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Propagation of a shock discontinuity in an elasto-plastic material: constitutive relations

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The shock discontinuity problem is analyzed in the case of elasto-plastic materials ; the jump relations for internal state variables cannot be exhibited directly. For this purpose, we solve the internal shock structure problem, assuming that the shock front is a continuous transition in a thin layer, taking account of dissipative effects. The shock generating function P is introduced. The canonical equations of the shock structure are determined in the general case when the evolution of plasticity is derived from a pseudo-potential of dissipation D. The plane wave is analyzed for an isotropic material obeying a von Mises criterion, assuming that inside the shock the material is under pure axial compression: the existence and uniqueness results are established.

1. Introduction

The description of the thermo-mechanical state of a body subjected to a shock loading process is an important issue in the case of explosive forming of metals. Such a process includes the propagation of a shock front which can be represented as a strong discontinuity of material characteristics. In this paper, we intend to solve the problem of moving discontinuities in the case of elasto-plastic solids, in order to derive the jump of internal state variables across the shock front; these variables are useful to describe the microstructural transformations of the material during the shock phase and its subsequent behavior.

The problem of propagation of strong discontinuities has already been solved in hydrodynamics: the Hugoniot curve is derived from the equation of state of the fluid and from the classical conservation laws written in terms of relations on discontinuities. Given the downstream state $(-)$ and the flux of mass through the discontinuity, there exist one and only one possible upstream state $(+)$ provided that the equation of state of the fluid satisfies the assumptions of Weyl [16]. For a proper choice of the mass flux, this state also satisfies the positivity of the jump of entropy. Gilbarg [8] proved that these upstream and downstream states were also the extremities of a continuous transition profile representing the shock front, in the case when diffusive mechanisms (like viscosity and heat conductivity) can no longer be neglected when gradients tend to become infinite.

This approach was extended by GERMAIN [4, 6] to magneto-hydrodynamics. Among the various possible states (+) satisfying the jump relations, he showed that the only admissible ones were those which were actually the extremities of such transition profiles and remained so when the diffusive coefficients tend towards zero in any manner.

In the case of elasto-plasticity, the shock problem is even more complex. The Hugoniot curve cannot be obtained unless strong hypotheses are made on both the equations of state and the evolution of internal state variables across the discontinuity. MANDEL [9] exhibited a Hugoniot curve for a nonlinear problem reduced to one-dimensional space. In this case the stress tensor remains radial during the loading path inside the discontinuity and characterisation of the strain hardening can be reduced to the axial plastic strain. GERMAIN and LEE [5] considered a transition profile in which total strain and plastic strain remained uniaxial. The shock solution was taken as the limit of this profile when diffusive coefficients linked to visco-plasticity tend to zero.

DRUGAN and SHEN [1] and NGUYEN and MAIGRE [10] studied elasto-plastic shocks in isothermal conditions, in the three-dimensional case and without so restrictive assumptions on the plastic behavior. They could not derive a unique solution. Nonetheless for generalized standard materials the existence of a transition profile ensures the definition of a lower bound for the dissipation inside the shock front, thanks to the maximum dissipation principle [10].

In this work, we give a solution for the shock problem in the case of elasto-plastic materials whose thermo-mechanical behavior is ruled by two potentials: a free energy and a dissipation pseudo-potential.

As an introduction, the equations of state of an elasto-plastic material are defined in a general manner. Then, the set of conservation laws, written in the form of jump relations along the discontinuity, is recalled. These relations are not sufficient to solve the problem of the moving discontinuity. It is necessary to introduce complementary relations for the jump of the internal state variables which describe microstructural changes due to plastic deformation.

In the second section, these additive relations are derived by studying the internal structure of the shock front and the evolution of the internal state variables in it. As in the previous works, the front is considered as a continuous transition in a thin layer between states (+) and (−), compatible with the presence of viscosity and thermal conductivity. At the scale of this transition, the problem is supposed to be plane and in a steady state. When inside the shock front the evolution of internal state variables is ruled by a pseudo-potential of dissipation, the equations of the shock transition profile can be written using a shock generating function P and a potential of dissipation D, as proposed in STOLZ [13, 14].

These results are applied in the uniaxial strain and isothermal case to an elasto-plastic material obeying a von Mises yield function and a normality law. Conditions for the existence and uniqueness of the solution are established: they are expressed in a compact form in terms of the shock generating function P. First, the case of one internal state variable is considered and then the results are extended to materials for which the free energy is a function of several internal state variables.

Finally we analyze briefly how the solution of the shock structure problem can be used to elaborate the constitutive laws for the activation of plasticity and strain hardening inside the shock. From the mechanical characterization of specimens shocked in plane configuration, we can estimate experimentally the jump relations for the internal state variables across the shock front; then, we go back to plasticity laws under shock by solving the inverse problem of the one of the internal shock structure. The case of shock-loaded copper was studied in Centre d'Etudes de Gramat and has been developed elsewhere [11].

2. Jump relations

2.1. Equations of state for an elasto-plastic material

In a small displacement analysis according to the local accompanying state hypothesis (GERMAIN [7]), the thermodynamical state of the material is described by a set of state variables:

- the total strain tensor $\boldsymbol{\varepsilon}$,
- the temperature T,
- a set of internal state variables α which describes the evolution of internal microstructure and of stored energy due to plastic deformation or other irreversible process.

Each of the internal parameters may be a scalar or a tensor, for example the plastic strain $\boldsymbol{\varepsilon}_p$ is an element of this set. Attached to the parameters are the internal energy density e and entropy density s.

For a thermodynamical description, as proposed in [15], the constitutive law is defined through a given free energy w defined per unit mass as a function of the state variables:

$$w = w(\boldsymbol{\varepsilon}, \alpha, T). \tag{2.1}$$

The local expression of energy conservation is then deduced

$$\rho \dot{e} = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\underline{v}) - \operatorname{div} \underline{q}. \tag{2.2}$$

ρ is the mass density, $\boldsymbol{\sigma}$ is the Cauchy stress tensor which verifies the momentum conservation, $\underline{v}$ is the rate of displacement, $\underline{q}$ is the heat flux.

In nonlinear mechanics the internal state is generally associated with irreversibility. Then the fundamental inequality of thermodynamics implies that the internal entropy production must be non-negative.

We assume that the choice of state parameters is a normal set of variables in the Gibb's sense: *in a thermodynamic transformation in which the evolution of temperature is imposed, the total strain being kept constant, the plastic strain and the internal state variables also remain constant.*

In this case, a variation of temperature doesn't induce variations of kinetic energy, so the internal production of entropy can be split into two parts: one due to internal mechanical irreversibility

$$D_m = \rho \dot{s} + \frac{1}{T} \operatorname{div} \underline{q} \geq 0 \tag{2.3}$$

and the second due to thermal conduction

$$D_{th} = -\frac{\underline{q} \cdot \nabla T}{T} \geq 0. \tag{2.4}$$

By introducing the conservation of energy, we can use the free energy w instead of internal energy $w = e + Ts$ in the expression of D_m:

$$D_m = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\underline{v}) - \rho(\dot{w} + s\dot{T}) \geq 0. \tag{2.5}$$

This inequality must be satisfied by any real evolution of the body from the state characterized by the actual value of the state variables $(\boldsymbol{\varepsilon}, \alpha, T)$. Then, we can deduce that

$$s = -\frac{\partial w}{\partial T}. \tag{2.6}$$

Moreover, assuming that the local behaviour doesn't depend on the strain rate $\boldsymbol{\varepsilon}(\underline{v})$, i.e. there is no viscosity, the inequality (2.5) implies:

$$\boldsymbol{\sigma} = \rho \frac{\partial w}{\partial \boldsymbol{\varepsilon}}. \tag{2.7}$$

In an analogous way, introducing the thermodynamic forces A associated with the internal state variables α through

$$\mathrm{A} = -\frac{\partial w}{\partial \alpha}, \tag{2.8}$$

the dissipation is then reduced to

$$D_m = \mathrm{A}\dot{\alpha} \geq 0. \tag{2.9}$$

2.2. Jump relations along a strong discontinuity front

Now, we study the properties of a moving surface inside the material. Let us consider a surface of discontinuity Γ. In any point M of this surface, ϕ and $\underline{\nu}$ denote respectively the shock velocity and the vector normal to Γ. The propagation of the surface is then governed by $\underline{c} = \phi\underline{\nu}$. The exponents $(+)$ and $(-)$ denote the values of state variables immediately behind and ahead of the discontinuity, respectively (Fig. 1).

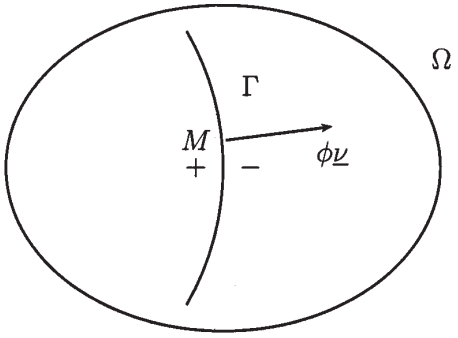

FIG. 1. Shock front represented as a discontinuity surface.

In any point M of Γ, the conservation laws are rewritten in the form of jump relations between the states $(+)$ and $(-)$. For any quantity b, $[\,b\,]_\Gamma = b^- - b^+$ denotes the jump of the quantity b.

When $[\,b\,]_\Gamma = 0$, the quantity b is conserved. If the value is known on one side and the value is the same on the other side, then the quantity is constant for the shock. At least three constants for the shock exist, each one is associated to the classical laws of conservation.

The mass conservation implies that if the mass flux $m = \rho\phi$ is continuous along Γ, then m is a constant for the shock.

The conservation of momentum has the form:

$$[\boldsymbol{\sigma}]_\Gamma.\underline{\nu} + m[\underline{v}]_\Gamma = 0 \tag{2.10}$$

then the quantity $m\underline{T}^d$ given by

$$-m\underline{T}^d = \boldsymbol{\sigma}^{\pm}.\underline{\nu} + m\underline{v}^{\pm} \tag{2.11}$$

is also a constant. From the expression of conservation of energy:

$$m\left[w + Ts + \frac{1}{2}\underline{v}^2\right]_\Gamma + \underline{\nu}.[\,\boldsymbol{\sigma}.\underline{v}\,]_\Gamma - [\,\underline{q}\,]_\Gamma.\underline{\nu} = 0, \tag{2.12}$$

where $\underline{q}$ denotes the heat flux, taking account of the equality[1)]:

$$\frac{1}{2}m[\underline{v}^2]_\Gamma = m\bar{\underline{v}}.[\underline{v}]_\Gamma \tag{2.13}$$

and using the conservation law of the momentum

$$m\underline{\nu}.[\boldsymbol{\sigma}.\underline{v}]_\Gamma - \frac{1}{2}m[\underline{v}^2]_\Gamma = m\underline{\nu}.\bar{\boldsymbol{\sigma}}.[\underline{v}]_\Gamma, \tag{2.14}$$

we obtain:

$$m[w + sT]_\Gamma + \underline{\nu}.\bar{\boldsymbol{\sigma}}.[\underline{v}]_\Gamma - [\underline{q}]_\Gamma.\underline{\nu} = 0. \tag{2.15}$$

The third constant, denoted by Q^d, is associated with the conservation of energy:

$$mQ^d = mw^\pm + m(Ts)^\pm + \underline{\nu}\bar{\boldsymbol{\sigma}}.\underline{v}^\pm - \underline{q}^\pm.\underline{\nu}. \tag{2.16}$$

The conservation laws are henceforth characterized by the values m, $\underline{T}^d$, Q^d which are named the constants of the shock.

These values and the jumps of any mechanical quantities must be compatible with the positivity of entropy production:

$$-m[s]_\Gamma + \left[\frac{\underline{q}}{T}\right]_\Gamma.\underline{\nu} \geq 0. \tag{2.17}$$

2.3. Compatibility condition

The continuity of the displacement along the surface Γ leads to classical relations between the gradient of displacement and velocity fields (HADAMARD [2]):

$$[\underline{v}]_\Gamma + \phi[\nabla\underline{u}]_\Gamma.\underline{\nu} = 0 \tag{2.18}$$

and

$$[\nabla\underline{u}]_\Gamma.\underline{e}_\alpha = 0 \tag{2.19}$$

The last property is true for all vectors $\underline{e}_\alpha$ tangent to the discontinuity surface Γ.

Contrary to the case of hydrodynamics, the set of jump relations is not sufficient to solve the shock problem in the following way: in one point M of the shock front, if we know the shock velocity ϕ, the normal $\underline{\nu}$ and the values of all state variables in state $(-)$, the constants of shock are determined, but an infinite number of values for the state $(+)$ exists.

This indetermination is essentially due to the presence of irreversibility.

[1)] The classical notation $\bar{f} = \frac{1}{2}\left(f^+ + f^-\right)$ is adopted.

The number of equations and unknown variables is such that we could find a solution for different choices of plastic strain and internal state variable. Each possible answer is parametrized by the internal state.

Therefore it is not possible to exhibit jump relations for the internal parameters directly since their own evolution laws are expressed in an incremental manner; the loading path inside the shock front must be known. We must determine the loading path during the transition from the $(-)$ state to the $(+)$ state.

The solution of the internal state variables across the shock can be determined by the resolution of a particular rate boundary value problem. It is the purpose of the next section.

For the sake of simplicity, we assume now that these shock waves are adiabatic, that is to say that there is no heat transfer immediately behind and ahead the discontinuity.

3. The internal structure of the shock front

3.1. The shock transition: change of scale

To determine the jump of internal variables we study the internal structure of the shock front by solving a particular rate boundary value problem.

At a lower scale, the shock front is described as a continuous transition between states $(+)$ and $(-)$, contained in a thin layer around the line of discontinuity. The existence of this continuous transition is justified by viscosity and thermal conductivity, which we have neglected up to now. In such a description, we implicitly make two hypotheses:

- the local accompanying state model is still valid; in other words, even if the strain rates are very high during the transition, each particle of the material is considered in a thermodynamic equilibrium at each time;
- even if the thickness of the transition profile is smaller than the size of heterogeneities, as the grain size of a metal for example, we use classical constitutive laws determined for an homogeneous material.

Let us consider a point M of the surface of discontinuities and an elementary volume around this point, with a space characteristic length x_{vef} during a short time interval of the characteristic time scale t_{vef} (see Fig. 2). These two scales are characteristic scales of the shock transition which characterize the continuous transition of matter from the downstream state $(-)$ to the upstream state $(+)$. Moreover, we admit that the following orders of magnitude are fulfilled:

- x_{vef} is one order of magnitude smaller than the radius of curvature of the discontinuity surface; so, on the width of scale x_{vef}, the discontinuity surface can be considered as a part of a plane;

- x_{vef} is one order of magnitude smaller than the characteristic length of variation of mechanical variables along the discontinuity surface near point M;
- t_{vef} is one order of magnitude smaller than the characteristic time of evolution of states $(+)$ and $(-)$.

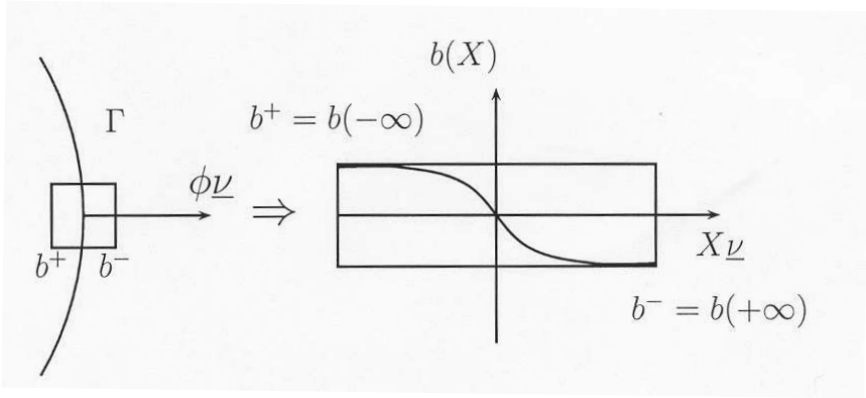

FIG. 2. Shock and boundary conditions.

The rate boundary value problem is based on the research of steady solution in the frame of the discontinuity. We write the equations of the shock transition in a referential moving at velocity ϕ in direction $\underline{\nu}$. So, the particles are located with the following time and space coordinates in referential:

$$X = x - \phi t, \qquad Y = y, \qquad Z = z. \tag{3.1}$$

Then any function $F(x, y, z, t) = f(X, Y, Z)$ has time derivative such that

$$\dot{F} = -\phi \frac{\partial f}{\partial X} = -\phi \frac{\partial F}{\partial x} = -\phi F_{,X}. \tag{3.2}$$

The steady state is subjected to a particular boundary condition. Far from the thin layer, for $X \to \infty$, the $(-)$ state is imposed, in the scale of time t_{vef} this state is a constant. The mechanical quantities are defined by

$$b^- = \lim_{X \to \infty} b(X, Y, Z), \tag{3.3}$$

then the constants of the shock are determined for a given propagation rate ϕ.

For $X \to -\infty$, the mechanical state is defined by $b^+ = \lim_{X \to -\infty} b(X, Y, Z)$ we are just looking for.

3.2. Matching conditions

The mechanical quantities must satisfy the jump relations written on the shock and then they are related with the two shock constants $\underline{T}^d$ and Q^d and

constraints with the Hadamard compatibility relations. The jump at the macroscopic scale $[b]_\Gamma = b^- - b^+$ is obtained by considering that at the scale of the layer $b^- = \lim_{X\to\infty} b(X,Y,Z)$ and $b^+ = \lim_{X\to-\infty} b(X,Y,Z)$. Then for the shock structure we have the boundary conditions:

$$\underline{v}^+ + \phi\nabla\underline{u}^+ .\underline{\nu} = \underline{v}^- + \phi\nabla\underline{u}^- .\underline{\nu}, \tag{3.4}$$

$$\nabla\underline{u}^+ .\underline{e}_\alpha = \nabla\underline{u}^- .\underline{e}_\alpha. \tag{3.5}$$

Moreover, the steady state conditions implies $\underline{v} + \phi\nabla\underline{u}.\underline{\nu} = 0$. Henceforth according to the adiabaticity condition $\underline{q}^\pm.\underline{\nu} = 0$, the equations of conservation are reduced to:

$$-m\underline{T}^d = \boldsymbol{\sigma}^\pm .\underline{\nu} - m\phi\nabla\underline{u}^\pm .\underline{\nu}, \tag{3.6}$$

$$mQ^d = mw^\pm + m(Ts)^\pm + \frac{1}{2}m(\phi\nabla\underline{u}^2)^\pm - \phi\underline{\nu}.(\boldsymbol{\sigma}.\nabla\underline{u})^\pm .\underline{\nu}. \tag{3.7}$$

We search a steady state solution for the transition profile, assuming that any state variables b depend only on the coordinate X.

3.3. The material behavior inside the shock front

To solve the problem for the shock transition, constitutive relations are choosen to govern the evolution of the internal state variables inside the shock front. Let us note that these relations, written in a formalism of mechanics of continuum, are different from those used outside the shock. This allows us to describe the microstructural mechanisms activated only in the shock transition due to high rates of loading. Moreover, both the viscosity and the heat conductivity which are neglected outside the shock, can no longer be neglected inside the front because the strain rates and temperature gradients are very high.

To describe the irreversible processes, we introduce a dissipation pseudo-potential Ω, characteristic for the behavior inside the shock, which is a convex function of the evolution rates of the state variables.

We assume then that the evolution is defined by the normality law:

$$\mathrm{A} = \frac{\partial\Omega}{\partial\dot{\alpha}}, \qquad \boldsymbol{\sigma}_{ir} = \frac{\partial\Omega}{\partial\dot{\boldsymbol{\varepsilon}}}. \tag{3.8}$$

The heat flux $\underline{q}$ is assumed to be proportional to the temperature gradient (Fourier's law):

$$\underline{q} = -\mathbf{K}.\nabla T. \tag{3.9}$$

Then the state of stress inside the layer is defined by:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_r + \boldsymbol{\sigma}_{ir} = \rho\frac{\partial w}{\partial\boldsymbol{\varepsilon}} + \frac{\partial\Omega}{\partial\dot{\boldsymbol{\varepsilon}}}. \tag{3.10}$$

3.4. The shock generating function

With the former assumptions on the material behavior inside the front, we search steady state solutions for the transition obeying the boundary conditions imposed by the jump relations and leading to the downstream state (–) in ∞. These solutions satisfy equations which can be written in a compact form using both the shock generating function and a potential of dissipation introduced by STOLZ [13]:

$$\mathcal{G}(\nabla \underline{u}, \alpha, T, \phi) = \frac{m}{T}\Big(- Q^d + \underline{T}^d.\underline{u}_{,X} + w(\boldsymbol{\varepsilon}(\underline{u}), \alpha, T) - \frac{1}{2}\phi^2 \underline{u}_{,X}^2\Big), \tag{3.11}$$

$$\mathrm{D}(\boldsymbol{\varepsilon}_{,X}, \alpha_{,X}, T_{,X}, \phi) = \frac{1}{T}\Big(\Omega(-\phi\boldsymbol{\varepsilon}_{,X}, -\phi\alpha_{,X}) + \frac{1}{2}\frac{K}{T}T_{,X}^2\Big). \tag{3.12}$$

The conservation of the momentum for a steady state evolution is given by

$$\nabla\boldsymbol{\sigma}.\underline{e}_x = \rho\phi^2 \underline{u}_{,xx}, \tag{3.13}$$

then by integration and taking account of the boundary condition we obtain:

$$\boldsymbol{\sigma}.\underline{e}_x - \rho\phi^2 \underline{u}_{,X} = -\rho\phi\underline{T}^d. \tag{3.14}$$

The problem being plane, the strain is:

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\underline{u}_{,X} \otimes \underline{e}_x + \underline{e}_x \otimes \underline{u}_{,X}) + \varepsilon_{\alpha\beta}\underline{e}_\alpha \otimes \underline{e}_{\beta_s}; \tag{3.15}$$

the components $\varepsilon_{\alpha\beta}$ are then uniform. We obtain finally, for any variations $\boldsymbol{\varepsilon}^* = \frac{1}{2}\left(\underline{u}^*_{,X} \otimes \underline{e}_x + \underline{e}_x \otimes \underline{u}^*_{,X}\right)$,

$$\frac{\partial \mathcal{G}}{\partial \underline{u}_{,X}}\underline{u}^*_{,X} = \frac{m}{T}\Big[\phi\underline{T}^d + \frac{\boldsymbol{\sigma}_r}{\rho}.\underline{e}_x - \phi^2\underline{u}_{,X}\Big]\underline{u}^*_{,X}, \tag{3.16}$$

$$\frac{\partial \mathrm{D}}{\partial \boldsymbol{\varepsilon}_{,X}} : \boldsymbol{\varepsilon}^* = -\frac{\phi}{T}\boldsymbol{\sigma}_{ir} : \Big[\underline{e}_x \otimes \underline{u}^*_{,X}\Big]. \tag{3.17}$$

Hence $\boldsymbol{\sigma} = \boldsymbol{\sigma}_r + \boldsymbol{\sigma}_{ir}$, then

$$\frac{\partial \mathcal{G}}{\partial \underline{u}_{,X}} - \frac{\partial \mathrm{D}}{\partial \boldsymbol{\varepsilon}_{,X}}.\underline{e}_x = \frac{m}{T}\Big[\phi\underline{T}^d + \frac{\boldsymbol{\sigma}}{\rho}.\underline{e}_x - \phi^2\underline{u}_{,X}\Big] = 0. \tag{3.18}$$

This is the first canonical equation for the shock, which expresses the conservation of the momentum.

In the same manner, the normality rule is rewritten as

$$\frac{\partial \mathcal{G}}{\partial \alpha} = -\frac{m}{T}\frac{\mathrm{A}}{\rho} = \frac{\partial \mathrm{D}}{\partial \alpha_{,X}}. \tag{3.19}$$

The conservation of energy is easily written as

$$\left(m\left(\mathrm{e} + \frac{1}{2}\underline{v}^2\right) + \underline{\nu}.\boldsymbol{\sigma}.\underline{v} - \underline{q}.\underline{\nu}\right)_{,X} = 0. \tag{3.20}$$

Besides:

$$\begin{aligned}\frac{\partial \mathcal{G}}{\partial T} &= -\frac{m}{T^2}\left(\left(w - T\frac{\partial w}{\partial T}\right) - Q^d \ + \phi\underline{T}^d.\underline{u}_{,X} - \frac{1}{2}\phi^2\underline{u}_{,X}^2\right) \\ &= -\frac{1}{T^2}\Big(m(w + Ts) - \phi\underline{\nu}.\boldsymbol{\sigma}.\underline{u}_{,X} + \frac{1}{2}m\phi^2\underline{u}_{,X}^2 - mQ^d\Big),\end{aligned} \tag{3.21}$$

$$\frac{\partial \mathcal{G}}{\partial T_{,X}} = \frac{KT_{,X}}{T^2} = \frac{\underline{q}.\underline{\nu}}{T^2}. \tag{3.22}$$

So, with the boundary condition, the conservation of energy is written:

$$\frac{\partial \mathcal{G}}{\partial T} = \frac{\partial \mathrm{D}}{\partial T_{,X}}. \tag{3.23}$$

The dissipation throughout the transition is linked to the evolution of $\mathcal{G}$:

$$-m[s]_\Gamma + \underline{\nu}.\left[\frac{\underline{q}}{T}\right]_\Gamma = \int_{-\infty}^{\infty} -\frac{\phi}{T}\left(\boldsymbol{\sigma}_{ir} : \boldsymbol{\varepsilon}_{,X} + A : \alpha_{,X} + \frac{KT_{,X}^2}{T^2}\right) dX. \tag{3.24}$$

So

$$-m[s]_\Gamma + \underline{\nu}.\left[\frac{\underline{q}}{T}\right]_\Gamma = [\mathcal{G}]_\Gamma. \tag{3.25}$$

The convexity of the pseudo-potentials D ensures the positivity of the entropy production.

Finally, the possible stationary solutions for the shock transition lead to upstream states (+) compatible with:

- the conservation laws across the front (jump relations);
- the positivity of the jump of entropy;
- the material behavior inside the shock front.

It now remains to show that such solutions actually exist. We restrict our analysis of existence of solution to a particular case. In the following paragraph, we develop the equations of the shock transition in a one-dimensional case. We study a material with an isotropic elastic behavior and a von Mises plastic criterion with isotropic strain hardening.

4. Existence and uniqueness of the shock transition in a one-dimensional isothermal case

4.1. Constitutive laws

In isothermal conditions we assume that the elastic behavior of the material is isotropic and that it is not affected by plastic deformation, but the elastic behavior is not linear: the hydrostatic elastic modulus increases in compression, the shear modulus being a constant:

$$w(\boldsymbol{\varepsilon}^e, \alpha) = w^e(\boldsymbol{\varepsilon}^e) + w^p(\alpha), \tag{4.1}$$

with

$$\rho w^e(\boldsymbol{\varepsilon}^e) = \rho w^H(\omega^e) + G\mathbf{e}^{\mathbf{e}} : \mathbf{e}^e, \tag{4.2}$$

where the notations correspond to:

$$\omega^e = \operatorname{Tr} \boldsymbol{\varepsilon}^e, \qquad \mathbf{e}^e = \boldsymbol{\varepsilon}^e - \frac{\omega_e}{3}\mathbf{I}, \tag{4.3}$$

and G is the shear modulus. The elastic moduli are chosen so that:

$$\frac{dK}{d\omega^e} \leq 0, \qquad \text{where} \quad K = \rho \frac{\partial^2 w^H}{\partial \omega^e \partial \omega^e}. \tag{4.4}$$

The stored energy is supposed to saturate with strain hardening:

$$H(\alpha) = \rho \frac{d^2 w^p}{d\alpha^2} = -\frac{dA}{d\alpha}, \tag{4.5}$$

so we suppose that

$$\frac{dH}{d\alpha} \leq 0. \tag{4.6}$$

We shall see further that the non-linearities of the behavior are essential to prove the existence of a steady-state profile.

The viscosity is supposed to be linear in the strain rate:

$$\boldsymbol{\sigma}_{ir} = \mathbb{H} : \dot{\boldsymbol{\varepsilon}}. \tag{4.7}$$

$\mathbb{H}$ is considered to be isotropic.

The plastic flow is defined by a von Mises criterion and a generalized normality law, with the yield surface defined by:

$$f(\boldsymbol{\sigma}, A) = J_2 + A - Y_o = 0, \qquad J_2 = \sqrt{\frac{3}{2} s_{ij} s_{ij}}\,. \tag{4.8}$$

The flow rule is given by:

$$\dot{\boldsymbol{\varepsilon}}_p = \lambda \frac{\partial f}{\partial \boldsymbol{\sigma}}, \qquad \dot{\alpha} = \lambda \frac{\partial f}{\partial A}\,. \tag{4.9}$$

4.2. Equations of the shock transition

We are interested in the propagation of plane shocks in a material in which the state $(-)$ is a state of uniaxial strain.

Moreover, this elasto-plastic shock is arriving after an elastic precursor so that in state $(-)$ the stress is on the yield surface but the plastic strain and internal variable are zero.

Even in this configuration, the study of all possible transition profiles representing the internal structure of the front still remains a difficult question. So, in the following, we choose to restrict ourselves to the profiles satisfying the following assumptions introduced by Mandel:

- the stress tensor always remains on the yield surface,
- the material is always in axial compression (without elastic release), the deviatoric stress tensor and plastic strain rate remaining transversally isotropic all over the profile.

In such profiles, it can be derived that the strain tensor is always uniaxial. So the strain, plastic strain, internal variable and the stress take the form:

$$\boldsymbol{\varepsilon} = \varepsilon \underline{e}_x \otimes \underline{e}_x, \qquad \boldsymbol{\varepsilon}_p = \varepsilon^p \Big(\underline{e}_x \otimes \underline{e}_x - \frac{1}{2} (\underline{e}_y \otimes \underline{e}_y + \underline{e}_z \otimes \underline{e}_z) \Big). \tag{4.10}$$

The hardening α is exactly $-\varepsilon^p$. The state of stresses is defined by

$$\boldsymbol{\sigma} = \sigma \underline{e}_x \otimes \underline{e}_x + \sigma_{tt} (\underline{e}_y \otimes \underline{e}_y + \underline{e}_z \otimes \underline{e}_z). \tag{4.11}$$

The free energy has a simple form

$$\rho w(\varepsilon, \alpha) = \rho w^H(\varepsilon) + \frac{3}{2} G \left(\frac{2}{3} \varepsilon + \alpha \right)^3 + \rho w^p(\alpha). \tag{4.12}$$

The axial stress satisfies the condition

$$\sigma = \rho \frac{\partial w}{\partial \varepsilon} + \eta \dot{\boldsymbol{\varepsilon}} = \sigma_r + \sigma_{ir}. \tag{4.13}$$

In this uniaxial configuration and with the hypothesis that the stress remains on the yield surface, there exists a pseudo-potential for plasticity, recalling that the thermodynamical force associated to α is

$$A = -\rho\frac{\partial w}{\partial \alpha} = \sigma^{eq} + \mathrm{A}. \tag{4.14}$$

The associated dissipation being $Y_o\dot{\alpha}$, we obtain:

$$A = Y_o. \tag{4.15}$$

For the steady-state transition profile, the shock generating function and the potential of dissipation are given by:

$$\mathcal{G}(\varepsilon,\alpha) \;=\; \rho\phi\Big(\phi\varepsilon \mathrm{T}^d + w(\varepsilon,\alpha) - \frac{1}{2}\phi^2\varepsilon^2\Big), \tag{4.16}$$

$$\mathrm{D}(\varepsilon_{,X},\alpha_{,X}) \;=\; \frac{1}{2}\eta(-\phi\varepsilon_{,X})^2 - Y_o\phi\alpha_{,X}. \tag{4.17}$$

In isothermal conditions, the equations of the profile are the conservation of momentum and the relation governing the plastic flow:

$$\frac{\partial \mathcal{G}}{\partial \varepsilon} = \frac{\partial \mathrm{D}}{\partial \varepsilon_{,X}}, \tag{4.18}$$

$$\frac{\partial \mathcal{G}}{\partial \alpha} = \frac{\partial \mathrm{D}}{\partial \alpha_{,X}}. \tag{4.19}$$

The condition of continuous loading in compression is $\alpha_{,X} < 0$. The constants of the shock are given by the boundary condition in $+\infty$. State $(-)$ is known and satisfies the condition

$$\frac{\partial \mathcal{G}}{\partial \varepsilon} = 0, \tag{4.20}$$

that is exactly the determination of T^d, state $(+)$ satisfies also the equality $\dfrac{\partial \mathcal{G}}{\partial \varepsilon} = 0$ and the boundary condition in $-\infty$ is then fulfilled.

4.3. Existence and uniqueness of the steady-state solution in visco-elasticity

In visco-elasticity, there remains only one state variable ε and one equation, the conservation of momentum:

$$\frac{\partial \mathcal{G}}{\partial \varepsilon} = \frac{\partial \mathrm{D}}{\partial \varepsilon_{,X}} = \eta\phi^2\varepsilon_{,X}, \tag{4.21}$$

where

$$\frac{\partial \mathcal{G}}{\partial \varepsilon} = \phi(\sigma_r - \rho\phi\varepsilon + m\underline{\mathrm{T}}^d) \tag{4.22}$$

such that $\dfrac{\partial \mathcal{G}}{\partial \varepsilon}(-) = 0$.

According to the hypotheses on the elastic modulus, the second derivative of $\dfrac{d\mathcal{G}}{d\varepsilon}$ is negative, which is $\dfrac{dK}{d\varepsilon}$. So, for a proper choice of ϕ, there exists one and only one possible state $(+)$ satisfying:

$$\varepsilon^+ < \varepsilon^-, \qquad \frac{\partial \mathcal{G}}{\partial \varepsilon}(+) = 0, \tag{4.23}$$

$$\frac{\partial \mathcal{G}}{\partial \varepsilon}(\varepsilon) > 0, \qquad \text{in } [\varepsilon^+, \varepsilon^-]. \tag{4.24}$$

The choice of ϕ and the associated state $(+)$ also satisfy the inequalities:

$$\frac{d^2\mathcal{G}}{d\varepsilon^2}(\varepsilon^+) > 0 \tag{4.25}$$

and

$$\frac{d^2\mathcal{G}}{d\varepsilon^2}(\varepsilon^-) < 0, \tag{4.26}$$

what means that the shock velocity is larger than the wave velocity in state $(-)$ and smaller than the wave velocity in state $(+)$.

With the initial condition $\varepsilon(0) = \dfrac{1}{2}(\varepsilon^+ + \varepsilon^-)$, according to the sign of $\dfrac{\partial \mathcal{G}}{\partial \varepsilon}$ in the interval $[\varepsilon^+, \varepsilon^-]$, there exists a solution with the following properties:

$$\varepsilon_{,X} > 0, \qquad \varepsilon(-\infty) = \varepsilon^+, \qquad \varepsilon(\infty) = \varepsilon^-. \tag{4.27}$$

The uniqueness of the profile is established in the following sense: let us consider two close solutions differing by an amount δu with boundary conditions $\delta u = 0$ at $\pm\infty$; these conditions exclude all translations of the former profile which are obviously other solutions for the shock structure. Then the perturbation δu satisfies:

$$\int \left(\frac{d^2\mathcal{G}}{d\varepsilon^2}\delta\varepsilon - \frac{d^2\mathrm{D}}{d\varepsilon_{,X}^2}\delta\varepsilon_{,X} \right) \delta u \, dX = 0, \tag{4.28}$$

which, integrated by parts, gives:

$$\int \left(\frac{d}{dX}\left(\frac{d^2\mathcal{G}}{d\varepsilon^2} \right) \right) \frac{1}{2}\delta\varepsilon^2 \, dX = \int \frac{d^2\mathrm{D}}{d\varepsilon_{,X}^2} \left(\delta\varepsilon\right)^2 \, dX \geq 0. \tag{4.29}$$

Since the wave velocity decreases with X, the first term is ≤ 0, what implies that $\delta u = 0$.

4.4. Extension to elasto-plasticity

The results of the former subsection can be extended to elasto-plasticity in the following way: the internal variable α is considered as a function of ε in the conservation of momentum and the previous results are applied; this is due essentially to the hypothesis of continuous loading.

To be more precise:

$$\frac{\partial}{\partial\alpha}\left(\frac{\partial\mathcal{G}}{\partial\alpha}-\frac{\partial \mathrm{D}}{\partial\alpha_{,X}}\right) = \frac{\partial^2\mathcal{G}}{\partial\alpha^2}=\phi\left(3G+\rho\frac{d^2w^p}{d\alpha^2}\right)>0, \tag{4.30}$$

$$\frac{\partial}{\partial\varepsilon}\left(\frac{\partial\mathcal{G}}{\partial\alpha}-\frac{\partial \mathrm{D}}{\partial\alpha_{,X}}\right) = \frac{\partial^2\mathcal{G}}{\partial\varepsilon\partial\alpha}=2\phi G>0. \tag{4.31}$$

The solution satisfies the conditions:

$$\frac{\partial\mathcal{G}}{\partial\alpha}=\frac{\partial \mathrm{D}}{\partial\alpha_{,X}} \tag{4.32}$$

which is an implicit relation between α and ε. The variations of ε and α are with opposite sign.

It can be derived from the former paragraph that there exists a unique profile which satisfies $\varepsilon_{,X}>0$ and $\alpha_{,X}<0$ under the condition that:

$$\frac{d^2}{d\varepsilon^2}\left(\frac{\partial\mathcal{G}}{\partial\varepsilon}(\varepsilon,\alpha(\varepsilon))\right)<0. \tag{4.33}$$

Let us develop this last condition:

$$\frac{d^2}{d\varepsilon^2}\left(\frac{\partial\mathcal{G}}{\partial\varepsilon}\right)=\frac{\partial^3\mathcal{G}}{\partial\varepsilon^3}+\frac{\partial^2\mathcal{G}}{\partial\varepsilon\partial\alpha}\frac{d^2\alpha}{d\varepsilon^2}. \tag{4.34}$$

From this relation, according to the properties of functional $\mathcal{G}$ we have:

$$\frac{d^2\alpha}{d\varepsilon^2}=-\frac{4G^2}{(3G+H)^3}\frac{dH}{d\alpha}>0. \tag{4.35}$$

This condition is not always satisfied. It is, when the increase of elastic modulus in compression dominates the effect of strain hardening saturation in plasticity.

4.5. Case of several internal state variables

The former results could be extended to the case of several internal state variables, each of them being associated with one particular irreversible mechanism of plastic deformation possessing its own yield stress (HALPHEN and NGUYEN [3].

Under isothermal uniaxial strain conditions, we admit that the axial plastic strain is the sum of $-\alpha^i$:

$$\boldsymbol{\varepsilon}_p = -\sum_i \alpha^i, \qquad \text{and} \qquad w^p = w^p(\alpha^i). \tag{4.36}$$

Each α^i satisfies normality rule of the same type as that previously defined by different threshold value Y_{io}.

Under the assumption that all these variables are activated continuously in the profile, the relations for plasticity are written:

$$\frac{\partial \mathcal{G}}{\partial \alpha^i} = \frac{\partial \mathrm{D}}{\partial \alpha^i_{,X}}. \tag{4.37}$$

The conditions on which the α^i can be written as functions of ε are linked to the second derivatives of w^p. Assume that the α^i are uniquely defined when the effects of self-strain hardening prevail on the effects of strain hardening due to coupling of the different mechanisms.

The existence and uniqueness of the shock profile is guaranteed when the following inequality is fulfilled:

$$\frac{d^2}{d\varepsilon^2}\left(\frac{\partial \mathcal{G}}{\partial \varepsilon}\right) = m\left(\frac{\partial^3 w^H}{\partial \varepsilon^3} + 2G\sum_i \frac{d^2\alpha_i}{d\epsilon^2}\right) < 0. \tag{4.38}$$

The analysis of implications of this relation is very complex, we can only guess that the inequality is satisfied when the effect of elastic stiffening is greater than the effect of saturation of the different strain hardening moduli.

In the last sections, we have managed to give a solution to the shock problem in the case of one-dimensional elasto-plasticity (when conditions for existence and uniqueness are fulfilled). Given the constitutive laws which apply inside the shock front, the result analogous to a Hugoniot curve can be deduced from the study of the internal shock structure; so, we define a constitutive relation for the shock front (relation between the jump of all the state variables and the shock velocity) and then we can deduce the initial conditions for the post-shock behavior.

5. Elaboration of constitutive laws for the behavior inside the shock

The elaboration of constitutive models for plasticity inside the shock fronts relies on the solution of the inverse problem. Various shock tests are carried out on the selected material. The simultaneous analysis of annealed and pre-shocked samples allows first to identify the variables which are necessary to characterize

the internal state of the material in both states and secondly, to establish jump relations for these variables across a shock front, as long as the loading path during the shock experiment is sufficiently simple. Then, one intends to go back to the constitutive laws which rule the evolution of the internal state variables inside the shock structure.

Such relations are derived to be used in the numerical solution of mechanical problems including both shock phases and other types of the loading phases. At the present time, numerical codes represent both the internal structure of shock fronts and the thermo-mechanical evolution outside the fronts.

This identification process has been carried out on copper in Centre d'Études de Gramat (C.E.G.) [11]. The thermo-mechanical formalism is slightly different from the one adopted above but the analysis has been conducted in the same spirit.

6. Conclusion

The first aim of this study on the propagation of strong discontinuities in elasto-plastic solids was to give a sort of a constitutive relation for the evolution of shock fronts: in each point of the front, this constitutive relation should link the shock velocity and the jump of all state variables. In other words, it would give a parametric equation to build a Hugoniot curve just like in hydrodynamics.

Because of the incremental formulation of plasticity laws, it is not possible to derive the jump relations for internal state variables and to exhibit directly such a constitutive relation. The solution adopted results from the assumption that the shock front is in fact a continuous transition which can be studied in the framework of thermo-mechanics, although the constitutive laws may be specific.

The fundations for the study of the internal shock structure were built in the general case of elasto-plastic materials for which irreversible processes are ruled by a pseudo- potential of dissipation. With the assumption that this local problem is plane and stationary, the equations of thermo-mechanics are written in a compact form with the help of the shock generating function (playing the role of a potential energy for steady-state problems) and a pseudo-potential of dissipation.

The derivations were carried out in one-dimensional conditions in the case of standard materials with a von Mises loading surface. The existence and uniqueness of a steady-state profile for a shock in compression were proved, when the non-linearity of the material behavior is such that the wave velocity increases continuously: in other words, the effect of elastic stiffening must dominate the opposite effect of strain hardening saturation. We find again the well-known result that the existence of a steady-state profile relies on the equilibrium between the increase of elastic modulus in compression, which tends to stiffen the

wave fronts, and the diffusive effects (just like viscosity and thermal conduction), which prevent gradients from being infinite. Moreover, we have established that the value of the viscosity coefficient does not affect the extremities of the profile, that is to say the upstream state (+), but only its thickness.

The experimental and numerical work using shock tests on copper illustrated the possibility of identifying the behavior inside the shock front by an inverse method. Generally, it is not possible to follow the evolution of state variables inside the shock fronts since their evolution rates are too high. We are only able to evaluate the mechanical characteristics (plastic behavior) in both unshocked and pre-shocked states. So, we are able to define an equation of state for the material in both states and to evaluate the changes of the internal state variables during the shock loading sequence. If the loading path during the test is simple enough, we manage to go back to state (+) of the shock front, state (−) being the state of the annealed material before the test.

Of course, the complete identification of plasticity laws which apply inside the shock is not possible when only state (+) and state (−) are known (it is obvious that the inverse problem formulated this way is not well-posed!). Nonetheless, bibliographic data on shock-loading can allow to postulate a general formulation for these laws; the values of various coefficients are then fitted in such a way that the solution of the shock structure problem gives the correct description of state (+). When the selected plasticity laws are quite complex, numerical simulation is useful to solve the problem of the internal shock structure.

However, in the future it would be useful to analyze more precisely how the jump of the different internal state variables across a shock discontinuity depends on different coefficients of the plasticity laws which apply inside the shock, especially when several strain hardening mechanisms are activated. Such an analysis would help to solve, in a more systematic way, the inverse problem giving the plastic laws for the shock structure from post-shock characterization.

Moreover, to justify the solutions exhibited here for the shock structure, we would have to prove that the steady-state solutions are stable and that the relaxation time for perturbations is smaller than the time characteristic for the shock transition.

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