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Brief Notes

On the admissibility of an isotropic, smooth elastic continuum

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Many studies of elasticity of inhomogeneous materials – in both elastostatics and elastodynamics – assume the existence of locally isotropic, smooth stiffness tensor fields. We investigate the correctness of such a model in the simplest setup of anti-plane classical elasticity. We work with the concept of mesoscale (or apparent) moduli for a finite-size window placed in such a material, in accordance with the Hill condition for the Hooke law. The limit from mesoscale down to infinitesimal windows is admissible within the model of an assumed smooth, locally isotropic continuum. However, this limit is not admissible from the standpoint of a microstructure, and, in order to set up an inhomogeneous elastic medium, one must introduce its anisotropy. A separate argument against the local isotropy stems from the representation of a correlation function of a wide-sense stationary and isotropic random field, whose realizations are smooth stiffness tensor fields.

1. Introduction

Studies of many natural and man-made materials necessitate the consideration of material microstructure. Examples are provided by practically all composites, polycrystals, granular matter, fibrous media, functionally graded media, etc. An additional aspect of such materials is their microstructural disorder, which typically leads to random field models of constitutive laws – these are models with spatial dependence (variability) of class C^1 (at least once differentiable, i.e. smooth) as required by the local (strong) form of field equations of continuum mechanics. Such models are a starting point in various researches of solid mechanics and wave propagation [1, 2], stochastic finite element methods [3, 4], and functionally graded materials [5] to name a few. A very often used Ansatz in classical elasticity of such inhomogeneous materials assumes the existence of locally isotropic, smoothly inhomogeneous stiffness tensor fields, such as depicted in Fig. 1b. Clearly, local isotropy simplifies the ensuing analysis and this is why most studies tend to employ it. In other words, one postulates

$$\sigma_{ik} = C_{ikjm}\varepsilon_{jm} \quad \text{with} \quad C_{ikjm}(\mathbf{x}) = \lambda(\mathbf{x})\delta_{ik}\delta_{jm} + \mu(\mathbf{x})\left[\delta_{ij}\delta_{km} + \delta_{im}\delta_{kj}\right], \tag{1.1}$$

where λ and μ are class C^1 fields of $\mathbf{x}$. [For all the tensors, we shall interchangeably use the symbolic ($\mathbf{V}$) and the subscript ($V_{ij\ldots}$) notations.]

In this paper we investigate the admissibility of such a model in the simplest setup of anti-plane elasticity. We shall consider Hooke's law of a linear elastic material in anti-plane shear

$$\sigma_i = C_{ij}\varepsilon_j, \tag{1.2}$$

where, for simplicity of notation, we denote $\sigma_i \equiv \sigma_{i3}$ and $\varepsilon_j \equiv \varepsilon_{j3}$, for $i, j = 1, 2$. We assume the anti-plane stiffness tensor C_{ij} ($\equiv \mathbf{C}$ in the symbolic notation) to be (i) locally isotropic, and (ii) a function of position x_i ($\equiv \mathbf{x}$), $i = 1, 2$. This is expressed as follows

$$C_{ij}(\mathbf{x}) = C(\mathbf{x})\,\delta_{ij} \quad \text{or} \quad \mathbf{C}(\mathbf{x}) = C(\mathbf{x})\,\mathbf{I} \tag{1.3}$$

The dilemma of correctness of this spatially dependent local isotropy postulate becomes relevant in many situations. For example, in the case of a finite element discretization of a material with such dependence, one should derive the stiffness matrix from the apparent stiffness tensor so as to have its mechanical and energetic definitions mutually consistent. If the model is not admissible, one then needs to accept jumps between contiguous elements placed along the gradient, each corresponding to a locally isotropic part. Alternatively, one needs to introduce finite elements having anisotropic stiffness matrices to assure the continuity of stiffness in the direction of its spatial gradient.

In this paper we inquire whether the anti-plane stiffness tensor field which is smooth in space can also be locally isotropic. Our strategy in examining the admissibility of the model (1.3) hinges on first recognizing that a smooth stiffness tensor field is really a continuum approximation

$$C_{ij}(\mathbf{x}, L/d) = C(\mathbf{x})\,\delta_{ij} \quad \text{or} \quad \mathbf{C}(\mathbf{x}, L/d) = C(\mathbf{x}, L/d)\,\mathbf{I} \tag{1.4}$$

on a *mesoscale* L (Fig. 1b) larger than the *microscale* d (heterogeneity size) of the underlying microstructure (Fig. 1a). That microstructure involves at least two distinct phases distributed in a disordered fashion; a periodic distribution is of no interest here, because the equivalent continuum is homogeneous. To pass from the microscale of Fig. 1a to the mesoscale of Fig. 1b we use finite size windows, subjected to boundary conditions dictated by the Hill condition (Sec. 2). An examination in Sec. 3 of these boundary conditions shows which particular type of mixed boundary condition is the proper way to make that passage. When conducting the same passage in the disordered microstructure, it becomes apparent that the approximating continuum cannot be isotropic. In Sec. 4 we offer another proof via *reductio ad absurdum* of the same result; we do it in the ensemble setting, where the materials of Fig. 1 are realizations (on micro and mesoscales) of a random medium.

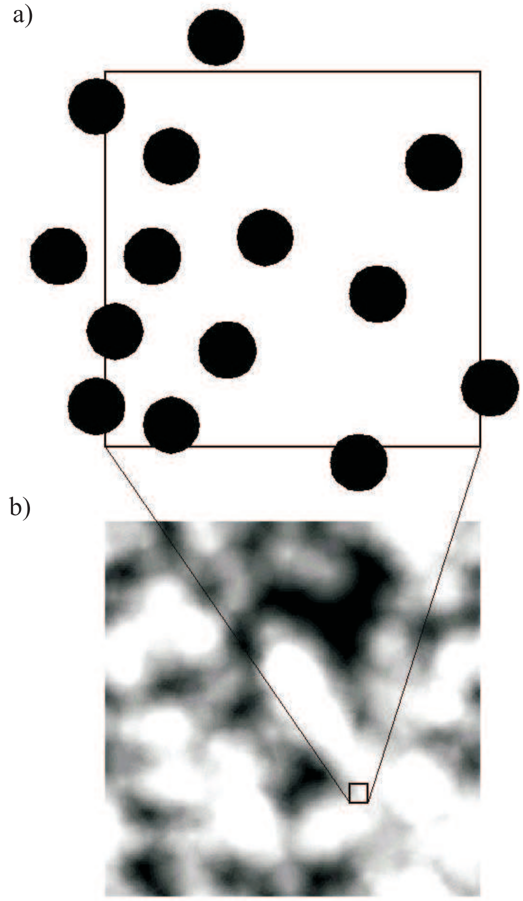

FIG. 1.

2. Preliminaries

With reference to Fig. 1a), let us consider a heterogeneous material B on microscale, i.e. a microstructure, made of two distinct phases $B^{(\alpha)}, \alpha = 1, 2$, distributed in a disordered fashion according to an indicator function:

$$\chi^{(\alpha)}(\mathbf{x}) = \begin{cases} 1 & \text{if} \quad \mathbf{x} \in B^{(\alpha)}, \\ 0 & \text{if} \quad \mathbf{x} \notin B^{(\alpha)}. \end{cases} \tag{2.1}$$

Here $B^{(1)}$ is the matrix and $B^{(2)}$ the inclusion, and the whole body $B = B^{(1)} \cup B^{(2)}$.

Equation (2.1) pertains to a particular realization $B(\omega)$ of the microstructure $\mathcal{B} = \{B(\omega); \omega \in \Omega\}$, so that $\chi^{(\alpha)}(\mathbf{x})$ in (2.1) really stands for $\chi^{(\alpha)}(\omega, \mathbf{x}), \omega \in \Omega$.

The ensemble of these functions makes up a random field: $\chi = \{\chi^{(\alpha)}(\omega, \mathbf{x}); \omega \in \Omega, \mathbf{x} \in X\}$. As usual, ω pertains to a single realization of the random medium $\mathcal{B} = \{B(\omega); \omega \in \Omega\}$. Formally, $\mathcal{B}$ is an ensemble of deterministic realizations $B(\omega)$, while Ω is the probability space endowed with a σ-algebra and a probability measure.

We assume χ to be a strict-sense stationary (SSS) and isotropic random field: all the n-point probability distributions of χ are invariant with respect to arbitrary shifts and rotations in the material domain. We also require χ to be mean-ergodic on domains of infinite extent.

Now, each phase is a homogeneous continuum described by an isotropic anti-plane stiffness tensor $\mathbf{C}^{(\alpha)}$, $\alpha = 1, 2$. It follows that

$$\mathbf{C}(\omega, \mathbf{x}) = \sum_{\alpha=1}^{2} \chi^{(\alpha)}(\omega, \mathbf{x})\mathbf{C}^{(\alpha)} \tag{2.2}$$

is a realization of a discrete-valued, continuous parameter random tensor field

$$\{\mathbf{C}(\omega, \mathbf{x}); \omega \in \Omega, \mathbf{x} \in \mathbb{R}^2\} \tag{2.3}$$

which, in view of the properties of (2.1), is also SSS, isotropic, and mean-ergodic.

To pass from the microscale of Fig. 1a to some larger mesoscale of Fig. 1b we determine a mesoscale stiffness $\mathbf{C}_\delta(\omega, \mathbf{x})$ for any finite (i.e. mesoscale) $L \times L$ domain $B_\delta(\omega)$ in (x_1, x_2)-plane according to the HILL (or Hill–Mandel) condition [6]

$$\overline{\sigma} \cdot \overline{\varepsilon} = \overline{\sigma \cdot \varepsilon}, \tag{2.4}$$

where the overbars denote spatial averages. The subscript δ, defined as

$$\delta = \frac{L}{d}, \tag{2.5}$$

is a non-dimensional mesoscale parameter, which covers the full range: from microscale ($\delta \to 0$, or continuum below the single inclusion level), up to the *macroscale* ($\delta \to \infty$). For any finite δ, we get a continuous-valued, continuous parameter random tensor field

$$\{\mathbf{C}_\delta(\omega, \mathbf{x}); \omega \in \Omega, \mathbf{x} \in \mathbb{R}^2\}. \tag{2.6}$$

It is well known that the necessary and sufficient condition for (2.4) to hold is

$$\int_{\partial B_\delta} (t - \overline{\sigma} \cdot \mathbf{n}(\mathbf{x})) \cdot (u - \overline{\varepsilon} \cdot \mathbf{x}) dS = 0. \tag{2.7}$$

This is satisfied by three different types of boundary conditions: *uniform displacement* (or Dirichlet) boundary condition

$$u(\mathbf{x}) = \overline{\varepsilon} \cdot \mathbf{x} \quad \forall \mathbf{x} \in \partial B_\delta; \tag{2.8}$$

uniform traction (or Neumann) boundary condition

$$t(\mathbf{x}) = \overline{\sigma} \cdot \mathbf{n}(\mathbf{x}) \quad \forall \mathbf{x} \in \partial B_\delta; \tag{2.9}$$

uniform displacement-traction (or *orthogonal-mixed*) boundary condition

$$(t(\mathbf{x}) - \overline{\sigma} \cdot \mathbf{n}(\mathbf{x})) \cdot (u(\mathbf{x}) - \overline{\varepsilon} \cdot \mathbf{x}) = 0 \quad \forall \mathbf{x} \in \partial B_\delta. \tag{2.10}$$

In (2.8)–(2.10) $\overline{\varepsilon}$ ($\equiv \overline{\varepsilon_j}$) and $\overline{\sigma}$ ($\equiv \overline{\sigma_i}$) denote constant tensors, prescribed *a priori*; $\mathbf{n}$ ($\equiv n_i$) is the outer unit normal to ∂B_δ.

Each of these boundary conditions, if applied to an arbitrary heterogeneous microstructure on a mesoscale $\delta < \infty$ results in a different stiffness or compliance tensors [7]; HUET and his co-workers use the adjective *apparent* [8, 9]. The limit $\delta \to \infty$ results in the random mesoscale properties converging towards the macroscopic stiffness tensor $\mathbf{C}^{\text{eff}}$ of the representative volume element (RVE). While most micromechanics studies [10–12] assume the RVE *a priori* without specifying its size, several studies indicate that the RVE may require extremely large length scales relative to the microscale, e.g. [7, 13]. Thus, mesoscale fluctuations shown in Fig. 1b cannot easily be ignored.

3. Micromechanics viewpoint

Evidently, we have three loadings to choose from, and hence, apparently, a non-uniqueness of response on mesoscale. Now, in place of the disordered microstructure of Fig. 1a described by equations (2.1)–(2.2), let us consider a special case of a smooth elastic continuum (1.3) with variability of $C(\mathbf{x})$ in x_1-direction only. By $C_{\delta(ij)}$ we denote the components of the mesoscale stiffness $\mathbf{C}_\delta$. We make analysis-type (A) observations:

A1: The mesoscale response $C_{\delta(11)}$ of $\mathbf{C}_\delta$ of the $L \times L$ window is calculated exactly under the assumption of a uniform stress $\sigma_1(\mathbf{x}) = \overline{\sigma_1}$, $\forall \mathbf{x} \in B_L$, because for this loading we have a smooth 'microstructure' of a series-type.

A2: The mesoscale response $C_{\delta(22)}$ of the $L \times L$ window is calculated exactly under the assumption of a uniform strain $\varepsilon_2(\mathbf{x}) = \overline{\varepsilon_2}$, $\forall \mathbf{x} \in B_L$, because for this loading we have a smooth 'microstructure' of a parallel-type.

A3: Loadings dictated by A1 and A2 jointly correspond to the special case of the boundary condition (2.7). Then, assuming any smooth function $C(\mathbf{x})$ in (1.3), $C_{\delta(ij)}$ on the left of equation (2.1) can be evaluated via integration of $C(\mathbf{x})$

over the mesoscale domain B_L. This would involve a calculation of compliance $S_{\delta(11)}$ and of stiffness $C_{\delta(22)}$, which, given the fact that the axes x_1 and x are oriented along the principal directions, would allow one to determine all the elements of $C_{\delta(ij)}$. Next, one can take the limit

$$\lim_{\delta \to 0} C_{\delta(ij)}(\mathbf{x}) = C\left(\mathbf{x}\right)\delta_{ij} \tag{3.1}$$

and recover the original $C\left(\mathbf{x}\right)\delta_{ij}$ [unfortunately, δ as the subscript stands for L/d, while δ_{ij} means Kronecker delta]. The foregoing shows which particular type of the mixed boundary condition is the proper way to make the passage from micro to mesoscale.

An *a priori* statement of the model such as (1.3) does not address the issue of a heterogeneous material microstructure from which the approximating inhomogeneous continuum – such as $C_{ij}\left(\mathbf{x}\right)$, or perhaps $C\left(\mathbf{x}\right)\delta_{ij}$ – should actually be derived. However, the observation A3 of the previous section suggests what loading should actually be introduced when passing from the microstructure to a mesoscale continuum. Let us denote this mesoscale response by $\mathbf{C}_\delta^{\mathrm{mix}}(\mathbf{x},\omega)$, and, in view of Fig. 1a), immediately remark that $\mathbf{C}_\delta^{\mathrm{mix}}(\mathbf{x},\omega)$ will be anisotropic. In particular, assuming the gradient of microstructural composition to be in x_1, the $C_{\delta(11)}^{\mathrm{mix}}$ response is softer than the $C_{\delta(2)}^{\mathrm{mix}}$ response! Additionally, because of the disorder and overall lack of symmetry, $C_{\delta(12)}^{\mathrm{mix}} \neq 0$.

Furthermore, using the boundary conditions (2.5) through (2.6) we obtain, respectively, mesoscale tensors $\mathbf{C}_\delta^D(x_1,\omega)$ and $\mathbf{S}_\delta^N(x_1,\omega)$; here ^{D} stands for Dirichlet (2.8) and ^{N} for Neumann (2.9) conditions. Now, the important thing to observe is that [8, 9]

$$\mathbf{C}_\delta^D(x_1,\omega) \leq \mathbf{C}_\delta^{\mathrm{mix}}(x_1,\omega) \leq \left[\mathbf{S}_\delta^N(x_1,\omega)\right]^{-1}. \tag{3.2}$$

We make micromechanics-type (M) observations:

M1: Each one of the three tensors in (3.2) is anisotropic.

M2: As the mesoscale $\delta \to \infty$, the tensors $\mathbf{C}_\delta^D(x_1,\omega)$, $\mathbf{C}_\delta^{\mathrm{mix}}(\mathbf{x},\omega)$ and $\left[\mathbf{S}_\delta^N(x_1,\omega)\right]^{-1}$ converge to $\mathbf{C}^{\mathrm{eff}}$, albeit far above the scale of all the fluctuations, i.e. larger than Fig. 1b). Of course, this is assured by the "separation of scales" limit (assured in turn by the ergodic and SSS properties of the microstructure) where the homogeneous continuum applies – a situation of no interest to us because fluctuations of Fig. 1b) arise below that limit.

M3: The ensemble average tensors $\left\langle\mathbf{C}_\delta^D(x_1)\right\rangle$, $\left\langle\mathbf{C}_\delta^{\mathrm{mix}}(x_1)\right\rangle$ and $\left\langle\mathbf{S}_\delta^N(x_1)\right\rangle^{-1}$ are orthotropic; a computed example was given in a study of thermal conductivities of functionally graded composites in 2-D [14], which, by a well-known analogy, is equivalent to anti-plane elasticity of the same microstructure.

Note that the inequalities (3.2) apply to all the 'well behaved' realizations of microstructure, except the extremely rare cases such as a microstructure covering a large extent with one phase only. Thus, we can say that (3.2) holds almost surely (i.e. with probability zero on Ω) and for any finite mesoscale ($L < \infty$).

4. Correlation theory viewpoint

Let us take an ensemble picture: the random medium $\mathcal{B}$ on the microscale (Fig. 1a) is described by statistics which are strict-sense stationary and isotropic. It follows that, the random medium homogenized on the mesoscale δ (Fig. 1b) is, at the very least, described by a wide-sense stationary (WSS) and isotropic random field $\mathcal{C}_\delta$. The mean-ergodic property is preserved at $\delta \to \infty$. We thus have a random tensor field $\mathcal{C}_\delta = \{\mathbf{C}_\delta(\mathbf{x},\omega); \mathbf{x} \in X, \omega \in \Omega\}$ with smooth realizations $\mathbf{C}_\delta(\mathbf{x},\omega)$. Then, the normalized correlation function of a second-rank tensor is a fourth-rank tensor:

$$\rho_{ijkl}(\mathbf{x}_1,\mathbf{x}_2) = \frac{\langle [C_{\delta(ij)}(\mathbf{x}_1) - \langle C_{\delta(ij)}(\mathbf{x}_1)\rangle][C_{\delta(kl)}(\mathbf{x}_2) - \langle C_{\delta(kl)}(\mathbf{x}_2)\rangle]\rangle}{\sigma_{ij}(\mathbf{x}_1)\sigma_{kl}(\mathbf{x}_2)}, \tag{4.1}$$

where $\sigma_{ij}(\mathbf{x}_1)$ and $\sigma_{kl}(\mathbf{x}_2)$ are the standard deviations of the pair $[C_{\delta(ij)}(\mathbf{x}_1), C_{\delta(kl)}(\mathbf{x}_2)]$ at respective points.

As is well known, there are two interpretations of stationarity and isotropy of random tensor fields:

(i) Conventional stationarity

$$\rho_{ijkl}(\mathbf{x}_1,\mathbf{x}_2) = \rho_{ijkl}(\mathbf{x}), \tag{4.2}$$

for any $\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1$, followed by conventional isotropy

$$\rho_{ijkl}(\mathbf{x}) = \rho_{ijkl}(x), \tag{4.3}$$

where x is the length of $\mathbf{x}$.

(ii) Stationarity and isotropy involving a simultaneous rotation of the coordinate system when rotating the distance vector $\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1$: a random tensor field is said to be stationary and isotropic when the mean $\langle C_{\delta(ij)}(\mathbf{x})\rangle$ (i.e. $\langle \mathbf{C}_\delta \rangle$) and the correlation $\rho_{ijkl}(\mathbf{x})$ do not change upon the rotation of $\mathbf{x}$ into $\mathbf{x}' = Q\mathbf{x}$, which is accompanied by an appropriate transformation of $\mathbf{C}_\delta$ into $\mathbf{C}'_\delta = QQ\mathbf{C}_\delta$, corresponding to this rotation. In other words,

$$\langle \mathbf{C}'_\delta \rangle = QQ \langle \mathbf{C}_\delta \rangle, \ \ \rho_{ijkl}(\mathbf{x}') = QQQQ\rho_{ijkl}(\mathbf{x}). \tag{4.4}$$

In the following we work with the latter interpretation. First, note that the symmetry $C_{\delta(ij)} = C_{\delta(ji)}$ implies these symmetries of ρ_{ijkl}

$$\rho_{ijkl} = \rho_{jikl} = \rho_{ijlk}. \tag{4.5}$$

Now, recall after ROBERTSON [15] and LOMAKIN [16] (also [2]) that $\rho_{ijkl}(x)$ in 3-D admits this representation

$$\begin{aligned}\rho_{ijkl}(x) = K_4(x)\,\delta_{ij}\delta_{kl} + K_6(x)\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right] \\ + \left[K_5(x) - K_6(x)\right]\left[n_j n_k \delta_{il} + n_i n_l \delta_{jk} + n_i n_k \delta_{jl} + n_j n_l \delta_{ik}\right] \\ + \left[K_3(x) - K_4(x)\right]\left[n_i n_j \delta_{kl} + n_k n_l \delta_{ij}\right] \\ + \left[K_1(x) + K_2(x) - 2K_3(x) - 4K_5(x)\right] n_i n_j n_k n_l,\end{aligned} \tag{4.6}$$

wherein the K_i's are

$$\begin{aligned} K_1 = \rho^1_{1111}, \qquad K_2 = \rho^1_{2222}, \qquad K_3 = \rho^1_{1122}, \\ K_4 = \rho^1_{2233}, \qquad K_5 = \rho^1_{1212}, \qquad K_6 = \rho^1_{2323}, \end{aligned} \tag{4.7}$$

and $n_i = x_i/x$. Here ρ^1_{ijkl} stands for the correlation between $C_{\delta(ij)}(\mathbf{x}_1)$ and $C_{\delta(kl)}(\mathbf{x}_2)$ in a coordinate system centered at $\mathbf{x}_1$ and directed to $\mathbf{x}_2$.

Now, if we assume local isotropy of each realization ($C_{\delta(ij)}(\omega, \mathbf{x}) = C_\delta(\omega, \mathbf{x})\,\delta_{ij}$) as per equation (1.3), we conclude that

$$\rho^1_{1212} = \rho^1_{2323} = 0, \tag{4.8}$$

with the obvious symmetries (4.5) present, and by (4.7) implies

$$K_5 = K_6 = 0. \tag{4.9}$$

Clearly, the 2-D setting of anti-plane elasticity is a special case of this, and we come to conclude that, in particular, $\rho_{1212}(x) = 0$; it must be because $C_{\delta(12)} = 0$ everywhere.

At this point we recall our computational mechanics studies of $\rho_{ijkl}(\mathbf{x})$ in three kinds of SSS and isotropic random two-phase materials [17, 18]:

- random chessboard;
- matrix-inclusion composite with circular inclusions being centered at the points of a 'Poisson point field with exclusion' (so as to prevent mutual disks' overlaps);
- matrix-inclusion composite with circular inclusions being centered at the points of a Poisson point field.

The figures presented in those papers clearly show that ρ_{ijkl} is strongly dependent on the particular pair $[C_{ij}, C_{kl}]$ as well as on the direction $\mathbf{x}$. Here, we make correlation-type (C) observations:

C1: As expected, ρ_{1111} transforms into ρ_{2222} upon the rotation of $\mathbf{x} = (x_1, 0)$ into $\mathbf{x}' = (0, x_2)$.

C2: Against expectation, $\rho_{1212} \neq 0$, which indeed can be understood from physical considerations alone, especially at $\mathbf{x} = \mathbf{0}$, without any recourse to numerics. This provides an argument via *reductio ad absurdum* against the admissibility of (1.3–1.4).

C3: Only ρ_{1112} and ρ_{2212} turn out to be null.

The $\rho_{ijkl}(\mathbf{x})$ function in [17, 18] was computed separately for tensors $\mathbf{C}_{\delta}^{D}$ and $\mathbf{S}_{\delta}^{N}$, but, since the mixed loading discussed in A3 involves less symmetric loadings than either uniform strain or uniform stress, the $\rho_{ijkl}(\mathbf{x})$ function for $\mathbf{C}_{\delta}^{mix}$ would exhibit even stronger anisotropic effects.

5. Conclusions

This quote from [19] might be a fitting motto of our paper "*Continuum mechanics presumes nothing regarding the structure of matter.*" Evidently, generalizing the notion of a uniform isotropic elastic continuum to an inhomogeneous, smooth and isotropic one does not appear to violate any principles of continuous media. However, when the micromechanics is brought into the analysis of anti-plane elastic response (1.3–1.4) – indeed, one of the simplest models in continuum mechanics – we arrive at two contradictions.

First, in terms of a single realization $\mathbf{C}_{\delta}(\omega)$ we see that the mesoscale response of a finite domain of a heterogeneous material is anisotropic, simply because the stiffness in the direction of the gradient in microstructural composition is lower than that in the direction orthogonal to that gradient (observation M1). This is consistent with the Hill condition.

Secondly, if we consider an ensemble $\mathcal{C}_{\delta}$ of smooth stiffness tensor fields $\mathbf{C}_{\delta}(\omega)$ with local isotropy, and adopt a wide-sense stationarity and isotropy property – which, actually, is much weaker than the strict-sense stationarity and isotropy of the underlying microstructure - we arrive at a correlation function inconsistent with the correlation function obtained directly by computational mechanics for several types of microstructures (observation C2). This conclusion is consistent with the Hill condition for every realization of $\mathcal{C}_{\delta}$.

We conclude with the following notes:

1. In order to set up a smoothly inhomogeneous elastic medium for any realization of the ensemble, one must introduce its local anisotropy.

2. By virtue of the well known mathematical analogies, the results presented here have related consequences in many other areas of continuum physics.

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