

# Energy-based limit criteria for anisotropic elastic materials with constraints

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THE MAIN aim of this paper is to investigate the influence of internal restrictions on the form of the energy-based limit condition. Some restrictions may be imposed in the elastic as well as in the limit states. Spectral decomposition of symmetric linear operators in a space with two different scalar products is applied. An algorithm for accounting for the considered restrictions in the limit condition is proposed.

It was shown that as long as the energy scalar product is defined properly in the elastic range, the limit condition having the energy-based interpretation can be found. Examining the material with such an internal structure that there are stresses which do not cause any strain, the space with passive stresses and locked strains has to be introduced. The limit condition in this case has two parts, one connected with active part of stresses which has energy-based interpretation and the second one connected with passive stresses. The algorithm how to introduce this part of stresses to the limit condition has been proposed. As examples, the energy-based form of the Schmid law for single slip is derived and fiber-reinforced materials are analyzed.

## 1. Introduction

THE LIMIT condition bounds in our paper the regime of applicability and validity of Hooke's law (the regime of linear elasticity)

$$(1.1) \quad \begin{aligned} \boldsymbol{\sigma} &= \mathbf{S} \cdot \boldsymbol{\epsilon} & \Leftrightarrow & \sigma_{ij} = S_{ijkl}\epsilon_{kl}, \\ \boldsymbol{\epsilon} &= \mathbf{C} \cdot \boldsymbol{\sigma} & \Leftrightarrow & \epsilon_{kl} = C_{klmn}\sigma_{mn}, \end{aligned}$$

where  $\boldsymbol{\sigma}$  and  $\boldsymbol{\epsilon}$  are the stress tensor and the small strain tensor (the symmetric part of the displacement gradient) while the fourth-order tensors  $\mathbf{S}$  and  $\mathbf{C}$  are the stiffness and the compliance tensor, respectively. They are connected by the relation

$$(1.2) \quad \mathbf{S} \circ \mathbf{C} = \mathbf{C} \circ \mathbf{S} = \mathbf{I}_s \Leftrightarrow S_{ijkl}C_{klmn} = C_{ijkl}S_{klmn} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}).$$

The elastic energy density  $\phi$  is then given by

$$(1.3) \quad \phi(\boldsymbol{\sigma}) = \frac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}(\boldsymbol{\sigma}) = \frac{1}{2}\boldsymbol{\sigma} \cdot \mathbf{C} \cdot \boldsymbol{\sigma} = \frac{1}{2}C_{ijkl}\sigma_{ij}\sigma_{kl}$$

that in the presented elasticity theory is equal to the internal energy density.

Reaching the limit state by the material in the considered point may be connected with different physical or engineering interpretations. It may be the passage from linear to nonlinear elasticity, appearing of the irreversible deformations (plasticity), appearing of viscosity, damage or other structural changes in the material. From engineering point of view this notion may be interpreted as a limit of functional quality of the designed part of structure usually connected with some prescribed value of strains or displacements. In this case the material may remain linearly elastic but the strains are too large to use the workpiece or part of structure properly or safely.

The most widely used limit condition, especially for metals, is the quadratic Huber–Mises criterion proposed as a yield condition for isotropic materials. Huber and earlier Maxwell<sup>1)</sup> gave it the energy-based interpretation basing on the concept of the critical distortion energy. In view of this criterion, the spherical part of stress is safe for the material and the material is insensitive to the sign of the stress state (particularly, the critical value of stress is the same for tension and compression).

Anisotropy is a feature of crystals, although a majority of modern materials such as composites and nanomaterials usually exhibit anisotropic properties. Also, metal that is initially isotropic can become anisotropic due to deformation-induced texture.

Most of the proposed yield conditions for anisotropic materials fall into two categories. On the one hand there are generalizations of the maximum shear stress condition (the Coulomb–Tresca type condition for isotropic materials), while on the other hand there are the various generalizations of the quadratic condition for isotropic materials by means of  $n$ -th degree equations depending on stress tensor components [3, 9]. Usually, for  $n = 2$  or  $n = 1$  they reduce to the Mises condition and for  $n \rightarrow \infty$  they generalize for anisotropic case the Coulomb–Tresca criterion (for some review of this aspect look at [10] or [2]). These extensions were introduced in order to better describe the experimental results reported for example in [7].

The direct extension of the approach proposed by Huber for isotropic solids is based on the assumption that only a part of the density of elastic energy  $\phi$  (1.3) is responsible for reaching the limit state. In the case of anisotropic materials, there is no physical reason to consider the spherical state as a state playing a decisive role in the formulation of the strength measure. The spherical part of the stress tensor may thus enter the limit condition. RYCHLEWSKI [19], generalizing the Mises condition for anisotropic materials, assumed that the safe part of stress tensor may be different for different types of anisotropy. Hence he considered the limit condition as follows:

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<sup>1)</sup>The historical background the reader can find in [21].

$$(1.4) \quad \boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} \leq 1.$$

Rychlewski considered two quadratic forms

$$(1.5) \quad \boldsymbol{\sigma} \cdot \mathbf{C} \cdot \boldsymbol{\sigma}, \quad \boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma}$$

first of which is positive definite and the second one is positive semidefinite. He proved that any stress measure of the form (1.4) has uniquely determined energy-based interpretation. The formulation and proof of the above theorem was based on the concepts of the spectral decomposition of the fourth order tensor [18, 22, 6] and the energy scalar product (see Appendix).

BURZYŃSKI [5] assumed, that the spherical part of stress is the proper state for all types of symmetry. It means, that he considered only the voluminally isotropic materials imposing additional restrictions on the compliance tensor  $\mathbf{C}$ . Influence of Burzyński's restrictions on the form of the spectral decomposition of  $\mathbf{C}$  was discussed in [13].

Certain attempts of formulating limit conditions of the form based on the spectral decomposition of  $\mathbf{H}$  for some classes of anisotropy were made in [15] and [16]. Assuming that tensors  $\mathbf{C}$  and  $\mathbf{H}$  are coaxial (they have the same eigen-subspaces), the limit conditions for cubic and transversally isotropic materials were proposed. The limit condition (1.4) in the case when tensors  $\mathbf{C}$  and  $\mathbf{H}$  are coaxial has an energy-based interpretation, because the proper states of  $\mathbf{H}$  are the proper states of  $\mathbf{C}$ . Another way to formulate the energy-based limit condition basing on the spectral theorem was proposed in [4] and [1]. There, the limit state is reached when the prescribed value of energy is attained separately for each eigen-subspace. In [1] also the so-called *complementary* Kelvin modes (proper states) were introduced in order to describe the difference in tension and compression stress states.

In general case the limit condition depends on the tensors  $\mathbf{C}$  and  $\mathbf{H}$ , which can be independent. This fact allows to consider different types of symmetry in elastic and limit states. However, when some form of coupling of elastic and plastic properties is assumed, e.g. coaxiality of these tensors, the solution of the problem becomes much simplified.

Rychlewski's approach was illustrated in [11] for transversally isotropic materials described by the quadratic Hill condition in the limit state and in [12] for cubic symmetry in the elastic range and orthotropic symmetry in the limit state. In both papers the energy-based conditions were proposed.

In the next section we shortly recall the energy-based formulation of the quadratic limit condition after [19] in order to introduce the needed notations. The energy-based interpretation of the limit criterion depends on the form of the compliance tensor  $\mathbf{C}$  by the definition of the energy scalar product. As long

as this tensor is positive definite, this new scalar product is defined properly and Rychlewski's approach may be applied. However, some modification is needed in the case when passive stresses described in [13] are present. This problem is discussed and a solution is proposed in Sec. 3. In that section also the safe stresses and their influence on the energy-based form of the limit condition are considered. Finally, in Sec. 4 the proposed approach to accounting for the internal restrictions in the elastic and limit states is illustrated by the derived energy-based form of the Schmid law for single slip activation in crystals and the examples of fiber-reinforced materials.

Energy-based interpretation of the limit condition becomes more attractive nowadays due to the development of the methods of calculating material constants by means of ab-initio calculations [14]. Energy is the quantity that can be transferable through the material scales so the results of quantum mechanics calculations are used to find out elastic constants describing the Hooke's law from the classical continuum mechanics [23]. The material parameters describing energy-based limit conditions can be in the same way obtained from the ab-initio calculations [8].

## 2. The limit condition of the Mises type

Let us consider a linearly elastic material, described by the compliance tensor  $\mathbf{C}$  and the limit tensor  $\mathbf{H}$ . In a general case the tensors  $\mathbf{C}$  and  $\mathbf{H}$  are mutually independent. However, some coupling of elastic properties with the limit ones is usually observed due to the influence of the internal structure of the material (fibres, crystallographic lattice) on its properties in the elastic and limit states.

Using the theorem on spectral decomposition for three fourth-order tensors:  $\mathbf{S}$ ,  $\mathbf{C}$  and  $\mathbf{H}$  with the standard scalar product (see Appendix) we obtain the following spectral forms of these tensors:

$$(2.1) \quad \mathbf{S} = \lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2 + \dots + \lambda_r \mathbf{P}_r, \quad r \leq 6,$$

$$(2.2) \quad \mathbf{C} = \frac{1}{\lambda_1} \mathbf{P}_1 + \frac{1}{\lambda_2} \mathbf{P}_2 + \dots + \frac{1}{\lambda_r} \mathbf{P}_r,$$

$$(2.3) \quad \mathbf{H} = \frac{1}{\chi_1^2} \mathbf{R}_1 + \frac{1}{\chi_2^2} \mathbf{R}_2 + \dots + \frac{1}{\chi_\nu^2} \mathbf{R}_\nu, \quad \nu \leq 6,$$

where  $\lambda_k$  are called the stiffness moduli (or the Kelvin moduli) and  $\chi_l$  are called the limit moduli. Coupling of elastic and limit properties may be observed by the fact that as a result of material symmetry, at least some of the orthogonal projectors  $\mathbf{P}_k$  and  $\mathbf{R}_l$  are the same for all three tensors, so one may say that  $\mathbf{C}$  (at the same time  $\mathbf{S}$ ) and  $\mathbf{H}$  are partially coaxial.

In [19] using the spectral theorem based on the energy scalar product for the tensor  $\mathbf{S} \circ \mathbf{H} \circ \mathbf{S}$  it was shown that for every limit condition of the Mises type, the energy-based form may be found. The proved theorem says that for each linearly elastic material, described by the positive definite compliance tensor  $\mathbf{C}$  (all  $\lambda_k > 0$ ) and the limit tensor  $\mathbf{H}$ , there is exactly one energy orthogonal decomposition of the stress space  $\mathcal{T}_2^s$

$$(2.4) \quad \mathcal{T}_2^s = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots \oplus \mathcal{H}_\varkappa, \quad \varkappa \leq 6$$

where  $\mathcal{H}_i \perp \mathcal{H}_j$ , and only one sequence of energy limits of elasticity  $h_1 < h_2 < \dots < h_\varkappa$  such that

$$(2.5) \quad \boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} = \frac{1}{h_1} \phi(\boldsymbol{\sigma}_1) + \dots + \frac{1}{h_\varkappa} \phi(\boldsymbol{\sigma}_\varkappa)$$

and

$$(2.6) \quad \frac{1}{2} \boldsymbol{\sigma} \cdot \mathbf{C} \cdot \boldsymbol{\sigma} = \phi(\boldsymbol{\sigma}) = \phi(\boldsymbol{\sigma}_1) + \dots + \phi(\boldsymbol{\sigma}_\varkappa),$$

where

$$(2.7) \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \dots + \boldsymbol{\sigma}_\varkappa, \quad \boldsymbol{\sigma}_k = \mathbf{H}_k \bullet \boldsymbol{\sigma} \in \mathcal{H}_k$$

and symbol  $\bullet$  denotes the energy scalar product. It means that the Mises limit criterion (1.4) bounds the weighted sum of stored elastic energies corresponding to uniquely defined, energy-orthogonal parts of stress.

The limit condition (1.4) in the case when tensors  $\mathbf{C}$  and  $\mathbf{H}$  are totally coaxial ( $r = \nu$ ,  $\mathbf{P}_i = \mathbf{R}_i$ ,  $i = 1, \dots, r$ ) has a straightforward energy-based interpretation, because the proper states of  $\mathbf{H}$  are then the proper states of  $\mathbf{C}$  and

$$(2.8) \quad \boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} = \sum_{i=1}^k \frac{\lambda_i}{\chi_i^2} \boldsymbol{\sigma}_i \cdot \mathbf{C} \cdot \boldsymbol{\sigma}_i = \sum_{i=1}^k 2 \frac{\lambda_i}{\chi_i^2} \phi(\boldsymbol{\sigma}_i) = \sum_{i=1}^k \frac{1}{h_i} \phi(\boldsymbol{\sigma}_i),$$

$$\text{where } h_i = \frac{\chi_i^2}{2\lambda_i} \quad (\text{no summation}) \quad \text{and} \quad \boldsymbol{\sigma}_i = \mathbf{P}_i \cdot \boldsymbol{\sigma} = \mathbf{R}_i \cdot \boldsymbol{\sigma}.$$

The decomposition of the space  $\mathcal{T}_2^s$  (2.4) in this case is not only energy orthogonal but also orthogonal in the standard sense. This kind of decomposition Rychlewski called the *main energy orthogonal* decomposition. The method of finding out an energy-based form of the limit condition (1.4) in the case of non-coaxiality of  $\mathbf{H}$  and  $\mathbf{C}$  was presented in [19] and [11].

### 3. Influence of internal restrictions on the form of limit criterions

Material symmetry is described in the elastic region by tensors  $\mathbf{C}$  or  $\mathbf{S}$  and in the limit state by tensor  $\mathbf{H}$ . Assuming their form, we introduce some restrictions on the way in which the material is deformed. For instance, a body of cubic symmetry will react only by changing its volume due to hydrostatic pressure while orthotropic body will react also by changing its shape.

Additional restrictions, apart from material symmetry, which are caused by material internal structure, can be also introduced. The restrictions of the type of strong fibers or other reinforcements that impose bounds on the admissible deformation modes, are called *internal restrictions* or *internal constraints*. They usually concern the possible deformation modes and are imposed on the strain tensor, for example, lack of reaction to hydrostatic pressure or incompressibility.

Some additional restrictions may be imposed in the elastic as well as in the limit states. The energy-based limit condition (2.5) proposed by Rychlewski depends on the form of the compliance tensor  $\mathbf{C}$  and on the form of the limit tensor  $\mathbf{H}$ . It means that any restrictions imposed on them change the form of the limit condition (2.5). Particularly, the restrictions that introduce the passive stress space cause that  $\mathbf{C}$  is not any more positive definite for all  $\boldsymbol{\sigma} \in T_2^s$ . The approach presented in the previous section must be in this case modified.

The main aim of this paper is to investigate the influence of internal restrictions on the form of energy-based limit conditions.

#### 3.1. Internal restrictions in the elastic region

The influence of some classes of internal restrictions imposed on deformation modes and on properties of linearly elastic anisotropic materials was discussed in [13]. Using the concept of eigen-states, Kelvin moduli and spectral decomposition of the compliance tensor  $\mathbf{C}$ , it was proved that the considered types of restrictions can make material symmetry higher or lower, because they have influence on the value of the Kelvin moduli and on the form of orthogonal projectors.

The energy-based limit condition depends on the form of the compliance tensor  $\mathbf{C}$  by the definition of the energy scalar product (Appendix, (A.1)). As long as this scalar product is defined properly in the elastic range, the limit condition having the energy interpretation can be found. The scalar product is defined properly if all axioms are satisfied.

Let us examine a material with such an internal structure that it hinders some modes of strains so strongly that one may regard them as negligibly small. When

$$(3.1) \quad \boldsymbol{\sigma}_p \neq \mathbf{0}, \quad \text{but} \quad \boldsymbol{\epsilon}_p = \mathbf{0},$$

the stress  $\boldsymbol{\sigma}_p$  is called passive or reactive because it does not cause any strain  $\mathbf{C} \cdot \boldsymbol{\sigma}_p = \boldsymbol{\epsilon}_p = \mathbf{0}$ . The set of all passive stresses is the kernel of the operator  $\mathbf{C}$

$$(3.2) \quad \boldsymbol{\sigma}_p \in \text{Ker} \mathbf{C}.$$

Each strain  $\boldsymbol{\epsilon}$  that is admissible in the material under consideration, is orthogonal to this kernel,

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_a \perp \text{Ker} \mathbf{C}.$$

The space  $\mathcal{P}_p \equiv \text{Ker} \mathbf{C}$  is called *the space of passive (reactive) stresses and locked strains* [21]. The space of the second order symmetric tensors  $\mathcal{T}_2^s$  can be orthogonally decomposed into two subspaces

$$(3.3) \quad \mathcal{T}_2^s = \mathcal{P}_p \oplus \mathcal{P}_a,$$

where the space  $\mathcal{P}_a$  will be the space of admissible strains and active stresses.

Hooke's law (1.1) connects admissible strain with active stress

$$(3.4) \quad \boldsymbol{\epsilon}_a = \mathbf{C}_a \cdot \boldsymbol{\sigma}_a, \quad \boldsymbol{\sigma}_a = \mathbf{S}_a \cdot \boldsymbol{\epsilon}_a,$$

where  $\mathbf{C}_a$  is an invertible operator that maps  $\mathcal{P}_a$  onto itself

$$(3.5) \quad \mathbf{C}_a = \mathbf{C} \quad \text{on} \quad \mathcal{P}_a.$$

The energy scalar product is now defined only on the subspace  $\mathcal{P}_a$

$$\boldsymbol{\sigma}_1 \bullet \boldsymbol{\sigma}_2 \equiv \boldsymbol{\sigma}_1 \cdot \mathbf{C}_a \cdot \boldsymbol{\sigma}_2, \quad \text{for} \quad \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in \mathcal{P}_a$$

due to the fact that  $\mathcal{P}_p = \text{Ker} \mathbf{C}$  consists of all proper states corresponding to the compliance modulus  $1/\lambda_p \rightarrow 0$  [13]. PIPKIN [17] examined in detail a particular case of the above restrictions when  $\dim \mathcal{P}_p = 1, \dim \mathcal{P}_a = 5$ . The constraints then have the form

$$(3.6) \quad \boldsymbol{\epsilon} \cdot \boldsymbol{\omega} = 0,$$

where for  $\boldsymbol{\omega} = \mathbf{1}$  incompressibility is described, for  $\boldsymbol{\omega} = \mathbf{k} \otimes \mathbf{k}$ , fiber inextensibility is investigated (see also [24]).

In this paper our interest is focussed on the problem how the passive stress  $\boldsymbol{\sigma}_p \in \mathcal{P}_p$  may be introduced to the limit condition.

From the relation (3.3), because

$$(3.7) \quad \mathbf{I}_s = \mathbf{P}_a + \mathbf{P}_p$$

it is concluded that

$$\bigwedge_{\boldsymbol{\sigma} \in \mathcal{T}_2^s} \mathbf{I}_s \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma} = \mathbf{P}_a \cdot \boldsymbol{\sigma} + \mathbf{P}_p \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma}_a + \boldsymbol{\sigma}_p,$$

$$\boldsymbol{\sigma}_a \in \mathcal{P}_a, \quad \boldsymbol{\sigma}_p \in \mathcal{P}_p,$$

where  $\mathbf{P}_a$  and  $\mathbf{P}_p$  are orthogonal projectors which map the space  $\mathcal{T}_2^s$  onto subspaces  $\mathcal{P}_a$  and  $\mathcal{P}_p$ , respectively. Both parts of  $\boldsymbol{\sigma}$  satisfy the following conditions:

$$(3.8) \quad \boldsymbol{\sigma}_a \cdot \boldsymbol{\sigma}_p = 0, \quad \boldsymbol{\sigma}_a \cdot \mathbf{C} \cdot \boldsymbol{\sigma}_p = 0,$$

since they belong to two different proper subspaces and the constraint  $\mathbf{C} \cdot \boldsymbol{\sigma}_p = \mathbf{0}$  is satisfied.

The limit condition (1.4) now may be written in the form

$$(3.9) \quad \begin{aligned} \boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} &= (\boldsymbol{\sigma}_a + \boldsymbol{\sigma}_p) \cdot \mathbf{H} \cdot (\boldsymbol{\sigma}_a + \boldsymbol{\sigma}_p) \\ &= \boldsymbol{\sigma}_a \cdot \mathbf{H} \cdot \boldsymbol{\sigma}_a + \boldsymbol{\sigma}_p \cdot \mathbf{H} \cdot \boldsymbol{\sigma}_p + 2\boldsymbol{\sigma}_a \cdot \mathbf{H} \cdot \boldsymbol{\sigma}_p \leq 1 \end{aligned}$$

or

$$(3.10) \quad \begin{aligned} \boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} &= \boldsymbol{\sigma}_a \cdot (\mathbf{P}_a \circ \mathbf{H} \circ \mathbf{P}_a) \cdot \boldsymbol{\sigma}_a + \boldsymbol{\sigma}_p \cdot \mathbf{P}_p \cdot (\mathbf{H} \cdot \boldsymbol{\sigma}_p) \\ &\quad + 2\boldsymbol{\sigma}_a \cdot \mathbf{P}_a \cdot (\mathbf{H} \cdot \boldsymbol{\sigma}_p) \leq 1. \end{aligned}$$

Using tensors  $\mathbf{C}_a, \mathbf{S}_a$  ( $\mathbf{C}_a \circ \mathbf{S}_a = \mathbf{S}_a \circ \mathbf{C}_a = \mathbf{P}_a$ ) we may write that

$$(3.11) \quad \begin{aligned} \boldsymbol{\sigma}_a \cdot \mathbf{H} \cdot \boldsymbol{\sigma}_a &= \boldsymbol{\sigma}_a \cdot (\mathbf{C}_a \circ \mathbf{S}_a \circ \mathbf{H} \circ \mathbf{S}_a \circ \mathbf{C}_a) \cdot \boldsymbol{\sigma}_a \\ &= \boldsymbol{\sigma}_a \bullet (\mathbf{S}_a \circ \mathbf{H} \circ \mathbf{S}_a) \bullet \boldsymbol{\sigma}_a = \frac{1}{h_1} \phi(\boldsymbol{\sigma}_1) + \dots + \frac{1}{h_{\varkappa}} \phi(\boldsymbol{\sigma}_{\varkappa}), \\ &\quad \varkappa \leq \dim \mathcal{P}_a \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{\sigma}_a &= \boldsymbol{\sigma}_1 + \dots + \boldsymbol{\sigma}_{\varkappa}, & \boldsymbol{\sigma}_K &= \mathbf{H}_K \bullet \boldsymbol{\sigma}_a \in \mathcal{H}_K, \\ \mathbf{S}_a &= \mathbf{H}_1 + \dots + \mathbf{H}_{\varkappa}. \end{aligned}$$

The last two expressions of the limit condition (3.10) depend only on the spectral decomposition of  $\mathbf{H}$  (2.3). We have then

$$(3.12) \quad \begin{aligned} \mathbf{H} \cdot \boldsymbol{\sigma}_p &= \left( \frac{1}{\chi_1^2} \mathbf{R}_1 + \frac{1}{\chi_2^2} \mathbf{R}_2 + \dots + \frac{1}{\chi_{\nu}^2} \mathbf{R}_{\nu} \right) \cdot \boldsymbol{\sigma}_p \\ &= \frac{1}{\chi_1^2} \boldsymbol{\sigma}_p^1 + \frac{1}{\chi_2^2} \boldsymbol{\sigma}_p^2 + \dots + \frac{1}{\chi_{\nu}^2} \boldsymbol{\sigma}_p^{\nu}, \end{aligned}$$

where

$$(3.13) \quad \begin{aligned} \boldsymbol{\sigma}_p &= \boldsymbol{\sigma}_p^1 + \dots + \boldsymbol{\sigma}_p^{\nu}, & \boldsymbol{\sigma}_p^i &= \mathbf{R}_i \cdot \boldsymbol{\sigma}_p \in \mathcal{R}_i \\ \mathbf{I}_s &= \mathbf{R}_1 + \dots + \mathbf{R}_{\nu} = \mathbf{P}_a + \mathbf{P}_p. \end{aligned}$$

Let us introduce the following notation:

$$(3.14) \quad \begin{aligned} \mathbf{P}_p \cdot (\mathbf{R}_i \cdot \boldsymbol{\sigma}_p) &= \mathbf{P}_p \cdot \boldsymbol{\sigma}_p^i = \boldsymbol{\sigma}_{pp}^i \in \mathcal{P}_p, & \text{for } i = 1, 2, \dots, \nu, \\ \mathbf{P}_a \cdot (\mathbf{R}_i \cdot \boldsymbol{\sigma}_p) &= \mathbf{P}_a \cdot \boldsymbol{\sigma}_p^i = \boldsymbol{\sigma}_{pa}^i \in \mathcal{P}_a, & \text{for } i = 1, 2, \dots, \nu. \end{aligned}$$

Now Eq. (3.12) may be presented in the form

$$(3.15) \quad \begin{aligned} \mathbf{H} \cdot \boldsymbol{\sigma}_p &= (\mathbf{I}_s \circ \mathbf{H}) \cdot \boldsymbol{\sigma}_p = [(\mathbf{P}_p + \mathbf{P}_a) \circ \mathbf{H}] \cdot \boldsymbol{\sigma}_p \\ &= [(\mathbf{P}_p + \mathbf{P}_a)] \cdot \left( \frac{1}{\chi_1^2} \mathbf{R}_1 + \frac{1}{\chi_2^2} \mathbf{R}_2 + \dots + \frac{1}{\chi_\nu^2} \mathbf{R}_\nu \right) \cdot \boldsymbol{\sigma}_p \\ &= \left( \frac{1}{\chi_1^2} \boldsymbol{\sigma}_{pp}^1 + \dots + \frac{1}{\chi_\nu^2} \boldsymbol{\sigma}_{pp}^\nu \right) + \left( \frac{1}{\chi_1^2} \boldsymbol{\sigma}_{pa}^1 + \dots + \frac{1}{\chi_\nu^2} \boldsymbol{\sigma}_{pa}^\nu \right). \end{aligned}$$

Since the subspaces  $\mathcal{P}_a$  and  $\mathcal{P}_p$  are orthogonal, the following equalities are fulfilled:

$$(3.16) \quad \begin{aligned} \boldsymbol{\sigma}_p \cdot \mathbf{H} \cdot \boldsymbol{\sigma}_p &= \left( \frac{1}{\chi_1^2} \boldsymbol{\sigma}_{pp}^1 + \dots + \frac{1}{\chi_\nu^2} \boldsymbol{\sigma}_{pp}^\nu \right) \cdot \boldsymbol{\sigma}_p \\ &= \frac{1}{\chi_1^2} \boldsymbol{\sigma}_{pp}^1 \cdot \boldsymbol{\sigma}_p + \dots + \frac{1}{\chi_\nu^2} \boldsymbol{\sigma}_{pp}^\nu \cdot \boldsymbol{\sigma}_p, \\ \boldsymbol{\sigma}_a \cdot \mathbf{H} \cdot \boldsymbol{\sigma}_p &= \left( \frac{1}{\chi_1^2} \boldsymbol{\sigma}_{pa}^1 + \dots + \frac{1}{\chi_\nu^2} \boldsymbol{\sigma}_{pa}^\nu \right) \cdot \boldsymbol{\sigma}_a \\ &= \frac{1}{\chi_1^2} \boldsymbol{\sigma}_{pa}^1 \cdot \boldsymbol{\sigma}_a + \dots + \frac{1}{\chi_\nu^2} \boldsymbol{\sigma}_{pa}^\nu \cdot \boldsymbol{\sigma}_a. \end{aligned}$$

From the relation

$$(3.17) \quad \mathbf{R}_i \cdot \boldsymbol{\sigma}_p = \boldsymbol{\sigma}_p^i = \boldsymbol{\sigma}_{pp}^i + \boldsymbol{\sigma}_{pa}^i$$

it is implied that if  $\boldsymbol{\sigma}_{pp}^i = \mathbf{0}$  ( $\boldsymbol{\sigma}_p^i \in \mathcal{P}_a$ ), then

$$(3.18) \quad \begin{aligned} \boldsymbol{\sigma}_p^i \cdot \boldsymbol{\sigma}_p^i &= (\mathbf{R}_i \cdot \boldsymbol{\sigma}_p) \cdot (\mathbf{R}_i \cdot \boldsymbol{\sigma}_p) \\ &= \boldsymbol{\sigma}_p \cdot (\mathbf{R}_i \circ \mathbf{R}_i) \cdot \boldsymbol{\sigma}_p = \boldsymbol{\sigma}_p \cdot \mathbf{R}_i \cdot \boldsymbol{\sigma}_p = \boldsymbol{\sigma}_p \cdot \boldsymbol{\sigma}_{pa}^i = 0. \end{aligned}$$

It means that  $\boldsymbol{\sigma}_p^i = \mathbf{0}$ .

Finally, the limit condition (3.10) takes the form

$$(3.19) \quad \begin{aligned} \boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} &= \frac{1}{h_1} \phi(\boldsymbol{\sigma}_1) + \dots + \frac{1}{h_\varkappa} \phi(\boldsymbol{\sigma}_\varkappa) \\ &+ \frac{1}{\chi_1^2} \boldsymbol{\sigma}_{pp}^1 \cdot \boldsymbol{\sigma}_p + \dots + \frac{1}{\chi_\nu^2} \boldsymbol{\sigma}_{pp}^\nu \cdot \boldsymbol{\sigma}_p + \frac{2}{\chi_1^2} \boldsymbol{\sigma}_{pa}^1 \cdot \boldsymbol{\sigma}_a + \dots + \frac{2}{\chi_\nu^2} \boldsymbol{\sigma}_{pa}^\nu \cdot \boldsymbol{\sigma}_a \leq 1. \end{aligned}$$

Some particular cases can be considered.

a) Let us assume that the subspace  $\mathcal{P}_p$  is the proper space of  $\mathbf{H}$  corresponding to the proper value  $1/\chi_p$

$$(3.20) \quad \mathbf{H} \cdot \boldsymbol{\sigma}_p = \frac{1}{\chi_p^2} \boldsymbol{\sigma}_p,$$

then

$$\boldsymbol{\sigma}_p \cdot \mathbf{H} \cdot \boldsymbol{\sigma}_p = \frac{1}{\chi_p^2} \boldsymbol{\sigma}_p \cdot \boldsymbol{\sigma}_p, \quad \boldsymbol{\sigma}_a \cdot \mathbf{H} \cdot \boldsymbol{\sigma}_p = \frac{1}{\chi_p^2} \boldsymbol{\sigma}_a \cdot \boldsymbol{\sigma}_p = 0.$$

The limit condition (3.19) in this case takes the form

$$(3.21) \quad \boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} = \frac{1}{h_1} \phi(\boldsymbol{\sigma}_1) + \dots + \frac{1}{h_\nu} \phi(\boldsymbol{\sigma}_\nu) + \frac{1}{\chi_p^2} \boldsymbol{\sigma}_p \cdot \boldsymbol{\sigma}_p \leq 1,$$

where

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{\sigma}_a + \boldsymbol{\sigma}_p = \boldsymbol{\sigma}_1 + \dots + \boldsymbol{\sigma}_\nu + \boldsymbol{\sigma}_p, \\ 2\phi(\boldsymbol{\sigma}_K) &= \boldsymbol{\sigma}_K \cdot \mathbf{C}_a \cdot \boldsymbol{\sigma}_K, \quad K = 1, 2, \dots, \nu. \end{aligned}$$

The last part of the condition (3.21) has the form of the Saint-Venant limit condition for rigid-perfectly plastic material.

When additionally, the space  $\mathcal{P}_p$  is the space of safe stresses then in (3.20)

$$(3.22) \quad \mathbf{H} \cdot \boldsymbol{\sigma}_p = \mathbf{H} \cdot \mathbf{0} \Rightarrow \frac{1}{\chi_p^2} \rightarrow 0.$$

In this case the limit condition (3.21) is as follows:

$$(3.23) \quad \boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma}_a \cdot \mathbf{H} \cdot \boldsymbol{\sigma}_a = \frac{1}{h_1} \phi(\boldsymbol{\sigma}_1) + \dots + \frac{1}{h_\varkappa} \phi(\boldsymbol{\sigma}_\varkappa) \leq 1.$$

b) Now we assume that the active part of stress  $\boldsymbol{\sigma}_a$  is the proper state of the limit tensor  $\mathbf{H}$

$$(3.24) \quad \mathbf{H} \cdot \boldsymbol{\sigma}_a = \frac{1}{\chi_a} \boldsymbol{\sigma}_a.$$

It means that

$$\boldsymbol{\sigma}_p \cdot \mathbf{H} \cdot \boldsymbol{\sigma}_a = \frac{1}{\chi_a} \boldsymbol{\sigma}_p \cdot \boldsymbol{\sigma}_a = 0$$

and from (3.19) we obtain the limit condition in the form

$$(3.25) \quad \frac{1}{h_1} \phi(\boldsymbol{\sigma}_1) + \dots + \frac{1}{h_\varkappa} \phi(\boldsymbol{\sigma}_\varkappa) + \frac{1}{\chi_1^2} \boldsymbol{\sigma}_p^1 \cdot \boldsymbol{\sigma}_p^1 + \dots + \frac{1}{\chi_\nu^2} \boldsymbol{\sigma}_p^\nu \cdot \boldsymbol{\sigma}_p^\nu \leq 1.$$

### 3.2. Internal restrictions in the limit state

The energy-based limit condition (2.5) derived by Rychlewski depends on the form of the stiffness tensor  $\mathbf{C}$  as well as the limit tensor  $\mathbf{H}$ . Let us assume that the energy scalar product is defined properly for the whole space  $\mathcal{T}_2^s$  ( $\mathcal{P}_p = \{0\}$ ), then the limit condition has the form (2.5).

The assumption that the space  $\mathcal{H}_k$  is safe for the material is equivalent to

$$\frac{1}{h_k} \rightarrow 0,$$

what means that a proper stress state  $\boldsymbol{\sigma}_k \in \mathcal{H}_k$  is never able to cause reaching the limit state by the material.

Another type of restrictions is described by relation

$$(3.26) \quad \boldsymbol{\sigma}^0 \cdot \mathbf{H} \cdot \boldsymbol{\sigma}^0 = 0$$

what means that there exists a prescribed state of stress  $\boldsymbol{\sigma}^0$ , not necessarily energy proper state, safe for the material.

Now the algorithm that optimizes the influence of restriction (3.26) on the limit properties of the material for calculating the moduli  $h_k$  and energy orthogonal subspaces  $\mathcal{H}_k$  has to be proposed. We may write that

$$(3.27) \quad \boldsymbol{\sigma}^0 = \boldsymbol{\sigma}_1^0 + \dots + \boldsymbol{\sigma}_{\varkappa}^0, \quad \text{where} \quad \boldsymbol{\sigma}_k^0 = \mathbf{H}_k \bullet \boldsymbol{\sigma}^0$$

and the condition (3.26) is equivalent to

$$(3.28) \quad \boldsymbol{\sigma}^0 \cdot \mathbf{H} \cdot \boldsymbol{\sigma}^0 = \frac{1}{h_1} \phi(\boldsymbol{\sigma}_1^0) + \dots + \frac{1}{h_{\varkappa}} \phi(\boldsymbol{\sigma}_{\varkappa}^0) = 0.$$

Trivial solution of Eq. (3.28) is obtained by assuming that all moduli fulfill the condition

$$\frac{1}{h_k} \rightarrow 0, \quad (k = 1, 2, \dots, \varkappa).$$

It means that the whole space  $\mathcal{T}_2^s$  is the safe space. However, the condition

$$\frac{1}{h_k} \phi(\boldsymbol{\sigma}_k^0) = 0$$

may be fulfilled in two ways:

$$(3.29) \quad \frac{1}{h_k} \rightarrow 0 \quad \text{or} \quad \phi(\boldsymbol{\sigma}_k^0) = 0.$$

The form of  $\boldsymbol{\sigma}^0$  and the type of considered symmetry of the material may cause that  $\boldsymbol{\sigma}^0$  is not projected onto some subspaces  $\mathcal{H}_m$ . It means that there are such projectors  $\mathbf{H}_m$  that

$$(3.30) \quad \boldsymbol{\sigma}_m^0 = \mathbf{H}_m \bullet \boldsymbol{\sigma}^0 = \mathbf{0} \implies \phi(\boldsymbol{\sigma}_m^0) = 0$$

and no restrictions are imposed on  $h_m$ .

When projectors  $\mathbf{H}_k$  give projections of  $\boldsymbol{\sigma}^0$  onto the whole subspaces  $\mathcal{H}_k$  then the subspaces  $\mathcal{H}_k$  sum up and become one subspace of safe stress states. This situation is always observed when subspaces  $\mathcal{H}_k$  are one-dimensional.

If subspaces  $\mathcal{H}_k$  have dimensions higher than one then there may exist such  $m$  that

$$(3.31) \quad \boldsymbol{\sigma}_m^0 = \mathbf{H}_m \bullet \boldsymbol{\sigma}^0 = \mathbf{H}_m^0 \bullet \boldsymbol{\sigma}^0 \in \mathcal{H}_m^0 \subset \mathcal{H}_m,$$

where

$$\mathbf{H}_m = \mathbf{H}_m^0 + \mathbf{H}_m^\perp \quad \text{and} \quad \mathcal{H}_m = \mathcal{H}_m^0 \oplus \mathcal{H}_m^\perp.$$

In this case the subspace  $\mathcal{H}_m$  is split into two subspaces  $\mathcal{H}_m^0$  and  $\mathcal{H}_m^\perp$  with two different moduli

$$\frac{1}{h_m^0} \rightarrow 0 \quad \text{and} \quad \frac{1}{h_m^\perp} = \frac{1}{h_m}.$$

The space  $\mathcal{H}_m^0$  is the safe space. Such a consideration may be applied for all projectors with property (3.31). All subspaces of the form  $\mathcal{H}_\mu^0$  sum up and constitute one subspace of safe stress states. The constraints of the form (3.26) can make material symmetry higher by reducing the number of moduli  $h_k$  and can also make material symmetry lower by subdivision of some subspaces.

### 3.3. Internal restriction in the elastic and limit state

In the general case some additional restrictions may be imposed in the elastic region as well as in the limit state. In Sec. 3.1, assuming that some state of stress is passive, the limit condition was presented in the form (3.19). Let us now assume additionally that the prescribed stress  $\boldsymbol{\sigma}_0$  is safe for the material

$$(3.32) \quad \boldsymbol{\sigma}^0 \cdot \mathbf{H} \cdot \boldsymbol{\sigma}^0 = 0 \quad \text{and} \quad \boldsymbol{\sigma}^0 = \boldsymbol{\sigma}_p^0 + \boldsymbol{\sigma}_a^0.$$

By means of (3.19) the condition (3.32) is then equivalent to

$$(3.33) \quad \begin{aligned} \boldsymbol{\sigma}^0 \cdot \mathbf{H} \cdot \boldsymbol{\sigma}^0 = & \frac{1}{h_1} \phi(\boldsymbol{\sigma}_1^0) + \dots + \frac{1}{h_\varkappa} \phi(\boldsymbol{\sigma}_\varkappa^0) \\ & + \frac{1}{\chi_1^2} (\boldsymbol{\sigma}_{pp}^0)^1 \cdot \boldsymbol{\sigma}_p^0 + \dots + \frac{1}{\chi_\nu^2} (\boldsymbol{\sigma}_{pp}^0)^\nu \cdot \boldsymbol{\sigma}_p^0 \\ & + \frac{2}{\chi_1^2} (\boldsymbol{\sigma}_{pa}^0)^1 \cdot \boldsymbol{\sigma}_a^0 + \dots + \frac{2}{\chi_\nu^2} (\boldsymbol{\sigma}_{pa}^0)^\nu \cdot \boldsymbol{\sigma}_a^0 = 0. \end{aligned}$$

Trivial solution of the above equation is obtained when all moduli

$$\frac{1}{h_k} \rightarrow 0 \quad \text{and} \quad \frac{1}{\chi_m} \rightarrow 0.$$

A particular case is considered when we assume that

$$(3.34) \quad \boldsymbol{\sigma}^0 = \boldsymbol{\sigma}_a^0, \quad \boldsymbol{\sigma}_p^0 = \mathbf{0};$$

then the following condition has to be satisfied

$$(3.35) \quad \boldsymbol{\sigma}^0 \cdot \mathbf{H} \cdot \boldsymbol{\sigma}^0 = \boldsymbol{\sigma}_a^0 \cdot \mathbf{H} \cdot \boldsymbol{\sigma}_a^0 = \frac{1}{h_1} \phi(\boldsymbol{\sigma}_1^0) + \dots + \frac{1}{h_{\varkappa}} \phi(\boldsymbol{\sigma}_{\varkappa}^0) = 0$$

and no restrictions are imposed on moduli  $\chi_m$ . The restrictions imposed on moduli  $h_k$  are obtained following the algorithm proposed in Sec. 3.2. In the case when

$$(3.36) \quad \boldsymbol{\sigma}^0 = \boldsymbol{\sigma}_p^0, \quad \boldsymbol{\sigma}_a^0 = \mathbf{0},$$

the equation

$$\boldsymbol{\sigma}^0 \cdot \mathbf{H} \cdot \boldsymbol{\sigma}^0 = \boldsymbol{\sigma}_p^0 \cdot \mathbf{H} \cdot \boldsymbol{\sigma}_p^0 = \frac{1}{\chi_1^2} (\boldsymbol{\sigma}_{pp}^0)^1 \cdot \boldsymbol{\sigma}_p^0 + \dots + \frac{1}{\chi_\nu^2} (\boldsymbol{\sigma}_{pp}^0)^\nu \cdot \boldsymbol{\sigma}_p^0 = 0$$

has to be solved. No restrictions are imposed on moduli  $h_k$ . The restrictions imposed on moduli  $\chi_m$  may be obtained by applying the algorithm proposed in [13]. They depend on the projections

$$\mathbf{R}_i \cdot \boldsymbol{\sigma}_p^0.$$

The influence of internal restrictions on the form of the energy-based limit condition will be discussed on the examples of some types of material symmetry.

#### 4. Examples

To illustrate the proposed approach let us consider the material of cubic symmetry in the elastic regime. In the first example the energy-based form of the Schmid law [10] for single slip initiations is derived. Two remaining examples concern the material modified in the elastic regime by the internal restriction imposing that the material is inextensible in some prescribed direction  $\mathbf{k}$ . Then, restrictions take the following form

$$(4.1) \quad \mathbf{C} \cdot (\mathbf{k} \otimes \mathbf{k}) = \mathbf{0}.$$

This kind of restrictions may be applied to describe the composites reinforced by thin fibers that are so stiff that extensions in the fiber direction can be negligible. They are discussed in the paper [13] (Sec. 4.3) for a more general case in which by putting  $a = b = 0$  condition (4.1) is obtained. We are considering the following two special cases for  $\mathbf{k}$ :

- a)  $\mathbf{k} \otimes \mathbf{k} = \mathbf{e}_1 \otimes \mathbf{e}_1$ ;
- b)  $\mathbf{k} \otimes \mathbf{k} = \frac{1}{3}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \otimes (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$ , where  $\mathbf{e}_i$  are along the edges of a cubic cell.

#### 4.1. Energy-based form of the Schmid law

Let us consider a single crystal in which plastic yielding is caused by slip on a single slip system defined by the unit normal  $\mathbf{n}$  to the slip plane and the unit vector  $\mathbf{m}$  along slip direction ( $\mathbf{m} \cdot \mathbf{n} = 0$ ). According to the classical Schmid law, the plastic yielding on the slip system is initiated when the resolved shear stress  $\tau$  reaches the critical value  $\tau_c$  that is

$$(4.2) \quad \tau = |\mathbf{m} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}| = \tau_c.$$

It is easy to show that Eq. (4.2) may be rewritten in the form of the Mises-type yield condition

$$(4.3) \quad \boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma} \cdot \left( \frac{1}{2\tau_c^2} \boldsymbol{\pi} \otimes \boldsymbol{\pi} \right) \cdot \boldsymbol{\sigma} = 1,$$

where  $\boldsymbol{\pi}$  is the pure shear

$$\boldsymbol{\pi} = \frac{1}{\sqrt{2}}(\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}).$$

One may note that condition (4.3) is the yield condition for the material with internal restrictions in the limit state. All stress states for which  $\boldsymbol{\sigma} \cdot \boldsymbol{\pi} = 0$  are safe in view of this criterion and constitute a five-dimensional subspace of stress space.

Now, we will derive the energy-based form of (4.3) for the material of cubic symmetry in the elastic regime. Since for such a material the subspace  $\mathcal{P}_I$  is one-dimensional,  $\mathcal{P}_{II}$  is two-dimensional and  $\mathcal{P}_{III}$  three-dimensional (for definition of subspaces look at [13]) let us introduce the following energy-orthonormal basis  $\{\boldsymbol{\eta}_K\}$  such that<sup>2)</sup>

$$\boldsymbol{\eta}_1 = \sqrt{\frac{\lambda_I}{3}} \mathbf{1}, \quad \boldsymbol{\eta}_2 = \sqrt{\lambda_{II}} \frac{\mathbf{P}_{II} \cdot \boldsymbol{\pi}}{\|\mathbf{P}_{II} \cdot \boldsymbol{\pi}\|}, \quad \boldsymbol{\eta}_4 = \sqrt{\lambda_{III}} \frac{\mathbf{P}_{III} \cdot \boldsymbol{\pi}}{\|\mathbf{P}_{III} \cdot \boldsymbol{\pi}\|}$$

and  $\boldsymbol{\eta}_3 \in \mathcal{P}_{II}$ ,  $\boldsymbol{\eta}_5, \boldsymbol{\eta}_6 \in \mathcal{P}_{III}$ . Let us notice that this basis is also orthogonal in a standard sense. In this basis the tensor  $\mathbf{S} \circ \mathbf{H} \circ \mathbf{S}$  has the form

$$\begin{aligned} \mathbf{S} \circ \mathbf{H} \circ \mathbf{S} = \frac{1}{2\tau_c^2} & \left[ \lambda_{II} \|\mathbf{P}_{II} \cdot \boldsymbol{\pi}\|^2 \boldsymbol{\eta}_2 \otimes \boldsymbol{\eta}_2 + \lambda_{III} \|\mathbf{P}_{III} \cdot \boldsymbol{\pi}\|^2 \boldsymbol{\eta}_4 \otimes \boldsymbol{\eta}_4 \right. \\ & \left. + \sqrt{\lambda_{II}\lambda_{III}} \|\mathbf{P}_{II} \cdot \boldsymbol{\pi}\| \|\mathbf{P}_{III} \cdot \boldsymbol{\pi}\| (\boldsymbol{\eta}_2 \otimes \boldsymbol{\eta}_4 + \boldsymbol{\eta}_4 \otimes \boldsymbol{\eta}_2) \right]. \end{aligned}$$

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<sup>2)</sup>  $\|\boldsymbol{\alpha}\| = \sqrt{\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}}$ .

Applying the spectral theorem (A.3) we obtain the following energy-orthogonal decomposition of the above tensor:

$$(4.4) \quad \mathbf{S} \circ \mathbf{H} \circ \mathbf{S} = \frac{1}{2h^*} \mathbf{H}^* = \frac{1}{2h^*} \boldsymbol{\chi}^* \otimes \boldsymbol{\chi}^*,$$

where

$$(4.5) \quad \frac{1}{h^*} = \frac{1}{\tau_c^2} (\lambda_{II} \|\mathbf{P}_{II} \cdot \boldsymbol{\pi}\|^2 + \lambda_{III} \|\mathbf{P}_{III} \cdot \boldsymbol{\pi}\|^2)$$

and

$$(4.6) \quad \boldsymbol{\chi}^* = \frac{\sqrt{\lambda_{II}} \|\mathbf{P}_{II} \cdot \boldsymbol{\pi}\| \boldsymbol{\eta}_2 + \sqrt{\lambda_{III}} \|\mathbf{P}_{III} \cdot \boldsymbol{\pi}\| \boldsymbol{\eta}_4}{\lambda_{II} \|\mathbf{P}_{II} \cdot \boldsymbol{\pi}\|^2 + \lambda_{III} \|\mathbf{P}_{III} \cdot \boldsymbol{\pi}\|^2}$$

The remaining energy eigenvalues  $1/2h \rightarrow 0$  and correspond to the five-dimensional space  $\mathcal{H}^\perp$  of safe stresses defined by the projector

$$\mathbf{H}^\perp = \mathbf{S} - \mathbf{H}^*.$$

In view of the Rychlewski theorem, the energy-based form of (4.3) is as follows:

$$(4.7) \quad |\mathbf{m} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}| = \tau_c \iff \frac{1}{h^*} \phi(\boldsymbol{\sigma}^*) = 1, \quad \text{where} \quad \boldsymbol{\sigma}^* = \mathbf{H}^* \bullet \boldsymbol{\sigma}.$$

When a single crystal that yields by multi-slip is considered, similarly to [4], the single energy-based yield condition will be replaced by the set of criterions

$$(4.8) \quad \frac{1}{h_r^*} \phi(\boldsymbol{\sigma}_r^*) = 1, \quad r = 1, \dots, M,$$

where  $M$  is the number of the slip systems.

#### 4.2. Fibers along the edge $\mathbf{e}_1$

In the considered case the spectral decomposition of the compliance tensor  $\mathbf{C}$  is as follows:

$$(4.9) \quad \begin{aligned} \mathbf{C} = \mathbf{C}_a &= \frac{1}{\lambda_{II}} \mathbf{P}_{II}^\perp + \frac{1}{\lambda_{III}} \mathbf{P}_{III}, \\ \mathbf{S} = \mathbf{S}_a &= \lambda_{II} \mathbf{P}_{II}^\perp + \lambda_{III} \mathbf{P}_{III}, \end{aligned}$$

where

$$(4.10) \quad \begin{aligned} \mathbf{P}_p &= \mathbf{P}_I + \mathbf{P}_{II}^*, \\ \mathbf{P}_a &= \mathbf{P}_{II}^\perp + \mathbf{P}_{III} \end{aligned}$$

project the space of stresses onto the subspace of the passive stresses and onto the subspace of active stresses. Let us remind after [13] that

$$\begin{aligned}
 \mathbf{P}_I &= \frac{1}{3} \mathbf{1} \otimes \mathbf{1}, \\
 \mathbf{P}_{II}^* &= \frac{3}{2} \left( \mathbf{k} \otimes \mathbf{k} - \frac{1}{3} \mathbf{1} \right) \otimes \left( \mathbf{k} \otimes \mathbf{k} - \frac{1}{3} \mathbf{1} \right), \\
 \mathbf{P}_{II}^\perp &= \mathbf{P}_{II} - \mathbf{P}_{II}^* = \mathbf{K} - \frac{1}{3} \mathbf{1} \otimes \mathbf{1} - \mathbf{P}_{II}^*, \\
 \mathbf{P}_{III} &= \mathbf{I}_s - \mathbf{K}.
 \end{aligned}
 \tag{4.11}$$

In this case the following decomposition of the stress space is helpful

$$\boldsymbol{\sigma} = (\boldsymbol{\sigma}_I + \boldsymbol{\sigma}_{II}^*) + (\boldsymbol{\sigma}_{II}^\perp + \boldsymbol{\sigma}_{III}) = \boldsymbol{\sigma}_p + \boldsymbol{\sigma}_a.
 \tag{4.12}$$

Due to the considered restriction the material becomes less symmetric and now is tetragonal.

Let us assume that the material has cubic symmetry in the limit state. It means that

$$\mathbf{H} = \frac{1}{\chi_1^2} \mathbf{P}_I + \frac{1}{\chi_2^2} \mathbf{P}_{II} + \frac{1}{\chi_3^2} \mathbf{P}_{III}.
 \tag{4.13}$$

The part of the limit condition (3.19) connected with the active part of stress (4.12) may be presented in the form

$$\boldsymbol{\sigma}_a \cdot \mathbf{H} \cdot \boldsymbol{\sigma}_a = \boldsymbol{\sigma}_a \bullet (\mathbf{S}_a \circ \mathbf{H} \circ \mathbf{S}_a) \bullet \boldsymbol{\sigma}_a,
 \tag{4.14}$$

where

$$\mathbf{S}_a \circ \mathbf{H} \circ \mathbf{S}_a = \frac{\lambda_{II}^2}{\chi_2^2} \mathbf{P}_{II}^\perp + \frac{\lambda_{III}^2}{\chi_3^2} \mathbf{P}_{III}.
 \tag{4.15}$$

The solutions of the appropriate characteristic equation (see Appendix (A.3)) are as follows:

$$2h_{II}^\perp = \frac{\chi_2^2}{\lambda_{II}}, \quad 2h_{III} = \frac{\chi_3^2}{\lambda_{III}}
 \tag{4.16}$$

and the corresponding energy orthogonal projectors have the form

$$\begin{aligned}
 \mathbf{H}_{II}^\perp &= \boldsymbol{\chi}^{(II)} \otimes \boldsymbol{\chi}^{(II)} = \lambda_{II} \mathbf{P}_{II}^\perp, \\
 \mathbf{H}_{III} &= \mathbf{S}_a - \mathbf{H}_{II}^\perp = \lambda_{III} \mathbf{P}_{III}.
 \end{aligned}
 \tag{4.17}$$

In view of the Rychlewski theorem (3.11) for active part of stress we have that

$$\boldsymbol{\sigma}_a \cdot \mathbf{H} \cdot \boldsymbol{\sigma}_a = \frac{1}{h_{II}^\perp} \phi(\boldsymbol{\sigma}_{II}^\perp) + \frac{1}{h_{III}} \phi(\boldsymbol{\sigma}_{III}),$$

where

$$\boldsymbol{\sigma}_{II}^\perp = \mathbf{H}_{II}^\perp \bullet \boldsymbol{\sigma}_a = \mathbf{P}_{II}^\perp \cdot \boldsymbol{\sigma}, \quad \boldsymbol{\sigma}_{III} = \mathbf{H}_{III} \bullet \boldsymbol{\sigma}_a = \mathbf{P}_{III} \cdot \boldsymbol{\sigma}.$$

The second part of the limit condition (3.19) is connected with the passive part of stress

$$\mathbf{H} \cdot \boldsymbol{\sigma}_p = \mathbf{H} \cdot (\boldsymbol{\sigma}_I + \boldsymbol{\sigma}_{II}^*) = \frac{1}{\chi_1^2} \boldsymbol{\sigma}_I + \frac{1}{\chi_2^2} \boldsymbol{\sigma}_{II}^*$$

and has the following form:

$$\begin{aligned} \boldsymbol{\sigma}_p \cdot \mathbf{H} \cdot \boldsymbol{\sigma}_p + 2\boldsymbol{\sigma}_a \cdot \mathbf{H} \cdot \boldsymbol{\sigma}_p &= \frac{1}{\chi_1^2} \boldsymbol{\sigma}_I \cdot \boldsymbol{\sigma}_I + \frac{1}{\chi_2^2} \boldsymbol{\sigma}_{II}^* \cdot \boldsymbol{\sigma}_{II}^* + 2 \cdot 0 \\ &= \frac{1}{\chi_1^2} \boldsymbol{\sigma}_I \cdot \boldsymbol{\sigma}_I + \frac{1}{\chi_2^2} \boldsymbol{\sigma}_{II}^* \cdot \boldsymbol{\sigma}_{II}^*. \end{aligned}$$

Finally the limit condition takes the form

$$(4.18) \quad \boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} = \frac{1}{h_{II}^\perp} \phi(\boldsymbol{\sigma}_{II}^\perp) + \frac{1}{h_{III}} \phi(\boldsymbol{\sigma}_{III}) + \frac{1}{\chi_1^2} \boldsymbol{\sigma}_I \cdot \boldsymbol{\sigma}_I + \frac{1}{\chi_2^2} \boldsymbol{\sigma}_{II}^* \cdot \boldsymbol{\sigma}_{II}^* \leq 1.$$

Some internal restrictions may be imposed in the limit state. Let us assume that the hydrostatic pressure is the safe state of stress. In such a case the condition (3.26) is as follows:

$$(4.19) \quad \boldsymbol{\sigma}_0 \cdot \mathbf{H} \cdot \boldsymbol{\sigma}_0 = p^2 \mathbf{1} \cdot \mathbf{H} \cdot \mathbf{1} = 0.$$

Introducing it to Eq. (4.18) it is obtained

$$(4.20) \quad \frac{1}{\chi_1^2} \mathbf{1} \cdot \mathbf{1} = \frac{3}{\chi_1^2} = 0 \implies \frac{1}{\chi_1^2} \rightarrow 0$$

and the modified yield condition (4.18) takes the form

$$(4.21) \quad \boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} = \frac{1}{h_{II}^\perp} \phi(\boldsymbol{\sigma}_{II}^\perp) + \frac{1}{h_{III}} \phi(\boldsymbol{\sigma}_{III}) + \frac{1}{\chi_2^2} \boldsymbol{\sigma}_{II}^* \cdot \boldsymbol{\sigma}_{II}^* \leq 1.$$

### 4.3. Fibers along the diagonal of cubic cell

In the considered case the spectral decomposition of the compliance tensor  $\mathbf{C}$  is as follows:

$$(4.22) \quad \begin{aligned} \mathbf{C} = \mathbf{C}_a &= \frac{1}{\lambda_{II}} \mathbf{P}_{II} + \frac{1}{\lambda_{III}} \mathbf{P}_{III}^\perp, \\ \mathbf{S} = \mathbf{S}_a &= \lambda_{II} \mathbf{P}_{II} + \lambda_{III} \mathbf{P}_{III}^\perp, \end{aligned}$$

where

$$(4.23) \quad \mathbf{P}_p = \mathbf{P}_I + \mathbf{P}_{III}^*, \quad \mathbf{P}_a = \mathbf{P}_{II} + \mathbf{P}_{III}^\perp$$

project the space of stresses onto the subspace of the passive stresses and onto the subspace of active stresses. Let us remind after [13] that

$$(4.24) \quad \begin{aligned} \mathbf{P}_I &= \frac{1}{3} \mathbf{1} \otimes \mathbf{1}, \\ \mathbf{P}_{III}^* &= \frac{3}{2} \left( \mathbf{k} \otimes \mathbf{k} - \frac{1}{3} \mathbf{1} \right) \otimes \left( \mathbf{k} \otimes \mathbf{k} - \frac{1}{3} \mathbf{1} \right), \\ \mathbf{P}_{II} &= \mathbf{K} - \frac{1}{3} \mathbf{1} \otimes \mathbf{1}, \\ \mathbf{P}_{III}^\perp &= \mathbf{I}_s - \mathbf{K} - \mathbf{P}_{III}^*. \end{aligned}$$

In this case the following decomposition of the stress space is helpful:

$$(4.25) \quad \boldsymbol{\sigma} = (\boldsymbol{\sigma}_I + \boldsymbol{\sigma}_{III}^*) + (\boldsymbol{\sigma}_{II} + \boldsymbol{\sigma}_{III}^\perp) = \boldsymbol{\sigma}_p + \boldsymbol{\sigma}_a.$$

Due to the considered restriction the material becomes less symmetric and now is trigonal.

Let us assume that the material has cubic symmetry in the limit state. It means that the limit tensor  $\mathbf{H}$  has the form (4.13).

The part of the limit condition connected with the active part of stress is given by (4.14), where

$$(4.26) \quad \mathbf{S}_a \circ \mathbf{H} \circ \mathbf{S}_a = \frac{\lambda_{II}^2}{\chi_2^2} \mathbf{P}_{II} + \frac{\lambda_{III}^2}{\chi_3^2} \mathbf{P}_{III}^\perp.$$

The solutions of appropriate characteristic equation are as follows:

$$(4.27) \quad 2h_{II} = \frac{\chi_2^2}{\lambda_{II}}, \quad 2h_{III}^\perp = \frac{\chi_3^2}{\lambda_{III}}$$

and the corresponding energy orthogonal projectors have the form

$$(4.28) \quad \begin{aligned} \mathbf{H}_{II} &= \lambda_{II} \mathbf{P}_{II}, \\ \mathbf{H}_{III}^\perp &= \mathbf{S}_a - \mathbf{H}_{II} = \lambda_{III} \mathbf{P}_{III}^\perp. \end{aligned}$$

Therefore, in view of the Rychlewski theorem we have that

$$\boldsymbol{\sigma}_a \cdot \mathbf{H} \cdot \boldsymbol{\sigma}_a = \frac{1}{h_{II}} \phi(\boldsymbol{\sigma}_{II}) + \frac{1}{h_{III}^\perp} \phi(\boldsymbol{\sigma}_{III}^\perp),$$

where

$$\begin{aligned} \boldsymbol{\sigma}_{II} &= \mathbf{H}_{II} \bullet \boldsymbol{\sigma}_a = \mathbf{P}_{II} \cdot \boldsymbol{\sigma}, \\ \boldsymbol{\sigma}_{III}^\perp &= \mathbf{H}_{III}^\perp \bullet \boldsymbol{\sigma}_a = \mathbf{P}_{III}^\perp \cdot \boldsymbol{\sigma}. \end{aligned}$$

The second part of the limit condition is connected with the passive part of stress

$$\mathbf{H} \cdot \boldsymbol{\sigma}_p = \mathbf{H} \cdot (\boldsymbol{\sigma}_I + \boldsymbol{\sigma}_{III}^*) = \frac{1}{\chi_1^2} \boldsymbol{\sigma}_I + \frac{1}{\chi_3^2} \boldsymbol{\sigma}_{III}^*$$

and has the following form:

$$\begin{aligned} \boldsymbol{\sigma}_p \cdot \mathbf{H} \cdot \boldsymbol{\sigma}_p + 2\boldsymbol{\sigma}_a \cdot \mathbf{H} \cdot \boldsymbol{\sigma}_p &= \frac{1}{\chi_1^2} \boldsymbol{\sigma}_I \cdot \boldsymbol{\sigma}_I + \frac{1}{\chi_3^2} \boldsymbol{\sigma}_{III}^* \cdot \boldsymbol{\sigma}_{III}^* + 2 \cdot 0 \\ &= \frac{1}{\chi_1^2} \boldsymbol{\sigma}_I \cdot \boldsymbol{\sigma}_I + \frac{1}{\chi_3^2} \boldsymbol{\sigma}_{III}^* \cdot \boldsymbol{\sigma}_{III}^*. \end{aligned}$$

Finally, the limit condition takes the form

$$(4.29) \quad \boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} = \frac{1}{h_{II}} \phi(\boldsymbol{\sigma}_{II}) + \frac{1}{h_{III}^\perp} \phi(\boldsymbol{\sigma}_{III}^\perp) + \frac{1}{\chi_1^2} \boldsymbol{\sigma}_I \cdot \boldsymbol{\sigma}_I + \frac{1}{\chi_3^2} \boldsymbol{\sigma}_{III}^* \cdot \boldsymbol{\sigma}_{III}^* \leq 1.$$

If the hydrostatic pressure is the safe state of stress like in the previous example, the conditions (4.19) and (4.20) are obtained and the modified yield condition takes the form

$$(4.30) \quad \boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} = \frac{1}{h_{II}} \phi(\boldsymbol{\sigma}_{II}) + \frac{1}{h_{III}^\perp} \phi(\boldsymbol{\sigma}_{III}^\perp) + \frac{1}{\chi_3^2} \boldsymbol{\sigma}_{III}^* \cdot \boldsymbol{\sigma}_{III}^* \leq 1.$$

The above examples show how the energy-based form of the limit condition depends on the elastic properties of the material. Comparing the forms of the limit conditions (4.18) and (4.29) for the material with the same limit tensor  $\mathbf{H}$  of cubic symmetry but with the different stiffness tensor  $\mathbf{C}$  due to different form of internal restriction in the elastic regime, one may notice that different parts of the limit conditions in these two cases have the interpretation connected with the internal energy density. The sequence of taking into account the restrictions imposed in the limit state is arbitrary.

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## Appendix A. Spectral theorem for the Euclidean space with an energy scalar product

Let us call the fourth-order tensor  $\mathbf{L}$  having all internal symmetries of the stiffness tensor ( $L_{ijkl} = L_{jikl} = L_{ijlk} = L_{klij}$ ) the Hooke tensor [19] and denote a set of such a tensors by  $\mathcal{L}$ .

Let us consider a scalar product  $\bullet$  satisfying the appropriate axioms defined as follows

$$(A.1) \quad \alpha \bullet \beta \equiv \alpha \cdot \mathbf{A} \cdot \beta = \alpha_{ij} A_{ijkl} \beta_{kl}, \quad \mathbf{A} \in \mathcal{L},$$

where  $\mathbf{A}$  is positive definite and fixed. This scalar product is called the *energy scalar product*. The energy orthogonality of two states  $\alpha$  and  $\beta$  means that

$$(A.2) \quad \alpha \bullet \beta = 0 \quad \Leftrightarrow \quad \alpha \perp \beta.$$

The six-dimensional Euclidean space  $\mathcal{T}_2^s$  with the energy scalar product obeys the general spectral theorem. Applying it we introduce the energy proper value  $1/2h$  being always real number and the energy proper state  $\chi$  for the symmetric linear operator  $\mathbf{L} \in \mathcal{L}$  satisfying the equation

$$(A.3) \quad \mathbf{L} \bullet \chi = \frac{1}{2h} \chi.$$

The symmetric linear operator  $\mathbf{L}$  is acting according to the formula

$$(A.4) \quad \mathbf{L} \bullet \alpha = \mathbf{L} \cdot (\mathbf{A} \cdot \alpha) = (\mathbf{L} \circ \mathbf{A}) \cdot \alpha.$$

Assuming in (A.4) that  $\mathbf{L} = \mathbf{A}^{-1}$  it is easy to verify that the tensor  $\mathbf{A}^{-1}$  is the identity operator for space  $\mathcal{T}_2^s$  with the energy scalar product.

For the scalar product (A.1), the so-called basic identity has the following form

$$(A.5) \quad \mathbf{L} = (\mathbf{L} \bullet \eta_K) \otimes \eta_K = (\mathbf{L} \bullet \eta_I) \otimes \eta_I + \dots + (\mathbf{L} \bullet \eta_{VI}) \otimes \eta_{VI},$$

where tensors  $\eta_K$  form the energy orthogonal basis in  $\mathcal{T}_2^s$  that is  $\eta_K \bullet \eta_L = \delta_{KL}$ . If as such an energy orthogonal basis  $\eta_K$  the energy proper tensors  $\chi_K$  (A.3) are taken, then the spectral decomposition of the operator  $\mathbf{L}$  is obtained

$$(A.6) \quad \mathbf{L} = \frac{1}{2h_1} \chi_I \otimes \chi_I + \dots + \frac{1}{2h_6} \chi_{VI} \otimes \chi_{VI}.$$

For multiple eigenvalues  $h_K$ , the unique form of the spectral decomposition is as follows:

$$(A.7) \quad \mathbf{L} = \frac{1}{2h_1} \mathbf{H}_I + \dots + \frac{1}{2h_{\varkappa}} \mathbf{H}_{\varkappa}, \quad \varkappa \leq 6.$$

The fourth-order tensors  $\mathbf{H}_K$  are called the *energy-orthogonal projectors*

$$(A.8) \quad \mathbf{H}_K \bullet \boldsymbol{\alpha} = \boldsymbol{\alpha}_K \in \mathcal{H}_K$$

and constitute spectral decomposition of the identity operator  $\mathbf{A}^{-1}$

$$(A.9) \quad \mathbf{A}^{-1} = \mathbf{H}_I + \mathbf{H}_{II} + \dots + \mathbf{H}_{\varkappa}.$$

Equation (A.9) implies that

$$(A.10) \quad \boldsymbol{\alpha} = \mathbf{A}^{-1} \bullet \boldsymbol{\alpha} = \mathbf{H}_I \bullet \boldsymbol{\alpha} + \mathbf{H}_{II} \bullet \boldsymbol{\alpha} + \dots + \mathbf{H}_{\varkappa} \bullet \boldsymbol{\alpha} = \boldsymbol{\alpha}_I + \boldsymbol{\alpha}_{II} + \dots + \boldsymbol{\alpha}_{\varkappa}.$$

The subspaces  $\mathcal{H}_K$  constitute the energy orthogonal decomposition of the space  $\mathcal{T}_2^s$

$$(A.11) \quad \mathcal{T}_2^s = \mathcal{H}_I \oplus \mathcal{H}_{II} \oplus \dots \oplus \mathcal{H}_{\varkappa}, \quad \mathcal{H}_K \perp \mathcal{H}_L \quad \text{if } L \neq K.$$

Let us remark that the standard scalar product called the geometrical scalar product also falls into the definition (A.1) with  $\mathbf{A} = \mathbf{I}_s = \mathbf{A}^{-1}$ . For the purpose of this work, the energy scalar product introduced in [19] will be used. In this case  $\mathbf{A} = \mathbf{C}$  and  $\mathbf{A}^{-1} = \mathbf{S}$  and clear mechanical interpretation of (A.2) may be given. Two stresses  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  are energy orthogonal if one of them does not work on the strain caused by the other. If in (A.1) we have  $\boldsymbol{\alpha} = \boldsymbol{\beta}$  then the square of energy norm is obtained that in this case is equal to the double elastic energy density accumulated due to the action of  $\boldsymbol{\alpha}$ .

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