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Electro-elastic fields of a plane thermal inclusion in isotropic dielectrics with polarization gradient

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Coupled electro-elastic fields excited by a thermal inclusion in an isotropic dielectric medium with polarization gradient are studied in the framework of the Mindlin theory. An inclusion represents a plane heated layer differing from the outside medium only by its temperature. A series of problems are considered: an inclusion in an infinite medium, at a surface of a half-infinite body, in the middle or at the surface of a plate with free or clamped faces. The found Mindlin's corrections to the fields of mechanical displacements, strains and stresses are expectably small. However, the arising accompanied electric fields localized in the narrow zones at surfaces and interfaces can have rather high amplitudes.

1. Introduction

The classical piezoeffects in their continual description do not represent a complete list of all possible electro-elastic couplings. It is evident that traditional continual theories must inevitably miss any effects arising from the discreteness of a crystal lattice. In particular, in the continual theory of elasticity a pre-surface material has the same properties as a material in the bulk and therefore there are no specific surface effects including a surface tension. The more so, this theory cannot describe the degrees of freedom related to intermolecular motions of the unit cell and therefore in this framework there are no optical phonons. Meanwhile such intermolecular motions can lead to electro-elastic coupling even in centro-symmetric materials. For example, in ionic crystals, like NaCl, where the unit cell represents a dipole (Na^+–Cl^-), the intermolecular displacements must be accompanied by a change of the corresponding electric polarization. It was Mindlin [1, 2]), who first found the way, how to describe continuously the above very specific coupling effects. He has extended the classical Voigt theory of piezoelectricity in Toupin's formulation [3] by supposing that the stored energy function depends not only on the strain tensor and polarization vector but also on the polarization gradient tensor.

As a result, he came to a new type of electro-elastic coupling, which may exist even in centro-symmetric crystals. The nonlinear version of this theory was developed by Suhubi [4]. Of course, such sort of coupling can manifest itself in rather specific phenomena related either to very high frequencies (optical phonons) or to a space scale comparable with a lattice parameter. And indeed, the Mindlin theory of elastic dielectrics with polarization gradient accommodates the observed and experimentally measured phenomena, such as electromechanical interaction in centro-symmetric materials, capacitance of thin dielectric films, surface energy of polarization, deformation and optical activity in quartz.

A linear theory of thermoelastic media with polarization gradient was developed by Chowdhury and Glockner [5] while a complete nonlinear theory for such materials was given by Chowdhury, Epstein and Glockner [6]. The related constitutive equations were derived by Chowdhury and Glockner [5] and by J.P. Nowacki and Glockner [7]. In [7] there was also solved the particular problem of electro-thermo-elastic fields excited in the layer by the heat sources.

The Mindlin formalism and its extensions have the only trouble – the much more complex equations of motion and constitutive relations, which provide quite serious technical difficulties in solving particular problems and especially in finding explicit results. These complexities are seriously simplified with the transition from a general anisotropy to centro-symmetric [1, 2] or isotropic [8] media. However, even after such simplifications the corresponding equations remain rather bulky and the number of explicitly solved problems is not large. The first examples of such solutions were given by Mindlin himself for cubic crystals: the description of surface tension [1], of a bulk acoustic wave dispersion [2], of coupled fields of a point charge [9], of electromagnetic radiation from a vibrating elastic sphere [10]. Later some of these results have been extended by Chowdhury and Glockner, who found coupled fields of a point charge in a half-space [11] and presented the theory of surface tension for a plate, and cylindrical or spherical shells [12]. It is worthwhile also to mention in this context the theories by Schwartz [13], Askar *et al.* [14] and Gou [15], who earlier considered the influence of surface curvature on the surface energy and other induced fields of cylindrical and spherical cavities. The other solution in this field has been obtained by Nowacki and Trimarco [16], who found the electro-elastic fields induced in an infinite medium by a cylindrical thermal inclusion.

Below, as a continuation of [16], we present the study of 1D coupled fields excited by a heated plane layer in a series of isotropic media: in an infinite dielectric space, at a surface of a semi-infinite medium, and in a middle or at a surface of a plate with free or clamped faces.

2. Equilibrium equations and constitutive relations

In this paper we shall study electro-elastic fields induced by the prescribed temperature distributions in elastic dielectrics with polarization gradient. The Mindlin theory [1, 2] provides for such sort of problems the following system of equilibrium equations:

$$\operatorname{div} \boldsymbol{\sigma} = 0, \tag{2.1}$$

$$\operatorname{div} (\varepsilon_0 \mathbf{E} + \mathbf{P}) = 0, \tag{2.2}$$

$$\operatorname{div} \boldsymbol{\epsilon} + \mathbf{E} + \mathbf{E}^L = 0. \tag{2.3}$$

Here $\boldsymbol{\sigma}$ is the mechanical stress tensor, $\mathbf{E}$ is the electric field, $\mathbf{P}$ is the electric polarization, ε_0 is the permittivity of vacuum, $\mathbf{E}^L$ and $\boldsymbol{\epsilon}$ are the specific quantities of Mindlin's theory: the local electric force vector and the electric tensor, respectively.

The corresponding constitutive relations [5, 7] for a particular case of isotropic media have the form

$$\sigma_{ij} = c_{12} u_{k,k} \delta_{ij} + c_{44}(u_{i,j} + u_{j,i}) + d_{12} P_{k,k} \delta_{ij} + d_{44}(P_{i,j} + P_{j,i}) - \gamma \delta_{ij} \theta, \tag{2.4}$$

$$E_i^L = a P_i, \tag{2.5}$$

$$\begin{aligned} \epsilon_{ij} = d_{12} u_{k,k} \delta_{ij} + d_{44}(u_{i,j} + u_{j,i}) + b_{12} P_{k,k} \delta_{ij} + b_{44}(P_{i,j} + P_{j,i}) \\ + b_{77}(P_{j,i} - P_{i,j}) - \eta \delta_{ij} \theta + b_0 \delta_{ij}, \end{aligned} \tag{2.6}$$

where $\mathbf{u}$ is the displacement vector, $\theta = T - T_0$ is the temperature measured from the reference level T_0, and coefficients $c_{ij}, d_{ij}, b_{ij}, b_0, a, \gamma, \eta$ are material constants.

Below we shall consider the fields $\mathbf{u}(\mathbf{r})$, $\mathbf{P}(\mathbf{r})$, $\mathbf{E}(\mathbf{r})$ excited in the infinite or semi-infinite media and in the infinite plate by the inhomogeneous step-like temperature field $\theta(\mathbf{r})$ arising due to occurrence in the medium of the heated layer parallel to the surface when it exists. Owing to the symmetry of the problem, we shall assume the above vector fields to have the only non-vanishing y-components dependent only on the y-coordinate orthogonal to the layer. Thus, if $\mathbf{n}$ is the unit normal to the layer, the unknown vector fields are looked for in the form

$$\mathbf{u} = \mathbf{n}\; u(y), \qquad \mathbf{P} = \mathbf{n}\; P(y), \qquad \mathbf{E} = \mathbf{n}\; E(y). \tag{2.7}$$

In these terms the combination of Eqs. (2.1)–(2.6) leads to the system

$$c \frac{d^2 u}{dy^2} + d \frac{d^2 P}{dy^2} = \gamma \frac{d\theta}{dy}, \tag{2.8}$$

$$d\frac{d^2u}{dy^2} + b\frac{d^2P}{dy^2} - aP + E = \eta\frac{d\theta}{dy}, \tag{2.9}$$

$$\frac{d}{dy}(\varepsilon_0 E + P) = 0, \tag{2.10}$$

where the notation is introduced

$$c = c_{12} + 2c_{44}, \qquad d = d_{12} + 2d_{44}, \qquad b = b_{12} + 2b_{44}. \tag{2.11}$$

Eq. (2.10) at any reasonable boundary conditions is trivially integrated providing the relation of the vanishing electric displacement

$$D \equiv \varepsilon_0 E + P = 0. \tag{2.12}$$

With (2.12) Eq. (2.9) simplifies to

$$d\frac{d^2u}{dy^2} + b\frac{d^2P}{dy^2} - \hat{a}P = \eta\frac{d\theta}{dy}, \tag{2.13}$$

where

$$\hat{a} = a + \varepsilon_0^{-1}. \tag{2.14}$$

Equations (2.8), (2.13) form a closed system of equations, which will be solved below for a series of particular boundary problems.

3. Plane thermal inclusion in unbounded dielectric medium

3.1. Statement of the problem

We consider an infinite plane layer of the thickness $d = 2h$ (Fig. 1), part of an isotropic elastic dielectric medium with the same material constants inside and outside the layer. The layer and the outside space are held at constant, but different temperatures. More precisely, we assume the temperature θ to be equal to θ_0 inside the layer and zero outside it. Such discontinuous temperature to be field to must excite in the medium thermoelastic deformations which, due to the supposed presence of the polarization gradient, should be accompanied by coupled electro-elastic fields.

In the one-dimensional coordinate system with the origin $y = 0$ chosen in the middle of the layer, the temperature distribution is given by

$$\theta(y) = \theta_0[\mathrm{H}(y+h) \ - \mathrm{H}(y-h)] = \theta_0 \begin{cases} 0, & y \geq h; \\ 1, & -h \leq y \leq h; \\ 0, & y \leq -h. \end{cases} \tag{3.1}$$

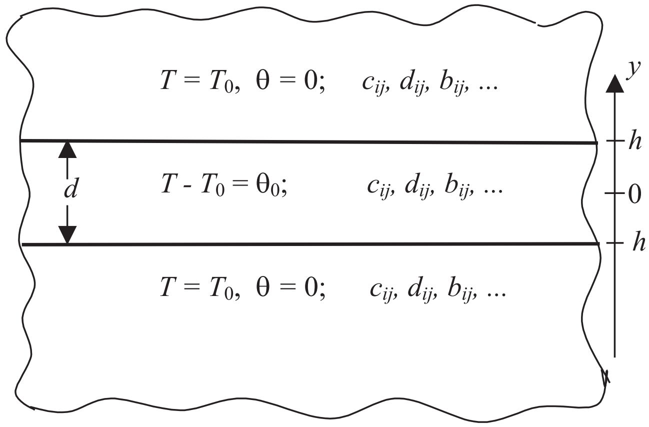

Fig. 1. Infinite medium with a plane thermal inclusion.

H(y) in (3.1) denotes the Heaviside step-function. The solutions for $u(y)$, $P(y)$ and $E(y)$ must be finite at both infinities $y \to \pm\infty$. More precisely, we demand

$$u|_{\pm\infty} \to \pm\text{const}, \qquad P|_{\pm\infty} \to 0, \qquad E|_{\pm\infty} \to 0. \tag{3.2}$$

At both interfaces $y = \pm h$ all three fields must be continuous

$$[u]_{\pm h} = 0, \qquad [P]_{\pm h} = 0, \qquad [E]_{\pm h} = 0, \tag{3.3}$$

together with the normal projections of mechanical stresses $\boldsymbol{\sigma}\mathbf{n} = [0,\ \boldsymbol{\sigma}(y),\ 0]$, electric displacements $\mathbf{D}{\cdot}\mathbf{n}{=}D(y)$ and electric tensor $\boldsymbol{\epsilon}\mathbf{n} = [0,\ \epsilon(y),\ 0]$,

$$[\sigma]_{\pm h} = 0, \qquad [D]_{\pm h} = 0, \qquad [\epsilon]_{\pm h} = 0. \tag{3.4}$$

It is clear that, due to the symmetry of the problem, the introduced scalar functions $\sigma(y){=}\ \mathbf{n}{\cdot}\boldsymbol{\sigma}\mathbf{n}$ and $\epsilon(y){=}\ \mathbf{n}{\cdot}\boldsymbol{\epsilon}\mathbf{n}$ are the only non vanishing components of the vectors $\boldsymbol{\sigma}\mathbf{n}$ and $\boldsymbol{\varepsilon}\mathbf{n}$. In fact, as we shall see, for a given problem the unknown functions $u(y)$, $P(y)$ and $E(y)$ are completely determined by the conditions at infinities (3.2). The above continuities at the interfaces, Eqs. (3.3) and (3.4), will be fulfilled automatically.

3.2. Equilibrium equations and their solutions

Thus, in the considered case the system of equilibrium equations (2.8), (2.13) reduces to

$$c\frac{d^2u}{dy^2} + d\frac{d^2P}{dy^2} = \gamma\theta_0[\delta(y+h) - \delta(y-h)], \tag{3.5}$$

$$d\frac{d^2u}{dy^2} + b\frac{d^2P}{dy^2} - \hat{a}P = \eta\theta_0[\delta(y+h) - \delta(y-h)], \tag{3.6}$$

where $\delta(y)$ is the Dirac delta function. Excluding the derivative $u_{,,yy}$ from the above equilibrium equations we obtain the following differential equation with respect to $P(y)$:

$$\frac{d^2P}{dy^2} - \lambda^2 P = K\theta_0[\delta(y+h) - \delta(y-h)]. \tag{3.7}$$

Here

$$\lambda = \sqrt{\frac{c\hat{a}}{cb-d^2}}, \qquad K = \frac{c\eta - d\gamma}{cb-d^2} = \lambda^2(\alpha_2 - \alpha_1)d/\hat{a}, \tag{3.8}$$

and we have introduced two coefficients of linear expansion α_1 and α_2,

$$\alpha_1 = \gamma/c, \qquad \alpha_2 = \eta/d. \tag{3.9}$$

As it was shown by Mindlin [2], the parameter λ (3.8) is real, which is a consequence of the positive definiteness of the energy. Now it is time to mention that the Mindlin theory does not contain any small dimensionless parameters. Instead it introduces the new moduli b, d, b_0 and η describing a space dispersion of the medium. Three of them (d, b_0 and η) must be linear in some length parameter l, which is expected to be of the order of the lattice discreteness measure, and the fourth modulus (b) – even quadratic in l. Accordingly, we shall assume that $\lambda^{-1} \sim l$. It is clear that in the limit $l \to 0$ all the above moduli (and λ^{-1}) are vanishing and we return to the ordinary formulation of the Voigt–Toupin theory of a dispersionless piezoelectricity.

The solution of Eq. (3.7) satisfying the conditions at infinities $(3.2)_2$ can be easily found:

$$P(y) = \frac{K\theta_0}{\lambda}\Big[e^{-\lambda h}\text{sh}\lambda y - \text{sh}\lambda(y+h)H(-y-h) - \text{sh}\lambda(y-h)H(y-h)\Big] = \frac{K\theta_0}{\lambda}\begin{cases} e^{-\lambda y}\text{sh}\lambda h, & y \ge h; \\ e^{-\lambda h}\text{sh}\lambda y, & -h \le y \le h; \\ -e^{\lambda y}\text{sh}\lambda h, & y \le -h. \end{cases} \tag{3.10}$$

The function $P(y)$ (3.10) is odd, $P(-y) = -P(y)$, which reflects the symmetry of the source in Eq. (3.7).

Now, let us substitute into Eq. (3.5) the derivative $P_{,yy}$ expressed from (3.7) with the function $P(y)$ defined by (3.10). This leads to the equation for the displacement field $u(y)$:

$$\frac{d^2u}{dy^2} = K_1\theta_0[\delta(y+h) - \delta(y-h)] - \lambda_1^2 P, \tag{3.11}$$

where

$$K_1 = \frac{b\gamma - d\eta}{cb - d^2}, \qquad \lambda_1^2 = \frac{d}{c}\lambda_1^2 = \frac{d\hat{a}}{cb - d^2}. \tag{3.12}$$

A displacement field $u(y)$ is defined up to an arbitrary constant. It is convenient to choose this constant so that $u(y)$ would be anti-symmetric as $P(y)$. This was the reason for a choice of the conditions at infinities in the form of Eq. $(3.2)_1$. The corresponding solution of Eq. (3.11) fitting $(3.2)_1$ is given by

$$u(y) = \alpha_1\theta_0 \begin{cases} [h - (k/\lambda)e^{-\lambda y}\mathrm{sh}\lambda h], & y \geq h; \\ [y - (k/\lambda)e^{-\lambda h}\mathrm{sh}\lambda y], & -h \leq y \leq h; \\ [-h + (k/\lambda)e^{\lambda y}\mathrm{sh}\lambda h], & y \leq -h. \end{cases} \tag{3.13}$$

Here k is a dimensionless parameter of the order of unity defined by

$$k = \frac{Kd}{\gamma} = \frac{\alpha_2/\alpha_1 - 1}{bc/d^2 - 1}. \tag{3.14}$$

It is easily checked that the found fields $u(y)$ and $P(y)$ are continuous at $y = \pm h$, which together with (2.12) fits all the three conditions (6.3). On the other hand, their derivatives defining the only non-vanishing components of distortions ($\beta = u_{,y}$) and polarization gradient ($\pi = P_{,y}$) are discontinuous:

$$\beta(y) = \frac{du}{dy} = \alpha_1\theta_0 \begin{cases} ke^{-\lambda y}\mathrm{sh}\lambda h, & y > h; \\ [1 - ke^{-\lambda h}\mathrm{ch}\lambda y], & -h < y < h; \\ ke^{\lambda y}\mathrm{sh}\lambda h, & y < -h; \end{cases} \tag{3.15}$$

$$\pi(y) = \frac{dP}{dy} = K\theta_0 \begin{cases} -e^{-\lambda y}\mathrm{sh}\lambda h, & y > h; \\ e^{-\lambda h}\mathrm{ch}\lambda y, & -h < y < h; \\ -e^{\lambda y}\mathrm{sh}\lambda h, & y < -h. \end{cases} \tag{3.16}$$

Fortunately, such behavior at the interfaces $y = \pm h$ of distortions and polarization gradient does not contradict any physical conditions. In accordance with Eq. (3.4) the functions, which cannot have any discontinuities, are the normal projections of electric displacements, $D(y)$, mechanical stresses, $\sigma(y)$, and electric tensor, $\varepsilon(y)$. As we have seen, the first one is continuous due to the identity (2.12).

However, the other two must be verified. Basing on Eqs. (2.4), (2.6) one has

$$\sigma(y) = c\beta(y) + d\pi(y) - \gamma\theta(y), \tag{3.17}$$

$$\boldsymbol{\epsilon}(y) = d\beta(y) + b\pi(y) - \eta\theta(y) + b_0. \tag{3.18}$$

Substituting here Eqs. (3.1), (3.15) and (3.16) we obtain

$$\sigma(y) = 0, \tag{3.19}$$

$$\boldsymbol{\epsilon}(y) = b_0 + d(\alpha_1 - \alpha_2)\theta_0 \begin{cases} -\lambda y \mathrm{sh}\lambda h, & y \geq h; \\ [1 - e^{-\lambda h}\mathrm{ch}\lambda y], & -h \leq y \leq h; \\ e^{\lambda y}\mathrm{sh}\lambda h, & y \leq -h. \end{cases} \tag{3.20}$$

An identically vanishing stress (3.19) is a rather expectable result for the problem without other sources apart from the temperature distribution (3.1). Anyway, this provides a trivial continuity of $\sigma(y)$. One can check that the function $\boldsymbol{\epsilon}(y)$ is also continuous at $y = \pm h$. Thus, the found solutions are mathematically correct.

3.3. Comments on the results

The considered problem in a classical statement, i.e. for a medium without Mindlin's couplings, has rather trivial solutions. Of course, there should be no electric fields, i.e. $E = 0$, $P = 0$, $D = 0$. Mechanical displacements outside the layer must be homogeneous and distortions must be distinct from zero only inside the layer:

$$u(y) = \alpha_1\theta_0 \begin{cases} h, & y \geq h; \\ y, & -h \leq y \leq h; \\ -h & y \leq -h; \end{cases} \qquad \beta(y) = \alpha_1\theta_0 \begin{cases} 0, & y > h; \\ 1, & -h < y < h; \\ 0, & y < -h. \end{cases} \tag{3.21}$$

And the corresponding mechanical stress must vanish everywhere:

$$\sigma(y) = c\beta(y) - \gamma\theta(y) = 0, \tag{3.22}$$

which is evident both physically and mathematically from (3.1), $(3.21)_2$ and $(3.9)_1$. As we have seen, vanishing $D(y)$ and $\sigma(y)$ fields remain even in Mindlin's medium. However, the electric field with account for Mindlin's couplings does not vanish any more,

$$E(y) = -\frac{K\Theta_0}{\varepsilon_0\lambda}\begin{cases} e^{-\lambda y}\,\mathrm{sh}\lambda h, & y \geq h; \\ e^{-\lambda h}\,\mathrm{sh}\lambda h, & -h \leq y \leq h; \\ -e^{\lambda y}\,\mathrm{sh}\lambda h, & y \leq -h; \end{cases} \tag{3.23}$$

The behavior of the fields $u(y)$ (3.13) and $\beta(y)$ (3.15) replacing the classical distributions (3.21) acquires new features. For instance, the medium outside the layer ($|y|>h$) is now not free of distortions: starting at the interfaces from the magnitudes comparable with the classical level $\alpha_1\theta_0$ (see (3.21)), the function $\beta(y)$ quickly decreases proportionally to $\exp(-\lambda|y\text{–}h|)$. The same factor characterizes the electric field (3.23) close to the interfaces. Taking into account that λ^{-1} is of the order of several lattice parameters, the obtained corrections are not negligible only in very narrow zones and could hardly be important for the strain fields. However, the electric field arises on the background of zero with the amplitude at $y \approx \pm h$, which can be rather high, as will be shown. Such field might create non-trivial physical consequences even in microscopically narrow regions. We shall return to this aspect of the problem at the end of the paper in the Conclusions.

4. Thermal inclusion at the surface of a half-infinite dielectric

4.1. Equations and boundary conditions

Let us now consider the analogous but less symmetric problem: the semi-infinite dielectric medium with a heated layer of the thickness h at the surface (Fig. 2). The temperature distribution in this case is determined by

$$\theta(y) = \theta_0\mathrm{H}(h-y) = \theta_0\begin{cases} 0, & y \geq h; \\ 1, & 0 \leq y \leq h. \end{cases} \tag{4.1}$$

The equations of equilibrium are very similar to (3.5)–(3.7):

$$c\frac{d^2u}{dy^2} + d\frac{d^2P}{dy^2} = -\gamma\theta_0\delta(y-h), \tag{4.2}$$

$$d\frac{d^2u}{dy^2} + b\frac{d^2P}{dy^2} - \hat{a}P = -\eta\theta_0\delta(y-h), \tag{4.3}$$

$$\frac{d}{dy}(\varepsilon_0E + P) = 0. \tag{4.4}$$

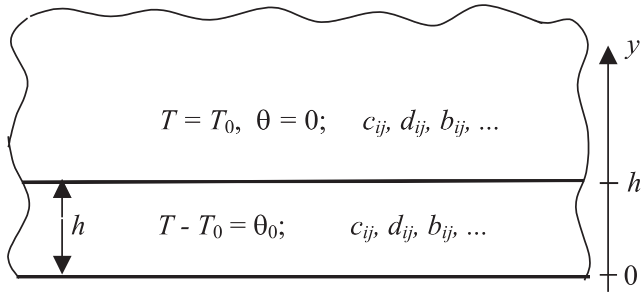

FIG. 2. Semi-infinite medium with a thermal inclusion at the surface.

Choosing the boundary conditions at the surface $y = 0$, we shall suppose that the surface is free of any sources. Mechanically, this means the vanishing tractions,

$$\sigma(0) = 0. \tag{4.5}$$

In fact, it is rather evident that in the given case for the same reasons as before, the thermoelastic stress $\sigma(y)$ must vanish. So, the condition (4.5) will hardly help us in solving the problem. On the other hand, in the absence of electrical sources at the surface, the problem becomes also insensitive to the ordinary electric conditions at $y = 0$. Indeed, for a considered one-dimensional problem, the surface is automatically equipotential and could be supposed to have a zero electric potential $\varphi(0) = 0$, which would be satisfied identically, without providing any additional condition for unknown functions. The same situation occurs for the condition of continuity of the normal components of electric displacements in the medium ($D = P + \varepsilon_0 E$) and in the adjoined vacuum (D_v), because both identically vanish at any y. In fact, the only non-trivial electric condition at the surface is provided by the standard Mindlin's restriction for the electric tensor,

$$\boldsymbol{\epsilon}(0) = 0. \tag{4.6}$$

By (3.18) this is equivalent to the equation

$$d\frac{du}{dy}\bigg|_{y=0} + b\frac{dP}{dy}\bigg|_{y=0} = \eta\theta_0 - b_0. \tag{4.7}$$

The conditions at infinity will be also very essential. They are similar to (3.2),

$$u|\infty \to 0, \qquad P|_\infty \to 0, \qquad E|_\infty \to 0. \tag{4.8}$$

At the interface we expect, as before, to obtain automatically the continuity

$$[u]_h = 0, \qquad [P]_h = 0, \qquad [E]_h = 0. \tag{4.9}$$

With the above conditions at infinity the solution of Eq. (4.4) again reduces to the identity (2.12), which indeed makes the requirement of continuity $D(0) = D_v(0)$ to be trivial: $0 = 0$. Combining as before the other two equations of the system, Eqs. (4.2) and (4.3), we arrive at transformed Eqs. (3.7) and (3.11),

$$\frac{d^2P}{dy^2} - \lambda^2 P = -K\theta_0\delta(y-h), \tag{4.10}$$

$$\frac{d^2u}{dy^2} = -K_1\theta_0\delta(y-h) - \lambda_1^2 P. \tag{4.11}$$

Here the parameters λ, K, λ_1 and K_1 are defined by the same Eqs. (3.8) and (3.12). In contrast to the previous consideration, where the function $P(y)$ was found independently of $u(y)$, now the above system, Eqs. (4.10), (4.11), cannot be solved in succession, because the boundary condition (4.7) combines both the unknown functions.

4.2. Solutions and analysis

The general solution of Eq.(4.10) is

$$P(y) = Ae^{\lambda y} + Be^{-\lambda y} - \frac{K\theta_0}{\lambda}\mathrm{sh}[\lambda(y-h)]\mathrm{H}(y-h), \tag{4.12}$$

where A and B are constants supposed to be found from the boundary conditions. One of these constants (A) can be found from Eq. $(4.8)_2$:

$$A = \frac{K\theta_0}{2\lambda}e^{-\lambda h}. \tag{4.13}$$

After integrating Eq. (4.11) and differentiating (4.12), it is easy to find

$$\frac{du}{dy} = \frac{\lambda_1^2}{\lambda}\left[Be^{-\lambda y} + \frac{K\theta_0}{2\lambda}e^{-\lambda|y-h|}\mathrm{sgn}(y-h)\right] + \frac{\gamma\theta_0}{c_{11}}\mathrm{H}(h-y), \tag{4.14}$$

$$\frac{dP}{dy} = -\lambda\left[Be^{-\lambda y} + \frac{K\theta_0}{2\lambda}e^{-\lambda|y-h|}\mathrm{sgn}(y-h)\right], \tag{4.15}$$

where $\mathrm{sgn}(y-h)$ means a sign of the argument. At the surface $y = 0$, Eqs. (4.13)–(4.15) give

$$\left.\frac{du}{dy}\right|_{y=0} = \frac{\gamma\theta_0}{c_{11}} - \frac{\lambda_1^2}{\lambda}(A-B), \quad \left.\frac{dP}{dy}\right|_{y=0} = \lambda(A-B). \tag{4.16}$$

By (4.16) the boundary condition (4.7) reduces to an equation with respect to $A - B$. The solution is

$$B = A - \frac{K\theta_0}{\lambda} + \frac{\lambda b_0}{\hat{a}}. \tag{4.17}$$

Knowing the derivatives (4.14), (4.15) and temperature distribution (4.1), we can calculate the stress $\sigma(y)$ (3.17) and the electric tensor $\boldsymbol{\epsilon}(y)$ (3.18):

$$\sigma(y) = 0, \tag{4.18}$$

$$\begin{aligned}\boldsymbol{\epsilon}(y) = b_0\left(1 - e^{-\lambda y}\right) + d_{11}(\alpha_2 - \alpha_1)\theta_0 \\ \cdot \left\{e^{-\lambda y} + \frac{1}{2}\left[e^{-\lambda|y-h|} - e^{-\lambda(y+h)}\right] - e^{-\lambda(y-h)}\mathrm{H}(y-h) - \mathrm{H}(h-y)\right\}.\end{aligned} \tag{4.19}$$

As it was expected, the body is free of stresses. On the other hand, the function $\boldsymbol{\epsilon}(y)$ vanishes only at the surface, in accordance with (4.6), being continuous at $y = h$.

Now let us return to the fieldsu(y) and $P(y)$. After the substitution of the found constants A and B, (4.13) and (4.17), into Eqs. (4.12), (4.14) and further integration of Eq. (4.14), with taking into account condition $(4.8)_1$ we arrive at the final solutions for $u(y)$ and $P(y)$:

$$\begin{aligned}u(y) = -\alpha_1\theta_0(h-y)\mathrm{H}(h-y) - \frac{\lambda d}{\hat{a}c}\left[b_0 - d(\alpha_2 - \alpha_1)\theta_0\right]e^{-\lambda y} \\ - \frac{\lambda d^2(\alpha_2 - \alpha_1)\theta_0}{\hat{a}c}\begin{cases} e^{-\lambda y}\mathrm{ch}\lambda h, & y \geq h; \\ e^{-\lambda h}\mathrm{ch}\lambda y, & y \leq h; \end{cases}\end{aligned} \tag{4.20}$$

$$\begin{aligned}P(y) = \frac{\lambda}{\hat{a}}\left\{\left[b_0 - d(\alpha_2 - \alpha_1)\theta_0\left(1 - \frac{1}{2}e^{-\lambda h}\right)\right]e^{-\lambda y} \right.\\ \left. + \frac{1}{2}d(\alpha_2 - \alpha_1)\theta_0 e^{-\lambda|y-h|}\right\}.\end{aligned} \tag{4.21}$$

In view of the proved identity (2.12), the corresponding electric field distribution $E(y) = -\varepsilon_0^{-1}P(y)$ again might be essential only in very narrow zones, this time close to the surface $(y = 0)$ and to the interface $(y = h)$:

$$\begin{aligned}E\Big|_{y\approx 0} &\approx -\frac{\lambda}{\varepsilon_0\hat{a}}\Big[b_0 - d(\alpha_2 - \alpha_1)\theta_0\Big]e^{-\lambda y}, \\ E\Big|_{y\approx h} &\approx -\frac{\lambda d(\alpha_2 - \alpha_1)\theta_0}{2\varepsilon_0\hat{a}}e^{-\lambda|y-h|}.\end{aligned} \tag{4.22}$$

It is natural that the magnitude of this field at the interface is determined by the same physical parameters as in the above infinite medium and could be rather high (see the Conclusions). On the other hand, at the surface we have additional source for electric effects – a surface tension [1], determined by the Mindlin modulus b_0. That is why the first expression in (4.22) for $E|_{y=0}$ has more complex structure than the formula for $E|_{y=h}$.

5. Plane thermal inclusion in the middle of a dielectric plate

5.1. A plate with faces free of tension

Now we consider the dielectric plate, shown in Fig. 3, with a symmetric position of the thermal inclusion in the middle of the plate, which represents some extension of the problem solved in Sec. 3.1. Indeed, the temperature distribution (3.1) and the corresponding equilibrium equations (3.5), (3.6) remain valid for the considered case of a plate. The continuity conditions (3.3), (3.4) at the interfaces $y = \pm h$ also can serve as good criteria of correct calculations. However, the boundary conditions should be formulated from the beginning. For the same reason as in Sec.4, again the only non-trivial electric boundary requirement is given by the Mindlin condition of vanishing of the normal components of the electric tensor $\boldsymbol{\epsilon}$ at the surfaces. In our case this means

$$\boldsymbol{\epsilon}(\pm H) = 0. \tag{5.1}$$

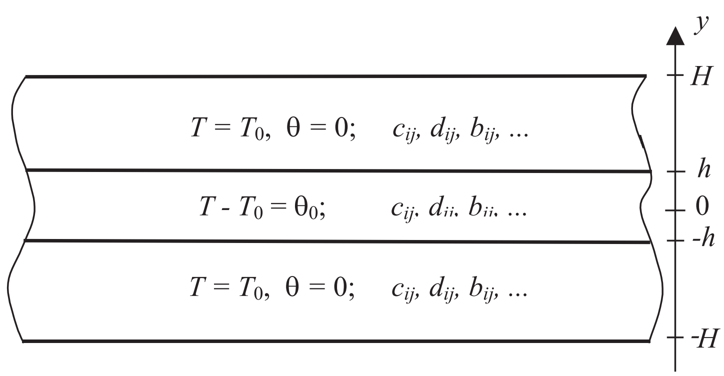

FIG. 3. Dielectric plate with a thermal inclusion in the middle.

In this section, for mechanical boundary conditions we choose the free surfaces,

$$\sigma(\pm H = 0. \tag{5.2}$$

It is obvious that Eqs. (3.7) and (3.11) retain their form for the plate. We are looking for a solution of Eq. (3.7) in the form

$$P(y) = \mathrm{Ash}\lambda y - \frac{K\theta_0}{\lambda}[\mathrm{sh}\lambda(y+h)\mathrm{H}(-y-h) + \mathrm{sh}\lambda(y-h)\mathrm{H}(y-h)], \tag{5.3}$$

where A is the unknown constant. The corresponding polarization gradient is equal to

$$\frac{dP}{dy} = A\lambda \mathrm{ch}\lambda y - K\theta_0[\mathrm{ch}\lambda(y+h)\mathrm{H}(-y-h) + \mathrm{ch}\lambda(y-h)\mathrm{H}(y-h)]. \tag{5.4}$$

The integration of Eq. (3.11) gives the distortion

$$\begin{aligned}\frac{du}{dy} = B - A\lambda\frac{d}{c}[\mathrm{ch}\lambda y - 1] - \frac{\gamma\theta_0}{c}[\mathrm{H}(-y-h) + \mathrm{H}(y-h)] \\ + \frac{d}{c}K\theta_0[\mathrm{ch}\lambda(y+h)\mathrm{H}(-y-h) + \mathrm{ch}\lambda(y-h)\mathrm{H}(y-h)]\end{aligned} \tag{5.5}$$

with the additional unknown constant B. Both the functions (5.4) and (5.5) are even and take equal values at the surfaces $y = \pm H$,

$$\left.\frac{du}{dy}\right|_{\pm H} = B - A\lambda\frac{d}{c}\Big[\mathrm{ch}\lambda H - 1\Big] - \frac{\gamma\theta_0}{c_{11}} + \frac{d}{c}K\theta_0\mathrm{ch}\lambda(H-h), \tag{5.6}$$

$$\left.\frac{dP}{dy}\right|_{\pm H} = A\lambda\mathrm{ch}\lambda H - K\theta_0\mathrm{ch}\lambda(H-h). \tag{5.7}$$

Substitution of (3.1), (5.6) and (5.7) into (3.17), (3.18) with (5.1), (5.2) leads to the linear system of two equations determining A and B:

$$\lambda dA + cB = \gamma\theta_0, \tag{5.8}$$

$$\left(\frac{\hat{a}}{\lambda d}\mathrm{ch}\lambda H + \frac{\lambda d}{c}\right)A + B = \alpha_1\theta_0 + (\alpha_2 - \alpha_1)\theta_0\mathrm{ch}\lambda(H-h) - \frac{b_0}{d}. \tag{5.9}$$

The solution is

$$A = -\frac{\lambda}{\hat{a}}\frac{[b_0 + d(\alpha_1 - \alpha_2)\theta_0\mathrm{ch}\lambda(H-h)]}{\mathrm{ch}\lambda H}, \qquad B = \frac{\gamma\theta_0}{c} - \frac{\lambda d}{c}A. \tag{5.10}$$

With the found values of A and B, Eqs. (5.4), (5.5) and (3.1) allow us to derive the functions $\sigma(y)$ and $\epsilon(y)$, defined by (3.17), (3.18):

$$\sigma(y) = -\gamma\theta_0 + c\left(B + \frac{\lambda d}{c}A\right) = 0, \tag{5.11}$$

$$\begin{aligned}\boldsymbol{\epsilon}(y) = d(\alpha_1 - \alpha_2)\theta_0\{1 - \mathrm{ch}\lambda(H-h) + [\mathrm{ch}\lambda(y+h) - 1] \\ \cdot \mathrm{H}(-y-h) + [\mathrm{ch}\lambda(y-h) - 1]\mathrm{H}(y-h)\} \\ - [b_0 + d(\alpha_1 - \alpha_2)\theta_0\mathrm{ch}\lambda(H-h)](\mathrm{ch}\lambda y/\mathrm{ch}\lambda H - 1).\end{aligned} \tag{5.12}$$

The customary identical zero for the stresses is a direct consequence of the free faces boundary conditions. As to the function $\boldsymbol{\epsilon}(y)$, it is continuous at the interfaces $y = \pm h$, which is the only feature important for us regarding this function.

The distribution of displacements $u(y)$ is obtained by integrating the distortion field (5.5) with A and B taken from (5.10):

$$u(y) = \alpha_1\theta_0[y - (y+h)\mathrm{H}(-y-h) - (y-h)H(y-h)] + \frac{d}{c}\left\{\left(\frac{\lambda b_0}{\hat{a}} - \frac{K\theta_0}{\lambda}\mathrm{ch}\lambda(H-h)\right)\frac{\mathrm{sh}\lambda y}{\mathrm{ch}\lambda H} + \frac{K\theta_0}{\lambda}\Big[\mathrm{sh}\lambda(y+h)\mathrm{H}(-y-h) + \mathrm{sh}\lambda(y-h)\mathrm{H}(y-h)\Big]\right\}. \tag{5.13}$$

This function is chosen to be even and its continuity at $y = \pm h$ is easily verified.

The corresponding continuous distribution of polarization $P(y)$ is found by combining Eqs. (5.3) and $(5.10)_1$:

$$P(y) = -\left\{\left(\frac{\lambda b_0}{\hat{a}} - \frac{K\theta_0}{\lambda}\mathrm{ch}\lambda(H-h)\right)\frac{\mathrm{sh}\lambda y}{\mathrm{ch}\lambda H} + \frac{K\theta_0}{\lambda}\Big[\mathrm{sh}\lambda(y+h)\mathrm{H}(-y-h) + \mathrm{sh}\lambda(y-h)\mathrm{H}(y-h)\Big]\right\}. \tag{5.14}$$

The electric field is again given by the identity (2.12),

$$E(y) = -\varepsilon_0^{-1}P(y). \tag{5.15}$$

As before, both the fields $P(y)$ and $E(y)$ are most interesting characteristics of the studied coupled fields, because close to the surfaces and to the interfaces their magnitudes could be rather high. For instance, the electric field in these narrow zones can be represented in the form

$$E\Big|_{y\approx\pm H} \approx \pm\frac{\lambda b_0}{\varepsilon_0\hat{a}}e^{-\lambda(H-|y|)}, \qquad E\Big|_{y\approx\pm h} \approx \pm\frac{\lambda d(\alpha_1-\alpha_2)\theta_0}{2\varepsilon_0\hat{a}}e^{-\lambda|h-|y||}. \tag{5.16}$$

In contrast to the case of the heated layer at the surface, this time the pre-surface electric field is completely determined by a surface tension (compare expressions $(5.16)_1$ and $(4.22)_1$).

5.2. A plate with clamped surfaces

In this sub-section we shall first meet the situation when thermal expansion is not free, being limited by clamped surfaces. Under such boundary condition one cannot expect any more the vanishing thermoelastic stresses. Thus, we shall solve the same problem, as in Sec. 5.1, with the requirement (5.2) replaced by

$$u(\pm H) = 0. \tag{5.17}$$

Equations (5.3)–(5.7) remain valid, however we should add to them the relations

$$\begin{aligned} u(y) = \left(B + A\lambda\frac{d}{c}\right) y - \alpha_1\theta_0[(y+h)\mathrm{H}(-y-h) + (y-h)\mathrm{H}(y-h)] \\ + \frac{d}{c}\left\{-A\mathrm{sh}\lambda y + \frac{K\theta_0}{\lambda}\Big[\mathrm{sh}\lambda(y+h)\mathrm{H}(-y-h) \right. \\ \left. + \mathrm{sh}\lambda(y-h)\mathrm{H}(y-h)\Big]\right\} \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} u(\pm H) = \pm\left\{\frac{d}{c}(\lambda H - \mathrm{sh}\lambda y)A + HB - \alpha_1\theta_0(H-h) \right. \\ \left. + \frac{d}{c}\frac{K\theta_0}{\lambda}\mathrm{sh}\lambda(H-h)\right\} \end{aligned} \tag{5.19}$$

Thus the system determining A and B is formed by the combination of Eqs. (5.17), (5.19) with (5.9). The solution of the system is given by

$$\begin{aligned} A = \frac{\lambda}{\hat{a}}\left\{\frac{(h/H)d\alpha_1\theta_0 - b_0}{\mathrm{ch}\lambda H + \Delta\mathrm{sh}\lambda H/\lambda H} \right. \\ \left. + \frac{d(\alpha_2-\alpha_1)\theta_0(\mathrm{ch}[\lambda(H-h)] + \Delta\mathrm{sh}[\lambda(H-h)]/\lambda H)}{\mathrm{ch}\lambda H + \Delta\mathrm{sh}\lambda H/\lambda H}\right\}, \end{aligned} \tag{5.20}$$

$$B = \alpha_1\theta_0\left(1 - \frac{h}{H}\right) - \frac{\lambda d}{c}\left(1 - \frac{\mathrm{sh}\lambda H}{\lambda H}\right) A - \frac{\Delta(\alpha_2-\alpha_1)\theta_0}{\lambda H}\mathrm{sh}\lambda(H-h). \tag{5.21}$$

Here the notation is introduced:

$$\Delta = \frac{d^2}{cb - d^2}. \tag{5.22}$$

As we know, the magnitude λ^{-1} is of the order of several parameters of a lattice, therefore for any macroscopic H, h and $H-h$, the parameters $\lambda H, \lambda h$ and $\lambda(H-h)$ must be very large. So we can neglect in the above bulky formulae

(5.20) and (5.21) the terms proportional to $\exp(-2\lambda H)$, $\exp[-2\lambda(H-h)]$ and $\exp(-\lambda H)/\lambda H$. This leads to more compact expressions

$$A \approx \frac{\lambda d}{\hat{a}}\left\{2\left(\alpha_1\theta_0\frac{h}{H}-\frac{b_0}{d}\right)e^{-\lambda H}+(\alpha_2-\alpha_1)\theta_0 e^{-\lambda h}\right\}, \tag{5.23}$$

$$B \approx -\frac{\lambda d}{c}A+\alpha_1\theta_0\left(1-\frac{h}{H}\right)+\frac{\Delta}{2\lambda H}\left(\alpha_1\theta_0\frac{h}{H}-\frac{b_0}{d}\right). \tag{5.24}$$

With (5.23), (5.24), Eq. (5.18) gives

$$\begin{aligned} u(y) \approx \alpha_1\theta_0\left\{\left(1-\frac{h}{H}\right)y-(y+h)\mathrm{H}(-y-h)-(y-h)\mathrm{H}(y-h)]\right\} \\ +\frac{\Delta}{2\lambda H}\left(\alpha_1\theta_0\frac{h}{H}-\frac{b_0}{d}\right)y \\ +\frac{\Delta}{\lambda}\left\{(\alpha_2-\alpha_1)\theta_0\left[\mathrm{sh}\lambda(y+h)\mathrm{H}(-y-h)+\mathrm{sh}\lambda(y-h)\mathrm{H}(y-h)-e^{-\lambda h}\mathrm{sh}\lambda y\right]\right. \\ \left.-2\left(\alpha_1\theta_0\frac{h}{H}-\frac{b_0}{d}\right)e^{-\lambda H}\mathrm{sh}\lambda y\right\}. \end{aligned} \tag{5.25}$$

Similarly, one obtains from (5.3)

$$\begin{aligned} P(y) \approx \frac{2\lambda d}{\hat{a}}\left(\alpha_1\theta_0\frac{h}{H}-\frac{b_0}{d}\right)e^{-\lambda H}\mathrm{sh}\lambda y+\frac{\lambda d}{\hat{a}}(\alpha_2-\alpha_1) \\ \cdot\,\theta_0\left[e^{-\lambda h}\mathrm{sh}\lambda y-\mathrm{sh}\lambda(y+h)\mathrm{H}(-y-h)-\mathrm{sh}\lambda(y-h)\mathrm{H}(y-h)\right]. \end{aligned} \tag{5.26}$$

These fields, $u(y)$ and $P(y)$, are clearly continuous everywhere including the interfaces $y=\pm h$, in contrast to the corresponding fields of distortions $\beta(y)=u_{,y}$ and polarization gradient $\pi(y)=P_{,y}$:

$$\begin{aligned} \frac{du}{dy} \approx \alpha_1\theta_0\left\{\left(1-\frac{h}{H}\right)-\mathrm{H}(-y-h)-\mathrm{H}(y-h)]\right\} \\ +\Delta\left(\alpha_1\theta_0\frac{h}{H}-\frac{b_0}{d}\right)\left[\frac{1}{2\lambda H}-2e^{-\lambda H}\mathrm{ch}\lambda y\right] \\ +\Delta(\alpha_2-\alpha_1)\theta_0\left\{-e^{-\lambda h}\mathrm{ch}\lambda y+\mathrm{ch}\lambda(y+h)\mathrm{H}(y-h)+\mathrm{ch}\lambda(y-h)\mathrm{H}(y-h)\right\}, \end{aligned} \tag{5.27}$$

$$\begin{aligned} \frac{dP}{dy} \approx \frac{\Delta c_{11}}{d_{11}}\left\{(\alpha_2-\alpha_1)\theta_0\left[e^{-\lambda h}\mathrm{ch}\lambda y-\mathrm{ch}\lambda(y+h)\mathrm{H}(-y-h)\right.\right. \\ \left.\left.-\mathrm{ch}\lambda(y-h)\mathrm{H}(y-h)\right]+2\left(\alpha_1\theta_0\frac{h}{H}-\frac{b_0}{d_{11}}\right)e^{-\lambda H}\mathrm{ch}\lambda y\right\}. \end{aligned} \tag{5.28}$$

They are discontinuous at $y = \pm h$. On the other hand, the linear superposition (3.17), (3.18) of these fields with the third discontinuous field of temperature distribution (3.1) form continuous fields $\sigma(y)$ and $Q(y)$. We shall present only the stress field, which does not vanish this time. The result is easily found by combining Eqs. $(5.11)_1$ and (5.24):

$$\sigma(y) \approx -\gamma\theta_0 \frac{h}{H} + \frac{\Delta}{2\lambda H}\left(\gamma\theta_0 \frac{h}{H} - \frac{cb_0}{d}\right). \tag{5.29}$$

The second term here represents Mindlin's correction to the result of classical theory given by the first term. Here we must note that a homogeneous character of the obtained field $\sigma(y)$ is an exact result, which has nothing to do with our neglecting the exponentially small terms. Indeed, as it follows from Eqs. (3.17), (5.4), (5.5) and (3.1),

$$\sigma(y) = cB + \lambda dA - \gamma\theta_0 = \text{const}. \tag{5.30}$$

Of course, the above small corrections to the thermoelastic stresses are much less interesting than Mindlin's electric coupled field (5.15). As before, it might be essential only in the vicinity of surfaces or interfaces, where again there are narrow zones of high electric field:

$$\begin{aligned} E_{y\approx\pm H} &\approx \pm\frac{\lambda}{\varepsilon_0 \hat{a}}\left(b_0 - d\alpha_1\theta_0 \frac{h}{H}\right) e^{-\lambda(H-|y|)}, \\ E_{y\approx\pm h} &\approx \pm\frac{\lambda d}{2\varepsilon_0 \hat{a}}\left(\alpha_1 - \alpha_2\right)\theta_0 e^{-\lambda|h-|y||}. \end{aligned} \tag{5.31}$$

It is very natural that the fields at the interfaces $(5.31)_2$ and $(5.16)_2$ coincide. Indeed, such very localized dependences could hardly "feel" boundary conditions at the surfaces, especially after our approximation relating to the transition from (5.20) to (5.23).

6. Plane thermal inclusion at the surface of a dielectric plate

6.1. A plate with one face free and the other face clamped

In this section we shall extend the problem solved in Sec. 4 for a semi-infinite dielectric medium, Fig. 2, by introducing into consideration the second surface, which earlier was at infinity (Fig. 4). The temperature distribution (4.1) and the equilibrium equations (4.2)–(4.4) and (4.10), (4.11) remain the same. We also inherit the boundary conditions (4.5) and (4.6). At the new surface $y = H$ we shall put

$$u(H) = 0, \qquad \boldsymbol{\epsilon}(H) = 0. \tag{6.1}$$

The first of this conditions is just a reformulation of $(4.8)_1$. It makes more definite the function $u(y)$, which is principally defined with the accuracy to an arbitrary constant. In fact, the requirement $(6.1)_1$ is trivially compatible with the condition $\sigma(H) = 0$, because, as will be shown, the demand (4.5) automatically provides the identity $\sigma(y) = 0$. That is why we shall not separately consider the boundary problem related to the case of both faces being free. On the other hand, the problem of the plate with both clamped faces, as we have just seen, eliminates identical vanishing of stresses and needs independent consideration (see the next subsection).

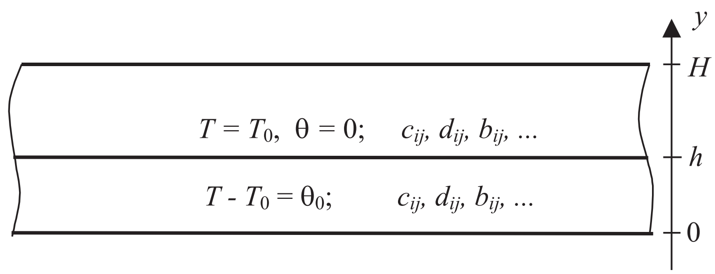

FIG. 4. Dielectric plate with a thermal inclusion at the surface.

Thus, we can again look for a solution in the form (4.12). After substitution of this function into Eq. (4.11) and integration of the latter, one finds the distortion

$$\frac{du}{dy} = C - \alpha_1\theta_0 \mathrm{H}(y-h) - \frac{\lambda d}{c}\left[Ae^{\lambda y} - Be^{-\lambda y} - \frac{K\theta_0}{\lambda}\mathrm{ch}\lambda(y-h)\mathrm{H}(y-h)\right], \tag{6.2}$$

where C is the third unknown constant. The differentiation of (4.12) gives additionally

$$\frac{dP}{dy} = \lambda\left[Ae^{\lambda y} - Be^{-\lambda y} - \frac{K\theta_0}{\lambda}\mathrm{ch}\lambda(y-h)\mathrm{H}(y-h)\right]. \tag{6.3}$$

We note that identically

$$\frac{du}{dy} + \frac{d}{c}\frac{dP}{dy} = C - \alpha_1\theta_0\mathrm{H}(y-h). \tag{6.4}$$

With this observation and Eq. (4.1), the stress $\sigma(y)$ (3.17) must be identically equal to a constant,

$$\sigma(y) = cC - \gamma\theta_0. \tag{6.5}$$

This general conclusion is very similar to the above analogous statement (5.30). So, if any face of the plate is free of tension, i.e. if $\sigma = 0$ at any of surfaces, as it is supposed to occur for the face $y = 0$, Eq. (4.5), then everywhere

$$\sigma(y) = 0 \tag{6.6}$$

and therefore

$$C = \alpha_1\theta_0. \tag{6.7}$$

With the known value of C and the boundary condition $(6.1)_1$ we can transform Eq. (6.4) into

$$u(y) = \alpha_1\theta_0(y-h)H(h-y) - \frac{d}{c}\Big[P(y) - P(H)\Big]. \tag{6.8}$$

Thus, now for evaluating the fields $u(y)$ and $P(y)$ we need to find the two unknown constants A and B. In view of (3.18), (4.1), the boundary conditions (4.6) and $(6.1)_2$ take the form

$$d\frac{du}{dy}\bigg|_{y=0} + b\frac{dP}{dy}\bigg|_{y=0} = \eta\theta_0 - b_0, \tag{6.9}$$

$$d\frac{du}{dy}\bigg|_{y=H} + b\frac{dP}{dy}\bigg|_{y=H} = -b_0. \tag{6.10}$$

Substituting here Eqs. (6.2) and (6.3) we obtain the system

$$A - B = \frac{K\theta_0}{\lambda} - \frac{\lambda b_0}{\hat{a}}, \tag{6.11}$$

$$Ae^{\lambda H} - Be^{-\lambda H} = \frac{K\theta_0}{\lambda}\mathrm{ch}\lambda(H-h) - \frac{\lambda b_0}{\hat{a}}, \tag{6.12}$$

with the solution

$$A = \frac{K\theta_0}{2\lambda}\frac{\mathrm{ch}\lambda(H-h) - e^{-\lambda H}}{\mathrm{sh}\lambda H} - \frac{\lambda b_0}{2\hat{a}}\frac{1-e^{-\lambda H}}{\mathrm{sh}\lambda H}, \tag{6.13}$$

$$B = \frac{K\theta_0}{2\lambda}\frac{\mathrm{ch}\lambda(H-h) - e^{\lambda H}}{\mathrm{sh}\lambda H} - \frac{\lambda b_0}{2\hat{a}}\frac{1-e^{\lambda H}}{\mathrm{sh}\lambda H}. \tag{6.14}$$

Substituting the found constants A and B into (4.12) one obtains the polarization distribution

$$P(y) = \frac{K\theta_0}{2\lambda}\left[\frac{\mathrm{ch}\lambda(H-h)\mathrm{ch}\lambda y - \mathrm{ch}\lambda(H-y)}{\mathrm{sh}\lambda H} - \mathrm{sh}\lambda(y-h)\mathrm{H}(y-h)\right] - \frac{\lambda b_0}{\hat{a}}\frac{\mathrm{ch}\lambda y - \mathrm{ch}\lambda(H-y)}{\mathrm{sh}\lambda H}, \tag{6.15}$$

which together with Eq. (6.8) determines also a continuous displacement field. One can check that the corresponding field $\varepsilon(y)$ is also continuous. We shall not present here its explicit form. As before, we are more interested in the electric field $E(y) = -\varepsilon_0^{-1}P(y)$ close to the surfaces $y = 0$, $y = H$ and the interface $y = h$, which can be found by some manipulations with Eq. (6.15):

$$
\begin{aligned}
E\big|_{y\approx 0} &\approx -\frac{\lambda}{\varepsilon_0 \hat{a}}\left[b_0 - d(\alpha_2-\alpha_1)\theta_0\right] e^{-\lambda y},\\
E\big|_{y\approx h} &\approx -\frac{\lambda d(\alpha_2-\alpha_1)\theta_0}{2\varepsilon_0 \hat{a}} e^{-\lambda|y-h|},\\
E\big|_{y\approx H} &\approx \frac{\lambda b_0}{\varepsilon_0 \hat{a}} e^{-\lambda(H-y)},
\end{aligned}
\tag{6.16}
$$

where the first two localized fields naturally coincide with those found in Sec. 4 for similar zones of the semi-infinite medium, Eq. (4.22). The field at the face H in our accuracy is completely determined by the Mindlin surface tension.

6.2. A plate with clamped faces

Below we shall solve for the same plate shown in Fig. 4 the boundary problem

$$
u(0) = 0, \qquad \boldsymbol{\epsilon}(0) = 0, \qquad u(H) = 0, \qquad \boldsymbol{\epsilon}(H) = 0. \tag{6.17}
$$

We can start from Eqs. (4.12), (6.2)–(6.4). Integration of Eq. (6.4) with making use of the boundary conditions $(6.17)_{1,3}$, leads to the relation

$$
u(y) = -\frac{d}{c}\Big[P(y) - P(0)\Big] + Cy - \alpha_1\theta_0(y-h)\mathrm{H}(y-h), \tag{6.18}
$$

where

$$
C = \alpha_1\theta_0\left(1 - \frac{h}{H}\right) + \frac{d}{cH}\left[P(H) - P(0)\right]. \tag{6.19}
$$

Combining Eqs. (6.2)–(6.4) with the boundary conditions (6.9), (6.10) we have

$$
\frac{\hat{a}}{\lambda^2}\left.\frac{dP}{dy}\right|_{y=0} = d\left(\alpha_2\theta_0 - C\right) - b_0, \tag{6.20}
$$

$$
\frac{\hat{a}}{\lambda^2}\left.\frac{dP}{dy}\right|_{y=H} = d\left(\alpha_1\theta_0 - C\right) - b_0. \tag{6.21}
$$

Substitution here of the corresponding derivatives of $P(y)$ from (6.3) gives the system similar to (6.11), (6.12):

$$Ae^{\lambda H} - Be^{-\lambda H} = \kappa_1 + \frac{K\theta_0}{\lambda}\mathrm{ch}\lambda(H-h), \tag{6.22}$$

$$A - B = \kappa_2, \tag{6.23}$$

where the coefficients κ_i (i =1, 2) are defined by

$$\kappa_i = \frac{\lambda}{\hat{a}}\Big[d\,(\alpha_i\theta_0 - C) - b_0\Big]. \tag{6.24}$$

The solution of the above system is given by

$$\begin{aligned} A &= \frac{\kappa_1 - \kappa_2 e^{-\lambda H} + (K\theta_0/\lambda)\mathrm{cth}\lambda(H-h)}{2\mathrm{sh}\lambda H}, \\ B &= \frac{\kappa_1 - \kappa_2 e^{\lambda H} + (K\theta_0/\lambda)\mathrm{cth}\lambda(H-h)}{2\mathrm{sh}\lambda H}. \end{aligned} \tag{6.25}$$

Now we can also find the unknown constant C. Combining (6.25) with (4.12) we arrive at the equation

$$\begin{aligned} P(H) - P(0) = -C\frac{2\lambda d}{\hat{a}}\mathrm{th}\frac{1}{2}\lambda H + \frac{\lambda}{\hat{a}}\Big[d(\alpha_1+\alpha_2)\theta_0 - 2b_0\Big]\mathrm{th}\frac{1}{2}\lambda H \\ + \frac{K\theta_0}{\lambda}\frac{\mathrm{sh}\lambda(h-\frac{1}{2}H)}{\mathrm{ch}\frac{1}{2}\lambda H}, \end{aligned} \tag{6.26}$$

which forms with Eq. (6.19) the necessary system allowing us to exclude the difference $P(H) - P(0)$ and to evaluate C. In fact, for our purposes there is no need to present a bulky exact solution. Omitting the small exponents and the terms of order of $(\lambda H)^{-2}$ one obtains

$$\begin{aligned} C &\approx \alpha_1\theta_0\left(1 - \frac{h}{H}\right) + \frac{\Delta}{\lambda H}\left\{\left[\alpha_2 - \alpha_1\left(1 - \frac{2h}{H}\right)\right]\theta_0 - \frac{2b_0}{d_{11}}\right\} \\ &\approx \alpha_1\theta_0\left(1 - \frac{h}{H}\right). \end{aligned} \tag{6.27}$$

Thus, with the known value of C, the parameters $\kappa_{1,2}$ become definite together with the fields $u(y)$ (6.18) and $P(y)$ (4.12). For instance, the explicit form of the latter is

$$\begin{aligned} P(y) = \frac{\kappa_1\mathrm{ch}\lambda y - \kappa_2\mathrm{ch}\lambda(H-y)}{\mathrm{sh}\lambda H} \\ + \frac{(K\theta_0/\lambda)\big[\mathrm{ch}\lambda y\,\mathrm{ch}\lambda(H-h) - \mathrm{sh}\lambda H\,\mathrm{sh}\lambda(y-h)H(y-h)\big]}{\mathrm{sh}\lambda H}. \end{aligned} \tag{6.28}$$

The found polarization distribution as well as the accompanied electric field $E(y) = -\varepsilon_0^{-1}P(y)$ are again concentrated basically close to the surfaces and the interface:

$$
\begin{aligned}
&E\Big|_{y\approx 0} \approx \frac{\kappa_2}{\varepsilon_0}e^{-\lambda y},\\
(6.29)\qquad &E\Big|_{y\approx h} \approx -\frac{\lambda d(\alpha_2-\alpha_1)\theta_0}{2\varepsilon_0\hat{a}}e^{-\lambda|y-h|},\\
&E\Big|_{y\approx H} \approx -\frac{\kappa_1}{\varepsilon_0}e^{-\lambda|H-y|},
\end{aligned}
$$

where $\kappa_{1,2}$ are given by Eqs. (6.24), (6.27). We note that the localized field $(6.29)_{2,3}$ are identical with $(5.31)_{2,1}$, respectively, and Eq. $(6.29)_1$ at $H \to \infty$ naturally coincides with $(4.22)_1$.

7. Conclusions

Thus, the theory of polarization gradient applied to a series of one-dimensional problems related to isotropic dielectrics predicts that a plane thermal inclusion must excite, apart from the ordinary thermoelastic strains, the additional electro-mechanical fields localized close to the surfaces and interfaces. It is interesting that with "switching off" the Mindlin's coupling effects ($d \to 0$, $b \to 0$, $b_0 \to 0$, $\eta \to 0$), the new fields disappear not due to vanishing of their amplitudes, but by means of decreasing of the zones, where they act. Let us look at this problem in the case of an electric component arising on the background of zero. Indeed, after such switching off of the polarization gradient coupling effects, the amplitudes $E_m \sim K\theta_0/\lambda\varepsilon_0$ of such fields do not disappear. Basing on (3.8) and supposing that $|bc - d^2| \sim d^2$, the magnitude of E_m can be estimated as

$$(7.1)\qquad E_m \sim \alpha_1\theta_0(c/\varepsilon_0)^{1/2}.$$

By the way, this is not at all a low field. At $\alpha_1\theta_0 \sim 10^{-4}$, $c \sim 10^3$ MPa, one obtains the estimate

$$(7.2)\qquad E_m \sim 10^5\,\mathrm{V/m}.$$

So, close to the interface $y = h$, where the temperature jump occurs, the field E can be expressed as

$$(7.3)\qquad E \sim E_m \exp[-\lambda|y-h|].$$

Thus, the formal disappearance of the field happens because the pre-interface zone of the field penetration $|y - h| \sim 1/\lambda$ at $\lambda \to \infty$ becomes infinitely narrow.

As we have already mentioned, in realistic conditions of ordinary dielectrics, the parameter λ^{-1} should be expected to be of order of several lattice parameters and the material constants d, b and η must be very small. This means that in ordinary dielectrics the considered thermal inclusion should excite rather high electric fields (7.2) in the very narrow zones around the interfaces. It appears that the accomplished analysis and the prediction made might be useful in some applications.

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