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Brief Notes

Flow induced by a constantly accelerating edge in a Maxwell fluid

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THE PAPER deals with the flow induced by a constantly accelerating edge in a Maxwell fluid. The solutions obtained satisfy both the associate partial differential equations and all imposed initial and boundary conditions. For $\lambda \to 0$ they reduce to those corresponding to a Navier–Stokes fluid.

1. Introduction

THE RAILEIGH–STOKES PROBLEM for an edge has attracted much attention due to its practical importance and to its fundamental value for the theory. An elegant solution, in the context of the Navier–Stokes theory, was given by ZIEREP [1]. Its extension to second grade fluids was established by FETECAU and ZIEREP [2] for a time-dependent boundary condition. Their solution contains, as special cases, both the flow due to an impulsively accelerated edge and the flow induced by a constantly accelerating edge. Similar solutions for the flat plate were obtained by BANDELLI *et. al.* [3] and ERDOGAN [4] using the Laplace and Fourier sine transforms. It can be easily observed that the corresponding solutions given in [4] are special cases of the solution (2.5) of [2]. For some extensions of the flow due to an impulsively accelerated edge see also FETECAU and ZIEREP [5] and FETECAU [6].

The purpose of this work is to present the exact solutions corresponding to the flow induced by a constantly accelerating edge in a Maxwell fluid. These solutions satisfy both the associated partial differential equations and all imposed initial and boundary conditions. Moreover, similar solutions corresponding to a Navier–Stokes fluid appear as limiting cases of our solutions.

2. Governing equations

The incompressible Maxwell fluid of B type is characterized by the constitutive equations [7] (see also [8])

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \qquad \mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T) = \mu\mathbf{A}, \tag{2.1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $-p\mathbf{I}$ denotes the indeterminate spherical stress, $\mathbf{S}$ is the extra stress tensor, $\mathbf{L}$ is the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin–Ericksen tensor, λ is the relaxation time, μ is the dynamic viscosity and superposed dot denotes the material time derivative.

The flows to be considered here have the velocity field [1, 2]

$$\mathbf{v} = v(y, z, t)\mathbf{i}, \tag{2.2}$$

where $\mathbf{i}$ denotes the unit vector along the x-direction of the Cartesian coordinate system x, y and z. Since the velocity field is independent of x, we expect that the extra stress field will be also independent of x.

Equations $(2.1)_2$ and (2.2) together with the natural condition [5–7]

$$\mathbf{S}(y, z, 0) = \mathbf{0}, \tag{2.3}$$

(the fluid being at rest up to the moment $t = 0$) lead to $S_{yy} = S_{yz} = S_{zz} = 0$ and

$$(1 + \lambda\partial_t)\tau_1 = \mu\partial_y v, \qquad (1 + \lambda\partial_t)\tau_2 = \mu\partial_z v, \tag{2.4}$$

$$(1 + \lambda\partial_t)\sigma = 2\lambda(\tau_1\partial_y v + \tau_2\partial_z v), \tag{2.5}$$

where $\tau_1 = S_{xy}$, $\tau_2 = S_{xz}$ are tangential tensions and $\sigma = S_{xx}$ is the normal stress.

For the flows given by (2.2), the constraint of incompressibility is automatically satisfied and the balance of linear momentum, in absence of the body forces and of a pressure gradient in the x-coordinate direction, reduces to

$$\partial_y\tau_1 + \partial_z\tau_2 = \rho\partial_t v, \tag{2.6}$$

where ρ is the constant density of the fluid.

Eliminating τ_1 and τ_2 between Eqs. (2.4) and (2.6) we attain to the following second-order linear partial differential equation

$$\lambda\partial_t^2 v(y, z, t) + \partial_t v(y, z, t) = \nu(\partial_y^2 + \partial_z^2)v(y, z, t); \qquad y, z, t > 0, \tag{2.7}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid.

Let us now consider an incompressible Maxwell fluid, at rest, occupying the quarter-space ($-\infty < x < \infty; y, z > 0$). Suppose that the infinitely extended edge is subject, after time zero, to a constant acceleration A. Owing to the shear the fluid is gradually moved. Its velocity field is of the form (2.2), the governing equation is given by (2.7) and the boundary and initial conditions are

$$\begin{aligned} &v(0,z,t) = v(y,0,t) = At, \qquad t > 0;\\ &v(y,z,0) = 0, \quad y > 0, \quad z > 0. \end{aligned} \tag{2.8}$$

The equation (2.7) being of a higher order in t than the similar equation for a Navier–Stokes fluid, the additional condition

$$\partial_t v(y,z,t) = 0 \quad \text{when} \quad t = 0, \tag{2.9}$$

has to be also satisfied (see also [8]). Furthermore, the natural condition [1]

$$v(y,z,t) \to 0 \qquad \text{as} \qquad y^2 + z^2 \to \infty, \qquad t > 0, \tag{2.10}$$

assures the fact that the fluid is quiescent far away from the edge.

3. The solutions of the problem

3.1. Calculation of the velocity field

Multiplying Eq. (2.7) by $(2/\pi)\sin(\xi y)\sin(\eta z)$, integrating then with respect to y and z from 0 to ∞ and having in mind the boundary and initial conditions (2.8)–(2.10), we attain to

$$\lambda \partial_t^2 v_s(\xi,\eta,t) + \partial_t v_s(\xi,\eta,t) + \nu(\xi^2+\eta^2)v_s(\xi,\eta,t) = \frac{2}{\pi}\nu A \frac{\xi^2+\eta^2}{\xi\eta} t; \qquad \xi,\eta,t > 0, \tag{3.1}$$

where $v_s(\xi,\eta,t)$, the double Fourier sine transform of $v(y,z,t)$, has to satisfy the initial conditions

$$v_s(\xi,\eta,0) = \partial_t v_s(\xi,\eta,0) = 0, \qquad \xi,\eta > 0. \tag{3.2}$$

The solution of the ordinary differential equation (3.1) with the initial conditions (3.2) is given by

$$
v_s(\xi,\,\eta,\,t) = \begin{cases} \dfrac{2}{\pi}\dfrac{A}{\xi\eta}\left[\dfrac{r_1^2\exp(r_2 t) - r_2^2\exp(r_1 t)}{\nu(\xi^2+\eta^2)(r_2-r_1)}\,\lambda \right. \\ \left. + \qquad t - \dfrac{1}{\nu(\xi^2+\eta^2)}\right] & \text{on } \mathcal{D}_1 \\ \dfrac{2}{\pi}\dfrac{A}{\xi\eta}\left\{\exp\left(-\dfrac{t}{2\lambda}\right)\left[\cos\left(\dfrac{\beta t}{2\lambda}\right)\right.\right. \\ \qquad \left. + \dfrac{1-2\nu\lambda(\xi^2+\eta^2)}{\beta}\,\sin\left(\dfrac{\beta t}{2\lambda}\right)\right] \\ \left. \cdot\,\dfrac{1}{\nu(\xi^2+\eta^2)} + t - \dfrac{1}{\nu(\xi^2+\eta^2)}\right\} & \text{on } \mathcal{D}_2. \end{cases} \tag{3.3}
$$

where

$$
r_{1,2} = \frac{-1\pm\sqrt{1-4\nu\lambda(\xi^2+\eta^2)}}{2\lambda}, \qquad \beta = \sqrt{4\nu\lambda(\xi^2+\eta^2)-1},
$$

$$
\mathcal{D}_1 = \left\{(\xi,\,\eta);\quad \xi,\,\eta>0;\quad \xi^2+\eta^2 \le \frac{1}{4\nu\lambda}\right\}
$$

and

$$
\mathcal{D}_2 = \left\{(\xi,\,\eta);\quad \xi,\,\eta>0;\quad \xi^2+\eta^2 > \frac{1}{4\nu\lambda}\right\}
$$

Inverting (3.3) by means of the Fourier sine formula, we get the next expression for the velocity field (see also the entry 1 of Table 5 of [9])

$$
\begin{aligned}
v(y,\,z,\,t) = At &- \frac{4A}{\nu\pi^2}\int_0^\infty\!\!\int_0^\infty \frac{\sin(y\xi)\sin(z\eta)}{\xi\eta\,(\xi^2+\eta^2)}\,d\xi\,d\eta \\
&+ \frac{4\lambda A}{\nu\pi^2}\iint_{\mathcal{D}_1}\frac{r_1^2\exp(r_2 t)-r_2^2\exp(r_1 t)}{r_2-r_1}\,\frac{\sin(y\xi)\sin(z\eta)}{\xi\eta\,(\xi^2+\eta^2)}\,d\xi\,d\eta \\
+ \frac{4A}{\nu\pi^2}\exp\left(-\frac{t}{2\lambda}\right)&\iint_{\mathcal{D}_2}\left[\cos\left(\frac{\beta t}{2\lambda}\right) + \frac{1-2\nu\lambda(\xi^2+\eta^2)}{\beta}\,\sin\left(\frac{\beta t}{2\lambda}\right)\right] \\
&\cdot\,\frac{\sin(y\xi)\sin(z\eta)}{\xi\eta\,(\xi^2+\eta^2)}\,d\xi\,d\eta\,.
\end{aligned} \tag{3.4}
$$

By letting now $\lambda \to 0$ in the above relation, we get

$$v(y,\, z,\, t) = At - \frac{4A}{\nu\pi^2} \int_0^\infty \int_0^\infty \Big[1 - \exp[-\nu(\xi^2+\eta^2)t]\Big] \cdot \frac{\sin(y\xi)\sin(z\eta)}{\xi\eta\,(\xi^2+\eta^2)}\, d\xi\, d\eta\,, \tag{3.5}$$

that represents the velocity field corresponding to a Navier–Stokes fluid.

3.2. Calculation of the tangential tensions

The solutions of the ordinary differential equations (2.4) with the initial conditions (2.3) are

$$\begin{aligned} \tau_1(y,\, z,\, t) &= \frac{\mu}{\lambda} \int_0^t \exp\left(\frac{\tau - t}{\lambda}\right)(1+\lambda_r\partial_\tau)\partial_y v(y,\, z,\, \tau)\, d\tau\,, \\ \tau_2(y,\, z,\, t) &= \frac{\mu}{\lambda} \int_0^t \exp\left(\frac{\tau - t}{\lambda}\right)(1+\lambda_r\partial_\tau)\partial_z v(y,\, z,\, \tau)\, d\tau\,. \end{aligned} \tag{3.6}$$

Introducing (3.4) in (3.6) we obtain

$$\begin{aligned} \tau_1(y,\, z,\, t) = &-\frac{4\rho A}{\pi^2} \int_0^\infty \int_0^\infty \frac{\cos(y\xi)\sin(z\eta)}{\eta\,(\xi^2+\eta^2)}\, d\xi\, d\eta \\ &+ \frac{4\rho A}{\pi^2} \iint_{\mathcal{D}_1} \frac{r_2\exp(r_1 t) - r_1\exp(r_2 t)}{r_2 - r_1}\, \frac{\cos(y\xi)\sin(z\eta)}{\eta\,(\xi^2+\eta^2)}\, d\xi\, d\eta \\ &+ \frac{4\rho A}{\pi^2}\exp\left(-\frac{t}{2\lambda}\right) \iint_{\mathcal{D}_2} \left[\cos\left(\frac{\beta t}{2\lambda}\right) + \frac{1}{\beta}\sin\left(\frac{\beta t}{2\lambda}\right)\right] \frac{\cos(y\xi)\sin(z\eta)}{\eta\,(\xi^2+\eta^2)}\, d\xi\, d\eta \end{aligned} \tag{3.7}$$

and

$$\tau_2(y,\,z,\,t) = -\frac{4\rho A}{\pi^2}\int\limits_0^\infty\int\limits_0^\infty \frac{\sin(y\xi)\cos(z\eta)}{\xi\,(\xi^2+\eta^2)}\,d\xi\,d\eta \tag{3.8}$$
$$+\frac{4\rho A}{\pi^2}\iint\limits_{\mathcal{D}_1}\frac{r_2\exp(r_1 t)-r_1\exp(r_2 t)}{r_2-r_1}\;\frac{\sin(y\xi)\cos(z\eta)}{\xi\,(\xi^2+\eta^2)}\,d\xi\,d\eta$$
$$+\frac{4\rho A}{\pi^2}\exp\left(-\frac{t}{2\lambda}\right)\iint\limits_{\mathcal{D}_2}\left[\cos\left(\frac{\beta t}{2\lambda}\right)+\frac{1}{\beta}\sin\left(\frac{\beta t}{2\lambda}\right)\right]\;\frac{\sin(y\xi)\cos(z\eta)}{\xi\,(\xi^2+\eta^2)}\,d\xi\,d\eta\,.$$

The tangential stresses

$$\tau_1(y,\,z,\,t) = -\frac{4\rho A}{\pi^2}\int\limits_0^\infty\int\limits_0^\infty \Big[1-\exp[-\nu(\xi^2+\eta^2)t]\Big]\;\frac{\cos(y\xi)\sin(z\eta)}{\eta\,(\xi^2+\eta^2)}\,d\xi\,d\eta \tag{3.9}$$

and

$$\tau_2(y,\,z,\,t) = -\frac{4\rho A}{\pi^2}\int\limits_0^\infty\int\limits_0^\infty \Big[1-\exp[-\nu(\xi^2+\eta^2)t]\Big] \tag{3.10}$$
$$\cdot\;\frac{\sin(y\xi)\cos(z\eta)}{\xi\,(\xi^2+\eta^2)}\,d\xi\,d\eta\,,$$

corresponding to the velocity field (3.5), can be also obtained as a limiting case of (3.7) and (3.8) for $\lambda \to 0$.

Finally, having τ_1, τ_2 and v, we can use (2.3) and (2.5) in order to determine the normal stress σ. The hydrostatic pressure p, as it results from the equations of motion, can be determined with the accuracy up to an arbitrary function of t.

4. Conclusions

In this paper we have determined the velocity field and the associated tangential tensions corresponding to the flow induced by a constantly accelerating edge in a Maxwell fluid. Direct computations show that $v(y,\,z,\,t)$, $\tau_1(y,\,z,\,t)$ and $\tau_2(y,\,z,\,t)$, given by (3.4), (3.7) and (3.8), satisfy both the associate partial differential equations (2.4), (2.6) and (2.7) and all the imposed initial and boundary conditions, the differentiation term by term in x, y and t being clearly permissible. Furthermore, the similar solutions (3.5), (3.9) and (3.10), for a Navier–Stokes fluid, appear as limiting cases of our solutions.

The solutions (3.4), (3.7) and (3.8), corresponding to a Maxwell fluid, contain sine and cosine terms. This indicates that in contrast with the Newtonian fluids, whose solutions (3.5), (3.9) and (3.10) do not contain such terms, the oscillations are set up in the fluid. The amplitudes of these oscillations decay exponentially in time, the damping being proportional to $\exp(-t/2\lambda)$.

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