Arch. Mech., **56**, 5, pp. 377–389, Warszawa 2004

Thermo-electrodynamics of rigid superconductors

V. A. CIMMELLI[(1)], F. OLIVERI[(2)], A. R. PACE[(1)]

[(1)] *Department of Mathematics, University of Basilicata,*
Campus Macchia Romana, 85100 Potenza – Italy
e-mail: `cimmelli@unibas.it`
e-mail: `pace@unibas.it`

[(2)] *Department of Mathematics, University of Messina*
Contrada Papardo, Salita Sperone 31, 98166 Messina – Italy
e-mail: `oliveri@mat520.unime.it`

This paper deals with a continuum model of rigid superconductors. It is assumed that their property of shifting to the superconductive state for suitable values of temperature and magnetic field is due to a vectorial internal variable, related to the superelectron current by a linear constitutive law. The compatibility of the model with the second law of thermodynamics is investigated. The propagation of thermo-electromagnetic waves through a one-dimensional conductor is analyzed as well. Comparison is made with different continuum approaches which may be found in the literature.

1. Introduction

The superconductivity, one of the most fascinating low temperature phenomena, consists in the non-dissipative current transport in metals [1]. A wide literature faced such a problem within the framework of both the quantum statistical mechanics and the macroscopic non-equilibrium thermodynamics [2]. A non-local theory of superconductivity has been developed in [3, 4], by modifying the classical London's model [5]. Moreover, in the last two decades several authors introduced certain complex internal variables to describe quantum effects at the macroscopic scale [6, 7]. The superconducting phase in metals was first observed in 1911 by Kamerlingh Onnes who discovered that, if the temperature is lowered until a certain critical value θ_{cr}, then the electrical resistance suddenly drops to zero and an electrical current flows without dissipation of energy [8]. Later on, Meissner and Ochsenfeld [1, 2] measured the magnetic induction $\mathbf{B}$ immediately outside a superconductor as it is cooled in an applied magnetic field $\mathbf{H}$. They observed a perfectly diamagnetic state inside the specimen which caused the internal field $\mathbf{B}$ to be expelled. Such a phenomenon is referred to as the Meissner effect. The existence of superconductive electrons flowing without resistance was first supposed by the London brothers [5]. In the presence of an electric field $\mathbf{E}$, these electrons obey the equation of motion

$$m\dot{\mathbf{v}}_s = e\mathbf{E}, \tag{1.1}$$

where m, e and $\mathbf{v}_s$ are the mass, the electric charge and the mean velocity of the electrons, respectively, and the superimposed dot stands for time derivative. The superconductive current flux $\mathbf{J}_s$ may be defined as

$$\mathbf{J}_s = n_s e\mathbf{v}_s, \tag{1.2}$$

where n_s is the density of superelectrons. Hence, (1.1) and (1.2) yield

$$\frac{m}{n_s e^2}\dot{\mathbf{J}}_s = \mathbf{E}. \tag{1.3}$$

Finally, by combining (1.3) with the Maxwell equations, we obtain

$$\nabla \times \left(\frac{m}{n_s e^2}\mathbf{J}_s\right) = -\mathbf{B}. \tag{1.4}$$

Relations (1.3)–(1.4), governing the evolution of the supercurrent in metals, are referred to as London's equations. The main London's hypothesis is that the total current density $\mathbf{J}$ may be split into the sum of a normal current density $\mathbf{J}_n$ and a supercurrent density $\mathbf{J}_s$, namely

$$\mathbf{J} = \mathbf{J}_n + \mathbf{J}_s. \tag{1.5}$$

In the present paper we consider a rigid body with a vectorial internal variable $\mathbf{a}$ which describes the intrinsic properties of the material under the application of an electromagnetic field. Weak memory effects, which characterize a superconductive state, are accounted for by an ordinary differential equation governing the evolution of the vector $\mathbf{a}$. Such a vector is assumed to be proportional to the flux of supercurrent $\mathbf{J}_s$. It is proved that $\mathbf{J}_s$ may circulate inside the conductor with a negligible energy dissipation.

The plan of the paper is the following. In Sec. 2 we derive the complete set of evolution equations for a rigid electromagnetic solid with an internal vectorial variable, by applying the modern approach to non-equilibrium thermodynamics with internal variables [9, 10]. Then, after postulating a suitable constitutive law for the supercurrent $\mathbf{J}_s$, some properties of such a vector on the boundary are derived. In Sec. 3 we investigate the compatibility of the constitutive equations with the second law of thermodynamics. A set of mathematical relations ensuring such a compatibility is obtained. In Sec. 4 the complete system of governing equations is derived. We prove that such a system, which is hyperbolic in general, becomes parabolic in the superconductive phase. Thence, in Sec. 5 we study the propagation of thermo-electrical waves along a one-dimensional rigid conductor which is in a state close to the superconductive phase. We find that the waves, transporting the internal variable, the electric and magnetic field and the temperature, may propagate through the metal. The expression of the wave velocities is derived as well. In Sec. 6, a final discussion leading to a comparison of the present theory with different approaches is developed.

2. Rigid electromagnetic solid with an internal variable

Let us consider a compact and simply connected region $\mathcal{C}$ of an Euclidean point space $\mathcal{E}_3$ representing an isotropic rigid body $\mathcal{B}$. The position of the points of $\mathcal{C}$ will be determined by a vector $\mathbf{x} = \sum_{i=1}^{3} x_i \mathbf{e}_i$ where the set $\{\mathbf{e}_i\}_{i=1,2,3}$ is a basis of the Euclidean vector space $\mathbf{E}_3$ associated with $\mathcal{E}_3$. We assume that a magnetic field $\mathbf{H}$ and an electrical field $\mathbf{E}$ act on $\mathcal{B}$ and that a part of the total electrical current inside $\mathcal{B}$ can flow without dissipation. This flux is represented by the vector $\mathbf{J}_s$, whereas the normal dissipative flux of current is given by the vector $\mathbf{J}_n$, and the total current takes the form (1.5). The space Σ of the thermodynamic states of $\mathcal{B}$ is spanned by the fields $\mathbf{E}$ and $\mathbf{H}$, the absolute temperature θ, together with its gradient $\mathbf{g} = \nabla\theta$, and an internal vector state variable $\mathbf{a}$ [11]. A thermodynamic process of $\mathcal{B}$ is an almost regular curve of Σ. We postulate that the evolution of $\mathbf{a}$ is controlled by the following kinetic equation

$$\tau\dot{\mathbf{a}} + \alpha\mathbf{a} + \mathbf{b}\dot{\theta} = \mathbf{g}, \tag{2.1}$$

accounting for the property of the material of shifting to the superconductive phase at suitable values of the temperature and of the magnetic field. In (2.1) the scalar material functions τ and α and the vector material function $\mathbf{b}$ are supposed to depend on the thermodynamic fields θ, $\mathbf{g}$, $\mathbf{E}$ and $\mathbf{H}$. Let us assume that both α and τ are positive and, moreover, $\alpha \ll \tau$ whenever the system is in a homothermal state ($\mathbf{g} = \mathbf{0}$) and both the temperature and the intensity of the magnetic field are below some given critical values θ_{cr} and H_{cr}. Under these experimental conditions, at the thermal equilibrium ($\dot{\theta} = 0, \mathbf{g} = \mathbf{0}$), the internal variable $\mathbf{a}$ decays very slowly, so that it remains almost constant for a very long time. We say that the system is in a weakly dissipative phase. The local balance of energy for a rigid system upon the action of an electromagnetic field takes the form [4]

$$\rho\dot{e} + \nabla\cdot\mathbf{q} = \rho r + \mathbf{H}\cdot\dot{\mathbf{B}} + \mathbf{E}\cdot\dot{\mathbf{D}} + \mathbf{J}\cdot\mathbf{E}, \tag{2.2}$$

where ρ is the mass density, e the specific internal energy, $\mathbf{q}$ the heat flux vector and r the radiative heat supply. Moreover, the fields $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, with the two scalar functions ϵ and μ representing the electrical permittivity and the magnetic permeability of the material, are the electric displacement and the magnetic induction, respectively. The evolution of the fields $\mathbf{E}$, $\mathbf{H}$, $\mathbf{B}$, $\mathbf{D}$, $\mathbf{J}$ is governed by the system of Maxwell equations

$$\begin{aligned} &\nabla \cdot \mathbf{D} = \rho_e, \\ &\nabla \cdot \mathbf{B} = 0, \\ &\dot{\mathbf{D}} = \nabla \times \mathbf{H} - \mathbf{J}, \\ &\dot{\mathbf{B}} = -\nabla \times \mathbf{E}, \end{aligned} \tag{2.3}$$

where ρ_e is the density of electric charges. Moreover, the absence of sources of superconductive charges inside $\mathcal{B}$ forces $\mathbf{J}_s$ to satisfy the boundary value problem

$$\begin{aligned} \nabla \cdot \mathbf{J}_s = 0 \qquad & \text{in } \mathcal{C}, \\ \mathbf{J}_s \cdot \mathbf{n} = 0 \qquad & \text{on } \partial\mathcal{C}, \end{aligned} \tag{2.4}$$

with $\mathbf{n}$ being the unitary vector normal to $\partial\mathcal{C}$. Since the system (2.1), (2.2) and (2.3) is not closed, a set of constitutive equations is needed [11]. These equations will be postulated in the form

$$\phi = \phi^*(\theta, \mathbf{a}, \mathbf{g}, \mathbf{E}, \mathbf{H}), \tag{2.5}$$

where ϕ is an element of the set of functions $\{e, r, \mu, \epsilon, \mathbf{q}, \mathbf{J}_s, \mathbf{J}_n\}$. In particular, for the current vectors $\mathbf{J}_s$ and $\mathbf{J}_n$ we assume

$$\begin{aligned} &\mathbf{J}_n = \sigma(\theta, \mathbf{a}, \mathbf{g}, \mathbf{H})\mathbf{E}, \\ &\mathbf{J}_s = j(\theta, \mathbf{g}, \mathbf{E}, \mathbf{H})\mathbf{a}, \end{aligned} \tag{2.6}$$

where j and σ are constitutive scalar quantities depending on the indicated arguments. Equation $(2.6)_1$ is the well-known Ohm's law, where σ is the electrical resistance of the normal state, whereas Eq. $(2.6)_2$, which constitutes our main hypothesis, links the supercurrent vector to the internal variable through a linear relation. In practice, vector $\mathbf{a}$ may be interpreted as the driving force moving the superelectrons inside $\mathcal{B}$. In thermal equilibrium, below the critical point, such a force can be assumed to remain constant for a very long time whereas above the critical point it decays exponentially to zero and the conventional current conduction is recovered. We call the class of materials which present such a behavior, the superconductors of weak dissipation. As a consequence of $(2.6)_2$, the boundary value problem (2.4) specializes to

$$\begin{aligned} \nabla j \cdot \mathbf{a} + j \nabla \cdot \mathbf{a} = 0 \qquad & \text{in } \mathcal{C}, \\ \mathbf{a} \cdot \mathbf{n} = 0 \qquad & \text{on } \partial\mathcal{C}. \end{aligned} \tag{2.7}$$

Let us observe that Eq. $(2.7)_2$ is necessary in order to satisfy Eq. $(2.4)_2$ since

$$j = 0 \quad \text{on} \quad \partial\mathcal{C} \tag{2.8}$$

would imply $j = 0$ in $\bar{\mathcal{C}}$. On the other hand, for a wide class of superconductors it results [12]

$$\mathbf{J}_s = \frac{1}{\lambda}\mathbf{n} \times \mathbf{H} \quad \text{on } \partial\mathcal{C}, \tag{2.9}$$

where $\lambda = \sqrt{m/n_s e^2 \mu}$ represents the penetration depth of the magnetic field in the interior of the superconductor. Then, we get the following boundary condition for j:

$$j = \frac{1}{\lambda a^2}[\mathbf{n} \times \mathbf{H}] \cdot \mathbf{a} \quad \text{on } \partial\mathcal{C}. \tag{2.10}$$

Let us remark that in a state of thermal equilibrium we have still $\mathbf{J_s} \neq \mathbf{0}$, according to the experimental evidence. Finally, let us take into account the dissipation principle, according to which the constitutive equations (2.5) must be compatible with the local form of the second law of thermodynamics (Clausius-Duhem inequality). In the present case such an inequality reads

$$-\rho(\dot{\psi} + s\dot{\theta}) - \frac{1}{\theta}\mathbf{q} \cdot \mathbf{g} + \mathbf{H} \cdot \dot{\mathbf{B}} + \mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{J} \cdot \mathbf{E} \geq 0, \tag{2.11}$$

where s denotes the specific entropy and $\psi = e - \theta s$ is the Helmholtz free energy. In order to investigate such a compatibility, a constitutive equation for ψ of the form (2.5) should be assigned as well.

3. Thermodynamic compatibility

The compatibility of (2.5) and (2.6) with (2.11) may be investigated by exploiting the classical procedures of non-equilibrium thermodynamics. To this end, let us calculate the time derivative of ψ appearing in (2.11) and, moreover, let us use (2.1) in the obtained expression to eliminate $\dot{\mathbf{a}}$. Then we are led to

$$\begin{aligned} &-\rho\left(s + \frac{\partial\mathcal{F}}{\partial\theta} - \frac{1}{\tau}\frac{\partial\mathcal{F}}{\partial\mathbf{a}} \cdot \mathbf{b}\right)\dot{\theta} - \rho\frac{\partial\mathcal{F}}{\partial\mathbf{g}} \cdot \dot{\mathbf{g}} \\ &\qquad - \left(\mathbf{B} + \rho\frac{\partial\mathcal{F}}{\partial\mathbf{H}}\right) \cdot \dot{\mathbf{H}} - \left(\mathbf{D} + \rho\frac{\partial\mathcal{F}}{\partial\mathbf{E}}\right) \cdot \dot{\mathbf{E}} \\ &\qquad - \left(\frac{\rho}{\tau}\frac{\partial\mathcal{F}}{\partial\mathbf{a}} + \frac{1}{\theta}\mathbf{q}\right) \cdot \mathbf{g} + \left(\frac{\rho\alpha}{\tau}\frac{\partial\mathcal{F}}{\partial\mathbf{a}} + j\mathbf{E}\right) \cdot \mathbf{a} + \sigma E^2 \geq 0, \end{aligned} \tag{3.1}$$

where

$$\mathcal{F} = \psi - \frac{\mu}{\rho} H^2 - \frac{\epsilon}{\rho} E^2. \tag{3.2}$$

Then, the classical procedure by COLEMAN and GURTIN [13] provides the following thermodynamic restrictions which ensure that (3.1) is satisfied along any process in Σ

$$s = -\frac{\partial \mathcal{F}}{\partial \theta} + \frac{1}{\tau}\frac{\partial \mathcal{F}}{\partial \mathbf{a}} \cdot \mathbf{b}, \tag{3.3}$$

$$\frac{\partial \mathcal{F}}{\partial \mathbf{g}} = \mathbf{0}, \tag{3.4}$$

$$\mathbf{B} = -\rho \frac{\partial \mathcal{F}}{\partial \mathbf{H}}, \tag{3.5}$$

$$\mathbf{D} = -\rho \frac{\partial \mathcal{F}}{\partial \mathbf{E}}, \tag{3.6}$$

$$-\left(\frac{\rho}{\tau}\frac{\partial \mathcal{F}}{\partial \mathbf{a}} + \frac{1}{\theta}\mathbf{q}\right) \cdot \mathbf{g} + \left(\frac{\rho\alpha}{\tau}\frac{\partial \mathcal{F}}{\partial \mathbf{a}} + j\mathbf{E}\right) \cdot \mathbf{a} + \sigma E^2 \geq 0. \tag{3.7}$$

The left-hand side of (3.7) presents the classical bilinear form of a product of generalized forces (affinities) times fluxes or rates, first pointed out by Onsager in non-equilibrium thermodynamics [11]. Moreover, it should be interpreted as the local entropy production inside the conductor. As we have pointed out in Sec. 2, in the fully superconductive phase such a production is negligible, provided the system is in thermal equilibrium below the critical point $(\theta_{\text{cr}}, H_{\text{cr}})$. If these experimental conditions are met, the inequality (3.7) yields

$$\left(\frac{\rho\alpha}{\tau}\frac{\partial \mathcal{F}}{\partial \mathbf{a}} + j\mathbf{E}\right) \cdot \mathbf{a} \cong 0, \tag{3.8}$$

which expresses the hypothesis of weak dissipation. Finally, as a consequence of (3.5), if

$$\frac{\partial \mathcal{F}}{\partial \mathbf{H}} = \mathbf{0}, \tag{3.9}$$

then a perfectly diamagnetic state is present inside the conductor (the Meissner effect). Hence we are allowed to consider Eqs. (3.8)–(3.9) as characterizing the superconductive phase. Let us observe that (3.8) does not imply $\mathbf{J}_s = \mathbf{0}$, so that we can have an electrical current circulating inside $\mathcal{B}$ without energy dissipation.

This fact proves that our model is able to describe the superconductive phase. Let us notice that, if the functions α, τ, σ, j and $\mathbf{q}$ do not depend on $\mathbf{g}$, then (3.7) splits in the potential relation

$$\mathbf{q} = -\frac{\rho\theta}{\tau}\frac{\partial \mathcal{F}}{\partial \mathbf{a}}, \tag{3.10}$$

together with the reduced entropy inequality

$$\left(\frac{\rho\alpha}{\tau}\frac{\partial \mathcal{F}}{\partial \mathbf{a}} + j\mathbf{E}\right)\cdot \mathbf{a} + \sigma E^2 \geq 0. \tag{3.11}$$

4. Governing equations

In this section we derive a set of evolution equations for the system at hand which allows finite speeds of propagation of thermo-electromagnetic perturbations. To accomplish this task we assume the heat flux to be independent of $\mathbf{g}$, according to (3.4) and (3.10), and so avoid to render the balance of the internal energy parabolic [14]. Hence let us suppose that (3.10) is true and let us postulate the following constitutive equation

$$\mathbf{q} = -\left(\chi_1(\theta)\mathbf{a} + \chi_2(\theta)\mathbf{E} + \chi_3(\theta)\mathbf{H}\right), \tag{4.1}$$

where $\chi_i(\theta)$ $(i = 1, \ldots, 3)$ represent some generalized heat conduction coefficients. Let us observe that, as the material functions τ and $\mathbf{b}$ tend to zero, then $\mathbf{a}$ becomes linearly related to the gradient of temperature and (4.1) yields

$$\mathbf{q} = -\left(\frac{\chi_1(\theta)}{\alpha}\mathbf{g} + \chi_2(\theta)\mathbf{E} + \chi_3(\theta)\mathbf{H}\right). \tag{4.2}$$

In such a way the classical Fourier's heat conduction law is recovered. Moreover, due to (4.1), the integration of (3.10) provides

$$\mathcal{F} = \frac{\tau}{\rho\theta}\left[\frac{\chi_1}{2}\mathbf{a}^2 + \chi_2\mathbf{E}\cdot\mathbf{a} + \chi_3\mathbf{H}\cdot\mathbf{a}\right] + \mathcal{F}_0(\theta, \mathbf{E}, \mathbf{H}), \tag{4.3}$$

where $\mathcal{F}_0$ is a regular function of the indicated arguments. From (3.3) and (3.10) it follows

$$s = -\left(\frac{1}{2}\frac{\partial\tilde{\chi}_1}{\partial\theta}\mathbf{a}^2 + \frac{\partial\tilde{\chi}_2}{\partial\theta}\mathbf{E}\cdot\mathbf{a} + \frac{\partial\tilde{\chi}_3}{\partial\theta}\mathbf{H}\cdot\mathbf{a}\right) - \frac{\partial\mathcal{F}_0}{\partial\theta} - \frac{\mathbf{q}\cdot\mathbf{b}}{\rho\theta}, \tag{4.4}$$

where

$$\tilde{\chi}_i = \frac{\tau\chi_i}{\rho\theta}, \qquad (i = 1, \ldots, 3). \tag{4.5}$$

Let us pursue our analysis under the additional hypothesis

$$\mathbf{b} = -\left(b_1(\theta)\mathbf{a} + b_2(\theta)\mathbf{E} + b_3(\theta)\mathbf{H}\right). \tag{4.6}$$

Then, from the thermodynamic relation

$$e = \psi + s\theta \tag{4.7}$$

we have

$$\begin{aligned} e = {} & \left[\frac{1}{2}\left(\tilde{\chi}_1 - \theta\frac{\partial\tilde{\chi}_1}{\partial\theta}\right) + \frac{b_1\chi_1}{\rho}\right] a^2 + \left[\tilde{\chi}_2 - \theta\frac{\partial\tilde{\chi}_2}{\partial\theta} + \frac{b_1\chi_2 + b_2\chi_1}{\rho}\right]\mathbf{E}\cdot\mathbf{a} \\ & + \left[\tilde{\chi}_3 - \theta\frac{\partial\tilde{\chi}_3}{\partial\theta} + \frac{b_1\chi_3 + b_3\chi_1}{\rho}\right]\mathbf{H}\cdot\mathbf{a} \\ & + \left[\mathcal{F}_0 - \theta\frac{\partial\mathcal{F}_0}{\partial\theta}\right] + \left[\frac{\epsilon + b_2\chi_2}{\rho}\right] E^2 + \left[\frac{\mu + b_3\chi_3}{\rho}\right] H^2 \\ & + \left[\frac{b_2\chi_3 + b_3\chi_2}{\rho}\right]\mathbf{E}\cdot\mathbf{H}. \end{aligned} \tag{4.8}$$

On the other hand, Debye's theory of specific heat in solids forces e to be independent of the non-equilibrium quantity $\mathbf{a}$; thus, we have to impose

$$\begin{aligned} \left[\frac{1}{2}\left(\tilde{\chi}_1 - \theta\frac{\partial\tilde{\chi}_1}{\partial\theta}\right) + \frac{b_1\chi_1}{\rho}\right] &= 0, \\ \left[\tilde{\chi}_2 - \theta\frac{\partial\tilde{\chi}_2}{\partial\theta} + \frac{b_1\chi_2 + b_2\chi_1}{\rho}\right] &= 0, \\ \left[\tilde{\chi}_3 - \theta\frac{\partial\tilde{\chi}_3}{\partial\theta} + \frac{b_1\chi_3 + b_3\chi_1}{\rho}\right] &= 0. \end{aligned} \tag{4.9}$$

Therefore, the field equations we are dealing with are the following:

$$\begin{aligned} & \rho\left(\frac{\partial e}{\partial\theta}\frac{\partial\theta}{\partial t} + \frac{\partial e}{\partial(E^2)}2\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} + \frac{\partial e}{\partial(\mathbf{E}\cdot\mathbf{H})}\frac{\partial(\mathbf{E}\cdot\mathbf{H})}{\partial t} + \frac{\partial e}{\partial(H^2)}2\mathbf{H}\cdot\frac{\partial\mathbf{H}}{\partial t}\right) \\ & \quad - \mathbf{g}\cdot\left(\frac{\partial\chi_1}{\partial\theta}\mathbf{a} + \frac{\partial\chi_2}{\partial\theta}\mathbf{E} + \frac{\partial\chi_3}{\partial\theta}\mathbf{H}\right) - \chi_1\nabla\cdot\mathbf{a} - \chi_2\nabla\cdot\mathbf{E} - \chi_3\nabla\cdot\mathbf{H} \\ & \quad = \rho r - \mathbf{H}\cdot\nabla\times\mathbf{E} + \mathbf{E}\cdot\nabla\times\mathbf{H}, \\ & \tau\frac{\partial\mathbf{a}}{\partial t} + \alpha\mathbf{a} + \mathbf{b}\frac{\partial\theta}{\partial t} = \mathbf{g}, \end{aligned} \tag{4.10}$$

(4.10) [cont.]
$$\frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H} - \mathbf{J},$$
$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E},$$
$$\nabla \cdot \mathbf{D} = \rho_e,$$
$$\nabla \cdot \mathbf{B} = 0.$$

5. Wave propagation

In what follows we investigate the propagation of thermal waves together with the electromagnetic ones through a one-dimensional rigid conductor which is in a quasi-superconductive state. We call quasi-superconductive the states which are close to the superconductive range, in which both the normal and superconductive phases coalesce. Hence the material functions μ and σ are assumed to be small but different from zero. Also, let us suppose

$$\mathbf{E} = E_1(x_1,t)\mathbf{e}_1 + E_3(x_1,t)\mathbf{e}_3, \quad \mathbf{H} = H_2(x_1,t)\mathbf{e}_2. \tag{5.1}$$

and moreover, according to experimental evidence,

$$\mu = \mu(\theta, H^2), \qquad \epsilon = \epsilon(\theta, E^2). \tag{5.2}$$

Finally, since we are considering a one-dimensional heat conductor, we take

$$\mathbf{a} = a_1(x_1,t)\mathbf{e}_1, \quad \mathbf{b} = -b_1(\theta)a_1\mathbf{e}_1. \tag{5.3}$$

The system of governing equations may be written in the compact matrix form:

$$\mathcal{A}_0(\mathbf{U})\frac{\partial \mathbf{U}}{\partial t} + \mathcal{A}_1(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x_1} = \mathbf{B}(\mathbf{U}), \tag{5.4}$$

where

$$\mathbf{U} = \begin{bmatrix} \theta \\ a_1 \\ E_1 \\ E_3 \\ H_2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \rho r \\ -\alpha a_1 \\ -\sigma E_1 - j a_1 \\ -\sigma E_3 \\ 0 \end{bmatrix},$$

$$\mathcal{A}_0 = \begin{bmatrix} \rho c_v & 0 & 2\rho c_E E_1 & 2\rho c_E E_3 & 2\rho c_H H_2 \\ b_1 a_1 & \tau & 0 & 0 & 0 \\ \epsilon_\theta E_1 & 0 & 2\epsilon_E E_1^2 + \epsilon & 2\epsilon_E E_1 E_3 & 0 \\ \epsilon_\theta E_3 & 0 & 2\epsilon_E E_1 E_3 & 2\epsilon_E E_3^2 + \epsilon & 0 \\ \mu_\theta H_2 & 0 & 0 & 0 & 2\mu_H H_2^2 + \mu \end{bmatrix},$$

$$\mathcal{A}_1 = \begin{bmatrix} -\frac{\partial \chi_1}{\partial \theta} a_1 - \frac{\partial \chi_2}{\partial \theta} E_1 & -\chi_1 & -\chi_2 & -H_2 & -E_3 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

and

$$c_v = \frac{\partial e}{\partial \theta}, \qquad c_E = \frac{\partial e}{\partial (E^2)}, \qquad c_H = \frac{\partial e}{\partial (H^2)},$$
$$\epsilon_\theta = \frac{\partial \epsilon}{\partial \theta}, \qquad \epsilon_E = \frac{\partial \epsilon}{\partial (E^2)},$$
$$\mu_\theta = \frac{\partial \mu}{\partial \theta}, \qquad \mu_H = \frac{\partial \mu}{\partial (H^2)}.$$

Moreover, we have to append to (5.4) the constraint

$$\epsilon_\theta E_1 \frac{\partial \theta}{\partial x_1} + 2\epsilon_E \left(E_1^2 \frac{\partial E_1}{\partial x_1} + E_1 E_3 \frac{\partial E_3}{\partial x_1} \right) + \epsilon \frac{\partial E_1}{\partial x_1} = \rho_e, \tag{5.5}$$

arising from $(2.3)_1$.

In general, the system (5.4) is hyperbolic in the t–direction if

$$\det(\mathcal{A}_0) \neq 0, \tag{5.6}$$

and the eigenvalue problem

$$(\mathcal{A}_1 - \lambda \mathcal{A}_0)\mathbf{r} = \mathbf{0} \tag{5.7}$$

has only real eigenvalues $\lambda(\mathbf{U})$ and a set of linearly independent right eigenvectors $\mathbf{r}$ spanning $\mathbf{E}^5$. To ascertain the conditions rendering the system (5.4) hyperbolic, let us consider the propagation of weak discontinuities [16], i.e., piecewise continuous solutions whose first–order derivatives suffer a jump across a curve Γ of equation $\varphi(x_1, t) = 0$ (wave front); it is well known [16] that the characteristic wave velocities $V = -\dfrac{\varphi_t}{\varphi_{x_1}}$ are given by the eigenvalues $\lambda(\mathbf{U}_0)$ satisfying (5.7) where the matrices $\mathcal{A}_0$ and $\mathcal{A}_1$ have to be evaluated in the unperturbed state $\mathbf{U}_0$.

Since the expression of the wave velocities is too long to allow for its complete writing here, in the sequel we shall restrict ourselves to consider the simplified case when the electrical permittivity is constant and the magnetic permeability does not depend on θ. By solving the eigenvalue problem (5.7) we get:

$$\lambda_0 = 0,$$

(5.8)
$$\lambda_{1,2} = \pm\frac{1}{\sqrt{\epsilon(\mu + 2\mu_H H_2^2)}},$$

$$\lambda_{3,4} = \frac{\chi \pm \sqrt{\chi^2 + 4\rho\tau c_v \chi_1}}{2\rho\tau c_v},$$

where $\chi = a_1 b_1 \chi_1 - \tau a_1 \chi_1' - \tau E_1 \chi_2'$. Thus we have a hyperbolic system if the Graffi's condition (see [17])

(5.9)
$$\mu + 2\mu_H H_2^2 \geq 0,$$

as well as the relation

(5.10)
$$\chi^2 + 4\rho\tau c_v \chi_1 \geq 0$$

hold true.

However, the constraint (5.5) imposes, in the simplified case we are analyzing, that the first-order derivatives of E_1 do not suffer a jump across the characteristic curves. Therefore, our model is compatible with the propagation of two electromagnetic waves travelling through the conductor in opposite directions with velocities λ_1 and λ_2, together with two thermo-electromagnetic waves, due to the presence of the internal variable, travelling through the conductor in opposite directions with velocities λ_3 and λ_4.

It is worth noticing the circumstance that across the wave fronts associated with the velocities λ_3 and λ_4, the only field components suffering a jump in their first order derivatives are θ and a_1. Also, in the fully developed superconductive state, the condition $\mu = 0$ renders the wave velocities λ_1 and λ_2 infinite and the Maxwell's governing system becomes parabolic.

6. Concluding remarks

In the present paper a phenomenological model of superconductivity in solids has been developed within the framework of thermodynamics with internal state variables. This model is compatible with both the Maxwell equations and the second law of thermodynamics. It differs from other theories of superconductivity with internal variables because the internal variable used assumes only real values. On the contrary, different authors recently introduced certain complex internal variables to account for quantum effects at a macroscopic level [7]. Thence, a comparison between these theories and the present one is difficult to realize. Let us observe that in our picture the classical LONDON'S model [5] is recovered if

$$j\mathbf{a} = n_s e \mathbf{v}_s, \tag{6.1}$$

i.e., if the vectorial internal variable $\mathbf{a}$ is related to the mean speed of the superelectrons. Moreover, a comparison with a macroscopic model proposed recently by FABRIZIO and co-workers [3, 4] can be performed as well. These authors postulate a non-local evolution equation for $\mathbf{J}_s$ having the form

$$\nabla \times (\Lambda_1 \mathbf{J}_s) + \Lambda_2 \dot{\mathbf{J}}_s = -\mu \mathbf{H}, \tag{6.2}$$

with Λ_1 and Λ_2 suitable material parameters. If we substitute the constitutive equation $(2.6)_2$ into (6.2) we get a nonlocal evolution equation for $\mathbf{a}$. Hence the two models result to be rather different, even if the nonlocal evolution equation may reduce to a simplified form of (2.1) under very particular hypotheses on Λ_1 and Λ_2.

Acknowledgments

The work has been supported by the "Istituto Nazionale di Alta Matematica" through "Gruppo Nazionale per la Fisica Matematica". Grants of the Departments of Mathematics of the University of Basilicata and of the University of Messina are acknowledged as well.

The authors would like to thank Prof. W. Kosiński, from Warsaw, for his valuable suggestions.

References

1. A.D.C. GRASSIE, *The Superconducting state*, Sussex University Press, London 1975.
2. H. THOMAS, *Some remarks in the history of superconductivity.* [in:] Earlier and Recent Aspects of Superconductivity, J. G. BEDNORZ and K. A. MULLER [Eds]., Springer-Verlag, 2–44, 1990.
3. M. FABRIZIO, C. GIORGI, *A nonlocal phenomenological theory in superconductivity*, Riv. Mat. Univ. Parma, **4**, 415–430, 1984.
4. M. FABRIZIO, G. GENTILI, B. LAZZARI, *A nonlocal thermodynamic theory of superconductivity*, Math. Mod. Meth. Appl. Sci., **7**, 345–362, 1997.
5. F. LONDON, H. LONDON, *The electromagnetic equations of the supraconductor*, Proc. Roy. Soc. London, **A149**, 72–87, 1935.

6. G. A. Maugin, *Irreversible thermodynamics of deformable superconductors.* C. R. Acad. Sci. Paris, **314**, 889–894, 1992.

7. C. S. Yeh, K. C. Chen, *A phenomenological theory for elastic superconductors*, Continuum Mech. Termodyn., **5**, 127–144, 1993.

8. H. Kamerlingh Onnes, *Report on the researches made in the Leiden cryogenic laboratory between the second and third international congress of refrigeration*, Commun. Kamerlingh Onnes Lab. Univ. Leiden, Suppl. **34B**, 55–70, 1913.

9. G.A. Maugin, W. Muschik, *Thermodynamics with internal variables, Part I. General concepts*, J. Non-Equilib. Thermodyn., **19**, 217–249, 1994.

10. J. Verhás, *Thermodynamics and rheology*, Kluver Academic Publishers, Dordrecht 1997.

11. W. Muschik, C. Papenfuss, H. Ehrentraut, *A sketch of continuum thermodynamics*, J. Non-Newtonian Fluid Mech., **96**, 255–290, 2001.

12. J. Bardeen, *Theory of superconductivity*, Handbuch der Physik, **XV**, S. Flugge [Ed.], Springer-Verlag, 274–369, 1956.

13. B. D. Coleman, M. E. Gurtin, *Thermodynamics with internal state variables*, J. Chem. Phys., **47**, 597–613, 1967.

14. C. Cattaneo, *Sulla conduzione del calore*, Atti Sem. Mat. Fis. Univ. Modena, **3**, 83–101, 1948.

15. V. A. Cimmelli, F. Oliveri, *A diffusive-hyperbolic model for heat conduction*, Mathl. Comput. Modelling, **39**, 11–12, 1413–1422, 2003.

16. G. Boillat, *La propagation des ondes*, Gauthier-Villars, Paris 1965.

17. A. Donato, T. Ruggeri, *Onde di discontinuità e condizioni di eccezionalità per materiali ferromagnetici*, Acc. Naz. Lincei, Rend. Cl. Sc. Fis. Mat. Nat., Ser. VIII, **LIII**, 208–214, 1972.

Received August 4, 2003; revised version February 9, 2004.