Arch. Mech., **56**, 6, pp. 485–463, Warszawa 2004

Brief Notes

Oblique wave scattering by undulations on the bed of an ice-covered ocean

B.N. MANDAL, P. MAITI

Physics and Applied Mathematics Unit,
Indian Statistical Institute,
203, B.T. Road, Kolkata 700 108, India

The problem of oblique wave scattering by cylindrical undulations on the bed of an ice-covered ocean is investigated by using a simplified perturbation analysis. The first-order potential function satisfies a boundary value problem (BVP) which is solved by employing the Green integral theorem after constructing an appropriate Green function. Analytical expressions for the first-order reflection and transmission coefficients are then obtained from the solution of this BVP, in terms of the integrals involving the shape function describing undulations. Three particular forms of the shape function are considered for which the reflection and transmission coefficients up to the first-order are obtained exactly.

1. Introduction

When a train of surface water waves is incident on an obstacle situated at the bottom of a laterally unbounded ocean of uniform finite depth, it is partially reflected by and transmitted over the obstacle. For an obstacle of arbitrary shape, the problem of determining the reflection and transmission coefficients is in general a difficult task. However, when the obstacle is in the form of a small deformation of the bottom (long-crested sea-bed undulations), then some approximate methods can be employed to obtain these coefficients approximately. For example, for small cylindrical deformation of the bottom, Miles [7] used a perturbation method followed by the finite cosine transform technique in the mathematical analysis to obtain the reflection and transmission coefficients up to the first order, when a train of surface waves is obliquely incident on the bottom deformation. When the obstacle is in the form of bottom undulations, such as sand ripples, Davies [3] considered normal incidence of a surface water wave train and treated the problem on the basis of linear perturbation theory. He introduced a linear friction term in the dynamical condition at the free surface so as to apply the Fourier transform technique in the mathematical analysis. The coefficient of friction was then made to tend to zero in the asymptotic results for the velocity potential far away from the undulations, so as to obtain the reflection and transmission coefficients up to the first order analytically. Later MANDAL and BASU [6] generalised the problem considered in [7] to include the effect of surface tension at the free surface. They also employed a simplified perturbation analysis followed by an appropriate use of Green's integral theorem in the mathematical analysis to obtain a general representation of the first-order potential function. Its asymptotic forms far away from the deformation at either side produce the first-order reflection and transmission coefficients in terms of the integrals involving the shape function describing the deformation.

All the works in [3, 6, 7] involve an ocean with a *free surface*. However, there is a considerable interest in recent times to investigate the wave propagation problems in an *ice-covered* ocean wherein the ocean is covered by a thin sheet of ice, modelled as an elastic plate, the ice-cover being virtually weightless. This has motivated us to consider the problem of oblique wave scattering by small undulations on the bottom of a laterally unbounded ocean with an *ice-cover* instead of a *free surface*. The ice-cover is modelled as a thin sheet of elastic plate of infinite extent having a very small thickness h_0.

Assuming linear theory and irrotational motion, the velocity potential function describing the time-harmonic motion of angular frequency σ, in water of *uniform* finite depth h and having an ice-cover at the top, can be represented by Re $(\phi e^{-i\sigma t})$, where ϕ satisfies the equation

$$\nabla^2\phi = 0, \qquad 0 \le y \le h, \tag{1.1}$$

the linearised ice-cover condition (GOLDSHTEIN and MARCHENKO [5], CHAKRABARTI [1])

$$K\phi + (D\nabla^4_{x,z} + 1)\phi_y = 0 \quad \text{on} \quad y = 0, \tag{1.2}$$

and the bottom condition

$$\phi_y = 0 \quad \text{on} \quad y = h. \tag{1.3}$$

The time-dependent factor $e^{-i\sigma t}$ will be dropped throughout the paper from now on. Here the y-axis is directed vertically downwards into the fluid region, (x, z)-plane is the rest position of the lower part of the ice-cover, ∇^2 denotes the three-dimensional Laplacian operator while $\nabla^4_{x,z}$ denotes the two-dimensional biharmonic operator in the (x, z)-plane, $K = \sigma^2/g$ where g is the gravity, D is the flexural rigidity of ice-cover and is given by

$$D = \frac{Eh_0^3}{12(1-\gamma^2)\rho g},$$

where E is the Young's modulus, γ is Poisson's ratio of the elastic material comprising the ice-cover and ρ is the density of water. In deriving the ice-cover condition (1.2), waves are assumed to be long compared to the thickness of the ice-cover. A possible solution for ϕ representing a train of time-harmonic waves propagating on the ice-cover and making an angle θ with the positive x-direction, is given by

$$\phi_0(x, y, z) = \cosh k_0(h - y)e^{ik_0(x\cos\theta + z\sin\theta)} \tag{1.4}$$

where k_0 is the unique real positive root of the transcendental equation

$$\Delta(k) \equiv k(Dk^4 + 1)\sinh kh - K\cosh kh = 0. \tag{1.5}$$

This equation has two real roots $\pm k_0$, two pairs of complex conjugate roots $\pm\mu, \pm\overline{\mu}(\mu = \alpha + i\beta,\ \overline{\mu} = \alpha - i\beta,\ \alpha > 0, \beta > 0$ and $\alpha > \beta)$ and an infinite number of purely imaginary roots $\pm ik_n(k_n > 0, n = 1, 2, \ldots)$ where $k_n(n = 1, 2, \ldots)$ are real and satisfy

$$k_n(Dk_n^4 + 1)\sin k_n h + K\cos k_n h = 0, \tag{1.6}$$

and $k_n \to \dfrac{n\pi}{h}$ as $n \to \infty$ (cf. CHUNG and FOX [2]).

To tackle the problem of oblique wave scattering by small cylindrical undulations of the bottom of an ocean with an *ice-cover*, here also we apply a perturbation technique directly to the governing partial differential equation and the boundary and infinity conditions for the potential function, after extracting the z-dependence by exploiting the geometry of the problem, to obtain a boundary value problem (BVP). A suitable use of Green's integral theorem produces the solution of this BVP, from which the first-order reflection and transmission coefficients are obtained in terms of integrals involving the shape function defining the undulations. For three different forms of the shape functions these coefficients are obtained in closed forms.

2. Formulation of the problem

The problem of oblique wave scattering by small cylindrical bottom undulations in an *ice-covered* ocean, assuming linear theory and irrotational motion, is mathematically equivalent to solving the following BVP. We solve the partial differential equation (PDE)

$$\nabla^2\phi = 0 \tag{2.1}$$

in the region $0 \le y \le h + \epsilon c(x),\ -\infty < x, z < \infty$, with the boundary conditions

$$K\phi + (D\nabla^4_{x,z} + 1)\phi_y = 0 \quad \text{on} \quad y = 0, \tag{2.2}$$

$$\phi_n = 0 \quad \text{on} \quad y = h + \epsilon c(x) \tag{2.3}$$

together with suitable conditions as $x \to \pm\infty$ which will be stated shortly.

Here $c(x)$ is a continuous and bounded function describing the shape of the undulations of the ocean bed and $c(x) \to 0$ as $|x| \to \infty$, so that the ocean is of uniform finite depth far away from the undulations on either side, and $\epsilon(> 0)$ is a small parameter giving a measure of the smallness of the undulations. ϕ_n in (2.3) denotes the normal derivative.

We assume that a water wave train represented by the velocity potential $\phi_0(x, y, z)$, given by (1.4), is obliquely incident upon the undulations from a large distance in the direction of negative x-axis, then it undergoes partial transmission and reflection by the undulations. Thus the asymptotic behaviour of $\phi(x, y, z)$ is given by

$$\phi \to \begin{cases} T\phi_0(x, y, z) & \text{as} \quad x \to \infty, \\ \phi_0(x, y, z) + R\phi_0(-x, y, z) & \text{as} \quad x \to -\infty, \end{cases} \tag{2.4}$$

where T and R are the transmission and reflection coefficients respectively and will have to be determined.

As ϵ is very small, we can approximate the bottom condition (2.3) after neglecting the $O(\epsilon^2)$ terms as

$$-\phi_y + \epsilon \left\{ c'(x)\phi_x - c(x)\phi_{yy} \right\} = 0 \quad \text{on} \quad y = h. \tag{2.5}$$

In view of the geometry of the problem, we can assume that

$$\phi(x, y, z) = \psi(x, y)e^{i\nu z} \tag{2.6}$$

where $\nu = k_0 \sin\theta$. Thus the z-dependence is extracted, and the function $\psi(x, y)$ satisfies the BVP described by

$$\begin{aligned} &\psi_{xx} + \psi_{yy} - \nu^2\psi = 0, \quad 0 \le y \le h, \quad -\infty < x < \infty, \\ &K\psi + \left\{ D\left(\frac{\partial^2}{\partial x^2} - \nu^2\right)^2 + 1 \right\}\psi_y = 0 \quad \text{on} \quad y = 0, \\ &-\psi_y + \epsilon\left\{ \frac{\partial}{\partial x}\left(c(x)\psi_x\right) - \nu^2 C(x) \right\} = 0 \quad \text{on} \quad y = h, \\ &\psi(x, y) \to \begin{cases} T\psi_0(x, y) & \text{as} \quad x \to \infty, \\ \psi_0(x, y) + R\psi_0(-x, y) & \text{as} \quad x \to -\infty \end{cases} \end{aligned} \tag{2.7}$$

where

$$\psi_0(x, y) = e^{ik_0 x \cos\theta} \cosh k_0(h - y). \tag{2.8}$$

This BVP is solved approximately up to the first order of ψ by using the perturbation analysis applied to the governing PDE, the boundary conditions and asymptotic conditions.

3. Method of solution

Because of the approximate boundary condition $(2.7)_3$ and the fact that a wave train propagating in an ocean of uniform finite depth h experiences no reflection, we may assume that ψ, T and R in (2.7) can be expanded in terms of the small parameter ϵ as

$$\begin{aligned} \psi(x,y) &= \psi_0(x,y) + \epsilon\psi_1(x,y) + O(\epsilon^2), \\ T &= 1 + \epsilon T_1 + O(\epsilon^2), \\ R &= \epsilon R_1 + O(\epsilon^2). \end{aligned} \tag{3.1}$$

Using the expansions (3.1) in Eqs. (2.7), we find that $\psi_1(x,y)$ satisfies the BVP described by

$$\begin{aligned} &\psi_{1xx} + \psi_{1yy} - \nu^2\psi_1 = 0, \quad 0 \le y \le h, \quad -\infty < x < \infty, \\ &K\psi_1 + \left\{ D\left(\frac{\partial^2}{\partial x^2} - \nu^2\right)^2 + 1 \right\}\psi_{1y} = 0 \quad \text{on} \quad y = 0, \\ &\psi_{1y} = ik_0\cos\theta\frac{\partial}{\partial x}\left(c(x)e^{ik_0 x\cos\theta}\right) - \nu^2 c(x) \\ &\qquad \equiv q(x) \quad \text{on} \quad y = h, \\ &\psi_1(x,y) \to \begin{cases} T_1\psi_0(x,y) & \text{as} \quad x \to \infty, \\ R_1\psi_0(-x,y) & \text{as} \quad x \to -\infty. \end{cases} \end{aligned} \tag{3.2}$$

We note that $\psi_1(x,y)$ behaves as an outgoing wave as $|x| \to \infty$.

By an appropriate use of Green's integral theorem, the solution of the BVP is obtained as

$$\psi_1(\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(x,h;\xi,\eta)q(x)dx, \tag{3.3}$$

where $G(x,h;\xi,\eta)$ is the corresponding Green's function and is given by (cf. EVANS and PORTER [4])

$$
\begin{aligned}
(3.4)\qquad & G(x,y;\xi,\eta) \\
&= -4\pi \sum_{n=1}^{\infty} \frac{k_n(Dk_n^4+1)\cos k_n(h-y)\cos k_n(h-\eta)}{2k_n h(Dk_n^4+1)+(5Dk_n^4+1)\sin 2k_n h} \frac{e^{-(K_n^2+\nu^2)^{1/2}|x-\xi|}}{(k_n^2+\nu^2)^{1/2}} \\
&\quad - 4\pi i \left[\frac{k_0(Dk_0^4+1)\cosh k_0(h-y)\cosh k_0(h-\eta)}{2k_0 h(Dk_0^4+1)+(5Dk_0^4+1)\sinh 2k_0 h} \frac{e^{i(k_0^2-\nu^2)^{1/2}|x-\xi|}}{(k_0^2-\nu^2)^{1/2}} \right. \\
&\qquad + \frac{\mu(D\mu^4+1)\cosh \mu(h-y)\cosh \mu(h-\eta)}{2\mu h(1+D\mu^4)+(5D\mu^4+1)\sinh 2\mu h} \frac{e^{i\mu'|x-\xi|}}{\mu'} \\
&\qquad \left. - \frac{\overline{\mu}(D\overline{\mu}^4+1)(\cosh \overline{\mu}(h-y)\cosh \overline{\mu}(h-\eta)}{2\overline{\mu} h(D\overline{\mu}^4+1)+(5D\overline{\mu}^4+1)\sinh 2\overline{\mu} h} \frac{e^{-i\overline{\mu}'|x-\xi|}}{\overline{\mu}'} \right],
\end{aligned}
$$

where $\mu' = (\mu^2-\nu^2)^{1/2}$ and $-\overline{\mu}' = \{(-\overline{\mu})^2-\nu^2\}^{1/2}$, and that branch of the square root has been chosen such that $\mu' = \mu, -\overline{\mu}' = -\overline{\mu}$ when $\nu = 0$.

Since μ' and $-\overline{\mu}'$ have positive imaginary parts, we find that, as $|x-\xi| \to \infty$,

$$
(3.5)\qquad G(x,y;\xi,\eta) \to -4\pi i \frac{k_0(Dk_0^4+1)\cosh k_0(h-y)\cosh k_0(h-\eta)}{2k_0 h(Dk_0^4+1)+(5Dk_0^4+1)\sinh 2k_0 h} \times \frac{e^{i(k_0^2-\nu^2)^{1/2}|x-\xi|}}{(k_0^2-\nu^2)^{1/2}}
$$

so that G behaves as an outgoing wave for $|x-\xi| \to \infty$.

To obtain the first-order transmission and reflection coefficients T_1 and R_1 respectively, we note from $(3.2)_4$ and (3.5) that

$$
(3.6)\qquad \psi_1(\xi,\eta) \to \begin{cases} T_1\psi_0(\xi,\eta) & \text{as} \quad \xi \to \infty, \\ R_1\psi_0(-\xi,\eta) & \text{as} \quad \xi \to -\infty \end{cases}
$$

and

$$
(3.7)\qquad G(x,0;\xi,\eta) \to -4\pi i \frac{e^{\mp ik_0 x\cos\theta}}{k_0\cos\theta} A\psi_0(\pm\xi,\eta) \quad \text{as} \quad \xi \to \pm\infty\,,
$$

where

$$
(3.8)\qquad A = \frac{1}{h + \dfrac{(1+5Dk_0^4)\sinh^2 k_0 h}{K}}\,.
$$

Using the asymptotic results (3.6) and (3.7) in the representation (3.3), we find that

$$
\begin{aligned}
T_1 &= -\frac{i}{k_0\cos\theta}A\int_{-\infty}^{\infty} e^{-ik_0x\cos\theta}q(x)dx \\
&= ik_0\sec\theta A\int_{-\infty}^{\infty} c(x)dx,
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
R_1 &= -\frac{i}{k_0\cos\theta}A\int_{-\infty}^{\infty} e^{ik_0x\cos\theta}q(x)dx \\
&= -ik_0\sec\theta\cos 2\theta A\int_{-\infty}^{\infty} e^{2ik_0x\cos\theta}q(x)dx.
\end{aligned}
\tag{3.10}
$$

The results for an ocean with a *free surface* are recovered by putting $D=0$ in (3.9) and (3.10) where then, however, k_0 denotes the unique real positive zero of the transcendental equation

$$
k\sinh kh - K\cosh kh = 0. \tag{3.11}
$$

It is also interesting to note that R_1 vanishes identically for $\theta=\pi/4$, independently of the shape function $c(x)$. This was also observed in [6, 7] in the case of an ocean with a free surface with or without surface tension.

We now consider three special types of undulations.

(i) $c(x)=ae^{-\lambda|x|}(\lambda>0)$. Here the bottom undulation reaches maximum at $(0,h)$ and decreases exponentially on either side of $(0,h)$. In this case

$$
\begin{aligned}
T_1 &= \frac{2iak_0A}{\lambda}\sec\theta\,, \\
R_1 &= -\frac{2iak_0A\lambda}{\lambda^2+4k_0^2\cos^2\theta}\sec\theta\cos 2\theta.
\end{aligned}
$$

(ii) $c(x)=ae^{-\lambda x^2}(\lambda>0)$. Here the undulation is of Gaussian type and has the maximum value at $(0,h)$. In this case

$$
\begin{aligned}
T_1 &= i\left(\frac{\pi}{\lambda}\right)^{1/2}k_0aA\sec\theta\,, \\
R_1 &= i\left(\frac{\pi}{\lambda}\right)^{1/2}k_0aA\sec\theta\cos 2\theta\; e^{-\frac{k_0^2\cos^2\theta}{\lambda}}\,.
\end{aligned}
$$

(iii) $c(x)=\begin{cases} a\sin\lambda x, & -\dfrac{m\pi}{\lambda}\le x\le\dfrac{m\pi}{\lambda}, \\ 0, & \text{otherwise.}\end{cases}$

This represents sinusoidal undulations of the bottom, having number m of patches and is of considerable physical interest. Davies [3] earlier made a somewhat elaborate study on the effect of sinusoidal undulations on the bottom of an ocean with a *free surface*, upon an incident surface water wave train. In this case

$$T_1 \equiv 0,$$

$$R_1 = \sec^2\theta\cos 2\theta B(-1)^m \frac{\alpha}{\alpha^2-1}\sin(\alpha m\pi)$$

where $B = aA$, $\alpha = \dfrac{2k_0}{\lambda}\cos\theta$. It is interesting to note that when $\alpha \approx 1$, i.e. $\lambda \approx 2k_0\cos\theta$,

$$R_1 \approx \frac{\pi}{2}\sec^2\theta\cos 2\theta Bm. \tag{3.12}$$

The result (3.11) has the implication that a somewhat large reflection of the incident wave energy occurs when the bed wave number λ is twice the wave number component of the incident wave field along the x-direction, if the integer m denoting the number of patches is made large. This phenomenon has a practical application in the construction of an efficient reflector of incident wave energy.

4. Discussion

A simplified perturbation analysis is employed to obtain the first-order transmission and reflection coefficients for the problem of oblique wave scattering by small cylindrical undulations on the bottom of an ocean with an *ice-cover* modelled as a thin elastic plate. The first-order reflection coefficient vanishes independently of the shape of the undulations if the angle of incidence is $\dfrac{\pi}{4}$. By making D equal zero, the results for an ocean with a *free surface* are recovered. For sinusoidal undulations having m patches, the first-order transmission coefficient vanishes identically, and the reflection coefficient becomes a constant multiple of the number of patches when the ocean-bed wave number is twice the x-component of the incident field wave number, what suggests that comparatively large reflection of the incident wave energy is possible by making the number of patches somewhat larger.

Acknowledgment

The authors thank the Reviewer for his comments to revise the paper in the present form.

References

1. A. CHAKRABARTI, *On the solution of the problem of scattering of surface water waves by the edge of an ice-cover*, Proc. R. Soc. Lond., **A456**, 1087–1099, 2000.

2. H. CHUNG and C. FOX, *Calculation of wave-ice interaction using the Wiener–Hopf technique*, New Zealand J. Math., **31**, 1–18, 2002.

3. A.G. DAVIES, *The reflection of wave energy by undulations in the seabed*, Dyn. Atmos. Oceans, **6**, 207–232, 1982.

4. D.V. EVANS and R. PORTER, *Wave scattering by narrow cracks in ice sheets floating on water of finite depth*, J. Fluid Mech., **484**, 143–165, 2003.

5. R.V. GOLDSHTEIN and A.V. MARCHENKO, *The diffraction of plane gravitational waves by the edge of an ice-cover*, PMM USSR, **53**, 731–736, 1989.

6. B.N. MANDAL and U. BASU, *A note on oblique water wave diffraction by a cylindrical deformation of the bottom in the presence of surface tension*, Arch. Mech., **42**, 723–727, 1990.

7. J.W. MILES, *Oblique surface wave diffraction by a cylindrical obstacle*, Dyn. Atmos. Oceans, **6**, 121–123, 1981.

Received May 13, 2003; revised version July 27, 2004.