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The planar crack problem for a dielectric medium in a uniform electric field

Y. SHINDO F. NARITA

Department of Materials Processing,
Graduate School of Engineering,
Tohoku University,
Aoba-yama 6-6-02, Sendai 980-8579, Japan

The theory for dielectric materials is applied to solve the planar problem of a Griffith crack in an infinite isotropic dielectric body subjected to a far-field tension and a uniform electric field. Fourier transforms are used to reduce the mixed boundary value problem to two simultaneous dual integral equations. The integral equations are then solved exactly, and the stress intensity factor and energy release rate under Mode I and Mode II loadings are expressed in closed form.

1. Introduction

Elastic dielectrics such as insulating materials are currently being used in construction of the spacecrafts which operate over long periods of time on Earth's orbit [1]. Insulating materials are reported to have poor mechanical properties. Mechanical failure of insulators is also a well-known phenomenon among insulation engineers. Expressions for the stress intensity factor and energy release rate may be useful for determining life expectancy of elastic dielectrics. Toupin [2] considered the isotropic elastic dielectric and obtained the form of constitutive relations for the stress and effective local fields. Kurlandzka [3] investigated a crack propagation problem of an elastic dielectric subjected to an electrostatic field. Pak and Herrmann [4, 5] also derived a material force in the form of a path-independent integral for the elastic dielectric, which is related to the energy release rate, and evaluated it for a crack placed in an infinite dielectric medium subjected to a far-field uniaxial tension and an external electric field. Shindo *et al.* [6] considered the scattering of normally incident waves by a crack in an isotropic dielectric body under a uniform electric field.

In the present paper, we investigate the planar problem for an infinite dielectric medium containing a Griffith crack subject to a uniform electric field. The problem is considered for a uniform electric field normal to the crack surface and the uniform loading (at infinity) forms an angle with the crack surface. By using Fourier transforms, the problem is reduced to that of solving a system of simultaneous dual integral equations. The simultaneous integral equations are solved exactly and explicit expressions for the Mode I and Mode II stress intensity factors are obtained. The solutions are then used in evaluating the path-independent integrals to find the material forces in explicit form for the Mode I and Mode II cases.

2. Basic Equations

In rectangular Cartesian coordinates $x_i(\mathrm{O}-x_1, x_2, x_3)$, we decompose the electric field intensity vector E_i, the polarization vector P_i and the electric displacement vector D_i into those representing the rigid body state, indicated by overbars and those for the deformed state, denoted by lower case letters,

$$E_i = \bar{E}_i + e_i\,, \qquad P_i = \bar{P}_i + p_i\,, \qquad D_i = \bar{D}_i + d_i. \tag{2.1}$$

We assume that the deformation will be small even with large electric fields and the second terms will have only a minor influence on the total fields. The quasi-linear formulations will then be linearized with respect to these unknown deformed state quantities.

The linearized field equations can be written as

$$\sigma^L_{ji,j} + \bar{E}_{i,j}p_j + \bar{P}_j e_{i,j} = 0, \tag{2.2}$$

$$\bar{D}_{i,i} = 0, \tag{2.3}$$

$$d_{i,i} = 0, \tag{2.4}$$

where σ^L_{ij} is the local stress tensor, a comma denotes partial differentiation with respect to the coordinate x_i, and the summation convention for repeated indices is employed.

The linearized constitutive equations become

$$\sigma^L_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}) + A_1(\bar{E}_k\bar{E}_k + 2\bar{E}_k e_k)\delta_{ij} + A_2(\bar{E}_i\bar{E}_j + \bar{E}_i e_j + \bar{E}_j e_i), \tag{2.5}$$

$$\sigma^M_{ij} = \varepsilon_0\varepsilon_r(\bar{E}_i\bar{E}_j + \bar{E}_i e_j + \bar{E}_j e_i) - \frac{1}{2}\varepsilon_0\left(\bar{E}_k\bar{E}_k + 2\bar{E}_k e_k\right)\delta_{ij}, \tag{2.6}$$

$$\bar{D}_i = \varepsilon_0\bar{E}_i + \bar{P}_i = \varepsilon_0\varepsilon_r\bar{E}_i, \qquad d_i = \varepsilon_0 e_i + p_i = \varepsilon_0\varepsilon_r e_i, \tag{2.7}$$

$$\bar{E}_i = \frac{1}{\varepsilon_0\eta}\bar{P}_i, \qquad e_i = \frac{1}{\varepsilon_0\eta}p_i, \tag{2.8}$$

where σ_{ij}^M is the Maxwell stress tensor, u_i is the displacement vector, λ and μ are the Lamé constants, A_1 and A_2 are the electrostrictive coefficients, ε_0 is the permittivity of free space, $\varepsilon_r = 1 + \eta$ is the specific permittivity, η is the electric susceptibility, and δ_{ij} is the Kronecker delta.

The linearized boundary conditions are

$$[|\ \sigma_{ji}^L\ |]\, n_j + \frac{1}{2\varepsilon_0}\Big[(\bar{P}_k n_k)^2 + 2\bar{P}_k p_l n_k n_l\Big] n_i = 0; \tag{2.9}$$

$$\begin{aligned} [|\ \bar{D}_i\ |] n_i &= 0,\\ e_{ijk} n_j [|\ \bar{E}_k\ |] &= 0, \end{aligned} \tag{2.10}$$

$$\begin{aligned} [|\ d_i\ |] n_i - [|\ \bar{D}_i\ |] u_{i,j} n_j &= 0,\\ e_{ijk}\{n_j [|\ e_k\ |] - n_l u_{l,j} [|\ \bar{E}_k\ |]\} &= 0, \end{aligned} \tag{2.11}$$

where n_i is the outer unit vector normal to an undeformed body, e_{ijk} is the permutation symbol and $[|\ f_i\ |]$ means the jump in any field quantity f_i across the discontinuity surface.

3. Problem statement

Let a Griffith crack be located in the interior of an infinite elastic dielectric. We consider a rectangular Cartesian coordinate system (x, y, z) such that the crack is placed on the x-axis from $-a$ to a as shown in Fig. 1, and assume plane

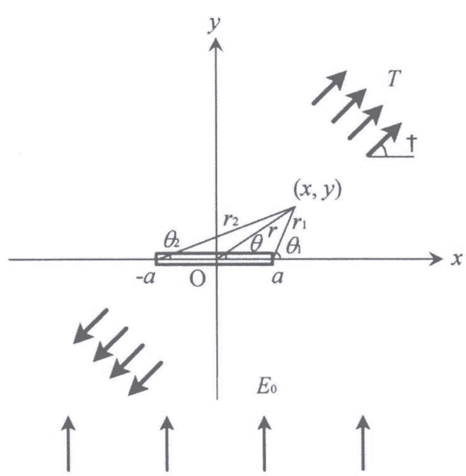

FIG. 1. An infinite isotropic dielectric medium with a Griffith crack.

strain perpendicular to the z-axis. A uniform electric field E_0 is applied perpendicularly to the crack surface. For convenience, all electric quantities outside the solid will be denoted by the superscript $+$. The solution for the rigid body state outside the solid is

$$\begin{aligned} &\bar{E}_y^+ = \varepsilon_r E_0, \qquad &\bar{D}_y^+ = \varepsilon_0 \varepsilon_r E_0, \qquad &\bar{P}_y^+ = 0, \\ &\bar{E}_y = E_0, \qquad &\bar{D}_y = \varepsilon_0 \varepsilon_r E_0, \qquad &\bar{P}_y = \varepsilon_0 \eta E_0. \end{aligned} \tag{3.1}$$

The governing equations in the x and y directions are then given by

$$\begin{aligned} &\nabla_1^2 u_x + \frac{1}{1-2\nu}\Big(u_{x,x} + u_{y,y}\Big)_{,x} + \frac{2A_1 E_0}{\mu} e_{y,x} + \frac{A_3 E_0}{\mu} e_{x,y} = 0, \\ &\nabla_1^2 u_y + \frac{1}{1-2\nu}\Big(u_{x,x} + u_{y,y}\Big)_{,y} + \Big(2A_1 + A_2 + A_3\Big)\frac{E_0}{\mu} e_{y,y} \\ &\qquad\qquad + \frac{A_2 E_0}{\mu} e_{x,x} = 0. \end{aligned} \tag{3.2}$$

where $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two-dimensional Laplace operator in the variables x and y, ν is Poisson's ratio, and $A_3 = A_2 + \varepsilon_0 \eta$. The electric field equations for the perturbed state are

$$e_{x,x} + e_{y,y} = 0, \qquad e_{x,x}^+ + e_{y,y}^+ = 0. \tag{3.3}$$

The electric field equations (3.3) are satisfied by introducing an electric potential ϕ such that

$$\begin{aligned} &e_i^+ = -\phi_{,i}^+, \qquad &\nabla_1^2 \phi^+ = 0, \\ &e_i = -\phi_{,i}, \qquad &\nabla_1^2 \phi = 0. \end{aligned} \tag{3.4}$$

The governing equations become

$$\begin{aligned} &\nabla_1^2 u_x + \frac{1}{1-2\nu}\Big(u_{x,x} + u_{y,y}\Big)_{,x} - \Big(2A_1 + A_3\Big)\frac{E_0}{\mu}\phi_{,xy} = 0, \\ &\nabla_1^2 u_y + \frac{1}{1-2\nu}\Big(u_{x,x} + u_{y,y}\Big)_{,y} - \Big(2A_1 + A_2 + A_3\Big)\frac{E_0}{\mu}\phi_{,yy} \\ &\qquad\qquad - \frac{A_2 E_0}{\mu}\phi_{,xx} = 0. \end{aligned} \tag{3.5}$$

When the uniform loading (at infinity) forms an angle ψ with the crack surface, the loading will cause mixed-mode deformation so that both Mode I and Mode II crack problems are considered. The problem will be split into two parts, and the mixed boundary conditions on the x-axis can be obtained from Eqs. (2.9)–(2.11) as

Mode I:

$$\sigma^L_{yx}(x,0) = 0 \qquad (0 \leq | x | < \infty), \tag{3.6}$$

$$\begin{cases} \phi_{,x}(x,0) = -\eta E_0 u_{y,x}(x,0) + \phi^+_{,x}(x,0) & (0 \leq | x | < a), \\ \phi(x,0) = 0 & (a \leq | x | < \infty), \end{cases} \tag{3.7}$$

$$\begin{cases} \sigma^L_{yy}(x,0) = \varepsilon_0 \eta^2 \left\{ \dfrac{E_0^2}{2} - E_0 \phi_{,y}(x,0) \right\} - T \sin^2\psi & (0 \leq | x | < a), \\ u_y(x,0) = 0 & (a \leq | x | < \infty). \end{cases} \tag{3.8}$$

Mode II:

$$\sigma^L_{yy}(x,0) = 0 \qquad (0 \leq | x | < \infty), \tag{3.9}$$

$$\begin{cases} \phi_{,x}(x,0) = -\eta E_0 u_{y,x}(x,0) + \phi^+_{,x}(x,0) & (0 \leq | x | < a), \\ \phi_{,y}(x,0) = 0 & (a \leq | x | < \infty), \end{cases} \tag{3.10}$$

$$\begin{cases} \sigma^L_{xy}(x,0) = -T \sin\psi \cos\psi & (0 \leq | x | < a), \\ u_x(x,0) = 0 & (a \leq | x | < \infty). \end{cases} \tag{3.11}$$

4. Symmetric problem

Symmetry arguments are used to reduce the consideration to only the first quadrant ($0 \leq x < \infty, 0 \leq y < \infty$) with appropriate boundary conditions (3.6)–(3.8) along the coordinate axes. By applying the Fourier transform with respect to x and the inversion theorem, it can be easily shown that the solutions u_{xs}, u_{ys}, ϕ_s, ϕ^+_s of the Eqs. (3.4) and (3.5) for $y \geq 0$ are

$$\begin{aligned} u_{xs} = \frac{2}{\pi} \int_0^\infty \Big\{ & A_s(\alpha) - (3 - 4\nu - y\alpha) B_s(\alpha) \frac{1}{\alpha} \\ & + \frac{E_0}{\mu} (1 - 2\nu)(2A_1 + A_3) a_s(\alpha) \Big\} e^{-\alpha y} \sin(\alpha x) d\alpha, \end{aligned} \tag{4.1}$$

$$u_{ys} = \frac{2}{\pi} \int_0^\infty \{ A_s(\alpha) + B_s(\alpha) y \} e^{-\alpha y} \cos(\alpha x) d\alpha,$$

(4.2) $$\phi_s = -\frac{2}{\pi}\int_0^\infty a_s(\alpha)e^{-\alpha y}\cos(\alpha x)d\alpha,$$

(4.3) $$\phi_s^+ = -\frac{2}{\pi}\int_0^\infty a_s^+(\alpha)\sinh(\alpha y)\cos(\alpha x)d\alpha,$$

where $A_s(\alpha)$, $B_s(\alpha)$, $a_s(\alpha)$ and $a_s^+(\alpha)$ are the unknowns to be found. The local stresses σ_{ijs}^L and Maxwell stresses σ_{ijs}^M can be derived from the displacement field (4.1) and electric field (4.2) which, when substituted into Eqs. (2.5) and (2.6), yield the expressions

(4.4)
$$\begin{aligned}
\sigma_{xxs}^L &= \frac{4\mu}{\pi}\int_0^\infty \Big[\alpha A_s(\alpha) - (3-2\nu-\alpha y)B_s(\alpha) \\
&\quad + \big\{(A_1+A_3) - \nu(2A_1+A_3)\big\}\frac{E_0}{\mu}\alpha a_s(\alpha)\Big]e^{-\alpha y}\cos(\alpha x)d\alpha + A_1E_0^2,\\
\sigma_{xys}^L &= \frac{2\mu}{\pi}\int_0^\infty \Big[-2\alpha A_s(\alpha) + 2(2-2\nu-\alpha y)B_s(\alpha) \\
&\quad - \big\{(1-2\nu)(2A_1+A_3)+A_2\big\}\frac{E_0}{\mu}\alpha a_s(\alpha)\Big]e^{-\alpha y}\sin(\alpha x)d\alpha,\\
\sigma_{yys}^L &= \frac{4\mu}{\pi}\int_0^\infty \Big[-\alpha A_s(\alpha) + (1-2\nu-\alpha y)B_s(\alpha) \\
&\quad - \big\{(A_1+A_2) - \nu(2A_1+A_3)\big\}\frac{E_0}{\mu}\alpha a_s(\alpha)\Big]e^{-\alpha y}\cos(\alpha x)d\alpha + (A_1+A_2)E_0^2,
\end{aligned}$$

(4.5)
$$\begin{aligned}
\sigma_{xxs}^M &= \frac{2\varepsilon_0 E_0}{\pi}\int_0^\infty \alpha a_s(\alpha)e^{-\alpha y}\cos(\alpha x)d\alpha - \frac{\varepsilon_0 E_0^2}{2},\\
\sigma_{xys}^M &= -\frac{2\varepsilon_0\varepsilon_r E_0}{\pi}\int_0^\infty \alpha a_s(\alpha)e^{-\alpha y}\sin(\alpha x)d\alpha,\\
\sigma_{yys}^M &= -\frac{2(1+2\eta)\varepsilon_0 E_0}{\pi}\int_0^\infty \alpha a_s(\alpha)e^{-\alpha y}\cos(\alpha x)d\alpha + \frac{\varepsilon_0 E_0^2(1+2\eta)}{2}.
\end{aligned}$$

The first of boundary conditions (3.6) leads to the following relation between the unknown functions:

$$-2\alpha A_s(\alpha) + 4(1-\nu)B_s(\alpha) - \Big[(1-2\nu)(2A_1 + A_3) + A_2\Big]\frac{E_0}{\mu}\alpha a_s(\alpha) = 0. \tag{4.6}$$

Using the relation (4.6), the two mixed boundary conditions of (3.7) and (3.8) are converted into the simultaneous dual integral equations,

$$\begin{cases} \int\limits_0^\infty \alpha[a_s(\alpha) - \eta E_0 A_s(\alpha)]\sin(\alpha x)d\alpha = 0 & (0 \le x < a), \\ \int\limits_0^\infty a_s(\alpha)\cos(\alpha x)d\alpha = 0 & (a \le x < \infty), \end{cases} \tag{4.7}$$

$$\begin{cases} \int\limits_0^\infty \Big\{\alpha A_s(\alpha) + \Big[2(1-2\nu)A_1 + 2(1-\nu)A_2 \\ \qquad - \varepsilon_0\eta - 2(1-\nu)\varepsilon_0\eta^2\Big]\dfrac{E_0}{2\mu}\alpha a_s(\alpha)\Big\}\cos(\alpha x)d\alpha \\ \qquad = \dfrac{\pi(1-\nu)}{2\mu}\Big[\Big(A_1 + A_2 - \dfrac{\varepsilon_0\eta^2}{2}\Big)E_0^2 + T\sin^2\psi\Big] \\ \qquad\qquad (0 \le x < a), \\ \int\limits_0^\infty A_s(\alpha)\cos(\alpha x)d\alpha = 0 \qquad (a \le x < \infty). \end{cases} \tag{4.8}$$

To solve the set of simultaneous dual integral equations, we make the integral representations for $a_s(\alpha)$ and $A_s(\alpha)$

$$\begin{aligned} a_s(\alpha) &= \int\limits_0^a g(\xi)J_0(\alpha\xi)d\xi, \\ A_s(\alpha) &= \int\limits_0^a h(\xi)J_0(\alpha\xi)d\xi, \end{aligned} \tag{4.9}$$

where $J_0(\ )$ is the zero-order Bessel function of the first kind. Having satisfied Eqs. (4.7) and (4.8) for $a \le x < \infty$, the remaining conditions for $0 \le x < a$ lead to

$$\frac{d}{dx}\int_x^a \frac{1}{(\xi^2-x^2)^{1/2}}\Big[g(\xi)-\eta E_0 h(\xi)\Big]d\xi = 0,$$

$$(4.10)\qquad \frac{d}{dx}\int_0^x \frac{1}{(x^2-\xi^2)^{1/2}}\Big[h(\xi) + \left\{(1-2\nu)(2A_1+\varepsilon_0)-2(1-\nu)(\varepsilon_0\eta^2+\varepsilon_0\eta-A_2)\frac{E_0}{2\mu}\right\}g(\xi)\Big]d\xi = \frac{\pi(1-\nu)}{2\mu}\left[\frac{2A_1+2A_2-\varepsilon_0\eta^2}{2}E_0^2+T\sin^2\psi\right].$$

The functions $g(\xi)$ and $h(\xi)$ are found to be

$$(4.11)\qquad g(\xi)-\eta E_0 h(\xi) = 0$$

$$h(\xi)+\left\{(1-2\nu)(2A_1+\varepsilon_0)-2(1-\nu)(\varepsilon_0\eta^2+\varepsilon_0\eta-A_2)\frac{E_0}{2\mu}\right\}g(\xi) = \frac{\pi(1-\nu)}{2\mu}\left[\frac{2A_1+2A_2-\varepsilon_0\eta^2}{2}E_0^2+T\sin^2\psi\right]\xi.$$

By substituting $h(\xi)$ and $g(\xi)$ which are obtained from Eq. (4.11) into Eq. (4.9), we can easily derive

$$(4.12)\qquad \begin{aligned} a_s(\alpha) &= \frac{\pi(1-\nu)}{2\mu y_0}\left[\frac{2A_1+2A_2-\varepsilon_0\eta^2}{2}E_0^2+T\sin^2\psi\right]\eta E_0\frac{a}{\alpha}J_1(\alpha a),\\ A_s(\alpha) &= \frac{\pi(1-\nu)}{2\mu y_0}\left[\frac{2A_1+2A_2-\varepsilon_0\eta^2}{2}E_0^2+T\sin^2\psi\right]\frac{a}{\alpha}J_1(\alpha a), \end{aligned}$$

where $J_1(\ )$ is the first order Bessel function of the first kind and

$$(4.13)\qquad y_0 = 1+\frac{1}{2}\Big[(1-2\nu)(2A_1+\varepsilon_0\eta)+2(1-\nu)(A_2-\varepsilon_0\eta^2-\varepsilon_0\eta)\Big]\frac{\eta}{\mu}E_0^2.$$

We introduce dimensionless variables as follows:

$$(4.14)\qquad E_\mu^2 = \frac{\varepsilon_0 E_0^2}{\mu},\qquad T_\mu = \frac{T}{\mu},\qquad A_{e1} = \frac{A_1}{\varepsilon_0},\qquad A_{e2} = \frac{A_2}{\varepsilon_0}.$$

Using the following results:

$$\int_0^\infty J_1(\alpha a)e^{-\alpha y}\begin{bmatrix}\cos(\alpha x)\\ \sin(\alpha x)\end{bmatrix}d\alpha = \begin{bmatrix}\frac{1}{a}\\ 0\end{bmatrix} - \frac{r}{a(r_1r_2)^{1/2}}\begin{bmatrix}\cos\left(\theta-\frac{\theta_1+\theta_2}{2}\right)\\ \sin\left(\theta-\frac{\theta_1+\theta_2}{2}\right)\end{bmatrix}, \tag{4.15}$$

$$\int_0^\infty \alpha J_1(\alpha a)e^{-\alpha y}\begin{bmatrix}\cos(\alpha x)\\ \sin(\alpha x)\end{bmatrix}d\alpha = -\frac{a}{(r_1r_2)^{3/2}}\begin{bmatrix}\sin\frac{3}{2}\left(\theta_1+\theta_2\right)\\ \cos\frac{3}{2}\left(\theta_1+\theta_2\right)\end{bmatrix},$$

the displacements near the crack tip, the singular parts of local stresses, Maxwell stresses, and electric fields may be written as

$$u_{xs} \sim \frac{K_I}{2z_s\mu}\left(\frac{r_1}{2\pi}\right)^{1/2}\Big\{2\,(1-2\nu) - \Big\{(1-2\nu)\,(A_{e1}+\eta)-2(1-\nu)A_{e2}\Big\}E_\mu^2\eta$$
$$+\left[2+\Big\{(1-2\nu)(A_{e1}+\eta)+2(1-\nu)A_{e2}\Big\}E_\mu^2\eta\right]\sin^2\left(\frac{\theta_1}{2}\right)\Big\}\cos\left(\frac{\theta_1}{2}\right), \tag{4.16}$$

$$u_{ys} \sim \frac{K_I}{2z_s\mu}\left(\frac{r_1}{2\pi}\right)^{1/2}\Big\{4(1-\nu)$$
$$+\left[2+\Big\{(1-2\nu)(A_{e1}+\eta)+2(1-\nu)A_{e2}\Big\}E_\mu^2\eta\right]\cos^2\left(\frac{\theta_1}{2}\right)\Big\}\sin\left(\frac{\theta_1}{2}\right),$$

$$\sigma^L_{xxs} \sim \frac{K_I}{2z_s}\Big\{\left[2+\Big\{2(1-2\nu)A_{e1}+2(1-\nu)A_{e2}-\eta\Big\}E_\mu^2\eta\right]$$
$$-\left[2+\Big\{(1-2\nu)(2A_{e1}+\eta)+2(1-\nu)A_{e2}\Big\}E_\mu^2\eta\right]\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{3\theta_1}{2}\right)\Big\}$$
$$\times\cos\left(\frac{\theta_1}{2}\right)\frac{1}{(2\pi r_1)^{1/2}}, \tag{4.17}$$

(4.17) [cont.]

$$\sigma^L_{xys} \sim \frac{K_I}{2z_s}\Big[2+\Big\{(1-2\nu)(2A_{e1}+\eta)+2(1-\nu)A_{e2}\Big\}E_\mu^2\eta\Big]\sin\left(\frac{\theta_1}{2}\right) \times \cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{3\theta_1}{2}\right)\frac{1}{(2\pi r_1)^{1/2}},$$

$$\sigma^L_{yys} \sim \frac{K_I}{2z_s}\Big\{\Big[2+\Big\{2(1-2\nu)A_{e1}+2(1-\nu)A_{e2}-\eta\Big\}E_\mu^2\eta\Big] + \Big[2+\Big\{(1-2\nu)(2A_{e1}+\eta)+2(1-\nu)A_{e2}\Big\}E_\mu^2\eta\Big]\sin\left(\frac{\theta_1}{2}\right) \times \sin\left(\frac{3\theta_1}{2}\right)\Big\}\cos\left(\frac{\theta_1}{2}\right)\frac{1}{(2\pi r_1)^{1/2}},$$

(4.18)

$$\sigma^M_{xxs} \sim -\frac{K_I}{z_s}(1-\nu)\eta E_\mu^2\cos\left(\frac{\theta_1}{2}\right)\frac{1}{(2\pi r_1)^{1/2}},$$
$$\sigma^M_{xys} \sim -\frac{K_I}{z_s}(1-\nu)\eta\varepsilon_r E_\mu^2\sin\left(\frac{\theta_1}{2}\right)\frac{1}{(2\pi r_1)^{1/2}},$$
$$\sigma^M_{yys} \sim \frac{K_I}{z_s}(1-\nu)(1+2\eta)\eta E_\mu^2\cos\left(\frac{\theta_1}{2}\right)\frac{1}{(2\pi r_1)^{1/2}},$$

(4.19)

$$E_{xs} \sim -\frac{K_I}{z_s\mu(2\pi r_1)^{1/2}}(1-\nu)\eta E_0\sin\left(\frac{\theta_1}{2}\right),$$
$$E_{ys} \sim \frac{K_I}{z_s\mu(2\pi r_1)^{1/2}}(1-\nu)\eta E_0\cos\left(\frac{\theta_1}{2}\right),$$

where $x = r\cos\theta = r_1\cos\theta_1 + a = r_2\cos\theta_2 - a$, $y = r\sin\theta = r_1\sin\theta_1 = r_2\sin\theta_2$ as defined in Fig. 1, and

(4.20) $$z_s = 1+\frac{1}{2}\Big\{(1-2\nu)(2A_{e1}+\eta)+2(1-\nu)(A_{e2}+\eta+1)\Big\}\eta E_\mu^2.$$

The Mode I stress intensity factor K_I is defined by

(4.21)

$$K_I = \lim_{x\to a^+}\{2\pi(x-a)\}^{1/2}(\sigma^L_{yys}+\sigma^M_{yys})_{y=0} = T(\pi a)^{1/2}\frac{z_s}{y_0}\left\{\frac{(2A_{e1}+2A_{e2}-\eta^2)}{2}\frac{E_\mu^2}{T_\mu}+\sin^2\psi\right\}.$$

5. Skew-symmetric problem

In the same way, utilizing the Fourier transform, the solutions u_{xa}, u_{ya}, ϕ_a and ϕ_a^+ are

$$
\begin{aligned}
u_{xa} &= \frac{2}{\pi}\int_0^\infty \Big\{ -A_a(\alpha) + (3-4\nu-y\alpha)B_a(\alpha)\frac{1}{\alpha} \\
&\qquad - \frac{E_0}{\mu}(1-2\nu)(2A_1+A_3)a_a(\alpha)\Big\} e^{-\alpha y}\cos(\alpha x)d\alpha, \\
u_{ya} &= \frac{2}{\pi}\int_0^\infty \{A_a(\alpha) + B_a(\alpha)y\} e^{-\alpha y}\sin(\alpha x)d\alpha,
\end{aligned} \tag{5.1}
$$

$$
\phi_a = -\frac{2}{\pi}\int_0^\infty a_a(\alpha)e^{-\alpha y}\sin(\alpha x)d\alpha, \tag{5.2}
$$

$$
\phi_a^+ = -\frac{2}{\pi}\int_0^\infty a_a^+(\alpha)\cosh(\alpha y)\sin(\alpha x)d\alpha, \tag{5.3}
$$

where $A_a(\alpha), B_a(\alpha), a_a(\alpha)$ and $a_a^+(\alpha)$ are the unknown functions. The local stresses and Maxwell stresses are obtained as

$$
\begin{aligned}
\sigma_{xxa}^L &= \frac{4\mu}{\pi}\int_0^\infty \Big\{ \alpha A_a(\alpha) - (3-2\nu-\alpha y)B_a(\alpha) \\
&\qquad + \Big[(A_1+A_3) - \nu(2A_1+A_3)\Big]\frac{E_0}{\mu}\alpha a_a(\alpha)\Big\} e^{-\alpha y}\sin(\alpha x)d\alpha, \\
\sigma_{xya}^L &= \frac{2\mu}{\pi}\int_0^\infty \Big\{ 2\alpha A_a(\alpha) - 2(2-2\nu-\alpha y)B_a(\alpha) \\
&\qquad + \Big[(1-2\nu)(2A_1+A_3) + A_2\Big]\frac{E_0}{\mu}\alpha a_a(\alpha)\Big\} e^{-\alpha y}\cos(\alpha x)d\alpha, \\
\sigma_{yya}^L &= \frac{4\mu}{\pi}\int_0^\infty \Big\{ -\alpha A_a(\alpha) + (1-2\nu-\alpha y)B_a(\alpha) \\
&\qquad - \Big[(A_1+A_2) - \nu(2A_1+A_3)\Big]\frac{E_0}{\mu}\alpha a_a(\alpha)\Big\} e^{-\alpha y}\sin(\alpha x)d\alpha,
\end{aligned} \tag{5.4}
$$

$$
\begin{aligned}
\sigma_{xxa}^{M} &= \frac{2\varepsilon_0 E_0}{\pi}\int_0^\infty \alpha a_a(\alpha)e^{-\alpha y}\sin(\alpha x)d\alpha,\\
\sigma_{xya}^{M} &= \frac{2\varepsilon_0\varepsilon_r E_0}{\pi}\int_0^\infty \alpha a_a(\alpha)e^{-\alpha y}\cos(\alpha x)d\alpha,\\
\sigma_{yya}^{M} &= -\frac{2(1+2\eta)\varepsilon_0 E_0}{\pi}\int_0^\infty \alpha a_a(\alpha)e^{-\alpha y}\sin(\alpha x)d\alpha.
\end{aligned}
\tag{5.5}
$$

The boundary condition (3.9) leads the following relation between unknown functions:

$$
B_a(\alpha) = \frac{\alpha}{2(1-\nu)}A_a(\alpha) + \frac{\alpha}{(1-\nu)}\Big[(A_1+A_2) - \nu(2A_1+A_3)\Big]\frac{E_0}{\mu}a_a(\alpha). \tag{5.6}
$$

We introduce a new unknown function $C_a(\alpha)$ as

$$
C_a(\alpha) = -2(1-\nu)A_a(\alpha) - \Big[(1-2\nu)A_1 + (2-3\nu)A_2 - (1-\nu)\varepsilon_0\eta\Big]\frac{E_0}{\mu}a_a(\alpha). \tag{5.7}
$$

Using these relations (5.6) and (5.7), the two mixed boundary conditions of (3.10) and (3.11) are converted into the simultaneous dual integral equations,

$$
\left\{
\begin{aligned}
&\int_0^\infty \alpha[\eta E_0 C_a(\alpha) + G_1 a_a(\alpha)]\cos(\alpha x)d\alpha = 0 && (0 \le x < a),\\
&\int_0^\infty a_a(\alpha)\sin(\alpha x)d\alpha = 0 && (a \le x < \infty);
\end{aligned}
\right.
\tag{5.8}
$$

$$
\left\{
\begin{aligned}
&\int_0^\infty \alpha[C_a(\alpha) - G_2 a_a(\alpha)]\cos(\alpha x)d\alpha\\
&\qquad = -\frac{\pi(1-2\nu)(1-\nu)}{2\mu}T\cos\psi\sin\psi && (0 \le x < a),\\
&\int_0^\infty C_a(\alpha)\cos(\alpha x) = 0 && (a \le x < \infty),
\end{aligned}
\right.
\tag{5.9}
$$

where

$$
\begin{aligned}
G_1 &= 2(1-\nu) + \Big[(1-2\nu)A_1 + (2-3\nu)A_2 - (1-\nu)\varepsilon_0\eta\Big]\frac{E_0^2\eta}{\mu},\\
G_2 &= (1-2\nu)\Big[(1-2\nu)A_1 - \nu A_2\Big]\frac{E_0}{\mu}.
\end{aligned}
\tag{5.10}
$$

The unknowns $a_a(\alpha)$ and $C_a(\alpha)$ can be found by the same method of approach as in the symmetric case. The results are

$$a_a(\alpha) = \int_0^a g(\xi) J_0(\alpha\xi) d\xi, \qquad C_a(\alpha) = \int_0^a h(\xi) J_0(\alpha\xi) d\xi. \tag{5.11}$$

The functions $g(\xi)$ and $h(\xi)$ in Eqs. (5.11) are the solutions of the following simultaneous equations:

$$\begin{aligned} &\eta E_0 h(\xi) - G_1 g(\xi) = 0, \\ &h(\xi) + G_2 g(\xi) = -\frac{\pi(1-\nu)(1-2\nu)}{2\mu} T\xi \cos\psi \sin\psi. \end{aligned} \tag{5.12}$$

The displacements near the crack tip are

$$\begin{aligned} u_{xa} &\sim \frac{K_{II}}{z_a \mu}\left(\frac{r_1}{2\pi}\right)^{1/2} \Big\{ 2(1-\nu) \\ &\qquad + \Big\{(1-2\nu)A_{e1} + (2-3\nu)A_{e2} - (1-\nu)\eta\Big\} E_\mu^2 \eta \\ &\qquad + \Big[1 + \Big\{(1-2\nu)A_{e1} + (1-\nu)A_{e2} - \nu\eta\Big\} E_\mu^2 \eta\Big] \cos^2(\frac{\theta_1}{2}) \Big\} \sin\left(\frac{\theta_1}{2}\right), \\ u_{ya} &\sim \frac{K_{II}}{z_a \mu}\left(\frac{r_1}{2\pi}\right)^{1/2} \Big\{ -(1-2\nu) \\ &\qquad + \Big[1 + \Big\{(1-2\nu)A_{e1} + (1-\nu)A_{e2} - \nu\eta\Big\} E_\mu^2 \eta\Big] \sin^2\left(\frac{\theta_1}{2}\right) \Big\} \cos\left(\frac{\theta_1}{2}\right), \end{aligned} \tag{5.13}$$

and the singular parts of local stresses, Maxwell stresses and electric fields are

$$\begin{aligned} \sigma^L_{xxa} &\sim -\frac{K_{II}}{z_a} \Big\{ 2\Big[1 + \Big\{(1-2\nu)A_{e1} + (1-\nu)A_{e2} - \frac{1}{2}\eta\Big\} E_\mu^2 \eta\Big] \\ &\quad + \Big[1 + \Big\{(1-2\nu)A_{e1} + (1-\nu)A_{e2} - \nu\eta\Big\} E_\mu^2 \eta\Big] \cos\left(\frac{\theta_1}{2}\right) \cos\left(\frac{3\theta_1}{2}\right) \Big\}, \\ &\qquad \times \sin\left(\frac{\theta_1}{2}\right) \frac{1}{(2\pi r_1)^{1/2}}, \\ \sigma^L_{xya} &\sim \frac{K_{II}}{z_a} \Big\{ \Big[1 + \Big\{(1-2\nu)A_{e1} + (1-\nu)A_{e2} - \frac{1}{2}\eta\Big\} E_\mu^2 \eta\Big] \\ &\quad - \Big[1 + \Big\{(1-2\nu)A_{e1} + (1-\nu)A_{e2} - \nu\eta\Big\} E_\mu^2 \eta\Big] \sin\left(\frac{\theta_1}{2}\right) \sin\left(\frac{3\theta_1}{2}\right) \Big\} \\ &\qquad \times \cos\left(\frac{\theta_1}{2}\right) \frac{1}{(2\pi r_1)^{1/2}}, \end{aligned} \tag{5.14}$$

$$\sigma^L_{yya} \sim \frac{K_{II}}{z_a}\left[1 + \left\{(1-2\nu)A_{e1} + (1-\nu)A_{e2} - \nu\eta\right\}E^2_\mu \eta\right] \sin\left(\frac{\theta_1}{2}\right) \times \cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{3\theta_1}{2}\right)\frac{1}{(2\pi r_1)^{1/2}}, \tag{5.14 [cont.]}$$

$$\begin{aligned}
\sigma^M_{xxa} &\sim \frac{K_{II}}{2z_a}E^2_\mu(1-2\nu)\frac{1}{(2\pi r_1)^{1/2}}\sin\left(\frac{\theta_1}{2}\right), \\
\sigma^M_{xya} &\sim -\frac{K_{II}}{2z_a}E^2_\mu(1-2\nu)(1+\eta)\frac{1}{(2\pi r_1)^{1/2}}\cos\left(\frac{\theta_1}{2}\right), \\
\sigma^M_{yya} &\sim -\frac{K_{II}}{2z_a}E^2_\mu(1-2\nu)(1+2\eta)\frac{1}{(2\pi r_1)^{1/2}}\sin\left(\frac{\theta_1}{2}\right),
\end{aligned} \tag{5.15}$$

$$\begin{aligned}
E_{xa} &\sim \frac{K_{II}}{z_a\mu\left(2\pi r_1\right)^{1/2}}\left(1-2\nu\right)\eta E_0\cos\left(\frac{\theta_1}{2}\right), \\
E_{ya} &\sim -\frac{K_{II}}{z_a\mu\left(2\pi r_1\right)^{1/2}}\left(1-2\nu\right)\eta E_0\sin\left(\frac{\theta_1}{2}\right),
\end{aligned} \tag{5.16}$$

where

$$z_a = 1 + \left[(1-2\nu)A_{e1} + (1-\nu)A_{e2} - (1-\nu)\eta - \frac{1}{2}(1-2\nu)\right]\eta E^2_\mu . \tag{5.17}$$

The stress intensity factor of Mode II, K_{II}, is defined by

$$\begin{aligned}
K_{II} &= \lim_{x\to a^+}\{2\pi(x-a)\}^{1/2}\left\{\sigma^L_{xya} + \sigma^M_{xya}\right\}_{y=0} \\
&= (\pi a)^{1/2}\,\frac{2\,(1-\nu)\,z_a}{G_1 + \eta E_0 G_2}\mu T_\mu \cos\psi\sin\psi .
\end{aligned} \tag{5.18}$$

6. Evaluation of the path-independent integral

PAK and HERRMANN have derived a path-independent integral for elastic dielectric materials [4]. It follows that

$$J = \int_\Gamma \left[(\rho\Sigma + \Phi)\delta_{jx} - (\sigma^L_{ij} + \sigma^M_{ij})u_{i,x} + D_j E_x\right]n_j dc , \tag{6.1}$$

where Γ is a contour in the undeformed configuration as shown in Fig. 2, n_i is a unit vector normal to Γ, ρ is density of mass in the deformed state, Σ denote the stored energy function of deformation and polarization, and

$$\Phi = -\frac{1}{2}\varepsilon_r\phi_{,i}\phi_{,i} + \phi_{,i}P_i , \tag{6.2}$$

$$\rho\Sigma(E_{ij}, P_i/\rho) = \frac{\rho^2}{2\varepsilon_0\eta}\delta_{ij}\Pi_i\Pi_j + \left\{\frac{\lambda}{2}\delta_{ij}\delta_{kl} + \frac{\mu}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right\} E_{ij}E_{kl}\,, \tag{6.3}$$

$$\begin{aligned} E_{ij} &= \frac{\delta_{Ai}\delta_{Bj}u_{A,B} + \delta_{Bi}\delta_{Aj}u_{B,A}}{2}\,, \\ \Pi_i &= \frac{(\delta_{Ai} + \delta_{Bi}u_{A,B})P_A}{\rho}\,. \end{aligned} \tag{6.4}$$

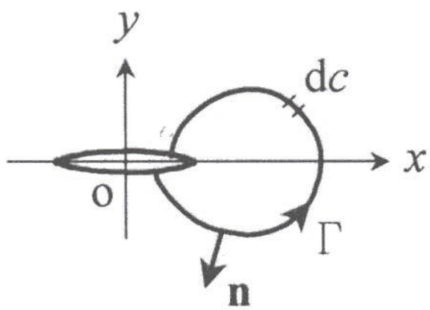

FIG. 2. Contours for the path-independent integrals.

Evaluating the integral J with the solutions given in Eqs. (4.16)–(4.19) along a small circular contour enclosing the crack tip, we obtain the following value of J for Mode I loading:

$$J = \frac{1}{(1-2\nu)}\frac{K_I^2}{128\mu z_s^2}\Big\{C_{1s}E_\mu^4 + C_{2s}E_\mu^2 + 64(1-\nu)(1-2\nu)\Big\}\,, \tag{6.5}$$

where

$$\begin{aligned} C_{1s} &= 2k_{2s}^2 + k_{3s}^2 + 4(1-2\nu)k_{1s}k_{3s} + 2(1-2\nu)k_{1s}k_{2s} + (1+4\nu)k_{3s}k_{2s} \\ &\qquad + 4(1-\nu)(1-2\nu)\eta\{k_{2s} - 2\eta k_{3s}\}, \\ C_{2s} &= 4(1-2\nu)\big[3k_{1s} - 4\nu k_{2s} - 3k_{3s} \\ &\qquad + 2(1-\nu)\{12 - 16\nu + (7-8\nu)\eta - 8(1-\nu)\eta^2\}\eta\big], \end{aligned} \tag{6.6}$$

$$\begin{aligned} k_{1s} &= \{2(1-2\nu)A_{e1} + 2(1-\nu)A_{e2} - \eta\}\eta, \\ k_{2s} &= \{(1-2\nu)(2A_{e1}+\eta) + 2(1-\nu)A_{e2}\}\eta, \\ k_{3s} &= \{(1-2\nu)(2A_{e1}+\eta) - 2(1-\nu)A_{e2}\}\eta. \end{aligned} \tag{6.7}$$

We also obtain the value of J for Mode II loading; that is

$$J = \frac{1}{(1-2\nu)}\frac{K_{II}^2}{128\mu z_a^2}\Big\{C_{1a}E_\mu^4 + C_{2a}E_\mu^2 + 64(1-\nu)(1-2\nu)\Big\}, \tag{6.8}$$

where

$$(6.9)\quad \begin{aligned} C_{1a} &= -2\Big\{2k_{2a}^2 + 2k_{3a}^2 - 24(1-2\nu)k_{1a}k_{3a} - 8\nu k_{2a}k_{3a} + 4(1-2\nu)k_{1a}k_{2a} \\ &\qquad - (1-2\nu)^2(1+\eta)\eta k_{2a} + 2(1-2\nu)^2(3+2\eta)\eta k_{3a}\Big\}, \\ C_{2a} &= 4(1-2\nu)\Big\{2(13-16\nu)k_{1a} - (4-5\nu)k_{2a} + 6k_{1a} \\ &\qquad - (1-2\nu)(11-16\nu)\eta - (1-2\nu)(9-16\nu)\eta^2\Big\}, \end{aligned}$$

$$(6.10)\quad \begin{aligned} k_{1a} &= \{(1-2\nu)A_{e1} + (1-\nu)A_{e2} - \eta/2\}\eta, \\ k_{2a} &= \{(1-2\nu)A_{e1} + (1-\nu)A_{e2} - \nu\eta\}\eta, \\ k_{3a} &= \{(1-2\nu)A_{e1} + (2-3\nu)A_{e2} - (1-\nu)\eta\}\eta. \end{aligned}$$

To examine the electroelastic interactions in the J-integral, we consider a crack of length $a = 1$ mm embedded in a polymethylmethacrylate (PMMA) material. The material properties are listed in Table 1 [5], and $\varepsilon_0 = 8.85 \times 10^{-12}$ C/Vm. Table 2 lists the J-integrals under Mode I and Mode II loadings for $T = 10$ MPa. It can be seen that the J-integral under the Mode I depends on the electric field. The effect of electric field on the J-integral under the Mode II is smaller than that under the Mode I, and the electric field effects are important in the case of the Mode I.

Table 1. Material properties of PMMA.

	μ (N/m^2)	ν	A_{e1}	A_{e2}	η	ε_r
PMMA	1.1×10^9	0.4	0	3.61	2	3

Table 2. J-integrals for the Mode I and Mmode II loadings.

	ψ	0	$\pi/4$	$\pi/2$
		J-integral (N/m)		
Mode I	$E_0 = 0$ GV/m	0	21.4	85.6
	$E_0 = 0.5$	11.1	64.1	161
	$E_0 = 1$	183	333	529
Mode II	$E_0 = 0$ GV/m	0	21.4	0
	$E_0 = 0.5$	0	21.5	0
	$E_0 = 1$	0	22.7	0

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