Arch. Mech., **57**, 5, pp. 405–416, Warszawa 2005

The unsteady Couette flow of a second grade fluid in a layer of porous medium

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In this work, the two Couette flows of a second grade fluid are discussed in a porous layer when (*i*) bottom plate moves suddenly (*ii*) bottom plate oscillates. Laplace transform method is used to determine the analytic solutions. Expressions for the velocities, volume fluxes and frictional forces are constructed.

Key words: second grade fluid, unsteady flow, exact analytic solutions.

1. Introduction

Recently, the interest in flows of non-Newtonian fluids through a porous medium has grown considerably because of their applications in engineering. Examples of these applications are filtration processes, biomechanics, packed bed reactors, insulation system, ceramic processing, enhanced oil recovery, chromatography and many others [1, 2] .

It is now generally recognized that in the industry, non-Newtonian fluids are more appropriate than Newtonian fluids. The equation that describes the Newtonian fluid flow is the Navier–Stokes equation. The exact solutions for Navier–Stokes equation are rare. This class of exact solutions further narrowed down for non-Newtonian fluids. Amongst the many non-Newtonian models, the fluids of differential type [3] have acquired a special status. In the case of differential type fluids, the equations of motion are one order higher than the Navier–Stokes equation and thus the adherence boundary condition is insufficient to determine the solution completely (see refs. [4–6] for a detailed discussion of the relevant issues). One particular subclass of differential-type fluids for which one can reasonably hope to obtain the exact solutions is the second grade fluid. The Cauchy stress tensor $\mathbf{T}$ for second grade fluid is

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1.1}$$

in which $-p\mathbf{I}$ is the spherical stress due to the constraint of incompressibility, μ is the dynamic viscosity, α_1 and α_2 are material moduli, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first two Rivlin–Ericksen tensors [7] defined by

$$\mathbf{A}_1 = (\mathrm{grad}\mathbf{V}) + (\mathrm{grad}\mathbf{V})^*, \tag{1.2}$$

$$\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1(\mathrm{grad}\mathbf{V}) + (\mathrm{grad}\mathbf{V})^*\mathbf{A}_1, \tag{1.3}$$

where $\mathbf{V}$ is the velocity, grad is the gradient operator, $(*)$ is the matrix transpose and d/dt denotes the material time derivative. The critical review on thermodynamic behavior of second grade fluids is given in the reference [8].

In a porous half-space, Jordan and Puri [9] made a study of Stokes' first problem in a second grade fluid. The unsteady flow of Stokes' first problem examines the diffusion of vorticity in a porous half-space filled with a viscous incompressible fluid that is set to motion when an infinite flat plate moves with a constant velocity in its own plane for $t > 0$ (t is the time). For unidirectional flow of second grade fluids, the solution obtained in reference [9] does not satisfy the initial condition of the model. More recently, Tan and Masuoka [10] studied the Stokes' first problem for a second grade fluid in a porous half-space with heated boundary. The motion of a viscous fluid caused by the sinusoidal oscillation of flat plate is termed as Stokes' second problem [11]. Exact solutions for unsteady Couette flow of a dipolar fluid have been given by Jordan and Puri [12]. In another paper, Jordan and Puri [13] considered the flow due to accelerated plate in a dipolar fluid. The study on the flow of a viscous fluid over an oscillating plate is not only of fundamental theoretical interest but it also occurs in many applied problems such as acoustic streaming around an oscillating body, or an unsteady boundary layer with fluctuation [14].

In this paper we extend the work of Jordan and Puri [9] and our purpose here is two-fold. Firstly, to discuss the flow of a second grade fluid in a porous layer between two plates when one of the plates is suddenly moved and other being at rest. Secondly, to examine the flow when one of the plates is oscillating and the other is at rest. The space between the two plates is porous with constant permeability and porosity. Modified Darcy's law is used to incorporate the effects of the pores on the velocity field. When the fluid motion is set up from rest, the velocity field contains transients determined by the initial conditions and these transients gradually disappear in time. For large times, the transient solution behaves like a steady solution. In order to obtain the unsteady solution, a transient solution must be added to the steady-state solution.

2. Mathematical formulation

Let us consider the flow of a second grade fluid between two plates at $y = 0$ and $y = h$ when the positive y-axis of a Cartesian coordinate system is in the upward direction. The second grade fluid fills the porous layer $0 < y < h$ (see Fig. 1). The x-coordinate is parallel the direction of the flow. Initially, both plates and fluid are at rest. At time $t = 0^+$, the lower plate suddenly starts to slide (or oscillate) in its own plane. The considered plates are rigid and infinite. Under these assumptions, the flow velocity at a given point in the porous layer depends only on its y-coordinate and time t and thus the velocity is

$$\mathbf{V} = (u(y,t), 0, 0) \tag{2.1}$$

in which u is the x-component of the velocity.

Since the flow is unsteady, the interaction terms depend upon the drag and virtual mass effect. The relation between the pressure drop and velocity for a second grade fluid in porous media is

$$\boldsymbol{\nabla} p = -\frac{\phi}{K}\left(\mu + \alpha_1 \frac{\partial}{\partial t}\right)\mathbf{V}, \tag{2.2}$$

where K (> 0) and ϕ $(0 < \phi < 1)$ are the (constant) permeability and porosity, respectively.

Note that Eq. (2.2) ignores the boundary effects on the flow and cannot be directly used to analyze flow problems in a porous space. Thus modified Darcy's law based on a local volume averaging technique [15–17] will be considered in a porous layer. Under consideration of the balance of forces acting on a volume element of fluid, the local volume average balance of linear momentum is given by [15, 17]

$$\rho \frac{d\mathbf{V}}{dt} = \operatorname{div}\mathbf{T} + \mathbf{r}, \tag{2.3}$$

in which ρ is the fluid density and $\mathbf{r}$ is the Darcy resistance for a second grade fluid in the porous space. Due to the volume averaging process, some information is lost, thus requiring supplementary empirical relation for the Darcy resistance [15] to be known as a measure of the resistance to the flow in the bulk of the porous space and $\mathbf{r}$ is also a measure of the flow resistance offered by the solid matrix; then $\mathbf{r}$ satisfies the following equation [15]:

$$\mathbf{r} = -\frac{\phi}{K}\left(\mu + \alpha_1 \frac{\partial}{\partial t}\right)\mathbf{V}. \tag{2.4}$$

Using (2.4) into (2.3) we have

$$\rho \frac{d\mathbf{V}}{dt} = \operatorname{div}\mathbf{T} - \frac{\phi}{K}\left(\mu + \alpha_1 \frac{\partial}{\partial t}\right)\mathbf{V}. \tag{2.5}$$

Substituting (1.1) into above equation, one obtains

$$\rho\frac{d\mathbf{V}}{dt} = -\nabla p + \mathrm{div}\mathbf{S} - \frac{\phi}{K}\left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\mathbf{V}, \tag{2.6}$$

where $\mathbf{S}$ is the extra stress tensor which for second grade fluid is

$$\mathbf{S} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2. \tag{2.7}$$

It is noted that if the terms $d\mathbf{V}/dt$ and div$\mathbf{S}$ are ignored then (2.6) reduces to (2.2).

Now from (2.1), (2.6) and (2.7) we can write

$$\frac{\partial u}{\partial t} - \nu\frac{\partial^2 u}{\partial y^2} - d^2\frac{\partial^3 u}{\partial y^2 \partial t} + \frac{\phi}{K}\left(\nu + d^2\frac{\partial}{\partial t}\right)u = 0, \tag{2.8}$$

in which $\nu = \mu/\rho$ is the kinematic viscosity, $\rho d^2 = \alpha_1 (d\ (\geq 0)$, the elastic coefficient, has the unit of length [9]) and pressure gradient in the x-direction is neglected which is reasonable when there is no applied pressure gradient. We are interested here in two initial-boundary value problems:

(i) Couette flow with sudden motion of bottom plate,

(ii) Couette flow with oscillation of bottom plate.

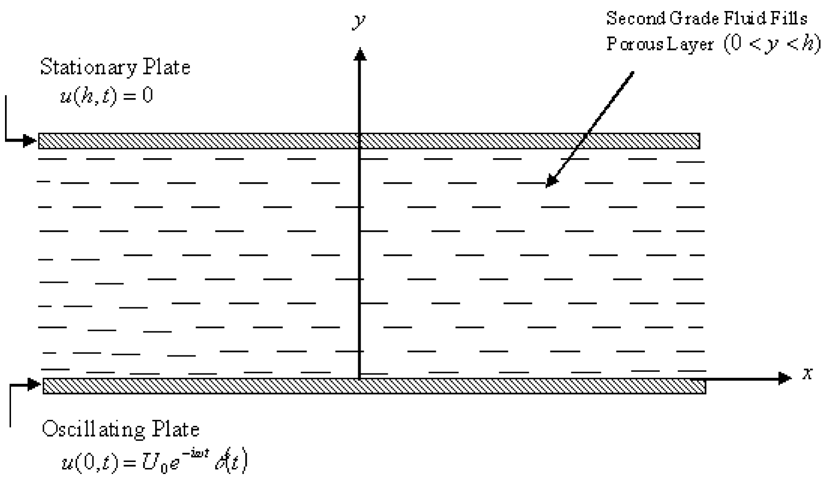

FIG. 1. Physical model under consideration.

3. First problem

This section deals with the solution of a second grade fluid in a porous layer in absence of the pressure gradient. The flow is induced due to motion of the lower plate i.e., for $t > 0$, the plate at $y = 0$ starts to slide in its own plane with a constant speed U_0, i.e. the velocity of the plate is given by $(U_0, 0, 0)$. The plate at $y = h$ is kept fixed. Under this situation, the boundary and initial conditions are given by

$$\begin{aligned} &u(0, t) = U_0\theta(t), \qquad u(h, t) = 0, \\ &u(y, 0) = 0 \qquad\qquad (y > 0), \end{aligned} \tag{3.1}$$

where $\theta(\cdot)$ denotes the Heaviside unit step function.

Defining the dimensionless quantities

$$u' = \frac{u}{U_0}, \quad y' = \frac{y}{h}, \quad t' = \frac{\nu t}{h^2}, \quad \omega' = \frac{h^2\omega}{\nu}, \quad l = \frac{d}{h}, \quad \beta = h\sqrt{\phi/K}, \tag{3.2}$$

the governing problem becomes

$$\left(1 + \beta^2 l^2\right)\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial y^2} - l^2\frac{\partial^3 u}{\partial y^2 \partial t} + \beta^2 u = 0, \tag{3.3}$$

$$\begin{aligned} &u(0, t) = \theta(t), \qquad (1, t) = 0, \\ &u(y, 0) = 0 \qquad\quad (y > 0), \end{aligned} \tag{3.4}$$

where the primes have been suppressed for simplicity.

For the solution of (3.3) subject to (3.4), we define

$$\overline{u}(y, s) = L[u(y, t)] = \int_0^\infty e^{-st} u(y, t)\, dt \tag{3.5}$$

as the Laplace transform of $u(y, t)$ (where s is a Laplace transform parameter).

Taking Laplace transform of (3.3) and (3.4), we arrive at

$$\frac{d^2\overline{u}}{dy^2} - \left(\frac{\beta^2 + s\left(1 + \beta^2 l^2\right)}{1 + sl^2}\right)\overline{u} = 0, \tag{3.6}$$

$$\overline{u}(0, s) = \frac{1}{s}, \qquad \overline{u}(1, s) = 0. \tag{3.7}$$

Solving the above problem we have

$$\overline{u}(y, s) = \frac{\sinh q(1 - y)}{s \sinh q}, \tag{3.8}$$

where

$$q=\left[\frac{\beta^{2}+s\left(1+\beta^{2}l^{2}\right)}{1+sl^{2}}\right]^{1/2}. \tag{3.9}$$

Taking inverse Laplace transform we obtain

$$u\left(y,\,t\right)=\theta\left(t\right)\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\frac{\sinh q\left(1-y\right)e^{st}}{s\sinh q}ds. \tag{3.10}$$

In order to obtain the solution, we have to solve the integral in (3.10). For that we use the complex variable theory. It is seen that $s=0$ is a simple pole. Therefore, the residue at $s=0$ is

$$\operatorname{Res}\left(0\right)=\frac{\sinh\beta\left(1-y\right)}{\sinh\beta}. \tag{3.11}$$

The other singular points are the zeros of

$$\sinh q=0. \tag{3.12}$$

Setting $q=i\lambda$, we find

$$\sin\lambda=0. \tag{3.13}$$

If $\lambda_n=n\pi$, $n=1,2,3,...,\infty$ are the zeros of (3.13), then

$$s_n=-\left[\frac{\beta^{2}+n^{2}\pi^{2}}{1+\left(\beta^{2}+n^{2}\pi^{2}\right)l^{2}}\right],\qquad n=1,2,3,...,\infty \tag{3.14}$$

are the poles. The residue at all these poles is obtained as

$$\operatorname{Res}\left(s_n\right)=\frac{2\left(-1\right)^{n}n\pi e^{s_n t}}{\left(\beta^{2}+n^{2}\pi^{2}\right)\left[1+\left(\beta^{2}+n^{2}\pi^{2}\right)l^{2}\right]}\sin n\pi\left(1-y\right). \tag{3.15}$$

Adding Res(0) and Res(s_n) , a complete solution is given by

$$\begin{aligned}u\left(y,t\right)=\theta\left(t\right)&\left[\frac{\sinh\beta\left(1-y\right)}{\sinh\beta}\right.\\&\left.+2\pi\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n}ne^{s_n t}}{\left(\beta^{2}+n^{2}\pi^{2}\right)\left[1+\left(\beta^{2}+n^{2}\pi^{2}\right)l^{2}\right]}\sin n\pi\left(1-y\right)\right].\end{aligned} \tag{3.16}$$

The steady-state solution is of the following form:

$$u = \left(\frac{\sinh \beta (1-y)}{\sinh \beta} \right), \tag{3.17}$$

which holds for large t.

The volume flux Q across a plane normal to the flow is

$$Q = \int_0^1 u dy \tag{3.18}$$

and inserting (3.16), one finds that

$$Q = \theta(t) \left[\frac{\cosh \beta - 1}{\beta \sinh \beta} + 2 \sum_{n=1}^{\infty} \frac{e^{s_n t}}{(\beta^2 + n^2\pi^2) [1 + (\beta^2 + n^2\pi^2) l^2]} \{(-1)^n - 1\} \right]. \tag{3.19}$$

Since the fluid is set into motion through the action of the stress at the plate, the calculation of the stress field is in need. The dimensionless stress can be represented by

$$\tau = \frac{\partial u}{\partial y} + l^2 \frac{\partial^2 u}{\partial y \partial t}. \tag{3.20}$$

On the moving plate at $y = 0$, we have

$$\tau_0 = \theta(t) \left[-\beta \coth \beta - 2\pi^2 \sum_{n=1}^{\infty} \frac{n^2 e^{s_n t}}{(\beta^2 + n^2\pi^2) [1 + (\beta^2 + n^2\pi^2) l^2]^2} \right] \tag{3.21}$$

and on the stationary plate at $y = 1$ we can write

$$\tau_1 = \theta(t) \left[\frac{-\beta}{\sinh \beta} - 2\pi^2 \sum_{n=1}^{\infty} \frac{(-1)^n n^2 e^{s_n t}}{(\beta^2 + n^2\pi^2) [1 + (\beta^2 + n^2\pi^2) l^2]^2} \right]. \tag{3.22}$$

4. Second problem

In this section, we discuss the flow of a second grade fluid in a porous layer $0 < y < h$ when no pressure gradient is applied. The plate at $y = 0$ is oscillating in its own plane with frequency ω. The oscillating plate at $y = 0$ is responsible for generating the motion. The plate at $y = h$ is at rest. The corresponding boundary and initial conditions are

$$\begin{aligned} u(0, t) &= U_0 e^{i\omega t}\theta(t), & u(h, t) &= 0, \\ u(y, 0) &= 0 & (y &> 0 \end{aligned} \tag{4.1}$$

After suppressing the primes, the dimensionless conditions are of the following form:

$$\begin{aligned} u(0, t) &= e^{i\omega t}\theta(t), & u(1, t) &= 0, \\ u(y, 0) &= 0 & (y &> 0). \end{aligned} \tag{4.2}$$

We note that the mathematical problem is the same as in Sec. 3 except that the boundary conditions (4.1) replace the conditions (3.1). Therefore, employing the same procedure as in Sec. 3, we obtain

$$u(y, t) = \theta(t)\left[\frac{\sinh m(1-y)\, e^{i\omega t}}{\sinh m} + 2\pi\sum_{n=1}^{\infty}\frac{(-1)^n n e^{s_n t}}{D}\sin n\pi(1-y)\right], \tag{4.3}$$

where

$$m = \left[\frac{\beta^2\left(1+\omega^2 l^4\right)+\omega^2 l^2 + i\omega}{1+\omega^2 l^4}\right]^{1/2},$$

$$D = \left[1+\left(\beta^2+n^2\pi^2\right)l^2\right]\left[\left(\beta^2+n^2\pi^2\right)\left(1+i\omega l^2\right)+i\omega\right].$$

The steady-state solution is of the following form

$$u = \left(\frac{\sinh m(1-y)\, e^{i\omega t}}{\sinh m}\right). \tag{4.4}$$

The volume flux Q across a plane normal to the flow is

$$Q = \theta(t)\left[\frac{(\cosh m - 1)\, e^{i\omega t}}{m\sinh m} + 2\sum_{n=1}^{\infty}\frac{e^{s_n t}}{D}\left\{(-1)^n - 1\right\}\right]. \tag{4.5}$$

The shear stress on the oscillating plate at $y = 0$ is

$$\tau_0 = \theta(t)\left[-m\coth m\left(1+i\omega l^2\right)e^{i\omega t} - 2\pi^2\sum_{n=1}^{\infty}\frac{n^2 e^{s_n t}}{D\left[1+\left(\beta^2+n^2\pi^2\right)l^2\right]}\right] \tag{4.6}$$

and on the stationary plate at $y = 1$ is

$$\tau_1 = \theta(t)\left[\frac{-m}{\sinh m}\left(1+i\omega l^2\right)e^{i\omega t} - 2\pi^2\sum_{n=1}^{\infty}\frac{(-1)^n n^2 e^{s_n t}}{D\left[1+\left(\beta^2+n^2\pi^2\right)l^2\right]}\right]. \tag{4.7}$$

5. Discussion of results

In this paper, we here considered the modified Stokes' first and second problems. The second grade fluid is between the two plates and there is a porous layer between the plates. A procedure of Laplace transform has been used to obtain the exact solutions of the problems. The solutions for the two problems are obtained in such a way that at time $t = 0$ these represent unsteady flows and for $t \to \infty$ these correspond to steady state.

The effects of various parameters such as l, β and t on the velocity distributions in the two cases have been studied and the results have been presented through several graphs. For the graphical results, we have considered 50 terms of the infinite series and used the standard package Mathematica 4.0. In Figs. 2 to 4, panel (a) is sketched for the first problem while panel (b) for the second problem when oscillation is of the type $\cos\omega t$.

a) $\beta = 2, \quad l = 0.1$ b) $\beta = 2, \quad l = 0.1, \quad \omega = 0.5$

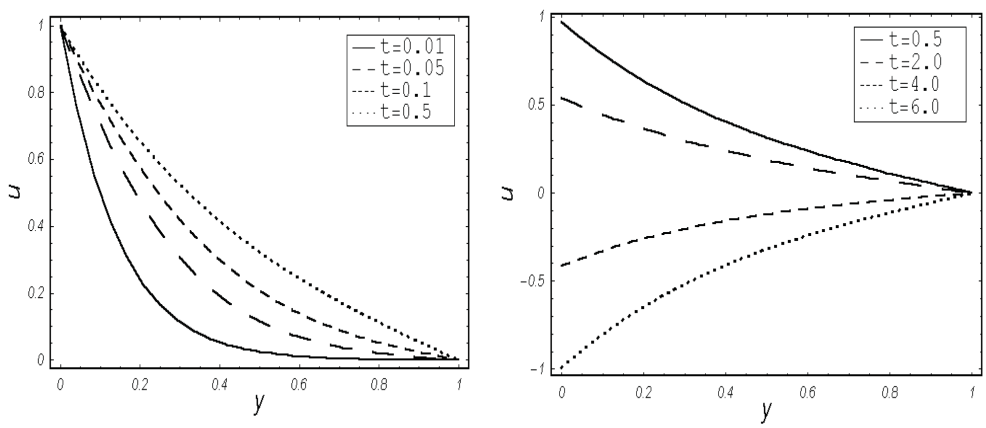

FIG. 2. Profiles of the dimensionless velocity $u\,(y)$ for various values of time t (panel (a): the first problem and panel (b): the second problem).

In order to study the effect of time on the velocity distribution, we have plotted u against y in Figs. 2a and 2b for fixed values of l, β and ω. From Fig. 2a, we see that with the increase in t, the velocity approaches a steady-state. It is worth mentioning here that in the first problem, the steady-state of the velocity distribution is obtained at $t = 0.5$. For the second problem, the transient part velocity will decay after a certain time. The long-time solution will be steady state periodic velocity profile. It is found that velocity (4.3) becomes the steady-state periodic velocity at $t = 6$. Thus, we conclude that steady-state in the first problem is achieved much earlier when compared with the second problem. Figures 3a and 3b are prepared to bring out the effects of β on the velocity distributions. From these figures, it is clear that the velocity profiles decrease with an increase in β. Figures 4a and 4b illustrate the effects of l on velocity distributions. It is interesting to note from these figures that u increases with an increase in l for fixed t, β and ω.

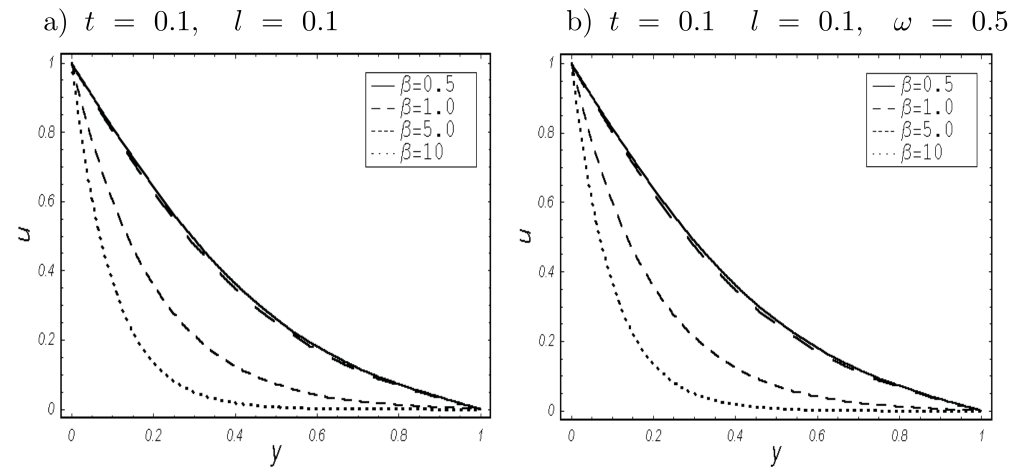

FIG. 3. Profiles of the dimensionless velocity $u\,(y)$ for various values of β (panel (a): the first problem and panel (b): the second problem).

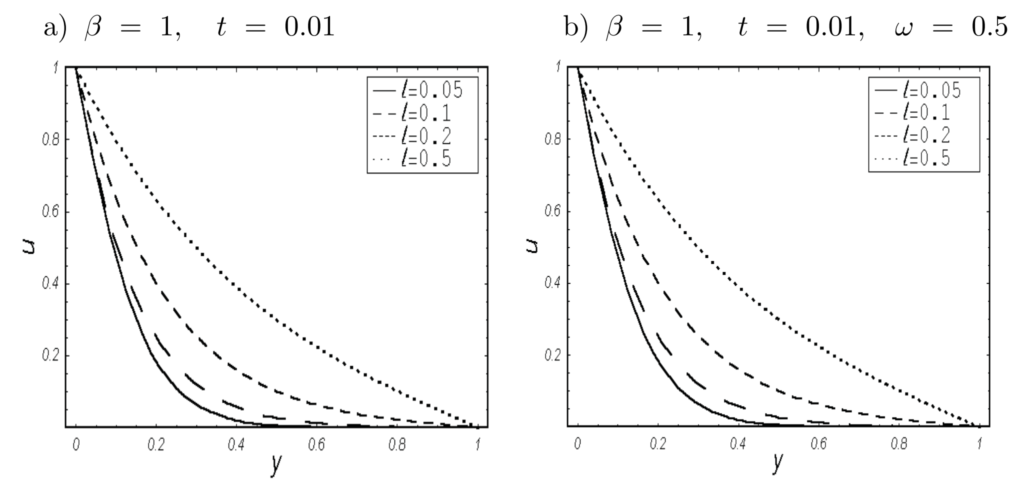

FIG. 4. Profiles of the dimensionless velocity $u\,(y)$ for various values of elastic co-efficient of second grade l (panel (a): first problem and panel (b): second problem).

In the special case, when elastic coefficient of the second grade fluid tends to zero, our solutions reduce to those corresponding to a Navier–Stokes fluid. The results for impermeable case can also be obtained as a limiting case for $K \to \infty$

or $\phi = 0$. It is also observed that steady-state solution in the first problem is the same in all types of fluid. In the second problem, periodic steady state solution is dependent upon the elastic coefficient. The transient solutions in both problems depend upon the nature of the fluids.

Acknowledgments

We are thankful to the referee for his valuable suggestions to improve the quality of the paper. The present work is supported by Quaid-i-Azam University Research Fund (URF scheme).

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Received August 23, 2004; revised version March 27, 2005.