}_i, \\
&h_{j,j} + g + l = \rho\chi\ddot{\phi}, \qquad \text{in} \quad B \times (0,\infty).
\end{aligned} \tag{2.1}
$$

In the above relations we have used the following notations: u_i are the components of the displacement vector field, ϕ is the change in volume fraction from the reference volume fraction, t_{ij} are the components of the stress tensor, h_i are the components of the equilibrated stress vector, g is the intrinsic equilibrated force, b_i are the components of the body force vector and l is the extrinsic equilibrated body force. Moreover, ρ and χ are the bulk mass density and the equilibrated inertia in the reference state.

The constitutive equations for the linear theory of isotropic and homogeneous elastic porous continuum are [3]

$$
\begin{aligned}
&t_{ij} = \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + \beta\phi\delta_{ij}, \qquad h_i = \alpha\phi_{,i}, \\
&g = -\beta e_{rr} - \xi\phi, \quad \text{in} \quad \bar{B} \times [0,\infty),
\end{aligned} \tag{2.2}
$$

where λ, μ, α, β and ξ are constitutive constants, δ_{ij} is the Kronecker's delta and

$$
e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right). \tag{2.3}
$$

The set $\bar{B}$ represents the closure of B.

The internal energy density $\mathcal{E}$ associated with the kinematic fields u_i, ϕ is defined by

$$
\mathcal{E} = \frac{1}{2}\lambda e_{rr}e_{ss} + \mu e_{ij}e_{ij} + \beta e_{rr}\phi + \frac{1}{2}\xi\phi^2 + \frac{1}{2}\alpha\phi_{,i}\phi_{,i}, \tag{2.4}
$$

and it is positive definite if and only if (see, e.g. COWIN and NUNZIATO [3])

$$
\mu > 0, \qquad \alpha > 0, \qquad \xi > 0, \qquad 3\lambda + 2\mu > 0, \qquad (3\lambda + 2\mu)\xi > 3\beta^2. \tag{2.5}
$$

We further note that, by assuming zero boundary displacements and volume fraction, the total internal energy associated with B,

$$
\mathcal{E}_B = \int_B \mathcal{E}\, dv, \tag{2.6}
$$

can be written in the following form:

$$
\mathcal{E}_B = \frac{1}{2}\int_B W\, dv, \tag{2.7}
$$

or in the form

$$\mathcal{E}_B = \frac{1}{2} \int_B W^* dv, \tag{2.8}$$

where

$$W = \mu u_{i,j} u_{i,j} + (\lambda + \mu)\, u_{r,r} u_{s,s} + 2\beta u_{r,r} \phi + \xi \phi^2 + \alpha \phi_{,i} \phi_{,i}, \tag{2.9}$$

$$W^* = \mu u_{i,j} u_{i,j} + (\lambda + \mu)\, u_{i,j} u_{j,i} + 2\beta u_{r,r} \phi + \xi \phi^2 + \alpha \phi_{,i} \phi_{,i}. \tag{2.10}$$

In this work we shall use the positive definiteness of the quadratic forms W and W^* in order to study the spatial behaviour of solutions in the linear theory of elastodynamics of materials with voids. As we will see later, the hypotheses of positive definiteness on the constitutive coefficients defined by (2.5) will be relaxed. To this end, we consider an isotropic, homogeneous elastic porous solid so that the basic equations (2.1) to (2.3) of the linear dynamic theory reduce to [3]

$$\begin{gathered} \mu u_{i,rr} + (\lambda + \mu) u_{r,ri} + \beta \phi_{,i} + b_i = \rho \ddot{u}_i, \\ \alpha \phi_{,rr} - \beta u_{r,r} - \xi \phi + l = \rho \chi \ddot{\phi}. \end{gathered} \tag{2.11}$$

With these equations we associate the following initial conditions

$$u_i = u_i^0 \qquad \dot{u}_i = \dot{u}_i^0, \qquad \phi = \phi^0, \qquad \dot{\phi} = \dot{\phi}^0, \quad \text{on} \quad \bar{B} \times \{0\}, \tag{2.12}$$

and the following boundary conditions

$$u_i = \tilde{u}_i, \qquad \phi = \tilde{\phi} \quad \text{on} \quad \partial B \times [0, \infty), \tag{2.13}$$

where u_i^0, $\dot{u}_i^0$, ϕ^0, $\dot{\phi}^0$, $\tilde{u}_i$, and $\tilde{\phi}$ are the prescribed continuous functions. Under the positive definiteness conditions (2.5) for the potential energy density $\mathcal{E}$ and by assuming $\rho > 0$, $\chi > 0$, spatial behaviour of the transient solutions of the initial-boundary value problem $\mathcal{P}$ defined by the relations (2.11)–(2.13), has been established by Scalia [11]. Throughout this paper we will consider classical solutions of the initial-boundary value problem $\mathcal{P}$, that is we will consider the couple of functions $\{u_i, \phi\}$ twice continuously differentiable with respect to the spatial and time variables, satisfying the relations (2.11)–(2.13). We have to stress that our method of proof can be used also for appropriate classes of weak solutions.

3. Spatial behaviour of the transient solutions

Throughout this section we will establish the spatial behaviour of the transient solutions of the initial-boundary value problem $\mathcal{P}$ under the hypotheses on the constitutive coefficients milder than those used in Scalia [11]. In fact, our results are established for the classes of porous elastic materials will be defined by relations (3.15) and (3.41). Clearly, these relations enlarge the class of porous elastic materials described by the relation (2.5). This fact is possible because we introduce two new measures associated with the solutions, essentially different from that used by Scalia [11]. Our analysis is motivated by the existence of the novel foam structures for which the internal energy density $\mathcal{E}$ is not always a positive definite quadratic form, but, under our hypotheses (3.15) and (3.41), we can assure the positive definiteness of one of the quadratic forms W and W^*.

3.1. First measure and related estimates

To this end, we follow an idea devised in Chiriţă [9] and write the system (2.11) in the form

$$S_{ji,j} + b_i = \rho \ddot{u}_i, \qquad h_{j,j} + g + l = \rho\chi\ddot{\phi}, \quad \text{in} \quad B \times (0,\infty), \tag{3.1}$$

with

$$S_{ji} = \mu u_{i,j} + (\lambda+\mu) u_{r,r}\delta_{ij} + \beta\phi\delta_{ij},$$

$$h_i = \alpha\phi_{,i}, \qquad g = -\beta u_{r,r} - \xi\phi. \tag{3.2}$$

Then the initial-boundary value problem $\mathcal{P}$ is defined by the Eqs. (3.1) and (3.2), the initial conditions (2.12) and the boundary conditions (2.13).

For fixed $T > 0$, we consider the support $\hat{D}_T$ of the initial and boundary data and the body forces in the time interval $[0,T]$ and further, we assume that it is a bounded set. Of course, $\hat{D}_T$ is the set of all $\mathbf{x} \in \bar{B}$ such that:

a) if $\mathbf{x} \in B$, then:

$$u_i^0(\mathbf{x}) \neq 0 \ \text{ or } \ \dot{u}_i^0(\mathbf{x}) \neq 0 \ \text{ or } \ \phi^0(\mathbf{x}) \neq 0 \ \text{ or } \ \dot{\phi}^0(\mathbf{x}) \neq 0$$

$$\text{or } \ b_i(\mathbf{x},\tau) \neq 0 \ \text{ or } \ l(\mathbf{x},\tau) \neq 0 \quad \text{for some} \quad \tau \in [0,T];$$

b) if $\mathbf{x} \in \partial B$, we have:

$$\tilde{u}_i(\mathbf{x},\tau) \neq 0 \quad \text{or} \quad \tilde{\phi}(\mathbf{x},\tau) \neq 0 \ \text{ for some } \ \tau \in [0,T].$$

Let $\hat{D}_T^*$ be a non-empty set such that:

(i) if $\hat{D}_T \cap B \neq \emptyset$ then we choose $\hat{D}_T^*$ to be the smallest bounded regular region in $\bar{B}$ that includes $\hat{D}_T$; in particular, we set $\hat{D}_T^* = \hat{D}_T$ if it also happens that $\hat{D}_T$ is a regular region;

(ii) if $\emptyset \neq \hat{D}_T \subset \partial B$, then we choose $\hat{D}_T^*$ to be the smallest regular subsurface of ∂B that includes $\hat{D}_T$; in particular, we set $\hat{D}_T^* = \hat{D}_T$ if $\hat{D}_T$ is a regular subsurface of ∂B;

(iii) if $\hat{D}_T = \emptyset$, then we choose $\hat{D}_T^*$ to be an arbitrary non-empty regular subsurface of ∂B.

Further, we introduce the following sets:

$$D_r = \left\{ \mathbf{x} \in \bar{B} : \ \overline{S(\mathbf{x}, r)} \cap \hat{D}_T^* \neq \emptyset \right\}, \tag{3.3}$$

$$B_r = B \backslash D_r, \quad r \geq 0, \quad B(r_1, r_2) = B_{r_2} \backslash B_{r_1}, \quad r_2 < r_1, \tag{3.4}$$

where $\overline{S(\mathbf{x}, r)}$ is the closure of the ball with radius r and center at $\mathbf{x}$. Finally, we denote by S_r the subsurface of ∂B_r contained inside B and whose outward unit normal vector is directed to the exterior of D_r. We can observe that the initial and boundary data and the body forces are null on B_r, S_r.

Corresponding to the solution $\mathbf{U} = \{u_i, \phi\}$ of the initial-boundary value problem $\mathcal{P}$, we introduce the following function

$$\Lambda(r, t) = -\int_0^t \int_{S_r} e^{-\sigma s} [\dot{u}_i(s) S_{ji}(s) + \dot{\phi}(s) h_j(s)] n_j da ds, \tag{3.5}$$

for a fixed positive parameter σ. This function is defined on $I \times [0, T]$, where I is the interval $[0, \infty)$ if B is an unbounded body, I is the interval $\left[0, \max_{\mathbf{x} \in \bar{B}} \left(\min_{\mathbf{y} \in \hat{D}_T^*} |\mathbf{y} - \mathbf{x}|^{1/2} \right) \right]$ if B is a bounded body.

From the definitions for S_r and $\Lambda(r, t)$, it results

$$\frac{\partial \Lambda}{\partial t}(r, t) = -\int_{S_r} e^{-\sigma t} [\dot{u}_i(t) S_{ji}(t) + \dot{\phi}(t) h_j(t)] n_j da. \tag{3.6}$$

By using the equations (3.1), (3.2), we get

$$[\dot{u}_i S_{ji} + \dot{\phi} h_j]_{,j} = \frac{1}{2} \frac{\partial}{\partial t} \Big[K + W \Big] - b_i \dot{u}_i - l \dot{\phi}, \tag{3.7}$$

where

$$K = \rho \dot{u}_i \dot{u}_i + \rho \chi \dot{\phi}^2, \tag{3.8}$$

$$W(u,\phi) = W_1(u,\phi) + W_2(u) + W_3(\phi), \tag{3.9}$$

$$\begin{aligned} W_1(u,\phi) = (\lambda+2\mu)\left(u_{1,1}^2 + u_{2,2}^2 + u_{3,3}^2\right) \\ + 2(\lambda+\mu)(u_{1,1}u_{2,2} + u_{2,2}u_{3,3} + u_{3,3}u_{1,1}) \\ + \xi\phi^2 + 2\beta\phi\left(u_{1,1} + u_{2,2} + u_{3,3}\right), \end{aligned} \tag{3.10}$$

$$W_2(u) = \mu(u_{1,2}^2 + u_{2,1}^2 + u_{2,3}^2 + u_{3,2}^2 + u_{1,3}^2 + u_{3,1}^2), \tag{3.11}$$

$$W_3(\phi) = \alpha\phi_{,j}\phi_{,j}. \tag{3.12}$$

The definition of S_r, B_r, the divergence theorem and the relations (3.5), (3.7) lead to

$$\begin{aligned} \varLambda(r_1,t) - \varLambda(r_2,t) &= -\int_0^t \int_{\partial B(r_1,r_2)} e^{-\sigma s}[\dot{u}_i(s)S_{ji}(s) + \dot{\phi}(s)h_j(s)]n_j\,da\,ds \\ &= -\frac{1}{2}\int_0^t \int_{B(r_1,r_2)} e^{-\sigma s}\frac{\partial}{\partial s}[K(s)+W(s)]\,dv\,ds, \qquad r_2 < r_1, \quad 0 \le t \le T. \end{aligned} \tag{3.13}$$

Taking into account (3.13) and the definition of S_r and B_r, it is a simple matter to obtain

$$\begin{aligned} \frac{\partial\varLambda}{\partial r}(r,t) = -\frac{1}{2}\int_{S_r} e^{-\sigma t}[K(t)+W(t)]\,da \\ - \frac{\sigma}{2}\int_0^t\int_{S_r} e^{-\sigma s}[K(s)+W(s)]\,da\,ds. \end{aligned} \tag{3.14}$$

Obviously, if $\rho > 0$ and $\chi > 0$ then the quadratic form K in the variables $\{\dot{u}_1, \dot{u}_2, \dot{u}_3, \sqrt{\chi}\,\dot{\phi}\}$ is positive definite and if $\alpha > 0$, then the form W_3 in the variables $\{\phi_{,1}, \phi_{,2}, \phi_{,3}\}$ is positive definite. Moreover, W_2 is a positive definite quadratic form in the variables $\{u_{1,2},\ u_{2,1},\ u_{2,3},\ u_{3,2},\ u_{1,3},\ u_{3,1}\}$ if and only if $\mu > 0$. On the other hand, the quadratic form W_1 in the variables $\{u_{1,1}, u_{2,2}, u_{3,3}, \phi\}$ is positive definite if and only if $\mu > 0$, $\xi > 0$, $4\mu + 3\lambda > 0$, $(4\mu+3\lambda)\xi > 3\beta^2$. The eigenvalues of the matrix $\mathcal{A}$ of the quadratic form W_1 are

$$\begin{aligned}
\kappa_1 &= \kappa_2 = \mu, \\
\kappa_3 &= \frac{1}{2}\left\{\xi + 4\mu + 3\lambda + \sqrt{[\xi - (4\mu + 3\lambda)]^2 + 12\beta^2}\right\}, \\
\kappa_4 &= \frac{1}{2}\left\{\xi + 4\mu + 3\lambda - \sqrt{[\xi - (4\mu + 3\lambda)]^2 + 12\beta^2}\right\}.
\end{aligned}$$

Throughout this subsection we will assume that

$$\begin{gathered}
\rho > 0, \qquad \chi > 0, \qquad \alpha > 0, \qquad \mu > 0, \\
\xi > 0, 4 \qquad 4\mu + 3\lambda > 0, \qquad (4\mu + 3\lambda)\xi > 3\beta^2,
\end{gathered} \tag{3.15}$$

so that K and W_i are positive definite. Consequently, the relations (3.13), (3.14) imply that $\Lambda(r,t)$ is a non–increasing function with respect to r, for all $t \in [0,T]$.

Let $\mathcal{F}[\mathcal{A};\psi,\gamma]$ be the symmetric bilinear form associated with the quadratic form W_1, that is

$$\begin{aligned}
\mathcal{F}[\mathcal{A};\psi,\gamma] = \psi\cdot\mathcal{A}\gamma = (\lambda + 2\mu)\,(\psi_1\gamma_1 + \psi_2\gamma_2 + \psi_3\gamma_3) + \xi\psi_4\gamma_4 \\
+ (\lambda + \mu)(\psi_1\gamma_2 + \psi_2\gamma_1 + \psi_2\gamma_3 + \psi_3\gamma_2 + \psi_3\gamma_1 + \psi_1\gamma_3) \\
+ \beta\left[\gamma_4\,(\psi_1 + \psi_2 + \psi_3) + \psi_4\,(\gamma_1 + \gamma_2 + \gamma_3)\right],
\end{aligned} \tag{3.16}$$

for every $\psi = \{\psi_1,\psi_2,\psi_3,\psi_4\}$, $\gamma = \{\gamma_1,\gamma_2,\gamma_3,\gamma_4\}$. Clearly, we have

$$\begin{gathered}
\kappa_m(\psi_1^2 + \psi_2^2 + \psi_3^2 + \psi_4^2) \le \mathcal{F}[\mathcal{A};\psi,\psi] \le \kappa_M(\psi_1^2 + \psi_2^2 + \psi_3^2 + \psi_4^2), \\
\mathcal{F}[\mathcal{A};\tilde{\psi},\tilde{\psi}] = W_1(u,\phi) \quad \text{for} \quad \tilde{\psi} = \{u_{1,1}, u_{2,2}, u_{3,3}, \phi\},
\end{gathered} \tag{3.17}$$

in which κ_m and κ_M are the lowest and the largest characteristic values of $\mathcal{A}$, respectively, that is

$$\begin{aligned}
k_m &= \min\left\{\mu,\ \frac{1}{2}\left[\xi + 4\mu + 3\lambda - \sqrt{[\xi - (4\mu + 3\lambda)]^2 + 12\beta^2}\right]\right\}, \\
k_M &= \max\left\{\mu,\ \frac{1}{2}\left[\xi + 4\mu + 3\lambda + \sqrt{[\xi - (4\mu + 3\lambda)]^2 + 12\beta^2}\right]\right\}.
\end{aligned} \tag{3.18}$$

Then, we can state the following theorem:

Theorem 1. *Let* $\mathbf{U} = \{u_i,\phi\}$ *be a solution of initial–boundary value problem* $\mathcal{P}$ *and* $\widehat{D}_T$ *be the bounded support of the external data in the time interval* $[0,T]$. *Under the hypotheses (3.15), we have*

$$\frac{\sigma}{c_1}\,|\Lambda(r,t)| + \frac{\partial\Lambda}{\partial r}(r,t) \le 0, \qquad (r,t) \in I \times [0,T], \tag{3.19}$$

and

$$\left|\frac{\partial \Lambda}{\partial t}(r,t)\right| + c_1 \frac{\partial \Lambda}{\partial r}(r,t) \leq 0, \qquad (r,t) \in I \times [0,T], \tag{3.20}$$

where

$$c_1 = \sqrt{\frac{\gamma_M}{\rho}}, \tag{3.21}$$

and

$$\gamma_M = \max\left\{\frac{\alpha}{\chi},\ \mu,\ \frac{1}{2}\left[\xi + 4\mu + 3\lambda + \sqrt{\left[\xi - (4\mu+3\lambda)\right]^2 + 12\beta^2}\right]\right\}. \tag{3.22}$$

P r o o f. By means of the Eqs. (3.2), (3.16), we have

$$\begin{aligned} S_{11}^2 + S_{22}^2 + S_{33}^2 + g^2 = {} & S_{11}\left[(\lambda+2\mu)\,u_{1,1} + (\lambda+\mu)\,(u_{2,2}+u_{3,3}) + \beta\phi\right] \\ & + S_{22}\left[\,(\lambda+2\mu)\,u_{2,2} + (\lambda+\mu)\,(u_{3,3}+u_{1,1}) + \beta\phi\right] \\ & + S_{33}\left[\,(\lambda+2\mu)\,u_{3,3} + (\lambda+\mu)\,(u_{1,1}+u_{2,2}) + \beta\phi\right] \\ & - g\,(\beta u_{r,r} + \xi\phi) = \mathcal{F}\big[\mathcal{A};\mathbf{S},\tilde{\psi}\big], \end{aligned} \tag{3.23}$$

where $\mathbf{S} = \{S_{11}, S_{22}, S_{33}, -g\}$ and $\tilde{\psi} = \{u_{1,1}, u_{2,2}, u_{3,3}, \phi\}$. Further, by using the Schwarz's inequality and the inequality (3.17), from (3.23) we obtain

$$\begin{aligned} \left[S_{11}^2 + S_{22}^2 + S_{33}^2 + g^2\right]^2 & \leq \mathcal{F}[\mathcal{A};\tilde{\psi},\tilde{\psi}]\mathcal{F}[\mathcal{A};\mathbf{S},\mathbf{S}] \\ & \leq W_1\kappa_M\left[S_{11}^2 + S_{22}^2 + S_{33}^2 + g^2\right], \end{aligned} \tag{3.24}$$

that is

$$S_{11}^2 + S_{22}^2 + S_{33}^2 + g^2 \leq \kappa_M W_1. \tag{3.25}$$

In view of the Eqs. $(3.2)_1$, (3.10), (3.11), (3.18) and (3.25), we deduce that

$$\begin{aligned} S_{ij}S_{ij} = {} & \mu^2(u_{1,2}^2 + u_{2,1}^2 + u_{2,3}^2 + u_{3,2}^2 + u_{1,3}^2 + u_{3,1}^2) + S_{11}^2 \\ & + S_{22}^2 + S_{33}^2 \leq \mu W_2 + \kappa_M W_1 \leq \kappa_M(W_2 + W_1). \end{aligned} \tag{3.26}$$

On the other hand, considering the Schwarz inequality and the arithmetic–geometric mean inequalities, the relations (3.2), (3.8), (3.12) and (3.26) imply

$$\begin{aligned} \left|\dot{u}_i S_{ji} n_j + \dot{\phi} h_j n_j\right| & \leq \frac{1}{2}\left(\varepsilon\rho\dot{u}_i\dot{u}_i + \frac{1}{\varepsilon\rho}S_{ij}S_{ij} + \varepsilon\rho\chi\dot{\phi}^2 + \frac{1}{\varepsilon\rho}\frac{\alpha}{\chi}\alpha\phi_{,j}\phi_{,j}\right) \\ & \leq \frac{1}{2}\left[\varepsilon K + \frac{1}{\varepsilon\rho}\gamma_M W\right], \end{aligned} \tag{3.27}$$

where ε is an arbitrary positive constant and γ_M is defined by (3.22). If we insert $\varepsilon = c_1$ into (3.27), then the relations (3.5), (3.6) imply

$$|\Lambda(r,t)| \leq \frac{c_1}{2} \int_0^t \int_{S_r} e^{-\sigma s}[K(s) + W(s)] da ds, \tag{3.28}$$

$$\left| \frac{\partial \Lambda}{\partial t}(r,t) \right| \leq \frac{c_1}{2} \int_{S_r} e^{-\sigma t}[K(t) + W(t)] da. \tag{3.29}$$

Under the hypotheses (3.15), the relations (3.14), (3.28), (3.29) lead to the equations (3.19), (3.20) and the proof is complete.

Theorem 1 forms the basis of the following theorems for a bounded or unbounded body:

THEOREM 2. *(Spatial behaviour): Provided the hypotheses of Theorem 1 hold, $\Lambda(r,t)$ is a measure associated with the solution $\mathbf{U} = \{u_i, \phi\}$ of $\mathcal{P}$:*

$$\Lambda(r,t) \geq 0 \qquad (r,t) \in I \times [0,T].$$

Moreover, at each fixed $t \in [0,T]$ we have

$$\begin{aligned} &\Lambda(r,t) = 0, \quad \text{that is} \quad u_i = 0, \qquad \phi = 0 \quad \text{for} \quad r \geq c_1 t, \\ &0 \leq \Lambda(r,t) \leq \Lambda(0,t)\, e^{(-\sigma/c_1)r} \quad \text{for} \quad 0 \leq r \leq c_1 t. \end{aligned} \tag{3.30}$$

THEOREM 3. *(Uniqueness): Assuming that the hypotheses of Theorem 1 hold, there exists at most one solution for the initial-boundary value problem $\mathcal{P}$.*

It is a simple matter to prove these theorems following the method established in SCALIA [11] and CHIRIŢĂ and CIARLETTA [17].

3.2. Second measure and related estimates

In this section, we describe a second method for analysing the spatial behaviour of the solutions for an appropriate class of materials. To this end, we write the system (2.11) in the following form:

$$T_{ji,j} + b_i = \rho \ddot{u}_i, \qquad h_{j,j} + g + l = \rho\chi\ddot{\phi}, \qquad \text{in} \quad B \times (0,\infty), \tag{3.31}$$

where

$$T_{ji} = \mu u_{i,j} + (\lambda + \mu) u_{j,i} + \beta\phi\delta_{ij}, \qquad h_i = \alpha\phi_{,i}, \qquad g = -\beta u_{r,r} - \xi\phi. \tag{3.32}$$

Thus, the initial-boundary value problem $\mathcal{P}$ is defined by the Eqs. (3.31) and (3.32) and the initial and boundary conditions (2.12) and (2.13). In what follows we denote by $\mathbf{U} = \{u_i, \phi\}$ a solution of the initial-boundary value problem $\mathcal{P}$.

Now, by introducing the notations D_r, B_r, S_r in the same manner as in the previous subsection, we define the function on $I \times [0,T]$:

$$\mathit{\Pi}(r,t) = -\int_0^t \int_{S_r} e^{-\sigma s}[\dot{u}_i(s)T_{ji}(s) + \dot{\phi}(s)h_j(s)]n_j \, dads. \tag{3.33}$$

The equations (3.31), (3.32) give

$$[\dot{u}_i T_{ji} + \dot{\phi} h_j]_{,j} = \frac{1}{2}\frac{\partial}{\partial t}\left[K + W^*\right] - b_i\dot{u}_i - l\dot{\phi}, \tag{3.34}$$

in which

$$W^*(u,\phi) = W_1^*(u,\phi) + W_2^*(u) + W_3(\phi), \tag{3.35}$$

$$\begin{aligned} W_1^*(u,\phi) &= (\lambda + 2\mu)\left(u_{1,1}^2 + u_{2,2}^2 + u_{3,3}^2\right) \\ &\qquad + 2\beta\phi\left(u_{1,1} + u_{2,2} + u_{3,3}\right) + \xi\phi^2, \\ W_2^*(u) &= \mu(u_{1,2}^2 + u_{2,1}^2 + u_{2,3}^2 + u_{3,2}^2 + u_{1,3}^2 + u_{3,1}^2) \\ &\qquad + 2(\lambda + \mu)(u_{1,2}u_{2,1} + u_{2,3}u_{3,2} + u_{3,1}u_{1,3}). \end{aligned} \tag{3.36}$$

Taking into account the definition for S_r, B_r and the divergence theorem, the relations (3.33), (3.34) imply that

$$\mathit{\Pi}(r_1,t) - \mathit{\Pi}(r_2,t) = -\frac{1}{2}\int_0^t \int_{B(r_1,r_2)} e^{-\sigma s}\frac{\partial}{\partial s}\left[K(s) + W^*(s)\right] dvds, \tag{3.37}$$

and

$$\begin{aligned} \frac{\partial \mathit{\Pi}}{\partial r}(r,t) &= -\frac{1}{2}\int_{S_r} e^{-\sigma t}[K(t) + W^*(t)]da \\ &\qquad -\frac{\sigma}{2}\int_0^t \int_{S_r} e^{-\sigma s}[K(s) + W^*(s)]dads, \\ \frac{\partial \mathit{\Pi}}{\partial t}(r,t) &= -\int_{S_r} e^{-\sigma t}[\dot{u}_i(t)T_{ji}(t) + \dot{\phi}(t)h_j(t)]n_j da. \end{aligned} \tag{3.38}$$

Clearly, K is positive definite if and only if $\rho > 0$, $\chi > 0$, while W_3 is positive definite if and only if $\alpha > 0$. Moreover, the quadratic form W_2^* in the variables $\{u_{1,2}, u_{2,1}, u_{2,3}, u_{3,2}, u_{3,1}, u_{1,3}\}$ is positive definite if and only if $\lambda < 0$, $\lambda+2\mu > 0$. The eigenvalues of the matrix $\mathcal{B}$ of this quadratic form are

$$\breve{\kappa}_1 = \breve{\kappa}_2 = \breve{\kappa}_3 = -\lambda, \qquad \breve{\kappa}_4 = \breve{\kappa}_5 = \breve{\kappa}_6 = \lambda + 2\mu. \tag{3.39}$$

Further, the quadratic form W_1^* in the variables $\{u_{1,1}, u_{2,2}, u_{3,3}, \phi\}$ is positive definite if and only if $\xi > 0$, $\lambda + 2\mu > 0$, $(\lambda + 2\mu)\xi > 3\beta^2$. The matrix $\mathcal{C}$ of the quadratic form W_1^* has the following eigenvalues:

$$\begin{aligned} &\hat{\kappa}_1 = \hat{\kappa}_2 = \lambda + 2\mu, \\ &\hat{\kappa}_{3,4} = \frac{1}{2}\left\{\xi + \lambda + 2\mu \pm \sqrt{[\xi - (\lambda + 2\mu)]^2 + 12\beta^2}\right\}. \end{aligned} \tag{3.40}$$

Throughout this subsection we will assume that

$$\begin{gathered} \rho > 0, \qquad \chi > 0, \qquad \alpha > 0, \qquad \mu > 0, \\ \xi > 0, \qquad -2\mu < \lambda < 0, \qquad (\lambda + 2\mu)\xi > 3\beta^2, \end{gathered} \tag{3.41}$$

so that K, W_1^*, W_2^*, and W_3 are positive definite. Thus, for all $t \in [0, T]$, we prove by means of the relation (3.32), that $\varPi(r, t)$ is a non-increasing function with respect to r.

We denote by $\mathcal{G}[\mathcal{B}; \breve{\psi}, \breve{\gamma}]$ and $\mathcal{I}[\mathcal{C}; \psi, \gamma]$ the bilinear forms associated with W_2^* and W_1^*, respectively, that is,

$$\begin{aligned} \mathcal{G}[\mathcal{B}; \breve{\psi}, \breve{\gamma}] = \breve{\psi}\cdot\mathcal{B}\breve{\gamma} =& \mu(\breve{\psi}_1\breve{\gamma}_1 + \breve{\psi}_2\breve{\gamma}_2 + \breve{\psi}_3\breve{\gamma}_3 + \breve{\psi}_4\breve{\gamma}_4 + \breve{\psi}_5\breve{\gamma}_5 + \breve{\psi}_6\breve{\gamma}_6) \\ &+ (\lambda + \mu)(\breve{\psi}_1\breve{\gamma}_2 + \breve{\psi}_2\breve{\gamma}_1 + \breve{\psi}_3\breve{\gamma}_4 + \breve{\psi}_4\breve{\gamma}_3 + \breve{\psi}_5\breve{\gamma}_6 + \breve{\psi}_6\breve{\gamma}_5), \\ &\breve{\psi} = \left\{\breve{\psi}_1, ..., \breve{\psi}_6\right\}, \quad \breve{\gamma} = \{\breve{\gamma}_1, ..., \breve{\gamma}_6\}, \end{aligned} \tag{3.42}$$

and

$$\begin{aligned} \mathcal{I}[\mathcal{C}; \psi, \gamma] = \psi\cdot\mathcal{C}\gamma =& (\lambda + 2\mu)(\gamma_1\psi_1 + \gamma_2\psi_2 + \gamma_3\psi_3) \\ &+ \beta(\psi_1\gamma_4 + \psi_4\gamma_1 + \psi_2\gamma_4 + \psi_4\gamma_2 + \psi_3\gamma_4 + \psi_4\gamma_3) + \xi\gamma_4\psi_4, \\ &\psi = \{\psi_1, ..., \psi_4\}, \quad \gamma = \{\gamma_1, ..., \gamma_4\}. \end{aligned} \tag{3.43}$$

Then, we have

$$\begin{aligned} &\breve{\kappa}_m(\breve{\psi}_1^2 + \breve{\psi}_2^2 + \breve{\psi}_3^2 + \breve{\psi}_4^2 + \breve{\psi}_5^2 + \breve{\psi}_6^2) \\ &\qquad \le \mathcal{G}[\mathcal{B}; \breve{\psi}, \breve{\psi}] \le \breve{\kappa}_M(\breve{\psi}_1^2 + \breve{\psi}_2^2 + \breve{\psi}_3^2 + \breve{\psi}_4^2 + \breve{\psi}_5^2 + \breve{\psi}_6^2), \\ &\hat{\kappa}_m(\psi_1^2 + \psi_2^2 + \psi_3^2 + \psi_4^2) \le \mathcal{I}[\mathcal{C}; \psi, \psi] \le \hat{\kappa}_M(\psi_1^2 + \psi_2^2 + \psi_3^2 + \psi_4^2), \end{aligned} \tag{3.44}$$

where $\breve{\kappa}_m$ and $\breve{\kappa}_M$ are the lowest and the largest characteristic values of $\mathcal{B}$, while $\hat{\kappa}_m$ and $\hat{\kappa}_M$ are the lowest and the largest characteristic values of $\mathcal{C}$, respectively, that is

$$\breve{\kappa}_m = \min\{-\lambda,\ \lambda+2\mu\}, \qquad \breve{\kappa}_M = \max\{-\lambda,\ \lambda+2\mu\}, \tag{3.45}$$

$$\begin{aligned} \hat{\kappa}_m &= \min\left\{\lambda+2\mu,\ \frac{1}{2}\left[\xi+\lambda+2\mu-\sqrt{[\xi-(\lambda+2\mu)]^2+12\beta^2}\right]\right\},\\ \hat{\kappa}_M &= \max\left\{\lambda+2\mu,\ \frac{1}{2}\left[\xi+\lambda+2\mu+\sqrt{[\xi-(\lambda+2\mu)]^2+12\beta^2}\right]\right\}. \end{aligned} \tag{3.46}$$

It is interesting to observe that

$$\begin{aligned} &\mathcal{G}[\mathcal{B};\breve{\boldsymbol{\Psi}},\breve{\boldsymbol{\Psi}}] = W_2^* \qquad \text{for} \quad \breve{\boldsymbol{\Psi}} = \{u_{1,2},u_{2,1},u_{2,3},u_{3,2},u_{3,1},u_{1,3}\},\\ &\mathcal{I}[\mathcal{C};\tilde{\boldsymbol{\Psi}},\tilde{\boldsymbol{\Psi}}] = W_1^* \qquad \text{for} \quad \tilde{\boldsymbol{\Psi}} = \{u_{1,1},u_{2,2},u_{3,3},\phi\}. \end{aligned} \tag{3.47}$$

We prove the following result

Theorem 4. *Let* $\mathbf{U}=\{u_i,\phi\}$ *be a solution of initial–boundary value problem* $\mathcal{P}$ *and* $\widehat{D}_T$ *be the bounded support of the external data in the time interval* $[0,T]$. *Provided the hypothesis (3.41) holds, it follows that*

$$\frac{\sigma}{c_2}\,|\Pi(r,t)| + \frac{\partial \Pi}{\partial r}(r,t) \le 0, \qquad (r,t)\in I\times[0,T], \tag{3.48}$$

and

$$\left|\frac{\partial \Pi}{\partial t}(r,t)\right| + c_2\frac{\partial \Pi}{\partial r}(r,t) \le 0, \qquad (r,t)\in I\times[0,T], \tag{3.49}$$

where

$$c_2 = \sqrt{\frac{\gamma_M^*}{\rho}}, \tag{3.50}$$

and

$$\gamma_M^* = \max\left\{\frac{\alpha}{\chi},-\lambda,\lambda+2\mu,\ \frac{1}{2}\left(\xi+\lambda+2\mu+\sqrt{[\xi-(\lambda+2\mu)]^2+12\beta^2}\right)\right\}. \tag{3.51}$$

P r o o f. We first note that the Eqs. (3.32), (3.42), (3.43) give

$$T_{12}^2+T_{21}^2+T_{23}^2+T_{32}^2+T_{31}^2+T_{13}^2 = \mathcal{G}[\mathcal{B};\breve{\mathbf{T}},\breve{\boldsymbol{\Psi}}], \tag{3.52}$$

and

$$T_{11}^2 + T_{22}^2 + T_{33}^2 + g^2 = \mathcal{I}[\mathcal{C}; \mathbf{T}^*, \tilde{\boldsymbol{\Psi}}], \tag{3.53}$$

where $\breve{\mathbf{T}} = \{T_{21}, T_{12}, T_{32}, T_{23}, T_{13}, T_{31}\}$, $\breve{\boldsymbol{\Psi}} = \{u_{1,2}, u_{2,1}, u_{2,3}, u_{3,2}, u_{3,1}, u_{1,3}\}$, $\mathbf{T}^* = \{T_{11}, T_{22}, T_{33}, -g\}$ and $\tilde{\boldsymbol{\Psi}} = \{u_{1,1}, u_{2,2}, u_{3,3}, \phi\}$.

By means of the Schwarz inequality and by using the relations (3.42), (3.43) and (3.44), we have

$$\begin{aligned}\left[T_{12}^2 + T_{21}^2 + T_{23}^2 + T_{32}^2 + T_{31}^2 + T_{13}^2\right]^2 &\le \mathcal{G}\,[\mathcal{B}; \breve{\boldsymbol{\Psi}}, \breve{\boldsymbol{\Psi}}]\mathcal{G}\,[\mathcal{B}; \breve{\mathbf{T}}, \breve{\mathbf{T}}] \\ &\le W_2^* \breve{\kappa}_M \left(T_{12}^2 + T_{21}^2 + T_{23}^2 + T_{32}^2 + T_{31}^2 + T_{13}^2\right),\end{aligned} \tag{3.54}$$

and

$$\begin{aligned}\left[T_{11}^2 + T_{22}^2 + T_{33}^2 + g^2\right]^2 &\le \mathcal{I}\,[\mathcal{C}; \tilde{\boldsymbol{\Psi}}, \tilde{\boldsymbol{\Psi}})\mathcal{I}\,[\mathcal{C}; \mathbf{T}^*, \mathbf{T}^*] \\ &\le W_1^* \hat{\kappa}_M \left[T_{11}^2 + T_{22}^2 + T_{33}^2 + g^2\right].\end{aligned} \tag{3.55}$$

Therefore, we obtain

$$T_{12}^2 + T_{21}^2 + T_{23}^2 + T_{32}^2 + T_{31}^2 + T_{13}^2 \le \breve{\kappa}_M W_2^*, \tag{3.56}$$

$$T_{11}^2 + T_{22}^2 + T_{33}^2 + g^2 \le \hat{\kappa}_M W_1^*. \tag{3.57}$$

Thus, we conclude that

$$T_{ij}T_{ij} \le \breve{\kappa}_M W_2^* + \hat{\kappa}_M W_1^*. \tag{3.58}$$

By means of the Cauchy-Schwarz's inequality and the arithmetic-geometric mean inequalities and the relations (3.8), (3.35) and (3.58), we get

$$\left|\dot{u}_i T_{ji} n_j + \dot{\phi} h_j n_j\right| \le \frac{1}{2}\left(\varepsilon K + \frac{1}{\varepsilon\rho}\gamma_M^* W^*\right), \tag{3.59}$$

where ε is an arbitrary positive constant and γ_M^* is defined by the relation (3.51). If we put $\varepsilon = c_2$, with c_2 defined by (3.50), into (3.59), then the relations (3.33), (3.59) imply that

$$|\Pi(r,t)| \le \frac{c_2}{2}\int_0^t \int_{S_r} e^{-\sigma s}[K(s) + W^*(s)]da\,ds, \tag{3.60}$$

$$\left|\frac{\partial \Pi}{\partial t}(r,t)\right| \le \frac{c_2}{2}\int_{S_r} e^{-\sigma t}[K(t) + W^*(t)]da. \tag{3.61}$$

These relations imply the differential inequalities (3.48), (3.49) and thus, we obtain the desired result.

Following the procedure developed in SCALIA [11] and CHIRIŢĂ and CIARLETTA [17] and using Theorem 4, we can prove that

THEOREM 5. *(Spatial behaviour): Provided the hypotheses of Theorem 4 hold, then* $\Pi(r,t)$ *is a measure associated with the solution* $\mathbf{U} = \{u_i, \phi\}$ *of* $\mathcal{P}$. *Further, at each fixed* $t \in [0,T]$ *we have*

$$\Pi(r,t) = 0, \quad \textit{that is} \quad u_i = 0, \phi = 0 \quad \textit{for} \quad r \geq c_2 t, \tag{3.62}$$

$$0 \leq \Pi(r,t) \leq \Pi(0,t)\, e^{(-\sigma/c_2)r}, \quad \textit{for} \quad 0 \leq r \leq c_2 t. \tag{3.63}$$

THEOREM 6. *(Uniqueness): Provided the hypotheses of Theorem 4 hold, there exists at most one solution for the initial-boundary value problem* $\mathcal{P}$.

4. Spatial behaviour of the steady-state solutions

Throughout this section we will discuss the problem of spatial behaviour of the steady vibrations. For this purpose we stipulate that the region B from here on refers to the interior of a right cylinder with parallel plane ends. The rectangular Cartesian coordinate frame is supposed to be chosen in such a way that one end of the cylinder lies in the x_1Ox_2 plane and contains the origin. We suppose that the length of the cylinder is L and that D_{x_3} represents the bounded cross–section at distance x_3 from the x_1Ox_2 plane. The boundary ∂D of each cross–section is assumed to be a piecewise smooth simple closed curve. The Greek subscripts range over 1,2. In what follows we shall assume that the external body force and the extrinsic equilibrated body force are absent. Moreover, we assume the following boundary conditions for all $t \in [0,T]$

$$u_i = 0 \quad \text{on} \quad (\partial D \times [0,L]) \cup D_L, \tag{4.1}$$

$$\begin{gathered} \phi = 0 \quad \text{on} \quad \Gamma_1 \times [0,L], \quad h_\alpha n_\alpha = 0 \quad \text{on} \quad \Gamma_2 \times [0,L], \\ \phi = 0 \quad \text{or} \quad h_3 = 0 \quad \text{on} \quad D_L, \end{gathered} \tag{4.2}$$

$$\begin{gathered} u_i(x_1,x_2,0,t) = \tilde{v}_i(x_1,x_2)e^{i\omega t}, \qquad \phi(x_1,x_2,0,t) = \tilde{\psi}(x_1,x_2)e^{i\omega t} \\ \text{or} \quad h_3(x_1,x_2,0,t) = \tilde{h}(x_1,x_2)e^{i\omega t}, \end{gathered} \tag{4.3}$$

where $\tilde{v}_i$, $\tilde{\psi}$ and $\tilde{h}$ are prescribed functions and Γ_1 and Γ_2 are subcurves of ∂D so that $\bar{\Gamma}_1 \cup \Gamma_2 = \partial D$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$ and ω is a positive prescribed parameter. Other boundary conditions can be viewed but essential for our considerations is to conserve the boundary condition (4.1). In this section we discuss the problem of steady-state vibrations assuming that

$$u_i = \Re e[v_i(\mathbf{x};\omega)e^{i\omega t}], \qquad \phi = \Re e[\psi(\mathbf{x};\omega)e^{i\omega t}], \tag{4.4}$$

where $\Re e[f]$ represents the real part of f. Then the equations of motion (2.11) reduce to

$$\mu v_{j,rr} + (\lambda+\mu)v_{r,rj} + \beta\psi_{,j} + \rho\omega^2 v_j = 0, \tag{4.5}$$

$$\alpha\psi_{,rr} - \beta v_{r,r} - \xi\psi + \rho\chi\omega^2\psi = 0, \tag{4.6}$$

while the boundary conditions (4.1), (4.2) and (4.3) reduce to

$$v_i = 0 \quad \text{on} \quad (\partial D \times [0,L]) \cup D_L, \tag{4.7}$$

$$\begin{gathered} \psi = 0 \quad \text{on} \quad \varGamma_1 \times [0,L], \quad H_\alpha n_\alpha = 0 \quad \text{on} \quad \varGamma_2 \times [0,L], \\ \psi = 0 \quad \text{or} \quad H_3 = 0 \quad \text{on} \quad D_L, \end{gathered} \tag{4.8}$$

$$\begin{gathered} v_i(x_1,x_2,0) = \tilde{v}_i(x_1,x_2), \qquad \psi(x_1,x_2,0) = \tilde{\psi}(x_1,x_2) \\ \text{or} \quad H_3(x_1,x_2,0) = \tilde{h}(x_1,x_2), \end{gathered} \tag{4.9}$$

where

$$H_i = \alpha\psi_{,i}. \tag{4.10}$$

In what follows we will study the spatial behaviour of the amplitude (v_i,ψ) of the harmonic vibration described by the relation (4.4). To this end we note that (v_i,ψ) is the solution of the boundary value problem $\mathcal{P}_0$ defined by the equations (4.5) and (4.6) and by the boundary conditions (4.7) to (4.9).

4.1. First measure

Throughout this subsection we will assume that the hypotheses described by the relation (3.15) hold true. Then we associate with the solution (v_i,ψ) of the problem $\mathcal{P}_0$ the function $\mathcal{J}$ on $[0,L]$ defined by

$$\mathcal{J}(x_3) = \int\limits_{D_{x_3}} (s_{3i}\bar{v}_i + \bar{s}_{3i}v_i + H_3\bar{\psi} + \bar{H}_3\psi)da, \qquad x_3 \in [0,L], \tag{4.11}$$

where

$$s_{ji} = \mu v_{i,j} + (\lambda+\mu)v_{r,r}\delta_{ij} + \beta\psi\delta_{ij}, \tag{4.12}$$

and a superposed bar denotes the complex conjugate. In view of the equations (4.5) and (4.6) and by using the relations (4.11) and (4.12), we get

$$\begin{aligned} \mathcal{J}'(x_3) = &- \int\limits_{D_{x_3}} \left(\bar{v}_i s_{\rho i,\rho} + v_i \bar{s}_{\rho i,\rho} + \bar{\psi} H_{\rho,\rho} + \psi \bar{H}_{\rho,\rho}\right) da \\ &- 2\omega^2 \int\limits_{D_{x_3}} \left(\rho v_i \bar{v}_i + \rho\chi\psi\bar{\psi}\right) da + \int\limits_{D_{x_3}} \Big[s_{3i}\bar{v}_{i,3} + \bar{s}_{3i} v_{i,3} + H_3 \bar{\psi}_{,3} + \bar{H}_3 \psi_{,3} \\ &+ 2\xi\psi\bar{\psi} + \beta\left(\bar{\psi} v_{r,r} + \psi \bar{v}_{r,r}\right) \Big] da. \end{aligned} \tag{4.13}$$

Furthermore, by an integration by parts and by using the boundary conditions (4.7) and (4.8), we obtain

$$\begin{aligned} \mathcal{J}'(x_3) = &\int\limits_{D_{x_3}} \Big[s_{ji}\bar{v}_{i,j} + \bar{s}_{ji} v_{i,j} + H_j \bar{\psi}_{,j} + \bar{H}_j \psi_{,j} \\ &+ 2\xi\psi\bar{\psi} + \beta\left(\bar{\psi} v_{r,r} + \psi \bar{v}_{r,r}\right) \Big] da - 2\omega^2 \int\limits_{D_{x_3}} \left(\rho v_i \bar{v}_i + \rho\chi\psi\bar{\psi}\right) da. \end{aligned} \tag{4.14}$$

In view of the relations (4.10) and (4.12), we deduce

$$\mathcal{J}'(x_3) = 2 \int\limits_{D_{x_3}} \left[w - \omega^2 (\rho v_i \bar{v}_i + \rho\chi\psi\bar{\psi}) \right] da, \tag{4.15}$$

where

$$w = \mu v_{i,j}\bar{v}_{i,j} + (\lambda+\mu) v_{r,r}\bar{v}_{s,s} + \xi\psi\bar{\psi} + \beta\left(\bar{\psi} v_{r,r} + \psi\bar{v}_{r,r}\right) + \alpha\psi_{,j}\bar{\psi}_{,j}. \tag{4.16}$$

We write w in the form

$$w = w_1 + w_2 + w_3, \tag{4.17}$$

where

$$\begin{aligned} w_1 = &(\lambda + 2\mu)(v_{1,1}\bar{v}_{1,1} + v_{2,2}\bar{v}_{2,2} + v_{3,3}\bar{v}_{3,3}) \\ &+ (\lambda+\mu)(v_{1,1}\bar{v}_{2,2} + \bar{v}_{1,1} v_{2,2} + v_{2,2}\bar{v}_{3,3} + \bar{v}_{2,2} v_{3,3} + v_{3,3}\bar{v}_{1,1} + v_{1,1}\bar{v}_{3,3}) \\ &+ \xi\psi\bar{\psi} + \beta\left[\bar{\psi}\left(v_{1,1} + v_{2,2} + v_{3,3}\right) + \psi\left(\bar{v}_{1,1} + \bar{v}_{2,2} + \bar{v}_{3,3}\right)\right], \end{aligned} \tag{4.18}$$

$$w_2 = \mu(u_{1,2}\bar{u}_{1,2} + u_{2,1}\bar{u}_{2,1} + u_{2,3}\bar{u}_{2,3} + u_{3,2}\bar{u}_{3,2} + u_{3,1}\bar{u}_{3,1} + u_{1,3}\bar{u}_{1,3}), \tag{4.19}$$

$$w_3 = \alpha\psi_{,j}\bar{\psi}_{,j}. \tag{4.20}$$

In view of the hypothesis (3.15) and by using the relation (3.17), we deduce that

$$k_m(v_{1,1}\bar{v}_{1,1} + v_{2,2}\bar{v}_{2,2} + v_{3,3}\bar{v}_{3,3} + \psi\bar{\psi}) \leq w_1 \leq k_M(v_{1,1}\bar{v}_{1,1} + v_{2,2}\bar{v}_{2,2} + v_{3,3}\bar{v}_{3,3} + \psi\bar{\psi}), \tag{4.21}$$

and hence we obtain

$$k_m(v_{j,k}\bar{v}_{j,k} + \psi\bar{\psi}) + \alpha\psi_{,j}\bar{\psi}_{,j} \leq w \leq k_M(v_{j,k}\bar{v}_{j,k} + \psi\bar{\psi}) + \alpha\psi_{,j}\bar{\psi}_{,j}. \tag{4.22}$$

In view of the boundary conditions (4.7), we have

$$\lambda_1 \int\limits_{D_{x_3}} v_j\bar{v}_j da \leq \int\limits_{D_{x_3}} v_{j,\alpha}\bar{v}_{j,\alpha} da, \qquad \lambda_1 \int\limits_{D_{x_3}} \psi\bar{\psi} da \leq \int\limits_{D_{x_3}} \psi_{,\alpha}\bar{\psi}_{,\alpha} da, \tag{4.23}$$

where λ_1 is the first eigenvalue corresponding to the membrane problem for the section D_{x_3}.

Then, the relations (4.15), (4.22) and (4.23) imply

$$\mathcal{J}'(x_3) \geq 2 \int\limits_{D_{x_3}} \left\{ k_m \left[\left(1 - \frac{\rho\omega^2}{\lambda_1 k_m}\right) v_{j,k}\bar{v}_{j,k} + \left(1 - \frac{\rho\chi\omega^2}{k_m}\right) \psi\bar{\psi} \right] + \alpha\psi_{,j}\bar{\psi}_{,j} \right\} da. \tag{4.24}$$

Further, we shall assume that

$$\omega < \omega_1, \tag{4.25}$$

where

$$\omega_1^2 = \frac{k_m}{\rho} \min\left\{\lambda_1, \frac{1}{\chi}\right\}. \tag{4.26}$$

Then, we have

$$\mathcal{J}'(x_3) \geq 2k_m \left(1 - \frac{\omega^2}{\omega_1^2}\right) \int\limits_{D_{x_3}} \left(v_{j,k}\bar{v}_{j,k} + \psi\bar{\psi}\right) da + 2\alpha \int\limits_{D_{x_3}} \psi_{,j}\bar{\psi}_{,j} da \geq 0. \tag{4.27}$$

On the other hand, by means of the Schwarz inequality and by using the relations (3.17), (3.25), (4.10), (4.12) and (4.21), we get

$$
\begin{aligned}
(4.28)\qquad |\mathcal{J}(x_3)| \leq \int\limits_{D_{x_3}} \Big\{ & \varepsilon_1 s_{3\rho}\bar{s}_{3\rho} + \frac{1}{\varepsilon_1} v_\rho \bar{v}_\rho + \varepsilon_2 s_{33}\bar{s}_{33} + \frac{1}{\varepsilon_2} v_3\bar{v}_3 \\
& + \varepsilon_3 \alpha \psi_{,3}\bar{\psi}_{,3} + \frac{\alpha}{\varepsilon_3}\psi\bar{\psi} \Big\}\, da \leq \int\limits_{D_{x_3}} \Big\{ \varepsilon_1 \mu^2 v_{\rho,3}\bar{v}_{\rho,3} + \frac{1}{\lambda_1\varepsilon_1} v_{\rho,\alpha}\bar{v}_{\rho,\alpha} \\
& + \varepsilon_2 k_M^2 (v_{1,1}\bar{v}_{1,1} + v_{2,2}\bar{v}_{2,2} + v_{3,3}\bar{v}_{3,3} + \psi\bar{\psi}) + \frac{1}{\lambda_1\varepsilon_2} v_{3,\rho}\bar{v}_{3,\rho} \\
& + \varepsilon_3 \alpha \psi_{,3}\bar{\psi}_{,3} + \frac{\alpha}{\lambda_1\varepsilon_3}\psi_{,\rho}\bar{\psi}_{,\rho} \Big\}\, da, \qquad \forall \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0.
\end{aligned}
$$

Thus, by setting $\varepsilon_1 = \varepsilon_2 = \dfrac{1}{k_M\sqrt{\lambda_1}}$ and $\varepsilon_3 = \dfrac{1}{\sqrt{\lambda_1}}$, we obtain

$$
(4.29)\qquad |\mathcal{J}(x_3)| \leq \int\limits_{D_{x_3}} \left\{ m_1 v_{j,k}\bar{v}_{j,k} + m_2\psi\bar{\psi} + m_3\psi_{,j}\bar{\psi}_{,j} \right\} da,
$$

where

$$
(4.30)\qquad m_1 = \frac{2k_M}{\sqrt{\lambda_1}}, \qquad m_2 = \frac{k_M}{\sqrt{\lambda_1}}, \qquad m_3 = \frac{2\alpha}{\sqrt{\lambda_1}}.
$$

On the basis of the relations (4.27) and (4.29), we can formulate the following result.

Theorem 7. *Let* $\mathbf{V} = \{v_i, \psi\}$ *be a solution of the boundary value problem* $\mathcal{P}_0$. *Provided the hypothesis (3.15) holds true and the frequency* ω *is lower than the critical value* ω_1 *defined by (4.26), the corresponding cross-sectional integral* $\mathcal{J}(x_3)$ *satisfies the following first-order differential inequality*

$$
(4.31)\qquad m^2\,|\mathcal{J}(x_3)| \leq \mathcal{J}'(x_3),
$$

where

$$
(4.32)\qquad \frac{1}{m^2} = \max\left\{ \frac{m_1}{2k_m\left(1-\dfrac{\omega^2}{\omega_1^2}\right)}, \frac{m_2}{2k_m\left(1-\dfrac{\omega^2}{\omega_1^2}\right)}, \frac{m_3}{2\alpha} \right\}.
$$

We now proceed to prove how the first-order differential inequality (4.31) can describe the spatial behaviour of the amplitude $\mathbf{V}$ of the considered vibration. To this end, we first suppose that $\mathcal{J}(0) > 0$. Since $\mathcal{J}'(x_3) \geq 0$, it follows that

$\mathcal{J}(x_3) > 0$ for all $x_3 \geq 0$, so that it results that $\mathcal{J}(L) > 0$. Then the differential inequality (4.31) becomes

$$\mathcal{J}'(x_3) \geq m^2 \mathcal{J}(x_3), \tag{4.33}$$

which, by an integration, gives

$$\mathcal{J}(0)e^{m^2 x_3} \leq \mathcal{J}(x_3) \leq \mathcal{J}(L)e^{-m^2(L-x_3)}, \qquad x_3 \in [0, L]. \tag{4.34}$$

Let us now consider the case $\mathcal{J}(0) = 0$. Then either $\mathcal{J}(L) = 0$ or $\mathcal{J}(L) > 0$. When $\mathcal{J}(L) = 0$ then it results that $\mathcal{J}(x_3) = \mathcal{J}'(x_3) = 0$ for $x_3 \in [0, L]$ and therefore, by the relations (4.1) and (4.27), we deduce that $v_i = 0$, $\psi = 0$ in B. When $\mathcal{J}(L) > 0$ it follows that there is $x_3^* = \inf\{x_3 \in [0, L]$ with $\mathcal{J}(x_3) > 0\} > 0$ and then the relation (4.31) implies

$$\begin{gathered} \mathcal{J}(x_3) = \mathcal{J}'(x_3) = 0 \quad \text{for} \quad x_3 \in (0, x_3^*), \\ \mathcal{J}(x_3^*)e^{m^2(x_3 - x_3^*)} \leq \mathcal{J}(x_3) \leq \mathcal{J}(L)e^{-m^2(L-x_3)}, \qquad x_3 \in [x_3^*, L]. \end{gathered} \tag{4.35}$$

Finally, we consider the case $\mathcal{J}(0) < 0$. Then we can have $\mathcal{J}(L) < 0$ or $\mathcal{J}(L) \geq 0$. If $\mathcal{J}(L) < 0$ then $\mathcal{J}(x_3) < 0$ for all $x_3 \in [0, L]$ and therefore, the relation (4.31) gives

$$-\mathcal{J}(L)e^{m^2(L-x_3)} \leq -\mathcal{J}(x_3) \leq -\mathcal{J}(0)e^{-m^2 x_3}, \qquad x_3 \in [0, L]. \tag{4.36}$$

For the case $\mathcal{J}(L) \geq 0$ we obtain a combination of situations discussed above by the relations (4.35) and (4.36).

We have to note that for a semi-infinite cylinder the cross-sectional measure has the following behaviour: for the case when $\mathcal{J}(0) < 0$, we obtain

$$-\mathcal{J}(x_3) \leq -\mathcal{J}(0)e^{-m^2 x_3}, \qquad x_3 \geq 0, \tag{4.37}$$

while for $\mathcal{J}(0) \geq 0$, we obtain

$$\mathcal{J}(x_3) \geq \mathcal{J}(0)e^{m^2 x_3}, \qquad x_3 \geq 0, \tag{4.38}$$

or

$$\mathcal{J}(x_3) \geq \mathcal{J}(x_3^*)e^{m^2(x_3 - x_3^*)}, \qquad x_3 \geq x_3^*. \tag{4.39}$$

4.2. Second measure

In the remainder of this paper we shall assume that the hypotheses described by the relation (3.41) hold true. Then we introduce the following measure:

$$\mathcal{I}(x_3) = \int_{D_{x_3}} (m_{3i}\bar{v}_i + \bar{m}_{3i}v_i + H_3\bar{\psi} + \bar{H}_3\psi)da, \qquad x_3 \in [0, L], \tag{4.40}$$

where

$$m_{jk} = \mu v_{k,j} + (\lambda + \mu)v_{j,k} + \beta\psi\delta_{jk}. \tag{4.41}$$

Further, we get

$$\mathcal{I}'(x_3) = 2\int_{D_{x_3}} \left[w^* - \omega^2(\rho v_i\bar{v}_i + \rho\chi\psi\bar{\psi})\right] da, \tag{4.42}$$

where

$$w^* = \mu v_{i,j}\bar{v}_{i,j} + (\lambda + \mu)v_{i,j}\bar{v}_{j,i} + \xi\psi\bar{\psi} + \beta\left(\bar{\psi}v_{r,r} + \psi\bar{v}_{r,r}\right) + \alpha\psi_{,j}\bar{\psi}_{,j}. \tag{4.43}$$

Thus, if we set $k_m^* = \min\left(\hat{k}_m,\ \breve{k}_m\right),\ k_M^* = \min\left(\hat{k}_M,\ \breve{k}_M\right)$ and define $\hat{\omega}_1$ by

$$\hat{\omega}_1^2 = \frac{k_m^*}{\rho}\min\left\{\lambda_1, \frac{1}{\chi}\right\}, \tag{4.44}$$

and assume that

$$\omega < \hat{\omega}_1, \tag{4.45}$$

then, by following the procedure used in the Subsec. 4.1, we get

$$\hat{m}^2\left|\mathcal{I}(x_3)\right| \leq \mathcal{I}'(x_3), \tag{4.46}$$

where

$$\frac{1}{\hat{m}^2} = \max\left\{\frac{\hat{m}_1}{2k_m^*\left(1 - \frac{\omega^2}{\hat{\omega}^2}\right)},\ \frac{\hat{m}_2}{2k_m^*\left(1 - \frac{\omega^2}{\hat{\omega}^2}\right)},\ \frac{\hat{m}_3}{2\alpha}\right\}, \tag{4.47}$$

$$\hat{m}_1 = \frac{1}{\sqrt{\lambda_1}}\left(\hat{k}_M + \breve{k}_M\right), \quad \hat{m}_2 = \frac{\hat{k}_M}{\sqrt{\lambda_1}}, \quad \hat{m}_3 = \frac{\alpha}{\sqrt{\lambda_1}}. \tag{4.48}$$

Thus, the first-order differential inequality (4.46) leads to spatial estimates of type described by the relations (4.34), (4.36), (4.38) or (4.39).

5. Concluding remarks

In the present paper we have introduced the new measures (3.5), (3.33) and (4.11), (4.40) in order to study the spatial behaviour of the transient and steady-state solutions, respectively. The measures defined by the relations (3.5) and (4.11) are suitable for the class of porous elastic materials described by the relation (3.15), while the measures (3.33) and (4.40) are useful in the class described by the relation (3.41). These classes of porous elastic materials are not coincident with the class defined by (2.5) and treated in [11]. The method presented here is believed to be used successfully for many novel foam structures with negative Poisson's ratio, because the relations (3.15) and (3.41) can include such materials, not included in the class defined by (2.5).

Our results expressed by the relations $(3.30)_1$ and (3.62) prove the idea of influence domain. The estimates $(3.30)_2$, (3.63) and (4.37) express the results of Saint–Venant type, while the estimates (4.38) and (4.39) give an alternative of the Phragmèn–Lindelöf type.

On the other hand, we have to stress that we can combine our estimates (3.30), (3.63), (4.37) with each other and also with those predicted in [11] in order to obtain complete information about the spatial behaviour of the transient and steady-state solutions.

In any case, our method allows us to study the spatial behaviour for a large class of special materials (useful in biomechanics) and characterized by essentially negative (or positive) Poisson's ratio. In particular, we outline the auxetic materials and the polymer foams.

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