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Integral representations at the boundary for Stokes flow and related symmetric Galerkin formulation

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A symmetric Galerkin boundary element formulation is given for the first time for two-dimensional, steady and incompressible flow. The formulation requires the derivation of certain integral representations (whose importance extends beyond the present application) for velocity gradient and pressure at the flow boundary; these turn out to be coupled at angular points of the contour profile.

1. Introduction

Integral equations and related numerical techniques are classical in linear fluid mechanics (Lamb, [9]). Among these numerical techniques, the boundary element method permits a successful treatment of flow incompressibility constraint and has been therefore thoroughly employed (Youngren and Acrivos [21], Wu and Wahbah [20] and, more recently, Pozrikidis [14, 16, 19]). However, direct boundary element methods involve an unsymmetrical coefficient matrix of the final solving algebraic system. This is certainly a drawback of the method, becoming particularly evident when boundary elements are coupled to finite elements. Following an approach known in solid mechanics, symmetry is recovered in the so-called symmetric Galerkin boundary element method (see Bonnet *et al.* [3] and references quoted therein), a technique apparently not developed in hydrodynamics. This development for a two-dimensional Stokes flow is the focus of the present article. To this purpose, integral representations of pressure and stress tensor at the flow boundary are derived, representing generalizations of formulations valid in interior points given by Ladyzhenskaya [8]. In particular, integral representations at corner points of the flow boundary are obtained, providing explicit expressions for the so-called "free term tensors". Interestingly, the integral representations for pressure and velocity gradient turn out to be coupled at corner points and – in the hypersingular form – also involve terms depending on the boundary curvatures at the corner. Analysis of corner points is believed to be relevant in relation to a number of problems in Stokes flow: for instance, flow past non-smooth rigid particles (PAKDEL AND KIM [12]) or cusp formation at fluid interfaces (POZRIKIDIS [17, 18]). In the special case of smooth boundary (the so-called "Lyapunov surfaces"), the equations de-couple and the integral formulations obtained by POZRIKIDIS [19] and by LIRON and BARTA [10] are recovered, respectively, when the boundary conditions model a gas bubble in a viscous liquid and when the boundary conditions pertain to flow past a rigid particle of arbitrary shape. The obtained integral equations are finally employed to establish, apparently for the first time, a symmetric Galerkin formulation for the Stokes flow. A few examples concluding the paper show possible advantages of the proposed method.

2. Basic equations

Two-dimensional viscous flow at small Reynold's number is considered, so that unsteady and inertial forces are negligible. For simplicity, body forces are also not included and reference is made to a two-dimensional domain Ω of boundary $\partial\Omega$. This boundary may correspond to a particle or a bubble of arbitrary shape in an infinite flow or to a domain, such as for instance a rectangular cavity, confining the flow. The boundary $\partial\Omega$ is divided into two non-overlapping portions $\partial\Omega_u$ and $\partial\Omega_f$ where velocities and tractions are prescribed; for simplicity, slip on fluid-solid interfaces or interfaces with involved mechanical properties (POZRIKIDIS [15] are not considered.

Denoting by $\mathbf{u}$ and p the velocity and the pressure in the fluid, the linear equations for Stokes flow are

$$\operatorname{div}\mathbf{u} = 0, \qquad -\nabla p + \mu\Delta\mathbf{u} = \mathbf{0}, \tag{2.1}$$

and at a point in the fluid, the stress tensor is given by

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu(\nabla\mathbf{u})_{\mathrm{sym}}, \tag{2.2}$$

where $(\cdot)_{\mathrm{sym}}$ denotes the symmetric part of a tensor.

With reference to two orthogonal coordinate axes singled out by the unit vectors $\mathbf{e}^g$, the two-dimensional free-space Green's function set $\{\mathbf{u}^g, p^g\}$, collecting the Stokeslet $\mathbf{u}^g$ and the associated pressure p^g (with the index g taking the values 1 and 2), is

$$\mathbf{u}^g = -\frac{1}{4\pi\mu}\left(\mathbf{e}^g \ln r - \frac{r_g}{r^2}\mathbf{r}\right), \qquad p^g = \frac{1}{2\pi}\frac{r_g}{r^2} \tag{2.3}$$

where $\mathbf{r} = \mathbf{x} - \mathbf{y}$ and $r = |\mathbf{r}|$. The Green's function set (2.3) satisfies the equations

$$\operatorname{div}\mathbf{u}^g = 0, \quad \mu\Delta\mathbf{u}^g(\mathbf{x},\mathbf{y}) - \nabla p^g(\mathbf{x},\mathbf{y}) + \delta(\mathbf{x}-\mathbf{y})\mathbf{e}^g = \mathbf{0}, \tag{2.4}$$

where $\delta(\mathbf{x}-\mathbf{y})$ is the Dirac delta function, $e_i^g = \delta_{ig}$ (δ_{ig} is the Kronecker delta) and all differentiations are carried out with respect to the variable $\mathbf{x}$, while $\mathbf{y}$ denotes the point where the concentrated force is applied. Note that – unless otherwise stated – differentiations are to be understood always in this sense throughout this paper. According to Eq. (2.2), the stress tensor $\boldsymbol{\sigma}^g$ associated to the fundamental solution (2.3) is given by:

$$\boldsymbol{\sigma}^g = -\frac{1}{\pi}\frac{r_g}{r^4}\mathbf{r}\otimes\mathbf{r}, \tag{2.5}$$

where we use the standard notation $(\mathbf{a}\otimes\mathbf{b})_{ij} = a_i b_j$, for every vector $\mathbf{a}$ and $\mathbf{b}$.

The boundary integral equations for velocity and pressure are (LADYZHENSKAYA [8])[1)]

$$u_g(\mathbf{y}) = \alpha u_g^\infty(\mathbf{y}) - \int\limits_{\partial\Omega} \mathbf{u}^g(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\sigma}\mathbf{n}\,dl_x + \int\limits_{\partial\Omega} \mathbf{u}\cdot\boldsymbol{\sigma}^g(\mathbf{x},\mathbf{y})\mathbf{n}\,dl_x, \tag{2.6}$$

$$p(\mathbf{y}) = \alpha p^\infty(\mathbf{y}) + \int\limits_{\partial\Omega} (\boldsymbol{\sigma}\mathbf{n})_g p^g(\mathbf{x},\mathbf{y})\,dl_x - 2\mu\int\limits_{\partial\Omega} u_g \nabla p^g(\mathbf{x},\mathbf{y})\cdot\mathbf{n}\,dl_x, \tag{2.7}$$

where a dot denotes scalar product of two vectors, $(\boldsymbol{\sigma}\mathbf{n})_g = \sigma_{gk}n_k$, in which $\mathbf{n}$ is the unit inward normal to $\partial\Omega$ (pointing into the fluid). The fields u_g^∞ and p^∞ are the velocity and pressure of the incident flow, so that $\alpha = 0$ for a flow in a bounded domain or $\alpha = 1$ for flow past a bubble or a rigid particle.

The gradient of Green's pressure in Eq. (2.7) is

$$\nabla p^g = \frac{1}{2\pi\, r^2}\left(\mathbf{e}^g - 2\frac{r_g}{r^2}\mathbf{r}\right). \tag{2.8}$$

The velocity gradient for interior points can be derived from Eq. (2.6) as

$$u_{g,k}(\mathbf{y}) = \alpha u_{g,k}^\infty(\mathbf{y}) + \int\limits_{\partial\Omega} \mathbf{u}_{,k}^g(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\sigma}^g\mathbf{n}\,dl_x - \int\limits_{\partial\Omega} \mathbf{u}\cdot\boldsymbol{\sigma}_{,k}^g(\mathbf{x},\mathbf{y})\mathbf{n}\,dl_x, \tag{2.9}$$

where

$$\mathbf{u}_{,k}^g = -\frac{1}{4\pi\mu\, r^2}\left(r_k\,\mathbf{e}^g - \delta_{gk}\mathbf{r} - r_g\,\mathbf{e}^k + 2\frac{r_g\, r_k}{r^2}\mathbf{r}\right), \tag{2.10}$$

$$\boldsymbol{\sigma}_{,k}^g = -\frac{1}{\pi\, r^4}\left(r_g\,\mathbf{e}^k\otimes\mathbf{r} + r_g\,\mathbf{r}\otimes\mathbf{e}^k + \delta_{gk}\,\mathbf{r}\otimes\mathbf{r} - 4\frac{r_g\, r_k}{r^2}\mathbf{r}\otimes\mathbf{r}\right). \tag{2.11}$$

[1)] A derivation of both equations has been given by BIGONI and CAPUANI [1] in a more general context, with a notation different from that employed in the present article.

3. Integral representations at the boundary

The velocity field at the boundary can be obtained from Eq. (2.6) moving the source point $\mathbf{y}$ on the boundary $\partial\Omega$, considering the integration contours sketched in Fig. 1 and taking the limit for $\varepsilon \to 0$ (and for $\rho \to \infty$ in the case of flow past a particle). Correspondingly,

$$C_i^g u_i(\mathbf{y}) = \alpha u_g^\infty(\mathbf{y}) - \int_{\partial\Omega} \mathbf{u}^g(\mathbf{x},\mathbf{y}) \cdot \boldsymbol{\sigma}\mathbf{n}\, dl_x + \overset{\mathrm{PV}}{\int_{\partial\Omega}} \mathbf{u} \cdot \boldsymbol{\sigma}^g(\mathbf{x},\mathbf{y})\mathbf{n}\, dl_x, \tag{3.1}$$

where PV denotes the Cauchy principal value and the $\mathbf{C}$-matrix is defined as

$$C_i^g = -\lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} (\boldsymbol{\sigma}^g\mathbf{n})_i\, dl_x = \alpha\delta_{gi} + \overset{\mathrm{PV}}{\int_{\partial\Omega}} (\boldsymbol{\sigma}^g\mathbf{n})_i\, dl_x, \tag{3.2}$$

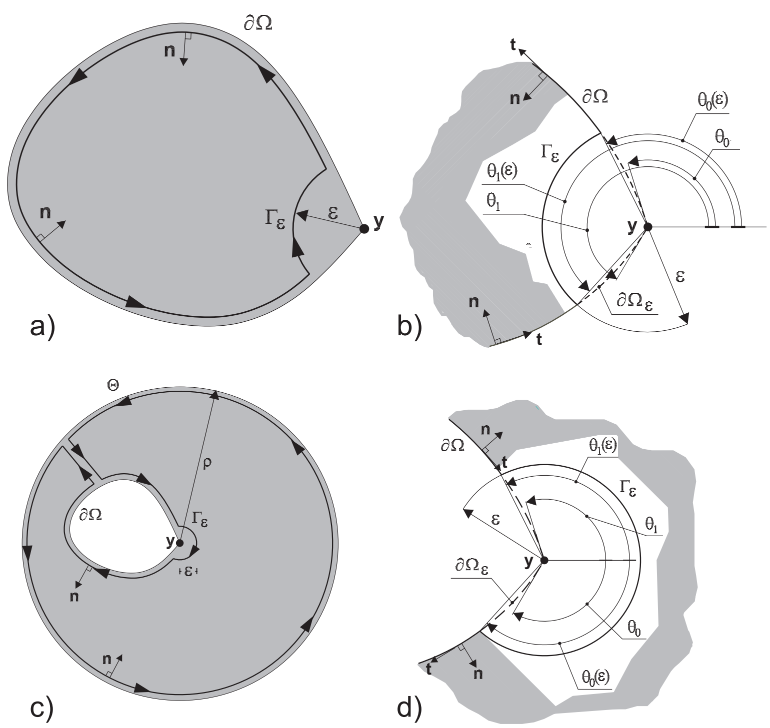

FIG. 1. Geometry of the problem and employed contours; a) Flow within a closed boundary; b) detail of geometry a) near the corner point $\mathbf{y}$; c) flow within an unbounded domain; d) detail of geometry c) near the corner point $\mathbf{y}$.

in which Γ_ε is the intersection of the circle of radius ε centered at $\mathbf{y}$ with the region occupied by the fluid, and the inward unit normal $\mathbf{n}$ points towards the fluid[2]. Using Eq. (2.5) we get the second-order free-term tensor

$$\mathbf{C} = \frac{1}{2\pi}\Big[(\theta_1 - \theta_0)\mathbf{I} - (\mathbf{n}(\theta_1) \otimes \mathbf{t}(\theta_1))_{\mathrm{sym}} + (\mathbf{n}(\theta_0) \otimes \mathbf{t}(\theta_0))_{\mathrm{sym}}\Big], \tag{3.3}$$

where $\mathbf{t}(\theta)$ is the unit tangent vector (note that a counterclockwise rotation is needed to superpose $\mathbf{t}$ on $\mathbf{n}$), and θ_0 and θ_1 are the angular coordinates of the half-tangents to the boundary at point $\mathbf{y}$ (Fig. 1). Note that at a smooth point of the boundary, where $\theta_1 = \theta_0 + \pi$, the $\mathbf{C}$-matrix reduces to a multiple of the identity, namely, $\mathbf{C} = (1/2)\mathbf{I}$.

In order to obtain a boundary integral representation for the pressure p, we note that, for a closed contour not enclosing the singularity point $\mathbf{y}$, the following condition holds:

$$\oint \nabla p^g(\mathbf{x}, \mathbf{y}) \cdot \mathbf{n}(\mathbf{x}) dl_x = 0. \tag{3.4}$$

The derivation here and in the following will be restricted to the case of a domain in an infinite flow (Fig. 1b), whereas the easier situation of the flow confined to a closed domain will be simply stated. An infinite flow $\mathbf{u}$ and its corresponding pressure p are decomposed into unperturbed components $\mathbf{u}^\infty$ and p^∞, that would prevail in the absence of any disturbance, and perturbed components $\mathbf{u}^D$ and p^D, so that $\mathbf{u} = \mathbf{u}^\infty + \mathbf{u}^D$ and $p = p^\infty + p^D$. As for the perturbed field, applying Eq. (2.7) together with condition (3.4) to the contour shown in Fig. 1b yields

$$\int\limits_{\partial\Omega - \partial\Omega_\varepsilon + \Gamma_\varepsilon + \Theta_\rho} \Big[\left(\boldsymbol{\sigma}^D \mathbf{n}\right)(\mathbf{x})_g p^g(\mathbf{x}, \mathbf{y}) \\ - 2\mu \, \nabla p^g(\mathbf{x}, \mathbf{y}) \cdot \mathbf{n}(\mathbf{x}) \, \left(u_g^D(\mathbf{x}) - u_g^D(\mathbf{y})\right) \Big] \, dl_x = 0, \tag{3.5}$$

where $\partial\Omega_\varepsilon$ is the boundary length intercepted by Γ_ε, and Θ_ρ is a circle of radius ρ centered at $\mathbf{y}$ (Fig. 1b). Taking the limit for ρ tending to infinity, considering the relation $\mathbf{r} = r\,\mathbf{n}$ and assuming that both $\boldsymbol{\sigma}^D(\rho)$ and $\mathbf{u}^D(\rho)/\rho$ tend to zero in the limit $\rho \to \infty$, the integral along Θ_ρ turns out to vanish. In order to evaluate the integral along Γ_ε in the limit $\varepsilon \to 0$, we introduce the first-order and zeroth-order

[2]It may be interesting to note that employing Eq. $(3.2)_2$ in Eq. (3.1) we get a regularized version of it $\alpha[u_g(\mathbf{y}) - u_g^\infty(\mathbf{y})] = -\int\limits_{\partial\Omega} \mathbf{u}^g(\mathbf{x}, \mathbf{y}) \cdot \boldsymbol{\sigma}\mathbf{n}\, dl_x + \overset{\mathrm{PV}}{\int\limits_{\partial\Omega}} [\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})] \cdot \boldsymbol{\sigma}^g(\mathbf{x}, \mathbf{y})\mathbf{n}\, dl_x$, holding true for points $\mathbf{y}$ lying internally, externally or at the (possibly non-smooth) boundary.

series expansions for the velocity $\mathbf{u}^D(\mathbf{x})$ and the stress $\boldsymbol{\sigma}^D(\mathbf{x})$

$$\begin{aligned}\mathbf{u}^D(\mathbf{x}) - \mathbf{u}^D(\mathbf{y}) &= \nabla\mathbf{u}^D(\mathbf{y})(\mathbf{x}-\mathbf{y}) + o(|\mathbf{x}-\mathbf{y}|^2),\\ \boldsymbol{\sigma}^D(\mathbf{x}) &= \boldsymbol{\sigma}^D(\mathbf{y}) + o(|\mathbf{x}-\mathbf{y}|).\end{aligned} \tag{3.6}$$

Using Eqs. (3.6) and taking the limit $\varepsilon \to 0$, Eq. (2.4) can be re-written as

$$\frac{\theta_1-\theta_0}{2\pi}p^D(\mathbf{y}) - 2\mu\mathbf{C}\cdot\nabla\mathbf{u}^D(\mathbf{y}) = \overset{\mathrm{PV}}{\int\limits_{\partial\Omega}} \left(\boldsymbol{\sigma}^D\mathbf{n}\right)_g p^g\, dl_x - 2\mu \overset{\mathrm{PV}}{\int\limits_{\partial\Omega}} \nabla p^g\cdot\mathbf{n}\left(u_g^D(\mathbf{x}) - u_g^D(\mathbf{y})\right) dl_x, \tag{3.7}$$

where $\mathbf{C}$ is defined by Eq. (3.3).

The unperturbed field p^∞ is now analyzed, considering a contour enclosing the inclusion and excluding the point $\mathbf{y}$ through a circle of radius ε. Writing the analogue of the Eq. (2.4) for the unperturbed fields and for the considered contour, gives in the limit $\varepsilon \to 0$

$$\left(\frac{\theta_1-\theta_0}{2\pi} - 1\right)p^\infty(\mathbf{y}) - 2\mu\mathbf{C}\cdot\nabla\mathbf{u}^\infty(\mathbf{y}) = \overset{\mathrm{PV}}{\int\limits_{\partial\Omega}} \left(\boldsymbol{\sigma}^\infty\mathbf{n}\right)_g p^g\, dl_x - 2\mu \overset{\mathrm{PV}}{\int\limits_{\partial\Omega}} \nabla p^g\cdot\mathbf{n}\left(u_g^\infty(\mathbf{x}) - u_g^\infty(\mathbf{y})\right) dl_x, \tag{3.8}$$

so that summing to (3.6) yields

$$\frac{\theta_1-\theta_0}{2\pi}p(\mathbf{y}) - 2\mu\mathbf{C}\cdot\nabla\mathbf{u}(\mathbf{y}) = \alpha p^\infty(\mathbf{y}) + \overset{\mathrm{PV}}{\int\limits_{\partial\Omega}} \left(\boldsymbol{\sigma}\mathbf{n}\right)_g p^g\, dl_x - 2\mu \overset{\mathrm{PV}}{\int\limits_{\partial\Omega}} \nabla p^g\cdot\mathbf{n}\left(u_g(\mathbf{x}) - u_g(\mathbf{y})\right) dl_x, \tag{3.9}$$

a formula that has been obtained for $\alpha = 1$ and that, for $\alpha = 0$, reduces to the case of flow confined to a finite domain.

Equation (3.9) is the integral equation representing the pressure p at points of the boundary. In this equation, the boundary values of p are coupled with those of the velocity gradient but, at smooth points of the contour, where $\theta_1 - \theta_0 = \pi$ and $\mathbf{C} = (1/2)\mathbf{I}$, the integral equation simplifies to

$$\frac{1}{2}p(\mathbf{y}) = \alpha p^\infty(\mathbf{y}) + \overset{\mathrm{PV}}{\int\limits_{\partial\Omega}} \left(\boldsymbol{\sigma}\mathbf{n}\right)_g p^g\, dl_x - 2\mu \overset{\mathrm{PV}}{\int\limits_{\partial\Omega}} \nabla p^g\cdot\mathbf{n}\left(u_g(\mathbf{x}) - u_g(\mathbf{y})\right) dl_x. \tag{3.10}$$

Note that when $\partial\Omega$ represents the boundary of a bubble, the traction is given by

$$\boldsymbol{\sigma}\mathbf{n} = (-p_B + \gamma\kappa)\mathbf{n}, \tag{3.11}$$

where p_B is the pressure, γ the surface tension and κ is the curvature of the bubble, so that condition (3.10) becomes equivalent to the integral equation obtained by POZRIKIDIS [19] (see Appendix A). Moreover, when $\partial\Omega$ represents the boundary of a rigid inclusion, the term due to the gradient of the Stokeslet pressure vanishes and a boundary equation implicitly derived by LIRON and BARTA [10] is recovered (see Appendix B).

An alternative form of (3.9) involving hypersingular integrals will be useful later and is simply obtained as follows. Considering the contour reported in Fig. 1b, we obtain when $\rho \to \infty$

$$\int\limits_{\partial\Omega-\partial\Omega_\varepsilon+\Gamma_\varepsilon+\Theta_\rho} \nabla p^g(\mathbf{x},\mathbf{y})\cdot\mathbf{n}(\mathbf{x})\,dl_x = \int\limits_{\partial\Omega-\partial\Omega_\varepsilon} \nabla p^g(\mathbf{x},\mathbf{y})\cdot\mathbf{n}(\mathbf{x})\,dl_x - \frac{1}{2\pi\varepsilon}\int\limits_{\theta_0(\varepsilon)}^{\theta_1(\varepsilon)} n_g(\theta)\,d\theta = 0, \tag{3.12}$$

where $\theta_0(\varepsilon)$ and $\theta_1(\varepsilon)$ are the angles singling out the initial and final edges of the arc Γ_ε (Fig. 1). A Taylor series expansion of the integral yields

$$\int\limits_{\theta_0(\varepsilon)}^{\theta_1(\varepsilon)} \mathbf{n}(\theta)\,d\theta = \mathbf{n}(\theta_1) + \mathbf{n}(\theta_0) - \varepsilon[\theta_1'(0)\mathbf{t}(\theta_1) + \theta_0'(0)\mathbf{t}(\theta_0)] + O(\varepsilon^2), \tag{3.13}$$

where $\theta_0 = \theta_0(0)$ and $\theta_1 = \theta_1(0)$ are the angular coordinates of the half-tangents to the boundary at point $\mathbf{y}$ (Fig. 1) and a prime denotes differentiation with respect to the argument. In the limit $\varepsilon \to 0$, we obtain

$$\overset{\mathrm{FP}}{\int\limits_{\partial\Omega}} \nabla p^g\cdot\mathbf{n}\,dl_x = -\frac{1}{2\pi}[\theta_1'(0)t_g(\theta_1) + \theta_0'(0)t_g(\theta_0)], \tag{3.14}$$

where FP denotes the Hadamard finite part of the integral (COURANT and HILBERT, [4]). Employing (3.14) in (3.9) we arrive at the expression

$$\frac{\theta_1-\theta_0}{2\pi}p(\mathbf{y}) - 2\mu\mathbf{C}\cdot\nabla\mathbf{u}(\mathbf{y}) + \frac{\mu}{\pi}\left[\theta_1'(0)\mathbf{t}(\theta_1) + \theta_0'(0)\mathbf{t}(\theta_0)\right]\cdot\mathbf{u}(\mathbf{y}) = \alpha p^\infty(\mathbf{y}) + \overset{\mathrm{PV}}{\int\limits_{\partial\Omega}}(\boldsymbol{\sigma}\mathbf{n})_g p^g\,dl_x - 2\mu\overset{\mathrm{FP}}{\int\limits_{\partial\Omega}}\nabla p^g\cdot\mathbf{n}\,u_g\,dl_x. \tag{3.15}$$

Note that in the case of piecewise linear boundary $\theta_1'(0) = \theta_0'(0) = 0$.

For a smooth boundary, where $\theta_1 - \theta_0 = \pi$, $\theta_1'(0) = -\theta_0'(0)$ and $\mathbf{t}(\theta_1) = \mathbf{t}(\theta_0)$, we get

$$\tag{3.16} \frac{1}{2}p(\mathbf{y}) = \alpha p^{\infty}(\mathbf{y}) + \overset{\mathrm{PV}}{\int\limits_{\partial\Omega}} (\boldsymbol{\sigma}\mathbf{n})_g p^g \, dl_x - 2\mu \overset{\mathrm{FP}}{\int\limits_{\partial\Omega}} \nabla p^g \cdot \mathbf{n}\, u_g \, dl_x,$$

which is an alternative to (3.10), involving a hypersingular kernel.

An integral equation for the velocity gradient at boundary points can be obtained starting from Eq. (2.9). In particular, we begin noting that for a closed contour not enclosing the singularity point $\mathbf{y}$, the following condition holds:

$$\tag{3.17} \oint \boldsymbol{\sigma}^g_{,k}(\mathbf{x},\mathbf{y})\mathbf{n}(\mathbf{x})\, dl_x = \mathbf{0}.$$

Restricting the derivation to the case of infinite flow past a particle, applying Eq. (2.9) and taking into account Eq. (2.7), we get for the perturbed and unperturbed fields

$$\tag{3.18} \int\limits_{\partial\Omega - \partial\Omega_\varepsilon + \Gamma_\varepsilon + \Theta_\rho} \left[\boldsymbol{\sigma}^D \mathbf{n} \cdot \mathbf{u}^g_{,k} - \boldsymbol{\sigma}^g_{,k}\mathbf{n} \cdot \left(\mathbf{u}^D(\mathbf{x}) - \mathbf{u}^D(\mathbf{y})\right)\right] dl_x = 0,$$

$$\tag{3.19} \int\limits_{\partial\Omega - \partial\Omega_\varepsilon + \Gamma_\varepsilon} \left[\boldsymbol{\sigma}^\infty \mathbf{n} \cdot \mathbf{u}^g_{,k} - \boldsymbol{\sigma}^g_{,k}\mathbf{n} \cdot \left(\mathbf{u}^\infty(\mathbf{x}) - \mathbf{u}^\infty(\mathbf{y})\right)\right] dl_x = 0,$$

where, according to expressions $(2.3)_1$ and (2.5), $\mathbf{u}^g_{,k}$ and $\boldsymbol{\sigma}^g_{,k}$ are singular as $1/r$ and $1/r^2$ respectively, when r tends to zero. Following a procedure analogous to that illustrated for the derivation of pressure representation and taking the limits for $\rho \to \infty$ and $\varepsilon \to 0$, the combination of Eqs. (2.8)–(2.9) gives

$$\tag{3.20} \begin{aligned}(\mathbb{H}\nabla\mathbf{u}(\mathbf{y}) + \mathbb{E}\mathbf{D}(\mathbf{y}) + p(\mathbf{y})\mathbf{F})_{gk} \\ = \alpha u^\infty_{g,k}(\mathbf{y}) - \overset{\mathrm{PV}}{\int\limits_{\partial\Omega}} \boldsymbol{\sigma}^g_{,k}\mathbf{n} \cdot (\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y}))\, dl_x + \overset{\mathrm{PV}}{\int\limits_{\partial\Omega}} \boldsymbol{\sigma}\mathbf{n} \cdot \mathbf{u}^g_{,k}\, dl_x,\end{aligned}$$

where

$$\mathbb{H}_{gkim} = \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} \sigma^g_{ij,k} n_j (x_m - y_m)\, dl_x, \tag{3.21}$$

$$\mathbb{E}_{gkim} = -\lim_{\varepsilon \to 0} 2\mu \int_{\Gamma_\varepsilon} n_m u^g_{i,k}\, dl_x, \tag{3.22}$$

$$F_{gk} = -\lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} \mathbf{n} \cdot \mathbf{u}^g_{,k}\, dl_x, \tag{3.23}$$

and $\mathbf{D}$ is the rate-of-strain tensor

$$\mathbf{D} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T). \tag{3.24}$$

Substituting the expressions for the velocity and stress gradients of the Green state given by Eqs. (2.10) and (2.2) and taking into account that $\mathbf{r} = r\mathbf{n}$ on Γ_ε, Eqs. (3.21)–(3.23) yield

$$\begin{aligned} &\mathbb{H}\nabla\mathbf{u} + \mathbb{E}\mathbf{D} \\ &\quad = \frac{1}{4}\Big[\nabla\mathbf{u}\mathbf{C} + \nabla\mathbf{u}^T\mathbf{C} - \mathbf{C}\nabla\mathbf{u} - 5\mathbf{C}\nabla\mathbf{u}^T - 6(\mathbf{C}\cdot\nabla\mathbf{u})\mathbf{I} + 16\mathbb{C}\nabla\mathbf{u}\Big], \end{aligned} \tag{3.25}$$

$$\mathbf{F} = -\frac{\theta_1 - \theta_0}{4\pi\,\mu}\mathbf{I} + \frac{1}{2\,\mu}\mathbf{C}, \tag{3.26}$$

where

$$\mathbb{C} = \frac{1}{\pi} \int_{\theta_0}^{\theta_1} \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n} d\theta. \tag{3.27}$$

In the special case of a smooth (Lyapunov) boundary

$$\mathbb{C} = \frac{1}{8}\Big(\mathbf{I} \otimes \mathbf{I} + 2\mathbb{S}\Big), \qquad \mathbb{E} = \frac{1}{4}\mathbb{S}, \qquad \mathbb{H} = \frac{1}{2}\mathbb{I} - \frac{1}{4}\mathbb{S}, \qquad \mathbf{F} = \mathbf{0}, \tag{3.28}$$

(where $\mathbb{I}$ is the identity and $\mathbb{S}$ is the symmetrizing fourth-order tensor) so that Eq. (3.20) reduces to

$$\frac{1}{2}u_{g,k}(\mathbf{y}) = \alpha u^\infty_{g,k}(\mathbf{y}) - \overset{\mathrm{PV}}{\int_{\partial\Omega}} \boldsymbol{\sigma}^g_{,k}\mathbf{n} \cdot (\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y}))\; dl_x + \overset{\mathrm{PV}}{\int_{\partial\Omega}} \boldsymbol{\sigma}\mathbf{n} \cdot \mathbf{u}^g_{,k}\, dl_x. \tag{3.29}$$

By means of Eqs. (3.10) and (3.29), the constitutive Eq. (2.2) provides the boundary integral representation for the stress

$$\begin{aligned}(3.30)\qquad \frac{1}{2}\,\sigma_{gk}\,(\mathbf{y}) &= \sigma^{\infty}_{gk}(\mathbf{y}) + \overset{\mathrm{PV}}{\underset{\partial\Omega}{\int}} (\boldsymbol{\sigma}^g\boldsymbol{\sigma}\mathbf{n})_k\; dl_x \\ &- \mu \overset{\mathrm{PV}}{\underset{\partial\Omega}{\int}} \Big[(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\cdot\Big(\boldsymbol{\sigma}^g_{,k}+\boldsymbol{\sigma}^k_{,g}\Big)\,\mathbf{n} + 2\,\delta_{gk}\,(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))_i\,\nabla p^i\cdot\mathbf{n}\Big]\; dl_x.\end{aligned}$$

In the special case of flow past a particle, the second integral in (3.30) vanishes and Eq. (3.30) reduces to a two-dimensional version of that obtained by LIRON and BARTA [10] for three-dimensional flow (see Appendix B).

An alternative form of Eq. (3.20), involving hypersingular integrals, can be obtained considering the contour shown Fig. 1c and taking the limit for $\rho \to \infty$, so that

$$(3.31)\qquad \int_{\partial\Omega-\partial\Omega_\varepsilon} \boldsymbol{\sigma}^g_{,k}(\mathbf{x},\mathbf{y})\,\mathbf{n}(\mathbf{x})\,dl_x + \frac{1}{\pi\varepsilon}\int_{\theta_0(\varepsilon)}^{\theta_1(\varepsilon)} \Big(n_g\,\mathbf{e}^k + \delta_{gk}\,\mathbf{n} - 3n_g n_k\,\mathbf{n}\Big)\,d\theta = 0.$$

Expanding the second integral in a Taylor series, taking the limit for $\varepsilon \to 0$, and substituting into Eq. (3.20), lead to

$$\begin{aligned}(3.32)\qquad &(\mathcal{B}\mathbf{u}(\mathbf{y}) + \mathbb{H}\nabla\mathbf{u}(\mathbf{y}) + \mathbb{E}\mathbf{D}(\mathbf{y}) + p(\mathbf{y})\mathbf{F})_{gk} \\ &\qquad = \alpha u^{\infty}_{g,k}(\mathbf{y}) - \overset{FP}{\underset{\partial\Omega}{\int}} \boldsymbol{\sigma}^g_{,k}\mathbf{n}\cdot\mathbf{u}\,dl_x + \overset{\mathrm{PV}}{\underset{\partial\Omega}{\int}} \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{u}^g_{,k}\,dl_x,\end{aligned}$$

where

$$\begin{aligned}(3.33)\qquad \mathcal{B} = \frac{1}{\pi}\,\big[&\theta_1'(0)(\mathbf{t}(\theta_1)\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{t}(\theta_1) - 3\mathbf{t}(\theta_1)\otimes\mathbf{t}(\theta_1)\otimes\mathbf{t}(\theta_1)) \\ &+ \theta_0'(0)(\mathbf{t}(\theta_0)\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{t}(\theta_0) + 3\mathbf{t}(\theta_1)\otimes\mathbf{t}(\theta_0)\otimes\mathbf{t}(\theta_0))\big]\,.\end{aligned}$$

The tensors $\mathcal{B}, \mathbb{C}, \mathbb{H}, \mathbb{E}$ and $\mathbf{F}$ collect the *free terms* of the integral equation representing the velocity gradient at points on the boundary. Note that $\mathcal{B}$ depends on the curvatures of the boundary and vanishes both for a smooth and piecewise rectilinear boundary. This dependence on the curvature has a correspondence in elasticity (GUIGGIANI, [6]). At smooth points of the contour, where $\mathbf{F}$ turns out to be zero, Eq. (3.32) simplifies as follows:

$$(3.34)\qquad \frac{1}{2}\,u_{g,k}(\mathbf{y}) = \alpha u^{\infty}_{g,k}(\mathbf{y}) - \overset{\mathrm{FP}}{\underset{\partial\Omega}{\int}} \boldsymbol{\sigma}^g_{,k}\mathbf{n}\cdot\mathbf{u}\,dl_x + \overset{\mathrm{PV}}{\underset{\partial\Omega}{\int}} \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{u}^g_{,k}\,dl_x.$$

Once the velocity gradient and the pressure are given, the integral representation of the stress tensor at smooth points of the boundary follows from Eq. (2.2):

$$\frac{1}{2}\sigma_{gk}(\mathbf{y}) = \alpha\sigma^{\infty}_{gk}(\mathbf{y}) + \overset{\mathrm{PV}}{\int_{\partial\Omega}} (\boldsymbol{\sigma}^g \boldsymbol{\sigma}\mathbf{n})_k \, dl_x - \mu \overset{\mathrm{FP}}{\int_{\partial\Omega}} \left[\mathbf{u}\cdot\left(\boldsymbol{\sigma}^g_{,k} + \boldsymbol{\sigma}^k_{,g}\right)\mathbf{n} + 2\delta_{gk}\, u_i \nabla p^i \cdot \mathbf{n}\right] dl_x. \tag{3.35}$$

Equation (3.35) coincides with the analogous known in elasticity (see for instance MANTIČ and PARIS [11]) in the special case when Poisson's ratio ν becomes equal to 0.5. This circumstance is not surprising, being continuity of the solution with ν often expected, but not trivial.

In closure of this section we note that Eqs. (3.34) and (3.35) permit the complete determination of velocity gradient and stress at the boundary when tractions and velocities are here known.

4. A simple example

To illustrate with a simple application the use of boundary integral equations (3.9) and (3.20), we refer here to a geometric situation involving right-angled corners, inclined at θ_0 with respect to a reference system, singled out by unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ (Fig. 2a).

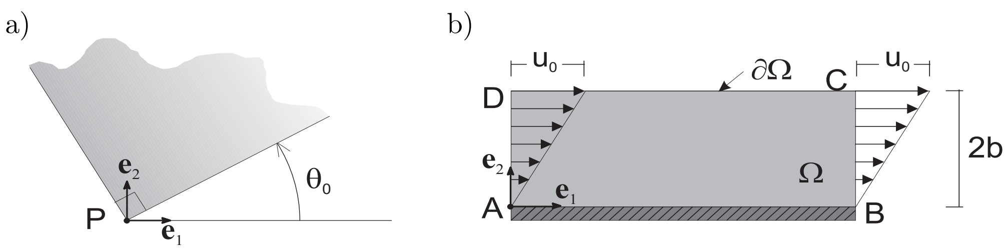

FIG. 2. Geometry of the considered corner a) and of the plane Couette flow b).

In this case, the free-term second-order (3.3) and fourth-order (3.27) tensors become

$$\mathbf{C} = \frac{1}{4}\mathbf{I} + \frac{1}{2\,\pi}\Big[-\sin 2\theta_0(\mathbf{e}_1\otimes\mathbf{e}_1 - \mathbf{e}_2\otimes\mathbf{e}_2) + \cos 2\theta_0(\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1)\Big], \tag{4.1}$$

and

$$\mathbb{C} = \frac{1}{16}\mathbf{I}\otimes\mathbf{I} + \frac{1}{8}\mathbb{S} + \frac{1}{2}\mathbf{I}\otimes\mathbf{C}. \tag{4.2}$$

Therefore, the left-hand side of Eq. (3.9) simplifies to

$$\frac{1}{4}p + \frac{\mu}{\pi}\Big[\sin 2\theta_0(u_{1,1} - u_{2,2}) - \cos 2\theta_0(u_{1,2} + u_{2,1})\Big], \tag{4.3}$$

whereas the four components of the tensor appearing on the left-hand side of Eq. (3.20) reduce to

$$\begin{aligned}
(\mathbb{H}\nabla\mathbf{u} + \mathbb{E}\mathbf{D} + p\mathbf{F})_{11} &= \frac{1}{4}\,u_{1,1} - \frac{\cos\,2\theta_0}{4\,\pi}(u_{1,2} - u_{2,1}) - \frac{\sin\,2\theta_0}{2\,\pi}p,\\
(\mathbb{H}\nabla\mathbf{u} + \mathbb{E}\mathbf{D} + p\mathbf{F})_{12} &= \frac{1}{4}\,u_{1,2} + \frac{\cos\,2\theta_0}{\pi}u_{1,1}\\
&\qquad + \frac{\sin\,2\theta_0}{4\,\pi}(u_{1,2} + 3u_{2,1}) + \frac{\cos\,2\theta_0}{2\,\pi}p,\\
(\mathbb{H}\nabla\mathbf{u} + \mathbb{E}\mathbf{D} + p\mathbf{F})_{21} &= \frac{1}{4}\,u_{2,1} - \frac{\cos\,2\theta_0}{\pi}u_{1,1}\\
&\qquad - \frac{\sin\,2\theta_0}{4\,\pi}(3u_{1,2} + u_{2,1}) + \frac{\cos\,2\theta_0}{2\,\pi}p,\\
(\mathbb{H}\nabla\mathbf{u} + \mathbb{E}\mathbf{D} + p\mathbf{F})_{22} &= \frac{1}{4}\,u_{2,2} + \frac{\cos\,2\theta_0}{4\pi}(u_{1,2} - u_{2,1}) + \frac{\sin\,2\theta_0}{2\,\pi}p.
\end{aligned} \tag{4.4}$$

Referring now for simplicity to the plane Couette flow sketched in Fig. 2b, characterized by the velocity field

$$\mathbf{u} = \frac{u_0\,x_2}{2\,b}\;\mathbf{e}_1, \tag{4.5}$$

the right-hand sides of Eqs. (3.9) and (3.20) can be integrated at each boundary point. In particular, these equations become at the point A indicated in Fig. 2b)

$$\begin{aligned}
\frac{\theta_1 - \theta_0}{2\pi}p - 2\mu\mathbf{C}\cdot\nabla\mathbf{u} \;&= \frac{1}{4}p_0 + \frac{\mu\,u_0}{2\,\pi\,b},\\
[\mathbb{H}\nabla\mathbf{u} + \mathbb{E}\mathbf{D} + p\mathbf{F}] &= \begin{bmatrix} -\dfrac{u_0}{8\,\pi\,b} & \dfrac{u_0}{8\,b}\\ 0 & \dfrac{u_0}{8\,\pi\,b}\end{bmatrix} + \frac{p_0}{2\,\pi}\begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix},
\end{aligned} \tag{4.6}$$

where the pressure p_0 takes an arbitrary value, uniform in the flow. Now, the expressions (4.3) and (4.4) can be substituted into (4.6) and solved for $\nabla\mathbf{u}$ and p, resulting in the expected values $\nabla\mathbf{u} = u_0/(2\text{b})\;\mathbf{e}_1\otimes\;\mathbf{e}_2$ and $p = p_0$.

5. Symmetric formulation of the boundary element method

Owing to Eqs. (3.1) and (3.35), the representations for the velocity $\mathbf{u}$ and the traction $\mathbf{f} = \boldsymbol{\sigma}\mathbf{n}$ (exerted by the fluid on the boundary) on a Lyapunov boundary are given by:

$$\frac{1}{2}\, u_g(\mathbf{y}) = \alpha u_g^{\infty}(\mathbf{y}) - \int\limits_{\partial\Omega} \mathbf{u}^g \cdot \boldsymbol{\sigma}\mathbf{n}\, dl_x + \overset{\mathrm{PV}}{\int\limits_{\partial\Omega}} \mathbf{u} \cdot \boldsymbol{\sigma}^g \mathbf{n}\, dl_x, \tag{5.1}$$

$$\begin{aligned} \frac{1}{2}\, f_g(\mathbf{y}) = \alpha f_g^{\infty}(\mathbf{y}) + \mathbf{n}(\mathbf{y}) \cdot \overset{\mathrm{PV}}{\int\limits_{\partial\Omega}} \boldsymbol{\sigma}^g \boldsymbol{\sigma}\mathbf{n}\, dl_x \\ - \mu n_k(\mathbf{y}) \overset{\mathrm{FP}}{\int\limits_{\partial\Omega}} \left[\mathbf{u} \cdot \left(\boldsymbol{\sigma}^g_{,k} + \boldsymbol{\sigma}^k_{,g} \right) \mathbf{n} + 2\delta_{gk}\, u_i \nabla p^i \cdot \mathbf{n} \right] dl_x. \end{aligned} \tag{5.2}$$

It is worth noting that either Eq. (5.1) or Eq. (5.2) can be employed to solve any boundary value problem through a collocation technique, thus obtaining in the former case the so-called direct method and, in the latter, the hypersingular boundary element method. In both cases the coefficient matrix of the solving system of algebraic equations is non-symmetric. A way to symmetrize the solving system is obtained below employing a Galerkin approach, in analogy to the methodologies used in solid mechanics (BONNET *et al.* [3]).

Let δu_g and δf_g be virtual velocity and traction fields satisfying the boundary conditions

$$\delta u_g = 0, \quad \text{on} \quad \partial\Omega_u, \qquad \delta f_g = 0, \quad \text{on} \quad \partial\Omega_f, \tag{5.3}$$

where $\partial\Omega_u$ and $\partial\Omega_f$ are the portions of the boundary where velocities and tractions are assigned, respectively.

Taking the scalar product of Eqs. (5.1), (5.2) by the virtual fields and integrating over the contour, the following equations are obtained:

$$\begin{aligned} \frac{1}{2} \int\limits_{\partial\Omega_{\mathbf{u}}} u_g(\mathbf{y}) \delta f_g(\mathbf{y})\, dl_y - \alpha \int\limits_{\partial\Omega_{\mathbf{u}}} u_g^{\infty}(\mathbf{y}) \delta f_g(\mathbf{y})\, dl_y \\ = - \int\limits_{\partial\Omega_{\mathbf{u}}} \delta f_g(\mathbf{y}) \left(\int\limits_{\partial\Omega} \mathbf{u}^g \cdot \mathbf{f}\, dl_x - \overset{\mathrm{PV}}{\int\limits_{\partial\Omega}} \mathbf{u} \cdot \mathbf{f}^g\, dl_x \right) dl_y, \end{aligned} \tag{5.4}$$

$$\frac{1}{2}\int\limits_{\partial\Omega_{\mathbf{f}}} \delta u_g(\mathbf{y}) f_g(\mathbf{y})\, dl_y - \alpha \int\limits_{\partial\Omega_{\mathbf{f}}} \delta u_g(\mathbf{y}) f_g^{\infty}(\mathbf{y})\, dl_y = \int\limits_{\partial\Omega_{\mathbf{f}}} \delta u_g(\mathbf{y}) \left[\mathbf{n}(\mathbf{y}) \cdot \overset{\mathrm{PV}}{\int\limits_{\partial\Omega}} \boldsymbol{\sigma}^g \mathbf{f}\, dl_x\right] dl_y - \mu \int\limits_{\partial\Omega_{\mathbf{f}}} \delta u_g(\mathbf{y}) \left\{ n_k(\mathbf{y}) \overset{\mathrm{FP}}{\int\limits_{\partial\Omega}} \left[\mathbf{u} \cdot \left(\boldsymbol{\sigma}^g_{,k} + \boldsymbol{\sigma}^k_{,g}\right) \mathbf{n} + 2\,\delta_{gk}\, u_i \nabla p^i \cdot \mathbf{n}\right] dl_x \right\} dl_y, \tag{5.5}$$

which, separating the unknowns and data, can be re-written as

$$\frac{1}{2}\int\limits_{\partial\Omega_{\mathbf{u}}} u_g(\mathbf{y}) \delta f_g(\mathbf{y})\, dl_y - \alpha \int\limits_{\partial\Omega_{\mathbf{u}}} u_g^{\infty}(\mathbf{y}) \delta f_g(\mathbf{y})\, dl_y + \int\limits_{\partial\Omega_{\mathbf{u}}} \delta f_g(\mathbf{y}) \left(\int\limits_{\partial\Omega_{\mathbf{f}}} \mathbf{u}^g \cdot \mathbf{f}\, dl_x - \overset{\mathrm{PV}}{\int\limits_{\partial\Omega_{\mathbf{u}}}} \mathbf{u} \cdot \mathbf{f}^g\, dl_x \right) dl_y = - \int\limits_{\partial\Omega_{\mathbf{u}}} \delta f_g(\mathbf{y}) \left(\int\limits_{\partial\Omega_{\mathbf{u}}} \mathbf{u}^g \cdot \mathbf{f}\, dl_x - \overset{\mathrm{PV}}{\int\limits_{\partial\Omega_{\mathbf{f}}}} \mathbf{u} \cdot \mathbf{f}^g\, dl_x \right) dl_y, \tag{5.6}$$

$$\frac{1}{2}\int\limits_{\partial\Omega_{\mathbf{f}}} \delta u_g(\mathbf{y}) f_g(\mathbf{y})\, dl_y - \alpha \int\limits_{\partial\Omega_{\mathbf{f}}} \delta u_g(\mathbf{y}) f_g^{\infty}(\mathbf{y})\, dl_y - \int\limits_{\partial\Omega_{\mathbf{f}}} \delta u_g(\mathbf{y}) \left[\mathbf{n}(\mathbf{y}) \cdot \overset{\mathrm{PV}}{\int\limits_{\partial\Omega_{\mathbf{f}}}} \boldsymbol{\sigma}^g \mathbf{f}\, dl_x\right] dl_y + \mu \int\limits_{\partial\Omega_{\mathbf{f}}} \delta u_g(\mathbf{y}) \left\{ n_k(\mathbf{y}) \overset{\mathrm{FP}}{\int\limits_{\partial\Omega_{\mathbf{u}}}} \left[\mathbf{u} \cdot \left(\boldsymbol{\sigma}^g_{,k} + \boldsymbol{\sigma}^k_{,g}\right) \mathbf{n} + 2\,\delta_{gk}\, u_i \nabla p^i \cdot \mathbf{n}\right] dl_x \right\} dl_y = \int\limits_{\partial\Omega_{\mathbf{f}}} \delta u_g(\mathbf{y}) \left[\mathbf{n}(\mathbf{y}) \cdot \overset{\mathrm{PV}}{\int\limits_{\partial\Omega_{\mathbf{u}}}} \boldsymbol{\sigma}^g \mathbf{f}\, dl_x\right] dl_y - \mu \int\limits_{\partial\Omega_{\mathbf{f}}} \delta u_g(\mathbf{y}) \left\{ n_k(\mathbf{y}) \overset{\mathrm{FP}}{\int\limits_{\partial\Omega_{\mathbf{f}}}} \left[\mathbf{u} \cdot \left(\boldsymbol{\sigma}^g_{,k} + \boldsymbol{\sigma}^k_{,g}\right) \mathbf{n} + 2\,\delta_{gk}\, u_i \nabla p^i \cdot \mathbf{n}\right] dl_x \right\} dl_y. \tag{5.7}$$

Equation (5.6) and (5.7) are the starting point to derive a symmetric formulation of the boundary element method. To this purpose, the boundary $\partial\Omega$ is divided into n_e elements $\partial\Omega^e$ $(e = 1, ..., n_e)$, with subsets n_e^u and n_e^f belonging to $\partial\Omega_u$ and $\partial\Omega_f$, respectively (i.e. $n_e = n_e^u + n_e^f$). Within each boundary element $\partial\Omega^e$, the following representations for velocities and tractions are chosen:

$$u_g(x) = \sum_{\alpha=1}^{n_\alpha} \varphi^\alpha(x)\, u_g^\alpha, \qquad f_g(x) = \sum_{\beta=1}^{n_\beta} \varphi^\alpha(x)\, f_g^\alpha, \tag{5.8}$$

where u_g^α and f_g^α are the nodal values of velocities and tractions, respectively and φ^α are the relevant shape functions. Assuming for δu_g and δf_g the *same shape functions* as in Eqs. (5.8), and taking into account that Eqs. (5.6), (5.7) hold true for every δu_g and δf_g, a linear algebraic system can be obtained in the generic form

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{D} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{f} \end{bmatrix} = \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix}, \tag{5.9}$$

representing the governing equations of the discrete model. In Eq. (5.9), vectors $\mathbf{u}$, $\mathbf{f}$ collect the unknown nodal values of velocity and tractions, so that the system matrix is obtained by assembling the element sub-matrices

$$\begin{aligned} A_{gi}^{\alpha\beta} &= -\mu \int_{\partial\Omega_f^e} \varphi^\alpha(\mathbf{y}) \Bigg\{ n_k(\mathbf{y}) \overset{\mathrm{FP}}{\int_{\partial\Omega_f^e}} \Big[\Big(\sigma_{ij,k}^g + \sigma_{ij,g}^k \Big)\, n_j \\ &\qquad\qquad + 2\delta_{gk}\, p_{,j}^i n_j \Big]\, \varphi^\beta(\mathbf{x}) dl_x \Bigg\} dl_y, \\ B_{gi}^{\alpha\beta} &= \int_{\partial\Omega_f^e} \varphi^\alpha(\mathbf{y}) \left[n_j(\mathbf{y}) \overset{\mathrm{PV}}{\int_{\partial\Omega_u^e}} \sigma_{ij}^g \varphi^\beta(\mathbf{x})\, dl_x \right] dl_y, \\ C_{gi}^{\alpha\beta} &= -\int_{\partial\Omega_u^e} \varphi^\alpha(\mathbf{y}) \left(\int_{\partial\Omega_u^e} u_i^g\, \varphi^\beta(\mathbf{x})\, dl_x \right) dl_y, \\ D_{gi}^{\alpha\beta} &= \int_{\partial\Omega_u^e} \varphi^\alpha(\mathbf{y}) \left(\overset{\mathrm{PV}}{\int_{\partial\Omega_f^e}} \sigma_{ij}^g n_j \varphi^\beta(\mathbf{x})\, dl_x \right) dl_y \end{aligned} \tag{5.10}$$

and vectors on the right-hand side are given, for node α of each element, by

$$
\begin{aligned}
(5.11)\qquad p_g^\alpha = {} & \frac{1}{2}\int\limits_{\partial\Omega_f^e} \varphi^\alpha(\mathbf{y}) f_g(\mathbf{y})\, dl_y - \alpha \int\limits_{\partial\Omega_f^e} \varphi^\alpha(\mathbf{y}) f_g^\infty(\mathbf{y})\, dl_y \\
& - \int\limits_{\partial\Omega_f^e} \varphi^\alpha(\mathbf{y}) \left[n_j(\mathbf{y}) \overset{\mathrm{PV}}{\int\limits_{\partial\Omega_f}} \sigma_{ij}^g f_i \, dl_x \right] dl_y \\
& + \mu \int\limits_{\partial\Omega_f^e} \varphi^\alpha(\mathbf{y}) \left\{ n_k(\mathbf{y}) \overset{\mathrm{FP}}{\int\limits_{\partial\Omega_u}} \left[u_i(\sigma_{ij,k}^g + \sigma_{ij,g}^k) n_j + 2\delta_{gk}\, u_i p_{,j}^i n_j \right] dl_x \right\} dl_y, \\
q_g^\alpha = {} & \frac{1}{2}\int\limits_{\partial\Omega_u^e} u_g(\mathbf{y}) \varphi^\alpha(\mathbf{y})\, dl_y - \alpha \int\limits_{\partial\Omega_u^e} u_g^\infty(\mathbf{y}) \varphi^\alpha(\mathbf{y})\, dl_y \\
& + \int\limits_{\partial\Omega_u^e} \varphi^\alpha(\mathbf{y}) \left(\int\limits_{\partial\Omega_f} u_i^g f_i \, dl_x - \int\limits_{\partial\Omega_u} u_i f_i^g \, dl_x \right) dl_y.
\end{aligned}
$$

Simple algebra, omitted for brevity, shows that $\mathbf{D} = \mathbf{B}^T$ so that the system matrix appearing in (5.9) turns out to be symmetric, a feature characteristic of the Galerkin boundary element formulation. The formulation is well-known in the context of compressible elasticity (BONNET *el al.* [3]), but it has never been extended to include the incompressible limit. More in detail, we have proved that the formulation of linear elasticity becomes coincident with that represented by linear system (5.9) in the limit when Poisson's ratio ν is 0.5, since also Eqs. (3.1) and (3.35) coincide in that limit, a circumstance occurring also in the usual unsymmetric collocation formulation. Therefore, since the solution of a problem involving Stokes flow can be obtained employing tractions and velocities at the boundary as unknowns [see Eqs. (5.9)–(5.11)], numerical codes developed for linear elasticity can be employed taking $\nu = 0.5$. Obviously, the calculation of the stress and pressure at the boundary cannot be pursued just using the above-mentioned codes, but it requires the use of Eqs. (3.10) and (3.30) or Eqs. (3.16) and (3.35).

6. Numerical examples

Examples of implementation of the symmetric Galerkin method in the context of elasticity have been given, among others, by BONNET [2], FRANGI and NOVATI [5], BONNET *et al.* [3], and PANZECA *et al.* [13]. To test the validity of the method proposed in the present paper a FORTRAN code, developed within the context of elasticity by FRANGI and NOVATI [5], has been modified to solve the Stokes viscous flow.

Results relative to the problem of shear flow over a rectangular cavity, with velocities prescribed on the whole boundary, are presented in Fig. 3; in the figure, computed velocity profiles are reported together with the employed boundary discretization and a sketch of the analyzed problem. Two geometries have been considered, corresponding to the two aspect ratios $a/b = 1/20$ and $a/b = 1/5$. The same problem was also solved numerically by HIGDON ([7], his Figs. 6b and 9a, respectively) and, in agreement with his results, our solution shows that a single eddy occupies the whole cavity for the aspect ratio $a/b = 1/20$, whereas a secondary eddy appears at the bottom of the cavity for $a/b = 1/5$.

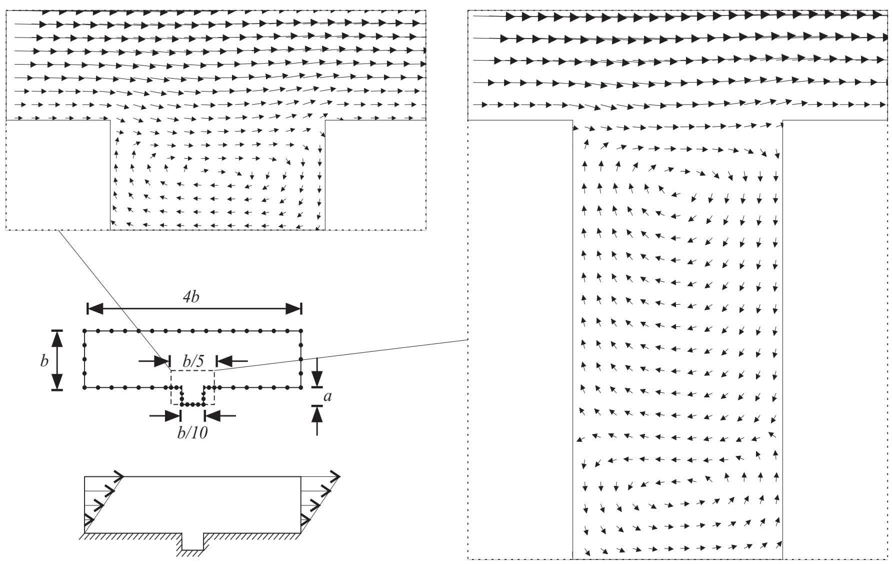

FIG. 3. Shear flow over rectangular cavities. Aspect ratios $a/b = 1/20$ and $a/b = 1/5$ have been considered.

The computation of the double integrals involved in the symmetric Galerkin formulation (SGBEM) is a computational difficulty, which does not arise in the collocation method (CBEM). On the other hand, the CBEM requires the storage and the inversion of the full unsymmetric matrix of the solving system. The differences between the two techniques become particularly important when the RAM limit is approached: in this case the SGBEM highlights its advantages. Moreover, the sum of the CPU times needed for the computation and for the solution of the final system becomes smaller for SGBEM than for CBEM, when the degrees of freedom are increased. To clarify this point with an example, the problem of a Couette flow in a square domain is considered, where a pure, uniform shear stress is applied on the boundary. The domain is discretised with an increasing number of elements and solved with CBEM and SGBEM. Results are reported in Fig. 4, showing the CPU time (expressed in seconds, of a Pentium III, 1GHz) required to build the final system (computation time) and to solve it (solution time), versus the number of degrees of freedom.

It can be concluded from Fig. 4 that the symmetric method results to be more convenient than the collocation method when the number of degrees of freedom exceeds ~ 3000.

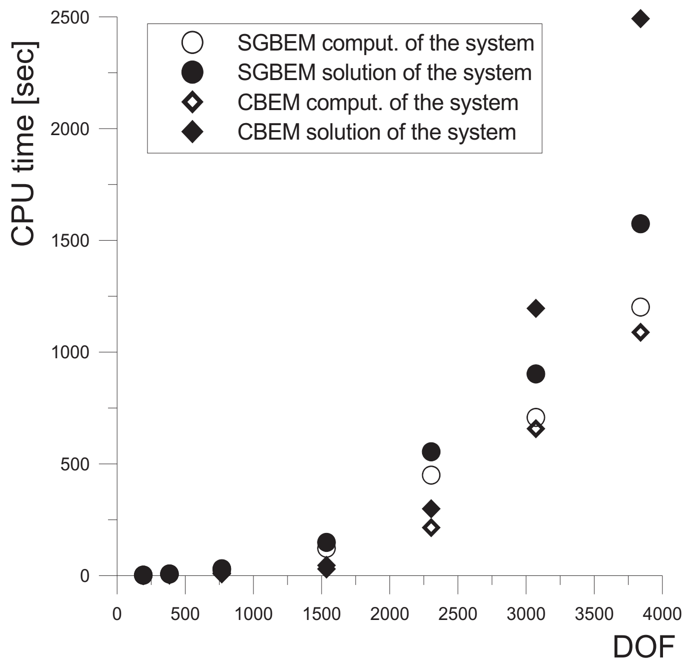

Fig. 4. CPU computation and solution times versus the number of degrees of freedom (DOF), for a Couette flow on a square domain.

7. Conclusions

A symmetric Galerkin formulation for the Stokes flow has been given for both cases of flow confined within a closed domain or flow past an inclusion (for instance in the form of a rigid particle or a bubble). The advantage of the formulation is the symmetry of the final system of equations, which can be easily coupled to the equation systems derived from finite elements. The development of the technique required the determination of integral representations at the boundary for pressure and velocity gradient (and thus for stress). These have been obtained in the general case of a boundary with corners and curved profile.

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Appendix A. A regularized expression for pressure in the case of a bubble

We consider here the special case of a bubble suspended in an ambient viscous liquid analyzed by POZRIKIDIS [19]. For a closed contour not enclosing the singularity point $\mathbf{y}$, the divergence theorem implies that the following condition holds

$$\oint p^g n_g \, dl_x = \int\limits_{\partial\Omega - \partial\Omega_\varepsilon + \Gamma_\varepsilon} p^g(\mathbf{x},\mathbf{y}) \, n_g(\mathbf{x}) \, dl_x = 0, \tag{A.1}$$

so that

$$\overset{\mathrm{PV}}{\int\limits_{\partial\Omega}} n_g p^g \, dl_x = \frac{1}{2} \,. \tag{A.2}$$

Therefore substituting Eq. (3.11) into Eq. (3.10) and using Eq. (A.2) yields

$$\begin{aligned} \frac{1}{2}\, p(\mathbf{y}) &= p^\infty(\mathbf{y}) - \frac{1}{2}\,(p_B - \gamma\kappa(\mathbf{y})) \\ &+ \gamma \overset{\mathrm{PV}}{\int\limits_{\partial\Omega}} (\kappa(\mathbf{x}) - \kappa(\mathbf{y})) \, n_g p^g \, dl_x - 2\mu \overset{\mathrm{PV}}{\int\limits_{\partial\Omega}} \nabla p^g \cdot \mathbf{n} \, \left(u_g^\infty(\mathbf{x}) - u_g^\infty(\mathbf{y})\right) \, dl_x. \end{aligned} \tag{A.3}$$

Keeping into account condition [3.11] of POZRIKIDIS [19], Eq. (A.3) coincides with Eq. (2.8) of POZRIKIDIS [19] [note the difference in the sign of our function ∇p^g, Eq. (2.8), with respect to the analogous definition (3.4b) of Pozrikidis].

Appendix B. Expressions for pressure and velocity gradient for a rigid particle

We consider here the case of viscous flow past a rigid particle analyzed, among others, by Youngren and Acrivos (1975) and Liron and Barta (1992). The velocity at the contact between fluid and particle satisfies Poisson's theorem of rigid body motion, so that it can be expressed in the form:

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{y}) + \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{y}), \tag{B.1}$$

where $\mathbf{u}$ and $\boldsymbol{\omega}$ are the translational and angular velocities, respectively, $\mathbf{x}$ and $\mathbf{y}$ are generic points on the rigid boundary $\partial\Omega$, so that $\mathbf{x} - \mathbf{y} = \mathbf{r}$ in our case. Standard arguments can be employed to show that

$$(C_i^g - \delta_i^g) u_i(\mathbf{y}) = \overset{\mathrm{PV}}{\int_{\partial\Omega}} (\mathbf{u}(\mathbf{y}) + \boldsymbol{\omega} \times \mathbf{r}) \cdot \boldsymbol{\sigma}^g(\mathbf{x}, \mathbf{y}) \mathbf{n} \, dl_x, \tag{B.2}$$

$$-\frac{1}{2\pi} [\theta_1'(0)\mathbf{t}(\theta_1) + \theta_0'(0)\mathbf{t}(\theta_0)] \cdot \mathbf{u}(\mathbf{y}) = \overset{\mathrm{FP}}{\int_{\partial\Omega}} (\mathbf{u}(\mathbf{y}) + \boldsymbol{\omega} \times \mathbf{r})_g \, \nabla p^g(\mathbf{x}, \mathbf{y}) \cdot \mathbf{n} \, dl_x, \tag{B.3}$$

for points $\mathbf{x}$ and $\mathbf{y}$ lying on $\partial\Omega$.

We consider now a contour along the boundary of the particle, but ruling out the point $\mathbf{y}$ with a small circle of radius ε. Along this circular contour $\mathbf{r} = r\,\mathbf{n}$, so that

$$\boldsymbol{\omega} \times \mathbf{r} \cdot \boldsymbol{\sigma}^g \mathbf{n} = (\boldsymbol{\omega} \times \mathbf{r})_g \nabla p^g \cdot \mathbf{n} = 0, \tag{B.4}$$

and thus taking the limit $\varepsilon \to 0$ we get

$$(C_i^g - \delta_i^g) u_i(\mathbf{y}) = \mathbf{u}(\mathbf{y}) \cdot \overset{\mathrm{PV}}{\int_{\partial\Omega}} \boldsymbol{\sigma}^g(\mathbf{x}, \mathbf{y}) \mathbf{n} \, dl_x, \tag{B.5}$$

$$-\frac{1}{2\pi} \left[\theta_1'(0)\mathbf{t}(\theta_1) + \theta_0'(0)\mathbf{t}(\theta_0)\right] \cdot \mathbf{u}(\mathbf{y}) = u_g(\mathbf{y}) \overset{\mathrm{FP}}{\int_{\partial\Omega}} \nabla p^g(\mathbf{x}, \mathbf{y}) \cdot \mathbf{n} \, dl_x. \tag{B.6}$$

Taking Eq. (B.5) into account, Eq. (3.1) simplifies to

$$u_g(\mathbf{y}) = u_g^{\infty}(\mathbf{y}) - \int_{\partial\Omega} \mathbf{u}^g(\mathbf{x},\mathbf{y}) \cdot \boldsymbol{\sigma}\,\mathbf{n}\, dl_x, \tag{B.7}$$

which *holds true also for corner points* and coincides with Eq. (2.3.30) of POZRIKIDIS [14], given for Lyapunov boundary.

Taking Eq. (B.6) into account, Eq. (3.15) reduces to

$$\frac{\theta_1 - \theta_0}{2\pi} p(\mathbf{y}) - 2\mu \mathbf{C} \cdot \nabla \mathbf{u}(\mathbf{y}) = p^{\infty}(\mathbf{y}) + \overset{\text{PV}}{\int_{\partial\Omega}} (\boldsymbol{\sigma}\mathbf{n})_g\, p^g\, dl_x, \tag{B.8}$$

which for smooth boundary becomes:

$$\frac{1}{2} p(\mathbf{y}) = p^{\infty}(\mathbf{y}) + \overset{\text{PV}}{\int_{\partial\Omega}} (\boldsymbol{\sigma}\mathbf{n})_g\, p^g\, dl_x, \tag{B.9}$$

a formula implicitly derived by LIRON and BARTA [10].

In addition to the above, we note now that *for smooth boundaries*

$$\overset{\text{PV}}{\int_{\partial\Omega}} (\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})) \cdot \left(\boldsymbol{\sigma}^g_{,k} + \boldsymbol{\sigma}^k_{,g} \right) \mathbf{n}\, dl_x = 0, \tag{B.10}$$

so that Eqs. (3.30) and (3.35) become

$$\frac{1}{2} \sigma_{gk}(\mathbf{y}) = \sigma_{gk}^{\infty}(\mathbf{y}) - \overset{\text{PV}}{\int_{\partial\Omega}} (\boldsymbol{\sigma}^g \boldsymbol{\sigma}\mathbf{n})_k\, dl_x, \tag{B.11}$$

from which multiplication by the inward unit normal $n_k(\mathbf{y})$ yields the expression proposed by LIRON and BARTA [10] [their Eq. (2.2) with an opposite normal to the boundary]

$$\frac{1}{2} f_g(\mathbf{y}) = f_g^{\infty}(\mathbf{y}) - \mathbf{n}(\mathbf{y}) \cdot \overset{\text{PV}}{\int_{\partial\Omega}} \boldsymbol{\sigma}^g \boldsymbol{\sigma}\mathbf{n}\, dl_x, \tag{B.12}$$

referred to a Lyapunov boundary, with $f_g = \sigma_{gk} n_k$.

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