

An asymptotic analysis of convection in boundary layer flows in the presence of a chemical reaction

S. SHATEYI⁽¹⁾, P. SIBANDA⁽²⁾, S. S. MOTSA⁽³⁾

⁽¹⁾*Department of Mathematics, Bindura University,
P Bag 1020, Bindura, Zimbabwe.*

⁽²⁾*School of Mathematics, University of KwaZulu-Natal,
Private Bag X01 Scottsville 3209, Pietermaritzburg, South Africa.*

⁽³⁾*Mathematics Department, University of Swaziland,
Private Bag 4 Kwaluseni, Swaziland.*

IN THIS STUDY we investigate the stability of two-dimensional disturbances imposed on a boundary layer flow over a semi-infinite flat plate in the presence of a reacting chemical species. Species concentration levels are assumed to be small, what is typical for many processes in water and in atmospheric air. We exploit the multi-deck structure of the flow in the limit of large Reynolds numbers to analyze asymptotically the perturbed flow. The neutral eigenrelations are obtained implicitly and limiting cases for large buoyancy and reaction kinematics are investigated. The results show some interesting effects of the Damkohler number on the wave number and wave speed of the disturbed flow.

Key words: asymptotic analysis, buoyancy, boundary layer, heat transfer.

1. Introduction

MOST RECENT studies on forced and free convection flows have used numerical approaches to investigate the influence of fluid buoyancy and other transport parameters on boundary layer flows. In most cases (see for instance, SELIM *et al.* [13]), the governing equations have been reduced to local non-similarity boundary layer equations using suitable transformation variables and then integrated, using a finite difference scheme and/or a Keller-box technique. The present work is motivated by the need to consider the convection flow problem from the framework of high Reynolds number asymptotic methods based on the triple deck and multi-deck ideas. The self-consistent asymptotic approach allows for a full solution of the Navier–Stokes equations and may enable to extend this study by permitting the incorporation of the effects of, *inter alia*, boundary layer non-parallelism, nonlinearity and unsteadiness. This approach has its genesis in the work of SMITH [14, 15], SMITH and BODONYI [16, 17] and

GAJJAR and SMITH [2]. For more recent contributions, see MOTSA *et al.* [9] and the references therein.

Natural convection is an important phenomenon in geophysics and astrophysics. Some natural convection flows in the atmosphere and micro-meteorological phenomena are often caused by combined effects of the temperature gradients and differences in concentrations of dissimilar chemical species. The heating of the earth by sunlight causes atmospheric thermal convection, which may be modified by the presence of moisture evaporated from the ground.

Convection in which the buoyant forces are due to both the temperature and chemical concentration gradients are generally referred to as thermosolutal or double diffusive convection. Various modes of convection are possible, depending on how the temperature and concentration gradients are oriented relative to one another, as well as to gravity, OSTRACH [12].

Thermosolutal convection caused by stable vertical concentration distribution with heating from the side or from the bottom is important in large water bodies such as in lakes and oceans. Consequently, thermosolutal convection has been widely investigated by oceanographers, see for instance, TURNER [19]. Here the stability theory has been used to explain the occurrence of layered structures observed in the ocean. TURNER [19] gave a systematic survey of the problems.

WANG *et al.* [21] experimentally investigated the physical phenomenon and obtained the heat and mass transfer data in a thermosolutal convection system. They found that the mass transfer rate increased with increasing thermal Grashof numbers. Their results showed that doubling temperature differences would increase the Sherwood number by 17% in their system.

Processes in which the buoyancy driving forces arise solely due to temperature differences and flows arising from differences in concentration or material constitution in conjunction with temperature effects, have received considerable attention for both the steady and transient internal flows, both laminar and turbulent.

Much information on simultaneous heat and mass transfer in laminar free convection boundary layer flows over plates can be found in the monograph by GEBHART *et al.* [3] and in the papers by KHAIR and BEJAN [5], LIN and YU [6, 7] and MONGRUEL *et al.* [8]. Many of the most important general characteristics and mechanisms of such flows have been clarified.

GEBHART and PERA [4] investigated natural convection flows caused by the simultaneous diffusion of thermal energy and chemical species. They assumed small species concentration levels and have shown that the Boussinesq approximations led to similarity solutions similar in form to those found for single buoyancy mechanism flows. MULONANI and RAHMAN [10] presented a theoretical study of laminar natural convection flow caused by chemical diffusion and reaction from a vertical plate surface. Their numerical results were based on the

fourth order Runge–Kutta method for the Schmidt number ranging from 0.0 to 100.0 and reaction orders of magnitude ranging from 0.0 to 1.5. Their results showed that when chemical reaction occurs, diffusion and velocity domains expand out from the plate.

In MUREITHI *et al.* [11], the effect of heat transfer on the upper-branch stability of Tollmien–Schlichting waves in accelerating boundary layers over a rigid surface in incompressible flows was investigated. Their analysis showed that buoyancy has generally a destabilizing effect on rigid surfaces. They also showed that in the presence of strong buoyancy forces, the five-zone structure is altered. However, for moderate buoyancy, the five-zone asymptotic structure of Smith and BODONYI [17] persists with some minor modifications. MOTSA *et al.* [9] showed that in the case of flow over a compliant boundary there are cases where large buoyancy leads to modes which are more stable than the instability modes which arise in the absence of buoyancy.

The present work presents an asymptotic analysis of the flow induced by buoyancy effects due to the diffusion of chemical species adjacent to horizontal surfaces. The flow has uniform surface conditions with the buoyancy effect primarily away from the surface. Our analysis is limited to processes which occur at low concentration gradients and moderate buoyancy. This allows the retention of the multi-deck flow structure and the use of the self-consistent asymptotic methods. We present an asymptotic investigation of the interactions between the chemical kinematics and the fluid hydrodynamics with the Damkohler number, Da , as the parameter of primary interest. The Damkohler number is the ratio of the flow time scale of the fluid to that of the chemical reaction time scale. In the limit of large Damkohler numbers, the chemical kinematics proceeds at a much faster rate compared to the fluid hydrodynamics. If the Damkohler number is close to zero, the chemical reactions are slow compared to the motion of the fluid. In this case, a non-reactive fluid can be assumed. Only when the Damkohler number is of the order of unity, we can anticipate the greatest interaction between chemical reactions and fluid dynamics.

In this study we primarily focus attention on the case when the Damkohler number is large. The objective is to determine the influence of fluid buoyancy that is due to the density stratification caused by the chemical concentration differences between the reacted and unreacted fluids and to determine the influence of the Damkohler number on the stability characteristics of the upper-branch Tollmien–Schlichting waves.

The essential difference between the current work and that of MUREITHI *et al.* [11] arises from the introduction of a chemical concentration and the absence of heating of the wall. The absence of wall compliance and wall heating or cooling, and the presence of a chemical species make the current work different from MOTSA *et al.* [9].

2. Mathematical formulation

We consider a two-dimensional, incompressible fluid flow over a flat plate which is composed of a reacting chemical species that is maintained at a fixed concentration. The chemical species is assumed to be diffusing into the nearby fluid inducing a buoyancy force. We shall also assume that the space and time scales of fluid dynamics and chemical reactions are much larger than those of thermodynamics. Thus, the thermodynamic process is always considered to be in equilibrium.

The equations governing a two-dimensional incompressible fluid flowing over a horizontal plate which is composed of a chemical species, expressed in dimensionless form are, under a Boussinesq-type approximation, given by:

$$\begin{aligned}
 & \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\
 (2.1) \quad & \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{\text{Re}} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \\
 & \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{\text{Re}} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + G_c C, \\
 & \frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = \frac{1}{\text{ReSc}} \left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} \right) - \text{Da}C,
 \end{aligned}$$

where x and y are the streamwise and normal coordinates respectively, u and v are the streamwise and normal velocity components respectively, t is the time, C is the species chemical concentration and p is the pressure. In addition, Sc ($= \mu/D\rho$) is the Schmidt number, μ is the dynamic viscosity of the fluid, ρ is the density of the fluid and Da ($= \tau_c/\tau_A$) is the Damkohler number, defined as the ratio of the flow time scale to the chemical time scale, where $\tau_c = 1/k_c$ with k_c being the chemical reaction rate and τ_A is the advective fluid time scale. The parameter G_c is a buoyancy parameter term defined by $G_c = \text{Gr}/\text{Re}^2$ where $\text{Gr} = \beta g L^3 (C_w - C_\infty)/\nu^2$ is the Grashof number, β is the volumetric coefficient of expansion with concentration, ν ($= \mu/\rho$) is the kinematic viscosity and g is the acceleration due to gravity and $\text{Re} = U_\infty L/\nu$ is the Reynolds number of the flow.

Equation (2.1) has been nondimensionalised by making the following substitutions: $(x^*, y^*) = L(x, y)$, $(u^*, v^*) = U_\infty(u, v)$, $p^* = \rho U_\infty^2 p$, $t^* = (L/U_\infty)t$, $C = (C^* - C_\infty)/(C_w - C_\infty)$, where L is the characteristic length scale (for example the distance measured from the leading edge of the plate). The asterisks denote dimensional quantities and the ‘ w ’ subscript refers to the values of the quantities at $y = 0$. Also the subscript ‘ ∞ ’ refers to the free stream values.

At the boundary $y = 0$, we assume no-slip conditions, therefore the boundary condition on velocity is

$$(2.2) \quad u = v = 0.$$

The horizontal plate is maintained at a uniform concentration, so we assume that the concentration is prescribed as C_{Bw} . In the absence of any disturbances and for large Reynolds number, the basic boundary-layer flow velocity, concentration and pressure have the form:

$$(2.3) \quad (u, v, p) = (U_B, \text{Re}^{-1/2}V_B, P_B)(x, Y) + \dots, \quad C = C_B(x, Y) + \dots,$$

where $Y = \text{Re}^{1/2}y$ is the boundary coordinate, U_B and V_B are the basic velocity components in the streamwise and normal directions, respectively, C_B is the species chemical concentration and p_B is the basic pressure within the boundary layer. Substituting Eqs. (2.3) into the Navier–Stokes equations (2.1) and taking the limit $\text{Re} \rightarrow \infty$, we obtain the following equations for the undisturbed boundary layer:

$$(2.4) \quad \frac{\partial U_B}{\partial x} + \frac{\partial V_B}{\partial Y} = 0,$$

$$(2.5) \quad U_B \frac{\partial U_B}{\partial x} + V_B \frac{\partial U_B}{\partial Y} = -\frac{\partial p_B}{\partial x} + \frac{\partial^2 U_B}{\partial Y^2},$$

$$(2.6) \quad U_B \frac{\partial C_B}{\partial x} + V_B \frac{\partial C_B}{\partial Y} = -\frac{1}{Sc} \frac{\partial^2 C_B}{\partial Y^2} - \text{Da}C_B,$$

$$(2.7) \quad \frac{\partial p_B}{\partial Y} = \text{Re}^{-1/2}G_c C_B,$$

with the non-slip boundary conditions on the velocity components

$$U_B = V_B = 0 \quad \text{and} \quad C_B = C_{Bw} \quad \text{on} \quad Y = 0.$$

For general pressure gradient boundary layers, we assume that the basic flow has the following additional properties:

$$\begin{aligned} U_B &\sim \lambda_1 Y + \lambda_2 Y^2 + \dots, & \text{as } Y \rightarrow 0, \\ C_B &\sim \mu_0 + \mu_1 Y + \dots, & \text{as } Y \rightarrow 0, \\ U_B &\rightarrow 1, \quad C_B \rightarrow 0 & \text{as } Y \rightarrow \infty. \end{aligned}$$

Details of basic flow properties can be found in STEWARTSON [18]. The coefficients $\lambda_1 = U_{By}|_{y=0} > 0$ and $\lambda_2 = U_{Byy}|_{y=0} < 0$ are respectively the skin friction and curvature of the basic flow profile. The coefficients μ_0 and μ_1 are the concentration transfer coefficients.

When the buoyancy term G_c is $O(1)$, the right-hand side of Eq. (2.7) is asymptotically small so that the velocity and concentration fields are decoupled, becoming fully coupled only when $G_c \sim O(\text{Re}^{1/2})$. For large buoyancy equations (2.4)–(2.7) reduce to the Taylor–Goldstein equation similar to that in DENIER and MUREITHI [1].

We next introduce infinitesimal perturbations to the basic flow of size σ (where $\sigma \ll 1$) and consider the stability of the basic flow. For $O(1)$ values of the buoyancy parameter, G_c , the momentum and concentration fields are decoupled. This allows us to adopt the five-tiered structure of SMITH and BODONYI [17] to investigate the stability of the flow in the presence of a chemical reaction. This structure (see Fig. 1) consists of five regions: the main part of the boundary layer $R1$ of thickness $O(\text{Re}^{-1/2})$, a thinner inviscid adjustment region $R2$ of thickness $O(\text{Re}^{-7/12})$ containing the critical-layer region $R3$, the viscous wall layer $R4$ of thickness $O(\text{Re}^{-2/3})$ and finally, the potential flow zone $R5$ of thickness $O(\text{Re}^{-5/12})$.

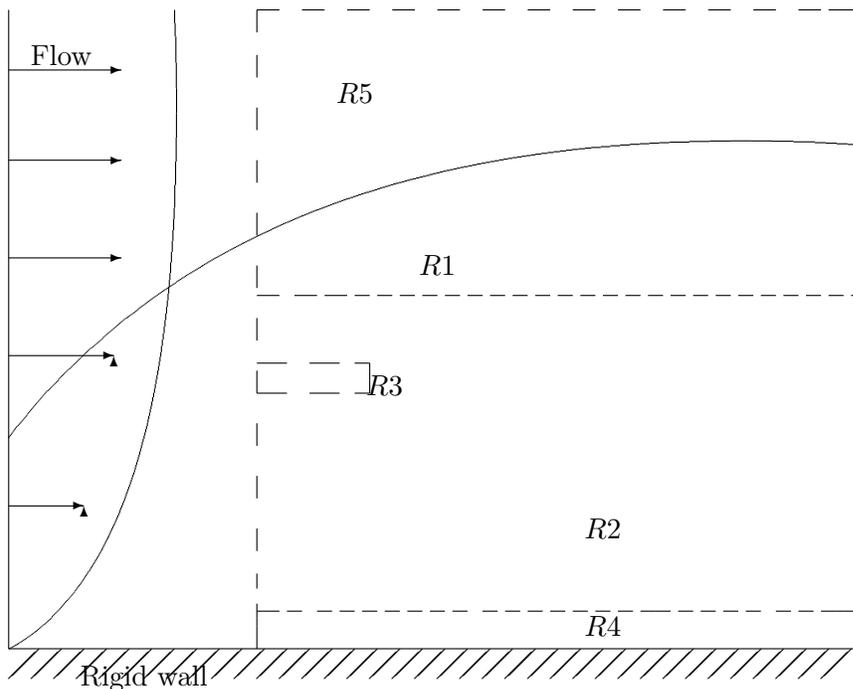


FIG. 1. Schematic sketch of flow structure showing the multi-layered nature of the boundary layer and the relative positioning of the five regions.

The upper branch scalings are now well known, and for large Reynolds number, Re , we define the small parameter $\epsilon = \text{Re}^{-1/12}$. To allow for wave modulation scaled streamwise and for temporal variables relevant to the upper branch of the

curve of the neutral stability, $x = \epsilon^5 X$, $t = \epsilon^4 \tau$ are introduced and we consider small disturbances that are proportional to $E = \exp[i(\alpha X - \alpha c \tau)]$, where α is the wave number and c is the wave speed of the perturbed wave. In MUREITHI *et al.* [11] it was shown that when $G_c = O(\epsilon^{-5})$, the Tollmien–Schlichting (TS) eigenrelation is first significantly altered. Following a similar analysis, we found that when $Da \sim O(\epsilon^{-4})$ that its effect becomes significant. In order to work with quantities of $O(1)$, we set $G_c = \epsilon^{-5} G_0$, $\alpha = \epsilon^{-5} \alpha_0$, $Da = \epsilon^{-4} D_0$, $\omega = -\epsilon^{-4} \omega_0$ and $c = -\epsilon^{-4} c_0$.

The derivatives $\partial/\partial x$, $\partial/\partial t$ in the governing equations are now replaced with $\epsilon^{-5} \alpha_0 \partial/\partial X$ and $-\epsilon^{-4} \omega_0 \partial/\partial X$, where ω_0 is the frequency of the disturbances and we have expanded the wave number α and the frequency ω as $\alpha = \epsilon^{-5} \alpha_0 + \dots$, $\omega = -\epsilon^{-4} \omega_0 + \dots$, since we are interested in wave numbers of $O(\epsilon^5)$ and frequencies of $O(\epsilon^4)$.

The disturbance expansions relevant to the upper branch are now well known, and are given for example in GAJJAR and SMITH [2] and in MOTSA *et al.* [9]. In each region we are going to present only the solutions which will be used to find the eigenrelations.

2.1. Region I: The main boundary layer

This region encompasses most of the boundary layer and is scaled on the thickness of the boundary layer. Here then we define $y = \epsilon^6 Y$, where $Y = O(1)$, and expand the disturbances quantities as:

$$(2.8) \quad \begin{aligned} \bar{u} &= U_B + \sigma(u_0 + \epsilon^2 u_1 + \dots), & \bar{v} &= \sigma \epsilon (v_0 + \epsilon^2 v_1 + \dots), \\ \bar{C} &= C_B + \sigma(C_0 + \epsilon^2 C_1 + \dots), & \bar{p} &= P_B + \sigma \epsilon^2 (p_0 + \epsilon^2 p_1 + \dots), \end{aligned}$$

where the u_i , v_i , etc. are functions of the boundary layer variable Y and of the slow spatial variable X ; σ is the measure of the size of disturbances. Substituting (2.8) into the linearized equations and solving yields the following first order solutions:

$$(2.9) \quad \begin{aligned} u_0 &= A_0 U_{BY}, & v_0 &= -\alpha_0 A_{0X} U_B, & C_0 &= A_0 C_{BY}, \\ p_0 &= P_0 + G_0 A_0 (\theta_B - R_0). \end{aligned}$$

At the second order, $O(\sigma^2)$, the solutions are found to be

$$(2.10) \quad \begin{aligned} v_1 &= \alpha_0 c_0 A_{0X} + \alpha_0 U_B \int_0^Y \frac{P_{0X}}{U_B^2} dY - \alpha_1 A_{0X} - \alpha_0 A_{1X} U_B \\ &\quad + \alpha_0 U_B G_0 A_{0X} \int_0^Y \frac{(C_B - \mu_0)}{U_B^2} dY, \end{aligned}$$

$$(2.10)_{\text{[cont.]}} \quad p_1 = P_1 - \alpha_0^2 A_0 \int_0^Y U_B^2 dY - \frac{G_0 A_0 D_0}{\alpha_0} \int_0^Y \frac{C_{BY}}{U_B} dY + G_0 A_1 C_B \\ - G_0 \int_{Y_1}^Y C_{BY} \left(\int_0^{Y_1} \frac{[P_0 + G_0 A_0 (C_B - \mu_0)]}{U_B^2} dY_1 \right) dY,$$

where $A_i = A_i(X)$, $P_i = P_i(X)$, ($i = 0, 1$) are unknown functions representing the displacement effect and the pressure effect. In the above results we set $A_i = \bar{A}_i e^{iX} + c.c$, $P_i = \bar{P}_i e^{iX} + c.c$, where $c.c$ denotes the complex conjugate.

2.2. Region II:

In this region we define $y = \epsilon^7 \bar{Y}$ with $\bar{Y} = O(1)$ and the expansions follow from Region I:

$$(2.11) \quad (\bar{u}, \bar{v}, \bar{C}, \bar{p})^T = \begin{cases} \lambda_1 \epsilon \bar{Y} + \epsilon^2 \lambda_2 \bar{Y}^2 + \sigma(u^{(0)} + \epsilon u^{(1)} + \dots), \\ \sigma \epsilon (\epsilon v^{(0)} + \epsilon^2 v^{(1)} + \dots), \\ \mu_0 + \epsilon \mu_1 \bar{Y} + \epsilon^2 \mu_2 \bar{Y}^2 + \sigma(C^{(0)} + \epsilon C^{(1)} + \dots), \\ p_B + \sigma(\epsilon p^{(0)} + \epsilon^2 p^{(1)} + \dots). \end{cases}$$

Substituting the above equations into the linearized governing equations and solving the resulting equations yields the following first order solutions:

$$u^{(0)} = \lambda_1 A_0, \quad v^{(0)} = -\frac{\alpha_0 p_X^{(0)}}{\lambda_1} - \alpha_0 A_{0X} \lambda_1 \xi, \\ p^{(0)} = P^{(0)}, \quad C^{(0)} = \frac{\mu_1}{\lambda_1^2} \left(A_0 \lambda_1^2 \xi + p^{(0)} \right) \left(\xi - \frac{i D_0}{\alpha_0 \lambda_1} \right)^{-1},$$

where $\xi = (Y - c_0/\lambda_1)$. Using the no-slip boundary conditions (2.2) gives:

$$P^{(0)} = c_0 \lambda_1 A_0.$$

At the next order we obtain:

$$(2.12) \quad v^{(1)} = -\frac{1}{\lambda_1} \alpha_0 P_X^{(1)} + \frac{\mu_1 G_0 \alpha_0 A_{0X}}{\lambda_1} (\xi \{\ln |\xi| + \phi^\pm\}) - \lambda_1 \alpha_0 \xi \bar{A}_{1X} \\ - \alpha_0 \lambda_2 A_{0X} \left(\xi^2 + \frac{2c_0}{\lambda_1} \xi \{\ln |\xi| + \phi^\pm\} - \frac{c_0^2}{\lambda_1^2} \right) - \frac{\alpha_0 \mu_1 c_0 G_0 A_{0X}}{\lambda_1^2} \\ - \frac{\mu_1 G_0 A_{0X}}{2\lambda_1^2} (\alpha_0 c_0 + i D_0) \ln \left| \xi^2 + \frac{D_0^2}{\alpha_0^2 \lambda_1^2} \right|$$

$$\begin{aligned}
(2.12) \quad & \underset{[\text{cont.}]}{+} \frac{\mu_1 \alpha_0 G_0 A_0 X}{D_0 \lambda_1} (\alpha_0 c_0 + i D_0) \xi \arctan \left(\frac{\alpha_0 \lambda_1 \xi}{D_0} \right) \\
& - \frac{i \mu_1 \alpha_0 G_0 A_0 X}{2 \lambda_1 D_0} (\alpha_0 c_0 + i D_0) \xi \left\{ \ln \left| \frac{D_0^2 + \alpha_0^2 \lambda_1^2 \xi^2}{\alpha_0^2 \lambda_1^2 \xi^2} \right| + \phi^\pm \right\} \\
& + \frac{i \mu_1 G_0 A_0 X}{\lambda_1^2} (\alpha_0 c_0 + i D_0) \left[\arctan \left(\frac{D_0}{\alpha_0 \lambda_1 \xi} \right) + \phi_t^\pm \right],
\end{aligned}$$

$$\begin{aligned}
(2.13) \quad & p^{(1)} = P^{(1)} + \mu_1 G_0 A_0 \left(\xi + \frac{c_0}{\lambda_1} \right) \\
& + \frac{\mu_1 G_0 A_0}{\alpha_0 \lambda_1} (\alpha_0 c_0 + i D_0) \left(\frac{1}{2} \ln \left| \xi^2 + \frac{D_0^2}{\alpha_0^2 \lambda_1^2} \right| \right) \\
& - \frac{i \mu_1 G_0 A_0}{\alpha_0 \lambda_1} (\alpha_0 c_0 + i D_0) \left[\tan^{-1} \left(\frac{D_0}{\alpha_0 \lambda_1 \xi} \right) + \phi_t^\pm \right].
\end{aligned}$$

The solutions in this region possess both the logarithmic and algebraic singularities in the limit $\xi \rightarrow 0$. At the point $\xi = 0$, the streamwise velocity U_B is equal to the local effective wave speed of the disturbances. The singularities are smoothed out by introduction of a thin viscous region, called the critical layer, in the neighbourhood of the critical level $\xi = 0$. The terms ϕ^\pm and ϕ_t^\pm represent the phase-shift introduced to connect the solutions in the normal velocity and pressure on either side of the critical level.

2.3. Region IV: The Wall Zone

The solutions found in Region II do not satisfy the no-slip conditions at the wall. We therefore introduce, as $y \rightarrow 0$, a thin viscous layer of thickness $O(\epsilon^8)$, in which the velocity components adjust to the no-slip condition at the wall. In this region, we then set $y = \epsilon^8 \zeta$ where ζ is an $O(1)$ coordinate and the flow expansions are:

$$(2.14) \quad (\bar{u}, \bar{v}, \bar{C}, \bar{p})^T = \begin{cases} \lambda_1 \epsilon^2 \zeta + \lambda_2 \epsilon^4 \zeta^2 + \cdots + \sigma \tilde{u}_0 + \cdots, \\ \sigma \epsilon^3 \tilde{v}_0 + \cdots, \\ \mu_0 + \mu_1 \epsilon^2 \zeta + \cdots + \sigma \tilde{C}_0 + \cdots, \\ p_B + \sigma \tilde{p}_0 + \cdots, \end{cases}$$

where $\tilde{u}_i, \tilde{v}_i, \tilde{p}_i, \tilde{C}_i$ for $i = 0, 1, 2, \dots$ are functions of ζ and X . Substituting these expansions into the governing equation (1.2) and then solving the resulting

disturbance differential equations, subject to the boundary conditions at the wall and matching (as $\zeta \rightarrow \infty$) with the results from R2 (as $\bar{Y} \rightarrow 0$) yields

$$\begin{aligned}\tilde{v}_0 &= \frac{i\alpha_0^2 \tilde{P}_0}{m\omega_0} (1 - m\zeta - e^{-m\zeta}), \\ \tilde{p}_0 &= \tilde{P}_0,\end{aligned}$$

where $m = (\alpha_0 c_0)^{1/2} e^{-i\pi/4}$.

2.4. Region V:

This is an outer potential-flow layer in which we define $y = \epsilon^5 \hat{y}$ where $\hat{y} \sim O(1)$. The expansions of the perturbations follow from the solutions of Region I in the limit $Y \rightarrow \infty$ and are given by:

$$(2.15) \quad (\bar{u}, \bar{v}, \bar{C}, \bar{p})^T = \begin{cases} 1 + \sigma\epsilon(\hat{u}_0 + \epsilon\hat{u}_1 + \dots), \\ \sigma\epsilon(\hat{v}_0 + \epsilon\hat{v}_1 + \dots), \\ \sigma\epsilon(\hat{C}_0 + \epsilon\hat{C}_1 + \dots), \\ p_B + \sigma\epsilon(\hat{p}_0 + \epsilon\hat{p}_1 + \dots). \end{cases}$$

From these expansions we obtain the following solutions:

$$(2.16) \quad \hat{u}_0 = -\hat{P}_0 e^{-\alpha_0 \hat{y}}, \quad \hat{v}_0 = -i\hat{P}_0 e^{-\alpha_0 \hat{y}} \quad \text{and} \quad \hat{p}_0 = \hat{P}_0 e^{-\alpha_0 \hat{y}},$$

where \hat{P}_0 is an unknown function which describes the disturbance pressure at the outer extreme of the boundary layer. The important solutions at the next order are:

$$(2.17) \quad \hat{v}_1 = -i[\hat{P}_1 - c_0 \hat{P}_0] e^{-\alpha_0 \hat{y}} \quad \text{and} \quad \hat{p}_1 = \hat{P}_1 e^{-\alpha_0 \hat{y}},$$

where \hat{P}_1 is an unknown function which describes the disturbance pressure at the outer extreme of the boundary layer.

3. Linear neutral results and eigenrelations

Asymptotically matching the velocity and pressure solutions in their respective overlap regimes, we obtain two important results. The first, found by matching the first order solutions across the entire boundary layer flow regime, is the dispersion relation

$$(3.1) \quad c_0 \lambda_1 = G_0 \mu_0 + \alpha_0.$$

Second order matching of pressure components between R1 (as $Y \rightarrow \infty$) and R5 (as $\hat{y} \rightarrow 0$) yields

$$(3.2) \quad \hat{P}_1 = P_1 - \alpha_0^2 A_0 I_0 - A_0 G_0 \left(J_0 + \frac{D_0}{\alpha_0} I_1 \right) + G_0 A_1 C_B^\infty,$$

where $C_B^\infty = \lim_{Y \rightarrow \infty} C_B$ and I_i, J_i for $i = 0, 1, 2$ are defined in the Appendix. Matching the pressure terms across R2 (as $\bar{Y} \rightarrow \infty$) and R1 (as $Y \rightarrow 0$) gives

$$(3.3) \quad P_1 = P^{(1)} - \frac{i\mu_1 G_0 A_0}{\alpha_0 \lambda_1} (c_0 \alpha_0 + iD_0) \phi_t^\pm - G_0 A_1 C_B^0,$$

where $C_B^0 = \lim_{Y \rightarrow 0} C_B$. Matching the pressure terms across R4 (as $\bar{Y} \rightarrow \infty$) and R2 (as $Y \rightarrow 0$) gives

$$(3.4) \quad \tilde{p}_1 = P^{(1)} + \frac{\mu_1 G_0 A_0}{2\alpha_0 \lambda_1} (iD_0 + \alpha_0 c_0) \ln \left| \frac{\alpha_0^2 \lambda_1^2 + D_0^2}{\alpha_0^2 \lambda_1^2} \right| \\ + \frac{i\mu_1 c_0 G_0 A_0}{\alpha_0 \lambda_1} (\alpha_0 c_0 + iD_0) \phi^- - \frac{\mu_1 G_0 A_0}{\alpha_0 \lambda_1} (\alpha_0 c_0 + iD_0) \tan^{-1} \left(\frac{D_0}{\alpha_0 c_0} \right).$$

Next, matching the normal velocity components between R1 (as $Y \rightarrow \infty$) and R5 (as $\hat{y} \rightarrow 0$) at the second order gives:

$$(3.5) \quad \hat{P}_1 = -c_0 \hat{P}_0 + \alpha_0 A_1 U_B^\infty - \alpha_0 c_0 A_0 - \alpha_0 U_B^\infty P_0 I_2 - \alpha_0 U_B^\infty G_0 A_0 J_1,$$

where $U_B^\infty = 1$ is the free stream velocity. Matching the normal velocity across regions R2 (as $\bar{Y} \rightarrow \infty$) and R1 (as $Y \rightarrow 0$) yields

$$(3.6) \quad B_0 A_{0X} + G_0 E_0 A_{0X} - \alpha_0 \lambda_1 A_{1X} = -\frac{2\alpha_0 \lambda_2 c_0 A_{0X}}{\lambda_1} \phi^+ \\ + \frac{\alpha_0 \mu_1 G_0 A_{0X}}{\lambda_1} \phi^+ - \alpha_0 \lambda_1 \bar{A}_{1X} - \frac{i\alpha_0 \mu_1 G_0 A_{0X} (\alpha_0 c_0 + iD_0) \phi_t^+}{2\lambda_1 D_0},$$

where the constants B_i and E_i $i = 0, 1, 2$ are defined in the Appendix.

Finally, matching of the normal velocity between R2 and R4 we obtain

$$(3.7) \quad \frac{i\alpha_0 \tilde{p}_0}{m c_0} = \frac{i\mu_1 \alpha_0 c_0 G_0 A_{0X}}{2\lambda_1^2 D_0} (\alpha_0 c_0 + iD_0) \phi^- \\ - \frac{\alpha_0 c_0 \mu_1 G_0 A_{0X}}{\lambda_1^2} \phi_t^- + \frac{i\mu_1 G_0 A_{0X}}{\lambda_1^2} (\alpha_0 c_0 + iD_0) \phi^- \\ - \frac{\alpha_0 c_0 \mu_1 G_0 A_{0X}}{D_0 \lambda_1^2} (\alpha_0 c_0 + iD_0) \tan^{-1} \left(\frac{\alpha_0 c_0}{D_0} \right)$$

$$\begin{aligned}
(3.7) \quad & - \frac{i\mu_1 G_0 A_{0X}}{\lambda_1^2} (\alpha_0 c_0 + iD_0) \tan^{-1} \left(\frac{D_0}{\alpha_0 c_0} \right) \\
& - \frac{\mu_1 G_0 A_{0X}}{2\lambda_1^2} (\alpha_0 c_0 + iD_0) \ln \left| \frac{D_0^2 + \alpha_0^2 c_0^2}{\lambda_1^2 \alpha_0^2} \right| \\
& + \frac{i\alpha_0 c_0 \mu_1 G_0 A_{0x}}{2\lambda_1^2 D_0} (\alpha_0 c_0 + iD_0) \ln \left| \frac{D_0^2 + \alpha_0^2 c_0^2}{\alpha_0^2 c_0^2} \right| \\
& - \frac{\alpha_0 P_X^{(1)}}{\lambda_1} + \frac{2\alpha_0 \lambda_2 c_0^2 A_{0X}}{\lambda_1^2} \phi^- + \alpha_0 c_0 \bar{A}_{1X} + A_{0X} (B_1 + G_0 E_1).
\end{aligned}$$

Eliminating \bar{A}_1 , $P^{(1)}$, P_0 and \hat{P}_0 gives a relation which determines the higher-harmonic components of A_1 . If we restrict our attention to the e^{iX} components, then after some algebra, Eq. (3.7) reduces to:

$$\begin{aligned}
(3.8) \quad & - \frac{\alpha_0 \lambda_1^2}{\sqrt{2\bar{m}}} - \frac{2\alpha_0 \lambda_2 c_0^2 \pi}{\lambda_1} + \frac{2c_0 \mu_1 \alpha_0 G_0 \pi}{\lambda_1} + \frac{\mu_1 \pi G_0 D_0}{\lambda_1} \\
& + \frac{D_0 \mu_1 G_0}{2\lambda_1} \ln \left| \frac{c_0^2 \alpha_0^2 + D_0^2}{\alpha_0^2 \lambda_1^2} \right| - \frac{\alpha_0^2 c_0^2 \mu_1 G_0}{2\lambda_1 D_0} \ln \left| \frac{D_0^2 + c_0^2 \alpha_0^2}{\alpha_0^2 c_0^2} \right| = 0,
\end{aligned}$$

where $\bar{m} = \sqrt{\alpha_0 c_0}$ and we have assumed the results for linear theory by taking the jump across the critical layer ϕ to be equal to $-i\pi$. We have also restricted our analysis to the real part of Eq. (3.7). Equations (3.1) and (3.8) are the crucial linear eigenrelations which fix the neutral wave number to the neutral wave speed.

4. Results and discussion

We begin by examining a certain interesting limiting behaviour of the neutral eigenrelations as the buoyancy parameter $G_0 \rightarrow +\infty$. The physical significance of the limit $G_0 \rightarrow +\infty$ corresponds to the increase in the buoyancy force through, for example, an increase in the density difference between the reacted fluid and the unreacted one. Solving the eigenrelations (3.1) and (3.8) we get, in the limit $G_0 \rightarrow +\infty$ with $D_0 \sim O(1)$

$$(4.1) \quad \alpha_0 = \left(\frac{\mu_1 \lambda_1}{\lambda_2} - \mu_0 \right) G_0 + \dots, \quad c_0 = \frac{\mu_1}{\lambda_2} G_0 + \dots.$$

These limiting case results are similar to the ones obtained by MOTSA *et al.* [9] when the buoyancy is due to the temperature differences. Solving the eigenrelations in the limit $D_0 \rightarrow +\infty$ with $G_0 \sim O(1)$, we get

$$(4.2) \quad \alpha_0 = \left(\frac{\mu_1 G_0 D_0 \ln |D_0|}{2\lambda_2 \pi} \right)^{1/3}, \quad c_0 = \left(\frac{\mu_1 \lambda_1^3 G_0 D_0 \ln |D_0|}{2\lambda_2 \pi} \right)^{1/3}.$$

This corresponds to the case when the chemical timescale is much more pronounced than that of the hydrodynamics of the flow. The asymptotic limit of the eigenrelations is

$$(\alpha_0, c_0) \sim O(D_0 \ln |D_0|)^{1/3}.$$

This limiting behaviour is confirmed by the results in Figs. 2–5. They show that the disturbances would grow without limit with increasing Damkohler numbers.

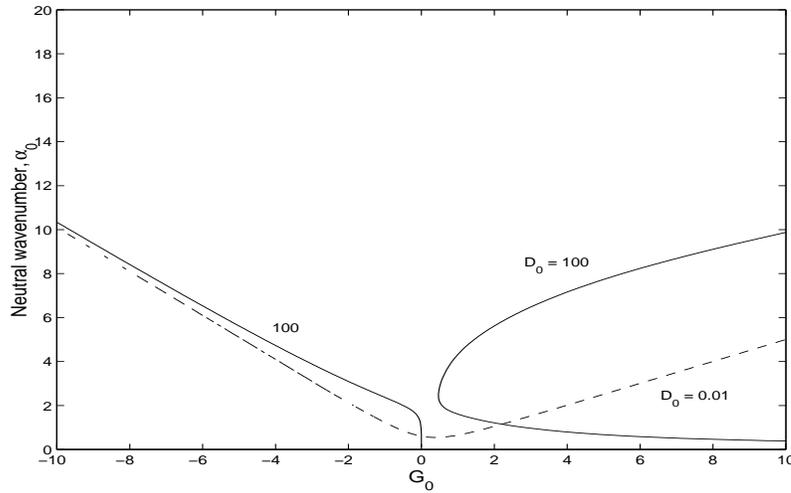


FIG. 2. Dependence of the linear neutral wave number α_0 on the buoyancy G_0 , for small and large Damkohler numbers. When the Damkohler number is large, the instability splits into two distinct branches.

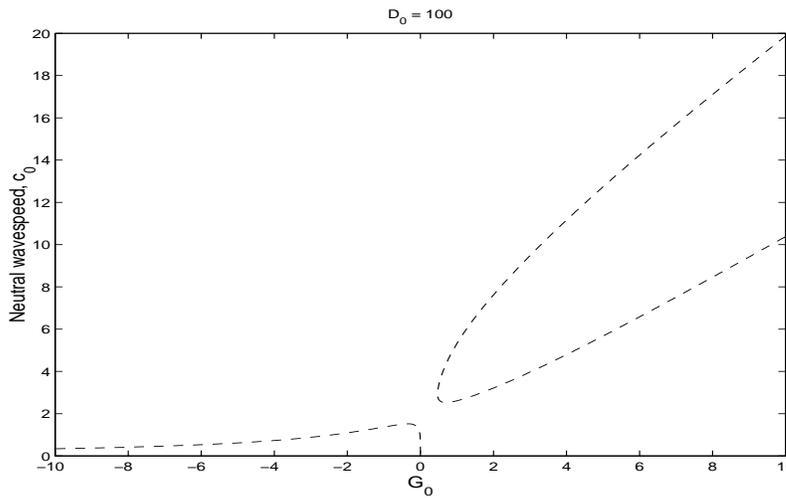


FIG. 3. Dependence of the linear neutral wave speed c_0 on the buoyancy when $D_0 = 100$.

Figure 2 shows the effect of increasing both the fluid buoyancy and the Damkohler number. For positive buoyancy, the effect of increasing the Damkohler numbers is to destabilize the flow. However, for large Damkohler numbers the instability is observed to have two branches. The second branch, with much smaller wavenumbers, appears to be stabilized by moderate increases in the fluid buoyancy. In the case of negative buoyancy, the wavenumbers converge to a fixed limit for increasing Damkohler numbers.

Figure 3 shows the response of the wave speed to increasing buoyancy for a fixed Damkohler number. In the limit $G_0 \rightarrow +\infty$, the wave speeds are enhanced in tandem with the increase in neutral wavenumbers. This increasing buoyancy is again confirmed to be destabilizing while for large negative buoyancy, the wave speeds reduce to zero for all Damkohler numbers.

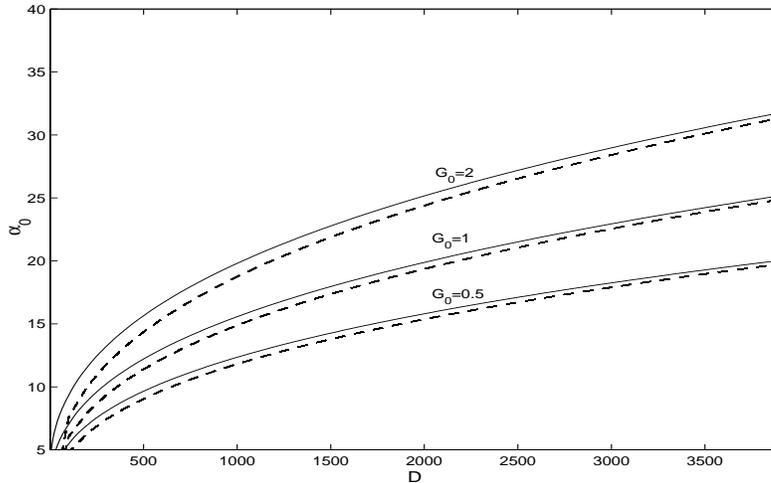


FIG. 4. Linear neutral wave number α_0 against D_0 , with $G_0 = 0.5$ to 2 and the respective asymptotic predictions (—).

Figure 4 shows the response of the wave number to increasing Damkohler numbers for different values of the buoyancy parameter G_0 . It is again observed that $\alpha_0 \rightarrow \infty$ as $D_0 \rightarrow \infty$. Increasing G_0 results in larger wavenumbers for a fixed Damkohler number. This shows that the Damkohler number on its own has weakly destabilizing effects.

Figure 5 shows the variation of the neutral wave speed c_0 against the scaled Damkohler number D_0 for selected values of the scaled buoyancy parameter G_0 . The results are qualitatively similar to those presented in Fig. 4. Hence the Damkohler number has the same effect on both the wave number and the wave speed.

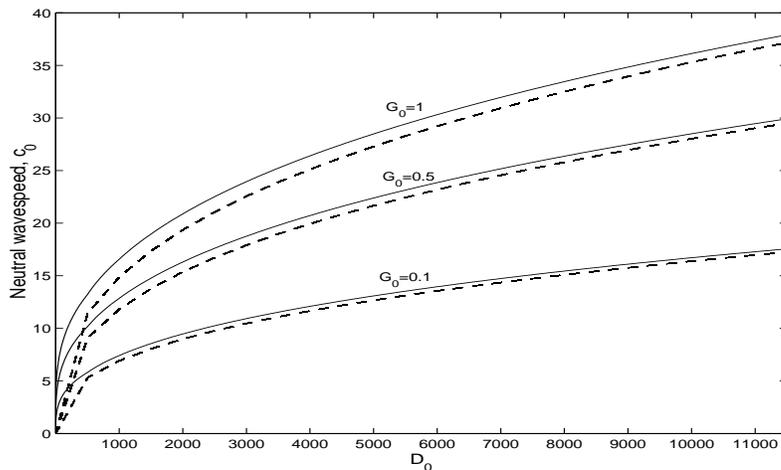


FIG. 5. Linear neutral wave speed c_0 against D_0 , with $G_0 = 0.1$ to 1 and the respective asymptotic result (—).

5. Conclusions

We have considered the effect of fluid buoyancy and chemical reaction between the chemical species and the fluid on the linear stability of two-dimensional disturbance wave modes. Asymptotic methods have been used to investigate the effects of increasing buoyancy and reaction kinematics. We have extended the well known theory of boundary layer flows over horizontal surfaces to include the chemical species and the effect of the Damkohler number. A steady-state natural and forced flow over a semi-infinite horizontal plate has been studied in which the plate is maintained at a given concentration in a chemical species and convection arises as a result of chemical reaction, diffusing within the ambient fluid and the slight disturbances applied to the flow.

When the wave number and speed number are varied against the scaled Damkohler numbers (D_0), the effect of increasing G_0 was shown to be destabilizing in line with conclusions in MOTSA *et al.* [9] and other earlier works. We have also shown that increasing reaction kinematics has weakly destabilizing effects for the TS waves. The obtained results were compared with the previously published work and were found to be in excellent agreement. It is hoped that the solutions presented in this paper with the various investigated effects would be useful for validation of future work to include the temperature differences.

Acknowledgment

SS would like to acknowledge the financial support from NUFU. PS would like to gratefully acknowledge the generous financial support from the University of KwaZulu-Natal.

Appendix

The constants and variables as used in the article are:

$$\begin{aligned}
 I_0 &= \int_0^{\infty} U_B^2 dY, & I_1 &= \int_{Y_0}^{\infty*} \frac{C_{BY}}{U_B} dY, \\
 I_2 &= \int_{Y_0}^{0*} \frac{1}{U_B^2} dY, & J_0 &= \int_{Y_0}^{0*} \frac{(C_B - \mu_0)}{U_B^2} dY, \\
 J_1 &= \int_0^{\infty} C_{BY} \left(\int_{Y_0}^Y \frac{G_0 \mu_0 + \alpha_0 + G_0(C_B - \mu_0)}{U_B^2} dY_1 \right) dY, \\
 B_0 &= \alpha_0 c_0 \lambda_1^2 I_2 - \frac{2\alpha_0 c_0 \lambda_2}{\lambda_1}, \\
 B_1 &= \frac{2\alpha_0 c_0^2 \lambda_2}{\lambda_1^2} \ln \left| \frac{c_0}{\lambda_1} \right|, \\
 B_2 &= -c_0 B_0 + \lambda_1 B_1 + \lambda_1 \omega_1 - \alpha_1 \lambda_1 c_0 + 2\alpha_0^2 c_0 + \alpha_0^2 (\lambda_1 I_2 - I_0), \\
 E_0 &= \alpha_0 \lambda_1 \int_{Y_0}^{\infty*} \frac{C_B - R_0}{U_B^2} dY, \\
 E_1 &= \frac{-\mu_1 \alpha_0 c_0 G_0}{\lambda_1}, \\
 E_2 &= c_0 E_0 + \lambda_1 E_1 - \alpha_0 J_0 - D_0 I_1 + \alpha_0^2 J_1 - \alpha_0 \mu_0.
 \end{aligned}$$

References

1. J. P. DENIER and E. W. MUREITHI, *Weakly nonlinear wave motions in thermally stratified boundary layer*, J. Fluid Mech., **315**, 293–316, 1996.
2. J. S. B. GAJJAR and F. T. SMITH, *On the global instability of free disturbances with time-dependent nonlinear viscous critical layers*, J. Fluid Mech., **157**, 53–77, 1985.
3. B. GEBHART, Y. JALURIA, R.L. MAHAJAN and B. SAMMAKIA, *Buoyancy-induced flow and transport*, Hemisphere, Washington DC 1988.
4. B. GEBHART and L. PERA, *The nature of vertical natural convection flows resulting from the combined buoyancy effects of thermal and mass diffusion*, Int. J. Heat Mass Transfer, **14**, 2025–2050, 1971.
5. K.R. KHAIR and A. BEJAN, *Mass transfer to natural convection boundary layer flow driven by heat transfer*, Int. J. Heat Mass Transfer, **30**, 369–376, 1985.

6. H. T. LIN and W. S. YU, *Combined heat and mass transfer by laminar natural convection from a vertical plate*, Heat and Mass Transfer, **30**, 369–376, 1995.
7. H. T. LIN and W. S. YU, *Combined heat and mass transfer by laminar natural convection from a vertical plate with uniform heat flux and concentration*, Heat and Mass Transfer, **32**, 293–299, 1995.
8. A. MONGRUEL, M. CLOITRE and C. ALLAN, *Scaling of boundary-layer flows driven by double diffusive convection*, Int. J. Heat Transfer, **1**, 69–85, 1996.
9. S. S. MOTSA, J. S. B. GAJJAR and P. SIBANDA, *On the upper-branch stability of boundary-layer flows over compliant surfaces with heating/cooling*, Computer Assisted Mechanics and Engineering Sciences, **9**, 163–181, 2002.
10. I. MULOLANI and R. RAHMAN, *Similarity analysis for natural convection from a vertical plate with distributed wall concentration*, Int. J. Math. and Math. Sci., **23**, 5, 319–334, 1998.
11. E. W. MUREITHI, J. P. DENIER and A. K. STOTT, *The effect of buoyancy on the upper-branch Tollmien–Schlichting waves*, IMA J. Appl. Math., **58**, 19–50, 1997.
12. S. OSTRACH, *Natural convection with combined buoyancy forces*, Physico-Chem. Hydrodyn, **1**, 233–247, 1980.
13. A. SELIM, M. A. HOSSAIN and D. A. S. REES, *The effect of surface mass transfer on mixed convection flow past a heated vertical flat permeable plate with thermophoresis*, Int. J. Therm. Sc., **42**, 973–982, 2003.
14. F. T. SMITH, *On the non-parallel flow stability of the Blasius boundary layer*, Proc. R. Soc. Lond. A, **366**, 91–109, 1979.
15. F. T. SMITH, *Nonlinear stability of boundary layers for disturbances of various sizes*, Proc. R. Soc. Lond. A, **368**, 573–589, 1979.
16. F. T. SMITH and R. J. BODONYI, *On the stability of the developing flow in a channel or circular pipe*, Q. J. Mech. Appl. Maths., **33**, 293–320, 1980.
17. F. T. SMITH and R. J. BODONYI, *Nonlinear critical layers and their development in streaming-flow stability*, J. Fluid Mech., **118**, 165–185, 1982.
18. K. STEWARTSON, *The theory of laminar boundary layers in compressible fluids*, Oxford University Press, Clarendon 1964.
19. J. S. TURNER, *Double-Diffusive phenomena*, Ann. Review of Fluid Mech., **6**, 37–56, 1974.
20. L. W. WANG, C. T. CHEN and J. J. CHEN, *Flow patterns of convection in enclosure with vertical temperature and concentration gradients*, Joint ASME/JSME, Thermal Engineering Conf. March 1987.
21. L. W. WANG, P. J. WANG, and Y. C. KUNG, *Thermosolutal convection in a rectangular enclosure with upper multi-partitions in Cooperate case*, Proceedings of PSFVIP – 3. 18–21, 2001 Maui, Hawaii, U.S.A. **F3050**.

Received April 7, 2004; revised version September 1, 2004.
