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On the foundations of ordinary and generalized rigid body dynamics and the principle of objectivity

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This article presents the foundations of Newton-Euler rigid body dynamics and its generalized forms in the light of the objectivity principle. We prove that most of the features of dynamics may be directly deduced from this principle and from properties of the group defining the geometry. In particular, these deductions seem to close the conjectures about the relevance of the objectivity principle to dynamics.

Notations

$\mathbb{R}$	real number field,
$\mathbb{G}$	connected Lie group (acting on $\mathbb{S}$ from the left), see Sec. 2,
$\mathbb{T}$	closed normal commutative Lie subgroup of $\mathbb{G}$ (translation group), see Sec. 4,
$\mathbb{S}$	configuration space of a rigid body, see Secs. 2 and 3,
$T\mathbb{S}$ and $T^*\mathbb{S}$	tangent and cotangent spaces of $\mathbb{S}$, see Sec. 2,
$\mathbf{G}_{\mathrm{kin}}$ and $\mathbf{G}_{\mathrm{gal}}$	group of kinematics and Galilée group, see Sec. 4,
$\mathfrak{g}$ and $\mathfrak{g}^*$	Lie algebra of $\mathbb{G}$ and its dual,
$\mathfrak{t}$	Lie algebra of $\mathbb{T}$ (a commutative ideal of $\mathfrak{g}$),
$[\cdot,\cdot]$	Lie bracket in $\mathfrak{g}$,
$[\![\cdot,\cdot]\!]$	Lie bracket of vector fields (on $\mathbb{S}$),
$\{\cdot,\cdot\}$	coadjoint Lie bracket in $\mathfrak{g}^*$, see Sec. 2,
Ad and Ad^*	adjoint and coadjoint representations of $\mathbb{G}$ in $\mathfrak{g}$ and $\mathfrak{g}^*$, see Sec. 2,
ϑ_r or ϑ	right Maurer–Cartan form of $\mathbb{S}$, see Secs. 2 and 3,
ϑ_ℓ	left Maurer–Cartan form of $\mathbb{S}$, see Secs. 2 and 3,
∇	natural connection of $\mathbb{S}$, see Sec. 2,
$\mathcal{F}$ ($\mathcal{F}_1$, $\mathcal{F}_2$ and so on)	frames of reference (inertial or kinematical), see Sec. 4,
H (resp. H_s)	inertia operator of a body (resp. in position $s \in \mathbb{S}$), see Secs. 6.1, and 6.2,
C (resp. C_s)	see Sec. 6.3,
$\mathbf{j}_{\mathcal{F}}(t)$ and $\mathbf{J}_{\mathcal{F}}(t)$	inertial forces observed with respect to a frame $\mathcal{F}$, see Sec. 6.3,
$\mathbb{D}$	Euclidean displacement group in dimension 3, see Sec. 7.1,
$\mathfrak{d}$	Lie algebra of $\mathbb{D}$, see Sec. 7.1,
$[\cdot \mid \cdot]$	Klein form on $\mathfrak{d}$, see Sec. 7.1.

1. Introduction

EXPOSITIONS OF MECHANICS based on the principle of frame indifference (in the sequel, we will say "principle of objectivity") appeared in W. NOLL [20] and [21]. Afterwards, this principle has played an important role in many works on constitutive relations in continuum mechanics. However, to our knowledge, it was not actually focused on the consequences of the principle regarding the mathematical form of inertial forces in rigid body dynamics. In fact, whether or not this principle is relevant to dynamics is a matter of controversy (see for example C. SPEZIALE [29]). On the one hand, as soon as we consider non-inertial frames of reference and want to strictly preserve its form, the well-known Newton's second law in dynamics of particles

$$m\frac{d^2\mathbf{x}}{dt^2} = \mathbf{f}\,, \tag{1.1}$$

is not frame indifferent. On the other hand, to express the laws of dynamics with respect to non-inertial frames necessitates a specification of the effect of a general change of frame on the mathematical representation of forces appearing to the right in Newton's law. This is actually the role of an *axiom* stating that forces are *objective* quantities and, in a consistent theory, it must concern all forces, including inertial forces. Note that the addition to the right of Coriolis and induced inertial forces, as explained in standard textbooks, leads to correct dynamic equations and preserves the mathematical form of the left hand side but does require such an axiom for specifying what becomes of $\mathbf{f}$ through the changes of frame. The remark extends to dynamics of a rigid body with Euler equations for rotations. To clarify this matter it is inevitable to return to the interpretation of the left-hand side of (1.1) not only for particles but for realistic models of body, with "forces" and "torques", what is much more complicated.

The natural framework of such investigation is the group theory, commonly used in other parts of mechanics and almost unavoidable here. After the article [1] by V. ARNOLD many articles appeared on Lie groups and dynamics of various models of generalized rigid bodies. The article by Y.N. FEDOROV and V.V. KOZLOV [10] emphasizes the relevance of such research and contains many historical references. The idea of a rigid body in n-dimensional space or of an affinely deformable body was in fact older (see for example CHETAYEV [3] and H. WEYL [31]), however, with [1] the subject was entering into a new mathematical framework. More recent works on these subjects have appeared, in particular V. V. KOZLOV and D. ZENKOV [15], J. SLAVIANOWSKI [27] and [28] and A. MARTENS [19], C. VALLÉE and coauthors [30], A.A. BOUROV and D.P. CHEVALLIER [2]. Dynamics of affinely-rigid body of infinitesimal size in Riemannian spaces was investigated by A. GOLUBOWSKA [11] and [12]. The Hamiltonian view point, with applications to various interesting physical models, were synthesized in C-M. MARLE [17]. Similar problems were treated by J. MARSDEN, T. RATIU and coauthors (see for example [18, 24] and [25]).

These works have dealt with the analysis of the mathematical form of given dynamic equations and their integrability but, to our knowledge, no attempt to point out links with the fundamental principles of Newtonian dynamics themselves and geometry of Lie groups has been made.

We will prove that the principle of objectivity, as a principle of invariance not only with respect to the Galilée group but with respect to the general group of kinematics, involves strong constraints on the description of the inertia of a rigid body (say the relation between velocity and momentum) and implies a lot of properties of dynamics such as the mathematical structure of dynamics in non-inertial frames and properties of gyroscopic and Coriolis forces. For an ordinary body in 3 dimension (as far as we consider symmetric positive definite inertia operators) the constraints involved in objectivity determine one and only one model: the Newton-Euler rigid body dynamics.

The paper is organized as follows:

Sections 2 and 3 expose the mathematical framework of generalized rigid body kinematics based on differential geometry. In it we attempt to sum up in a few lines a complete set of tools for reasoning and calculating with Lie groups and dynamics of rigid body.

Section 4 aims at endowing the word "objectivity" with a precise meaning, to explain the mathematical description of the various objects we have to consider and the relations between their representations with respect to different frames of reference. To take geometry into account, no more than a Lie group $\mathbb{G}$ and a commutative Lie subgroup, the translation group $\mathbb{T}$ will be used. With $\mathbb{G}$, $\mathbb{T}$ and time (the absolute time of Newton mechanics) we can construct the general group of kinematics $\mathbf{G}_{\text{kin}}$ and its subgroup $\mathbf{G}_{\text{gal}}$, the Galilée group. We can also construct the configuration space of a rigid object, within the meaning of the transformation group $\mathbb{G}$, that is a manifold $\mathbb{S}$ with a transitive and free left action of $\mathbb{G}$ on $\mathbb{S}$ (a "principal homogeneous space of $\mathbb{G}$").

Section 5 summarizes, in the very compact form provided by Lie groups and principal homogeneous spaces, some formulas in kinematics of composed motions of ordinary or generalized rigid bodies for further use. In particular we introduce a new formalism for composition of accelerations.

Section 6 contains the main results of the paper regarding ordinary or generalized bodies: the condition for the existence of objective inertial forces (Theorem 1), the structure of the inertial forces in non-inertial frames (Theorem 2) and their consequences. In particular we show how some kinds of "absolute" kinematical quantities may be pointed out in the universe of Newton and that objectivity of inertial forces means that they are functions these absolute quantities (Theorem 5).

Inertia of a rigid body may be described by standard mathematical objects of differential geometry. Either a left invariant Riemannian structure on $\mathbb{S}$ defining the kinetic energy as in [1] or, which is slightly more general, an "invariant vector bundle homomorphism" $H: T\mathbb{S} \to T^*\mathbb{S}$; if a velocity of a body is described by a tangent vector $\mathbf{v}$ of the configuration space $\mathbb{S}$, then the momentum is a cotangent vector $H(\mathbf{v})$ and the relation between $\mathbf{v}$ and $H(\mathbf{v})$ has two properties: first it is invariant by the action of the group $\mathbb{G}$ and second, for a given position of the body in $\mathbb{S}$, it is linear. At this stage nothing more is assumed on the mathematical form of H. Of course each of the words velocity, momentum and force must be understood now in a broad sense. For an ordinary rigid body $\mathbf{v}$ describes linear and angular velocity, $H(\mathbf{v})$ describes linear and angular momentum, forces mean ordinary forces and torques. For a generalized rigid body, as an affinely deformable body, they describe more general kinematical or dynamical objects.

The law of dynamics with respect to an inertial frame may be similar to Newton's second law, "the derivative of the momentum equal the force", and turns out to be an equality in $T^*\mathbb{S}$

$$\frac{\nabla}{dt} H(\mathbf{v}) = \mathbf{f} \;\text{ with }\; \mathbf{v} = \frac{ds}{dt},$$

where $t \mapsto s(t) \in \mathbb{S}$ describes the motion of the body, ∇ is a well-defined canonical connection on $\mathbb{S}$ and $\mathbf{f}$ is the force acting on the body. The objectivity of this law raises the following questions: is it possible to associate with any motion an objective inertial force $\mathbf{j}\,(t)$ defined in any frame such that $\mathbf{j}\,(t) = -\dfrac{\nabla}{dt} H(\mathbf{v})$ whenever the frame is inertial? What is the expression of $\mathbf{j}\,(t)$ with respect to non-inertial frames? The necessary and sufficient objectivity condition is reduced to an additional algebraic condition to be verified by H, namely:

for all $\mathbf{u}\in\mathfrak{t}$, all $\mathbf{v}\in\mathfrak{g}$ and all $s\in\mathbb{S}$: $\{\mathbf{u}, H_s(\mathbf{v})\} + \{\mathbf{v}, H_s(\mathbf{u})\} + H_s([\mathbf{u}, \mathbf{v}]) = 0 \; (\star)$

(where $\mathfrak{g}$ and $\mathfrak{t}$ are the Lie algebras of $\mathbb{G}$ and $\mathbb{T}$ and, for $s \in \mathbb{S}$, the H_s are linear operators from $\mathfrak{g}$ to $\mathfrak{g}^*$ describing inertia in a given position s of the body, the bracket $\{.,.\}$ is defined in Sec. 2).

According to Secs. 5 to 6 many features of dynamics require no more assumptions on geometry of "space" and "spacetime" than the groups $\mathbb{G}$, $\mathbb{T}$ and $\mathbf{G}_{\text{kin}}$, $\mathbf{G}_{\text{gal}}$, and no more information on the configurations of our rigid bodies than the action of $\mathbb{G}$ on $\mathbb{S}$.

In Sec. 7 we apply the previous results to the particular case of the three-dimensional Euclidean group $\mathbb{D}$ and we prove that the sole (symmetric positive) solutions to $(\star)$ are operators H_s defined by a number, the position of a point and a tensor, that are rightfully interpreted as the mass, the center of inertia and the tensor of inertia of the body as in the Newton–Euler model (this connection was mentionned in [6]). This result appears here as a pure consequence of the principle of objectivity and of some features of the Euclidean group. In particular, no model of a mass distribution or of aggregate of massive particles in space was required.

Let us make some additional remarks. After [1], most of the works on rigid body mechanics described the positions of the body in a group. Here they are described as points of a principal homogeneous space $\mathbb{S}$ of a group, what is slightly more technical but provides a more accurate picture of the properties of a rigid body than the group itself.

In this article, the same group $\mathbb{G}$ defines rigidity for the frames and for our rigid object. It would also be relevant to consider objects that are "rigid" in the sense of a larger group as in Chevallier [4] where $\mathbb{S}$ must be replaced by a principal fiber bundle of $\mathbb{G}$. However, this leads to a more complicated mathematical treatment which is beyond the scope of this article.

If we understand that $\mathbb{G}$ is always the Euclidean group, this article provides a complete and new analysis of the principles of dynamics of the ordinary rigid body in the light of group theory and the objectivity principle. When more general $\mathbb{G}$ are considered, the article points out the conditions permitting to conceive extensions preserving most of the features of ordinary rigid body dynamics.

Henri Poincaré in [22] and [23] explained that the origin of geometry lies in a group detected through the changes of position of (perfectly) rigid objects and that space and geometry are deduced from properties of this group. Finally, this article explains how such an idea applies to a direct deduction of dynamics.

It is noteworthy that group-theoretical background is also present in modern mechanics of continua. Following the fundamental work of Cosserat, various models, such as micromorphic media introduced by A.C. Eringen, use orthogonal or, more generaly, linear or affine groups in order to describe the positions of the elements of a continuum. Complete references regarding those connections between groups and mechanics is beyond the scope of this article. We only mention the proceedings [16], including many contributions where groups and differential geometry appear explicitly or not, and J.C. Simo and L. Vu–Quoc [26] for the theory of rods.

2. Geometry of a principal homogeneous space

In this section we summarize the mathematical background used for developing rigid body dynamics in the framework of differential geometry (see also the article [9]). The mechanical interpretation will be explained in Sec. 3.

Let $\mathbb{G}$ be a (real) Lie group, $\mathfrak{g}$ its Lie algebra, $[\cdot,\cdot]$ the Lie bracket in $\mathfrak{g}$ (that is to say: $\mathbb{G}$ is a group endowed with a manifold structure such that the operations of the group are differentiable. As a set, $\mathfrak{g} = T_e\mathbb{G}$ is the tangent vector space of $\mathbb{G}$

at the identity and the theory of Lie groups proves that there exists a natural Lie algebra structure on $T_e\mathbb{G}$).

Let Ad and Ad* be the adjoint and coadjoint representations of $\mathbb{G}$ in $\mathfrak{g}$ and $\mathfrak{g}^*$ (for $g \in \mathbb{G}$, $\mathrm{Ad}\, g$ is the linear operator in $\mathfrak{g}$ such that $\mathrm{Ad}\,.\mathbf{u}$ is the value on $\mathbf{u} \in \mathfrak{g}$ of the tangent map to $x \mapsto gxg^{-1}$ at $x = e$). Here, the definition of coadjoint representation is $\mathrm{Ad}^* g = {}^t\left(\mathrm{Ad}\, g^{-1}\right)$ (and differs from that of [1]). If $\mathbf{u} \in \mathfrak{g}$ we note $\mathrm{ad}\,\mathbf{u}$ the map $\mathbf{v} \mapsto [\mathbf{u}, \mathbf{v}]$ and $\mathrm{ad}^*\mathbf{u} = -{}^t(\mathrm{ad}\,\mathbf{u})\colon \mathfrak{g}^* \to \mathfrak{g}^*$ the transposed operator of $-\mathrm{ad}\,\mathbf{u}$ so that $< \mathrm{ad}^*\mathbf{u}.z, \mathbf{v} >= - < z, \mathrm{ad}\,\mathbf{u}.\mathbf{v} >$ for $\mathbf{v} \in \mathfrak{g}$, $z \in \mathfrak{g}^*$. In the following we will also use the coadjoint bracket notation defined by

$$\{\mathbf{u}, z\} = \mathrm{ad}^*\mathbf{u}.z \qquad \text{for} \qquad \mathbf{u} \in \mathfrak{g},\ z \in \mathfrak{g}^*.$$

A principal homogeneous space of $\mathbb{G}$ is a set $\mathbb{S}$ (in due form a pair $(\mathbb{S}; \mathbb{G})$!) where $\mathbb{G}$ is acting *transitively* and *freely* by the left on $\mathbb{S}$. More specifically, each $g \in \mathbb{G}$ defines a transformation of $\mathbb{S}$ denoted by $L_g\colon s \mapsto L_g(s) = g.s$ such that $e.s = s$ and $h.(g.s) = (h.g).s$ and, for any fixed element s of $\mathbb{S}$, the map $\sigma_s\colon g \mapsto g.s$ is one-one and onto.

There exists a unique manifold structure on $\mathbb{S}$ such that, for all s, the bijection σ_s is an analytic diffeomorphism (then the action of $\mathbb{G}$ on $\mathbb{S}$ is analytic). When $\mathbb{S}$ will be considered as a manifold we will always refer to this structure. As manifolds $\mathbb{G}$ and $\mathbb{S}$ are equivalent, the difference lies in the algebraic structure: in some sense $\mathbb{S}$ is "a Lie group whose unit element has been lost". Since $\sigma_s \circ L_g = L_g \circ \sigma_s$ (noting also L_g the left translation by g in $\mathbb{G}$), the principal homogeneous spaces $(\mathbb{S}; \mathbb{G})$ and $(\mathbb{G}; \mathbb{G})$, where $\mathbb{G}$ acts on itself by left translations, are isomorphic by the map σ_s.

In the following, $T\mathbb{S}$ (resp. $T^*\mathbb{S}$) will denote the tangent (resp. cotangent) space of $\mathbb{S}$ and $< \cdot, \cdot >$ the duality bracket between $T\mathbb{S}$ and $T^*\mathbb{S}$. These spaces have natural vector-bundle structures with base $\mathbb{S}$ and with projections o from $T\mathbb{S}$ (resp. $T^*\mathbb{S}$) onto $\mathbb{S}$; o maps each tangent vector $\mathbf{x}$ (resp. tangent covector $\mathbf{z}$) onto its origin $o(\mathbf{x}) = s$ (resp. $o(\mathbf{z}) = s$). The action of $\mathbb{G}$ on $\mathbb{S}$ may be lifted into left actions of $\mathbb{G}$ on $T\mathbb{S}$ and $T^*\mathbb{S}$; we will note $L_g^T(\mathbf{x}) = g.\mathbf{x}$ (resp. $(L_g^T)^*(\mathbf{z}) = g.\mathbf{z}$) the action of $g \in \mathbb{G}$ on $\mathbf{x} \in T\mathbb{S}$ (resp. $\mathbf{z} \in T^*\mathbb{S}$), where L_g^T is the tangent map. If $\mathbf{x}$ and $\mathbf{z}$ have the same origin s in $\mathbb{S}$, then $< \mathbf{z}, \mathbf{x} > = < g.\mathbf{z}, g.\mathbf{x} >$.

2.1. Maurer–Cartan differential forms

The infinitesimal action of $\mathbb{G}$ on $\mathbb{S}$ takes the form $\mathfrak{g} \times \mathbb{S} \to T\mathbb{S}$ and is defined as:

$$\mathbf{u}.s = \left(\frac{d}{dt} \exp(t\mathbf{u}).s\right)_{t=0}$$

where exp denotes the exponential map of $\mathbb{G}$. An equivalent definition of $\mathbf{u}.s$ is the value of the tangent map of $g \mapsto g.s$ at $g = e$ on $\mathbf{u} \in T_e\mathbb{G}$ $(= \mathfrak{g})$. Since for each s, σ_s is a diffeomorphism from $\mathbb{G}$ onto $\mathbb{S}$ mapping e to s, we deduce that the map $\mathbf{u} \mapsto \mathbf{u}.s$ is a linear isomorphism from $\mathfrak{g} = T_e\mathbb{G}$ to $T_s\mathbb{S}$; in other words every element $\mathbf{x}$ of $T_s\mathbb{S}$ is of the form $\mathbf{u}.s$ with a uniquely defined $\mathbf{u} \in \mathfrak{g}$. Right and left Maurer-Cartan forms ϑ_r and ϑ_ℓ are the $\mathfrak{g}$-valued differential forms defined for $\mathbf{x} \in T\mathbb{S}$ by:

$$\begin{cases} \vartheta_r(\mathbf{x}) = \mathbf{u} \;\Leftrightarrow\; \mathbf{x} = \mathbf{u}.s \\ \vartheta_\ell(\mathbf{x}) = \mathbf{u} \;\Leftrightarrow\; g^{-1}.\mathbf{x} = \mathbf{u}.s_o \end{cases} \quad \text{if} \quad s = o(\mathbf{x}) = g.s_o\,. \tag{2.1}$$

In other words $g^{-1}.\mathbf{x} \in T_{s_o}\mathbb{S}$ and ϑ_ℓ is defined by $\vartheta_\ell(\mathbf{x}) = \vartheta_r(g^{-1}.\mathbf{x})$. The definition of the form ϑ_ℓ assumes the choice of an origin s_o in $\mathbb{S}$ ("reference position") whereas ϑ_r is intrinsically defined; in the following we will often write ϑ for ϑ_r. Maurer–Cartan forms and their exterior differentials verify the following properties:

$$\begin{cases} \vartheta_\ell(h.\mathbf{x}) = \vartheta_\ell(\mathbf{x}), \qquad \vartheta_r(h.\mathbf{x}) = \mathrm{Ad}\,h.\vartheta_r(\mathbf{x}), \quad h \in \mathbb{G}, \\ \vartheta_r(\mathbf{x}) = \mathrm{Ad}\,g.\vartheta_\ell(\mathbf{x}) \quad \text{when} \quad o(\mathbf{x}) = g.s_o, \end{cases} \tag{2.2}$$

$$\mathrm{d}\vartheta_r = [\vartheta_r, \vartheta_r], \quad \mathrm{d}\vartheta_\ell = -[\vartheta_\ell, \vartheta_\ell] \;\; \text{(Maurer–Cartan formulas)} \tag{2.3}$$

(For example, the first formula (2.3) means that $d\vartheta_r(\mathbf{x}, \mathbf{y}) = [\vartheta_r(\mathbf{x}), \vartheta_r(\mathbf{y})]$ if $\mathbf{x}$ and $\mathbf{y} \in T_s\mathbb{S}$).

2.2. Parallelizations of $T\mathbb{S}$ and $T^*\mathbb{S}$

The Maurer–Cartan forms lead to parallelizations of the tangent and cotangent bundles of $\mathbb{S}$ corresponding to the top and the bottom of the following diagrams:

$$\begin{array}{ccccccc} & \nearrow & \mathbb{S} \times \mathfrak{g} \;\; \mathbf{x} \mapsto (s, \vartheta_r(\mathbf{x})) & & & \nearrow & \mathbb{S} \times \mathfrak{g}^* \;\; \mathbf{z} \mapsto (s, \vartheta_r^*(\mathbf{z})) \\ T\mathbb{S} & & \uparrow & & T^*\mathbb{S} & & \uparrow \\ & \searrow & \mathbb{G} \times \mathfrak{g} \;\; \mathbf{x} \mapsto (g, \vartheta_\ell(\mathbf{x})) & & & \searrow & \mathbb{G} \times \mathfrak{g}^* \;\; \mathbf{z} \mapsto (g, \vartheta_\ell^*(\mathbf{z})) \end{array} \tag{2.4}$$

The vertical arrows represent the maps $(g, \mathbf{u}) \mapsto (g.s_o, \mathrm{Ad}\,g.\mathbf{u})$ and $(g, \mu) \mapsto (g.s_o, \mathrm{Ad}^*g.\mu)$. The upper part of the diagrams is intrinsical whereas the lower part depends on the choice of s_o. The right-hand diagram is the dual of the left hand one and the maps ϑ_r^* and ϑ_ℓ^* are determined by:

$$< \mathbf{z}, \mathbf{x} > = < \vartheta_r^*(\mathbf{z}), \vartheta_r(\mathbf{x}) > = < \vartheta_\ell^*(\mathbf{z}), \vartheta_\ell(\mathbf{x}) > \quad \text{for} \;\; \mathbf{z} \in T_s^*\mathbb{S},\; \mathbf{x} \in T_s\mathbb{S},$$

so that, on each cotangent fiber $T_s^*\mathbb{S}$, ϑ_r^* (resp. ϑ_ℓ^*) is the contragredient isomorphism of the isomorphism from $T_s\mathbb{S}$ onto $\mathfrak{g}$ defined by ϑ_r (resp. ϑ_ℓ). Relations (2.2) imply:

$$\begin{cases} \vartheta_r^*(h.\mathbf{z}) = Ad^*h.\vartheta_r^*(\mathbf{z}), & \vartheta_\ell^*(h.\mathbf{z}) = h\vartheta_\ell^*(\mathbf{z}), \quad h \in \mathbb{G} \\ \vartheta_r^*(\mathbf{z}) = Ad^*g.\vartheta_\ell^*(\mathbf{z}) & \text{when} \quad o(\mathbf{z}) = s = g.s_o. \end{cases} \tag{2.5}$$

A consequence of the parallelizations is that any differentiable tensor field (we will merely say "tensor" rather than tensor field) on $\mathbb{S}$ may be represented by a map from $\mathbb{S}$ to a tensor space on $\mathfrak{g}$ (or to a space of multilinear maps built with $\mathfrak{g}$ and $\mathfrak{g}^*$). So a contravariant and covariant of degree 1 tensor H is described by a differentiable map $s \mapsto H_s \in \mathcal{L}(\mathfrak{g})$. A contravariant of degree 1 and covariant of degree 2 tensor B is described by a map $s \to B_s \in \mathcal{L}_2(\mathfrak{g} \times \mathfrak{g}; \mathfrak{g})$. The tensors themselves may be reconstructed with the relations $\vartheta_r\big(H(\mathbf{x})\big) = H_s\big(\vartheta_r(\mathbf{x})\big)$ or $\vartheta_r\big(B(\mathbf{x},\mathbf{y})\big) = B_s\big(\vartheta_r(\mathbf{x}), \vartheta_r(\mathbf{y})\big)$ if $\mathbf{x}$ and $\mathbf{y} \in T_s\mathbb{S}$.

2.3. Left invariant vector fields and fundamental vector fields

As for any manifold we can define the set $\mathfrak{X}(\mathbb{S})$ of differentiable vector fields which classically is a Lie algebra under the Lie bracket of vector fields. In the present case, due to the particular structure of $\mathbb{S}$, $\mathfrak{X}(\mathbb{S})$ contains two important vector subspaces:

i) Fundamental vector fields, defined through the infinitesimal action of $\mathfrak{g}$ on $\mathbb{S}$, are of the form $X_\mathbf{u}\colon s \mapsto \mathbf{u}.s = X_\mathbf{u}(s)$ where $\mathbf{u} \in \mathfrak{g}$ is fixed. They make a Lie algebra $\mathfrak{X}_r(\mathbb{S})$, with the bracket $[\![\cdot,\cdot]\!]$, that is isomorphic to the opposite of $\mathfrak{g}$, in fact (see [13] chap. I.4):
$$[\![X_\mathbf{u}, X_\mathbf{v}]\!] = X_{-[\mathbf{u},\mathbf{v}]}, \quad (\mathbf{u},\ \mathbf{v} \in \mathfrak{g}).$$
In other words $[\![X_\mathbf{u}, Y_\mathbf{u}]\!](s) = -[\mathbf{u},\mathbf{v}].s$ (with the bracket of $\mathfrak{g}$ on the right-hand side).

ii) Left invariant vector fields $X\colon \mathbb{S} \to T\mathbb{S}$ verifying $X(g.s) = L_g^T(X(s))$ for all $s \in \mathbb{S}$ and $g \in \mathbb{G}$ and making a Lie subalgebra $\mathfrak{X}_\ell(\mathbb{S})$ of $\mathfrak{X}(\mathbb{S})$.

The Lie algebra $\mathfrak{X}_\ell(\mathbb{S})$ is isomorphic with $\mathfrak{g}$ and, for each $s \in \mathbb{S}$, there is a natural Lie algebra structure on $T_s\mathbb{S}$. The construction is similar to the classical definition of the Lie algebra of a Lie group $\mathbb{G}$ as a structure on $T_e\mathbb{G}$ isomorphic to the Lie algebra of left invariant vector fields on $\mathbb{G}$, except that we now consider $T_s\mathbb{S}$ at the current point of $\mathbb{S}$. On the one hand, for each fixed s in $\mathbb{S}$ the evaluation map $X \mapsto X(s)$ is a linear isomorphism between $\mathfrak{X}_\ell(\mathbb{S})$ and $T_s\mathbb{S}$. On the other hand, transporting the structure, $T_s\mathbb{S}$ turns into a Lie algebra with Lie bracket $[\cdot,\cdot]_s$ such that

$$[\mathbf{x},\mathbf{y}]_s = [\![X,Y]\!](s) \ \text{ for } \ X \ \text{ and } \ Y \in \mathfrak{X}_\ell(\mathbb{S}) \ \text{ such that } X(s) = \mathbf{x} \text{ and } Y(s) = \mathbf{y}.$$

(Of course, the relation $[\mathbf{x},\mathbf{y}]_s = [\![X,Y]\!](s)$ should be untrue for vector fields which are not left invariant.) It is easy to prove that the Lie algebra $T_s\mathbb{S}$ is isomorphic with $\mathfrak{g}$ and that

$$\vartheta_r\left([\mathbf{x},\mathbf{y}]_s\right) = [\vartheta_r(\mathbf{x}),\vartheta_r(\mathbf{y})] \ \text{ for } \ \mathbf{x} \ \text{ and } \ \mathbf{y} \in T_s\mathbb{S}.$$

As on the Lie algebra of a Lie group we also define the coadjoint bracket $\{\cdot,\cdot\}_s$

$$< \{\mathbf{x},z\}_s,\mathbf{y} > = - < z,[\mathbf{x},\mathbf{y}]_s > \ \text{ for } \mathbf{x},\mathbf{y} \in T_s\mathbb{S},\ z \in T_s^*\mathbb{S}.$$

2.4. Left and right differentials

The parallelizations of $T\mathbb{S}$, allowing to identify $T\mathbb{S}$ with a product of manifolds $\mathbb{S}\times\mathfrak{g}$, simplify the differential calculus on $\mathbb{S}$. In the sequel we only consider the left and right derivatives of a derivable motion $t \mapsto s(t)$ ($t \in \mathbb{R}$ is the time) in $\mathbb{S}$

$$\frac{d^r}{dt}\,s(t) = \vartheta_r\left(\frac{ds}{dt}\,(t)\right), \qquad \frac{d^\ell}{dt}\,s(t) = \vartheta_\ell\left(\frac{ds}{dt}\,(t)\right).$$

The derivative inside the parenthesis is a vector of $T_{s(t)}\mathbb{S}$ which is transformed by ϑ_r and ϑ_ℓ into elements of $\mathfrak{g}$. So, the first order differential calculus with "motions" is reduced to operations on functions from $\mathbb{R}$ to the fixed vector-space $\mathfrak{g}$ what is much more simple than to calculate on maps from $\mathbb{R}$ to $T\mathbb{S}$.

2.5. Canonical connections on a principal homogeneous space

There exist two natural connections $\nabla^{(\mathrm{r})}$ and $\nabla^{(\ell)}$ on $\mathbb{S}$ which may be defined by the following conditions (where X denotes any vector field):

- $\nabla^{(\mathrm{r})}_X Y = 0$ whenever Y is a fundamental vector field on $\mathbb{S}$,
- $\nabla^{(\ell)}_X Y = 0$ whenever Y is a left invariant vector field on $\mathbb{S}$.

The proof is straightforward: there exist bases $\mathcal{B} = \{Y_1,\dots,Y_n\}$ of the module $\mathfrak{X}(\mathbb{S})$ of all the differentiable vector fields made with fundamental or with left invariant vector fields and every $Y \in \mathfrak{X}(\mathbb{S})$ is expanded in the form $Y = f_1Y_1 + \cdots + f_nY_n$ where $f_1,\dots,f_n$ are differentiable functions on $\mathbb{S}$. A connection ∇ is completly defined as soon as the $\nabla_X Y_k$ are known, in particular if they vanish.

Other forms of the definition of $\nabla^{(\mathrm{r})}$ and $\nabla^{(\ell)}$, useful for calculations, are

$$\vartheta_r\big(\nabla^{(\mathrm{r})}_X Y\big) = X.\vartheta_r(Y), \qquad \vartheta_\ell\big(\nabla^{(\ell)}_X Y\big) = X.\vartheta_\ell(Y),$$

$$\vartheta_r\left(\nabla^{(\mathrm{r})}_{\mathbf{x}} Y(s)\right) = d\left(\vartheta_r Y\right)(\mathbf{x}) \ \text{ for } \ \mathbf{x} \in T_s\mathbb{S} \ \text{ and } \ Y \in \mathfrak{X}(\mathbb{S})$$

(for example, $\vartheta_r(Y)$ denotes the map $s \mapsto \vartheta_r(Y(s))$ from $\mathbb{S}$ to $\mathfrak{g}$ and $X.\vartheta_r(Y) = d\vartheta_r(Y)(X)$ the derivative of this map evaluated on the vector field X. The vector field $\nabla^{(\mathrm{r})}_X Y$ or the tangent vector $\nabla^{(\mathrm{r})}_{\mathbf{x}} Y(s)$ are well defined by these relations and, as it is readily verified, a connection on $\mathbb{S}$ is well defined in this way). Note two important properties we will use in the following (where $\nabla = \nabla^{(\mathrm{r})}$):

- If X and $Y \in \mathfrak{X}_\ell(\mathbb{S})$ then $\nabla_X Y = [\![X, Y]\!]$,
- If $\mathbf{x} \in T_s\mathbb{S}$ and $Y \in \mathfrak{X}_\ell(\mathbb{S})$ then $\nabla_{\mathbf{x}} Y(s) = [\mathbf{x}, Y(s)]_s$.

2.6. Particular case

The previous mathematical developments may be applied to a Lie group $\mathbb{G}$ acting on itself by left translations (that is to the principal homogeneous space $(\mathbb{G}; \mathbb{G})$). Maurer–Cartan forms are then the classical right and left forms $\vartheta_r g.g^{-1}$, $\vartheta_\ell = g^{-1}.dg$ of Lie group theory:

$$\vartheta_r(\mathbf{x}) = \mathbf{x}.g^{-1}, \quad \vartheta_\ell(\mathbf{x}) = g^{-1}.\mathbf{x} \qquad \text{when} \qquad \mathbf{x} \in T_g\mathbb{G},$$

and, the fundamental vector fields are nothing but right invariant vector fields on $\mathbb{G}$. Differentiation of the adjoint and coadjoint representations leads to very important relations (where $\mathbf{x} \in T_g\mathbb{G}$):

$$d\mathrm{Ad}\,(\mathbf{x}) \;=\; \mathrm{ad}\,\vartheta_r(\mathbf{x}) \circ \mathrm{Ad}\,g \equiv \mathrm{Ad}\,g \circ \mathrm{ad}\,\vartheta_\ell(\mathbf{x}), \tag{2.6}$$

$$d\mathrm{Ad}^*(\mathbf{x}) \;=\; \mathrm{ad}^*\vartheta_r(\mathbf{x}) \circ \mathrm{Ad}^*g \equiv \mathrm{Ad}^*g \circ \mathrm{ad}^*\vartheta_\ell(\mathbf{x}). \tag{2.7}$$

3. Mechanical interpretation

After a frame of reference was choosen, two given positions of a rigid object are related by one and only one transformation belonging to a group and, when s_o is a fixed reference position, each position s is determined by the element of the group transforming s_o into s. In other words, the positions of the body are described in a principal homogeneous space.

For instance, the right-hand orthonormal frames in an Euclidean affine space make a concrete principal homogeneous space that is commonly used to describe positions of rigid bodies in mechanics. The parameters used in practice to describe the positions of a rigid body are nothing but coordinates on $\mathbb{S}$. For example $s \leftrightarrow (x, y, z, \theta, \varphi, \psi)$ where x, y, z are the Cartesian coordinates of a point of the body and θ, φ, ψ are Euler angles specifying the position of a body fixed frame with respect to a reference position. In fact the meaning of this process is that $(x, y, z, \theta, \varphi, \psi)$ are coordinates, in the Euclidean group, of the displacement transforming the reference position s_o into s. However, it is possible to carry on all the reasoning intrinsically without coordinates as in the following.

Most of the works on dynamics of rigid body and geometry, to say the least, assume that the configuration space itself is a Lie group, what comes down to assuming the choice of a reference position s_o and to identifying $(\mathbb{S}, \mathbb{G})$ and $(\mathbb{G}, \mathbb{G})$

by the isomorphism σ_{s_o}. The picture referring to a principal homogeneous space rather than a Lie group is more faithful since the principles of dynamics are independent of such a choice and it turns out that they rely on the canonical objects associated with the principal homogeneous space structure (Maurer–Cartan form $\vartheta = \vartheta_r$, canonical connection $\nabla = \nabla^{(\mathrm{r})}$, left invariant Riemannian structures or left invariant homomorphisms to describe inertia, which are specific of the action of the group on $\mathbb{S}$).

For an ordinary rigid body, $\mathbb{G}$ is the Euclidean group of a three-dimensional affine Euclidean space $\mathcal{E}$. Generalizations rely on other groups, for instance $\mathbb{G}$ is an affine group for affinely deformable bodies. Mechanics in n-dimensional space also deserves research. In all these cases the Lie algebra $\mathfrak{g}$ is isomorphic with an algebra of vector fields on a space $\mathcal{E}$ (moment fields or others).

Tangent vectors of $\mathbb{S}$ describe velocities $\mathbf{v} = \dot{s}$ of motions $t \mapsto s(t)$ in $\mathbb{S}$. In concrete situations the right and left derivatives $\vartheta_r(\mathbf{v}) = V$ and $\vartheta_\ell(\mathbf{v}) = W$ represent vector-fields on a space $\mathcal{E}$ ("twists" or "screws" comprising linear and angular velocities, see Sec. 7.1). Cotangent vectors $\mathbf{f}$ may for example describe forces (say "torsors", "motors" or "wrenches" comprising ordinary forces and torque) more precisely that is $\vartheta_r^*(\mathbf{f})$ and $\vartheta_\ell^*(\mathbf{f})$ which correspond to those familiar things. An element δg of $\mathfrak{g} = T_e\mathbb{G}$ may be interpreted as an "infinitesimal" element of the group $\mathbb{G}$ and $\delta s = \delta g.s$, such as $\vartheta_r(\delta s) = \delta g$, describes an "infinitesimal displacement of the body" in position s.

From the Eulerian standpoint, that is the picture of mechanics with respect to "space-fixed" (inertial) frames, kinematics relies on the upper isomorphisms of diagram (2.4), the left action of $\mathbb{G}$ on $T\mathbb{S} \simeq \mathbb{S}\times\mathfrak{g}$ and $T^*\mathbb{S} \simeq \mathbb{S}\times\mathfrak{g}^*$, the canonical connection $\nabla = \nabla^{(\mathrm{r})}$ and so on. For instance, the connection is involved in the statement of the fundamental law of dynamics for taking the derivative of the momentum with respect to inertial frames. From the Lagrangian standpoint, the picture of mechanics with respect to body-fixed frames, a reference configuration s_o is fixed and one uses the lower isomorphisms of (2.4), the right action of $\mathbb{G}$ on $\mathbb{G} \times \mathfrak{g}$ and $\mathbb{G} \times \mathfrak{g}^*$ and so on. The connection $\nabla^{(\ell)}$ and other left invariant connections play a role in Levi–Civita connections for left invariant metrics. (The mathematical aspects of relations between the Eulerian and Lagrangian points of view are studied in more detail in [9].)

4. Galilean and kinematic groups. Frames of reference and objects in the Newtonian Dynamics

The statement of the Galilean properties of invariance requires further properties of the group $\mathbb{G}$. In fact we need a subgroup to play the role of the translation group and, from now on, to develop kinematics we will assume that

$$\begin{gathered}\mathbb{G} \textit{ is a connected Lie group,}\\ \mathbb{T} \textit{ is a closed normal commutative Lie subgroup of } \mathbb{G}.\end{gathered} \tag{4.1}$$

The consequence at the level of Lie algebras is:

$\mathfrak{t}$ *(the Lie algebra of* $\mathbb{T}$*) is a commutative ideal of* $\mathfrak{g}$.

Note that, since $\mathbb{G}$ is connected, $\mathfrak{t}$ is an invariant subspace for the adjoint representation of $\mathbb{G}$ in $\mathfrak{g}$. A map of the form $t \mapsto \exp(t\mathbf{u})$ with $\mathbf{u} \in \mathfrak{t}$ describes a uniform motion of translation and if A and $B \in \mathbb{G}$ and $\mathbf{u}$ and $\mathbf{v} \in \mathfrak{t}$, then for all $t \in \mathbb{R}$:

$$A \exp(t\mathbf{u}) = \exp(t \,\mathrm{Ad}\, A.\mathbf{u})A,$$
$$(B \exp(t\mathbf{v}))^{-1} A \exp(t\mathbf{u}) = B^{-1} A \exp(t\mathbf{w}) \text{ with } \mathbf{w} = \mathbf{u} - \mathrm{Ad}\,(A^{-1}B).\mathbf{v} \;(\in \mathfrak{t}).$$

□ The first formula is standard in the Lie group theory. The second one follows from:

$$(B \exp(t\mathbf{v}))^{-1} A \exp(t\mathbf{u}) = \exp(-t\mathbf{v}).(A^{-1}B)^{-1} \exp(t\mathbf{u})$$
$$= B^{-1} A \exp\big(- t\mathrm{Ad}\,(A^{-1}B).\mathbf{v}\big) \exp(t\mathbf{u}) = B^{-1} A \exp\big(t\mathbf{u} - t\mathrm{Ad}\,(A^{-1}B).\mathbf{v}\big)$$

(this transformation of the exponentials is licit according to a general result since $\mathbf{u}$ and $\mathrm{Ad}\,(A^{-1}B).\mathbf{v}$ are in $\mathfrak{t}$ and their Lie bracket vanishes). ∎

Those formulas justify the statement of the following definitions:

Definition 1.

1) *The group of kinematics is the group* $\mathbf{G}_{\mathrm{kin}} = \mathcal{C}^{\infty}(\mathbb{R}; \mathbb{G})$ *(with the natural group structure).*
2) *The subgroup* $\mathbf{G}_{\mathrm{gal}}$ *of* $\mathbf{G}_{\mathrm{kin}}$ *whose elements are of the form* $t \mapsto A.\exp(t\mathbf{u})$ *or, what is equivalent, of the form* $t \mapsto \exp(t\mathbf{u}).A$ *with* $A \in \mathbb{G}$, $\mathbf{u} \in \mathfrak{t}$, *is the Galilée group.*

It is worth noting that our definition of the group of kinematics and the Galilée group are not exact generalizations of the standard definitions. According to the standard definitions, those groups should operate on a space-time, in the present case $\mathbb{R} \times \mathbb{S}$. For instance the form of the elements of the Galilean group should be $(t, s) \mapsto (kt + a, A.\exp(t\mathbf{u}).s)$ $(k > 0)$ including changes of unit and origin of time. However, such a group contains a normal subgroup isomorphic with $\mathbf{G}_{\mathrm{gal}}$ and, in the sequel, invariance under translations of time or change of time unit will be obvious. In other words, according to the assumption of an absolute time in Newtonian mechanics, we may assume that a measure of time was choosen once for all, that our frames will always use synchroneous clocks, so that the interesting invariance groups to consider are $\mathbf{G}_{\mathrm{gal}}$ and $\mathbf{G}_{\mathrm{kin}}$.

A frame of reference is thought of as a material system, say axes linked with a rigid body, including a clock and used by an "observer" to attach time and position to various objects (or "events"). Two classes of frames of reference are used for observing the events in the universe: *kinematical* frames and *inertial* (or *Galilean*) frames. For the mathematical development of dynamics it is sufficient to know the following outline of the theory of frames and observations (the way to a more complete formalization was presented in [5]):

1. To any pair $(\mathcal{F}_1, \mathcal{F}_2)$ of kinematical frames is associated a unique element A_{12} of $\mathbf{G}_{\text{kin}}$.
2. If $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_2$ are kinematical frames then $A_{13} = A_{23}.A_{12}$ (in the group $\mathbf{G}_{\text{kin}}$).
3. If $\mathcal{F}_1$ and $A \in \mathbf{G}_{\text{kin}}$ are given, there exists a unique frame $\mathcal{F}_2$ such that $A_{12} = A$.
4. The class of inertial frames is a non-empty subclass of the kinematical frames.
5. If $\mathcal{F}_1$ is inertial and $\mathcal{F}_2$ is any frame, then $\mathcal{F}_2$ inertial $\Leftrightarrow A_{12} \in \mathbf{G}_{\text{gal}}$.

Observers translate their observations into a common mathematical language and two observers can decide whether or not they are observing the same object and measuring the same thing according to the following rules, which are the cornerstone of the concept of objectivity:

6. To each object is associated a space $\mathbb{X}$ left operated by the group $\mathbb{G}$.
7. With respect to a (kinematical) frame, "events" regarding an object are described by pairs $(t, x) \in \mathbb{R} \times \mathbb{X}$ (time and "location" of the object observed from the frame).
8. If (t_1, x_1) and (t_2, x_2) describe the same event relative to frames $\mathcal{F}_1$ and $\mathcal{F}_2$ then
$$t_2 = t_1, \quad x_2 = A_{12}(t_1).x_1$$
(where . means the action of $\mathbb{G}$ on $\mathbb{X}$ according to 1).

It should be possible, and necessary for other purposes, to take into account more general laws for the changes of frames including the changes of origin or unit of time (and perhaps of unit of length).

Example 1. When the "events" are positions of a rigid body, the space $\mathbb{X}$ will be a principal homogeneous space $\mathbb{S}$. With respect to a frame, a position of the body at some time will be described by a pair $(t, s) \in \mathbb{R} \times \mathbb{S}$ and the positions with respect to frames $\mathcal{F}_1$ and $\mathcal{F}_2$ will be related by $s_2 = A_{12}(t).s_1$. This relation creates the opportunity to specify the meaning of A_{12}: if the body is fixed with respect to $\mathcal{F}_1$ in position s_1, then s_2 is a function $\mathbb{R} \to \mathbb{S}$ which describes the motion of the body seen from $\mathcal{F}_2$. Hence, the function $t \mapsto A_{12}(t)$ describes the motion of the frame $\mathcal{F}_1$ observed with respect to $\mathcal{F}_2$.

Example 2. When the "events" we consider are forces acting on a rigid body, the space $\mathbb{X}$ will be $T^*\mathbb{S}$; with respect to a frame, a force will be described by an element $\mathbf{f} \in T^*_s\mathbb{S}$ where the origin s in $\mathbb{S}$ is the position of the body subjected to the force (Here force again means generalized force, including ordinary forces and torques and described by "wrenches" or "motors"). A fundamental principle of dynamics states that *forces are objective quantities*. Objectivity of forces means that if $(t_1, \mathbf{f}_1)$ and $(t_2, \mathbf{f}_2)$ describe the same force relative to two given frames $\mathcal{F}_1$ and $\mathcal{F}_2$ then

$$t_1 = t_2 = t, \qquad \mathbf{f}_2 = A_{12}(t).\mathbf{f}_1$$

where, to the right, the operation is the natural left action of $\mathbb{G}$ on $T^*\mathbb{S}$. Using the parallelization (2.4) of the cotangent bundle of $\mathbb{S}$, from the Eulerian standpoint, a force is also described by the pair $(s, \mathbf{F}) \in \mathbb{S} \times \mathfrak{g}^*$ $(s = o(\mathbf{f}\,), \mathbf{F} = \vartheta(\mathbf{f}\,))$. The law of change of frames now reads

$$\mathbf{F}_2 = \operatorname{Ad}^* A_{12}(t).\mathbf{F}_1.$$

Another definition of forces could rely on the spaces $\mathbb{X} = T\mathbb{S}$ and $\mathbb{S} \times \mathfrak{g}$ and leads to $\mathbf{f}_2 = A_{12}(t).\mathbf{f}_1$ with the natural left action of $\mathbb{G}$ on $T\mathbb{S}$ and $\mathbf{F}_2 = \operatorname{Ad} A_{12}(t).\mathbf{F}_1$ with the adjoint action. The former definition fits the D'Alembert-Lagrange form of mechanics where forces are defined through the work or power they expand in virtual displacements or virtual velocities. The latter definition fits the Newton-Euler form of mechanics where forces are described as "motors". Both definitions are equivalent when $\mathbb{G}$ has a left and right invariant pseudo-Riemannian structure as the Euclidean group.

EXAMPLE 3. Inertia operators generalizing the inertia tensors will be defined as objects described in $\mathbb{X} = \mathcal{L}(\mathfrak{g}, \mathfrak{g}^*)$ with the left action of $\mathbb{G}$ defined as $g.\mathbf{H} = \operatorname{Ad}^* g \circ \mathbf{H} \circ \operatorname{Ad} g^{-1}$ (for more details see Secs. 7 and 7.3).

5. Composition of motions in the space $\mathbb{S}$

In this section we summarize the laws of composition of velocities and accelerations in the framework of Sec. 2 and 3. With respect to kinematical frames $\mathcal{F}_1$ and $\mathcal{F}_2$, a motion of a rigid body is observed as $t \mapsto s_1 = s_1(t)$ and $t \mapsto s_2 = s_2(t)$ so that $s_2 = A_{12}(t).s_1$ (in the rest of this section we will generally omit to mention the time-dependance for positions, velocities and accelerations). Taking the derivative we obtain the relation between velocities with respect to $\mathcal{F}_1$ and $\mathcal{F}_2$ as an equality in $T_{s_2}\mathbb{S}$

$$\mathbf{v}_2 = L^T_{A_{12}}(t).\mathbf{v}_1 + U_{12}.s_2 \ \text{ with } \ U_{12} = \vartheta\left(\frac{dA_{12}}{dt}\right). \tag{5.1}$$

We recognize the velocity relative to $\mathcal{F}_1$ transformed into an observation from $\mathcal{F}_2$ and the induced velocity of $\mathcal{F}_1$ relative to $\mathcal{F}_2$ (velocity with respect to $\mathcal{F}_2$ of a body which should be at the same position and fixed relative to $\mathcal{F}_1$). Putting $\vartheta(\mathbf{v}_1) = V_1$ and $\vartheta(\mathbf{v}_2) = V_2$, relations (5.1) and (2.2) imply

$$V_2 \;=\; \operatorname{Ad} A_{12}(t).V_1 + U_{12}, \tag{5.2}$$

$$\dot{V}_2 \;=\; \operatorname{Ad} A_{12}(t).\dot{V}_1 + \dot{U}_{12} + [U_{12}(t), \operatorname{Ad} A_{12}(t).V_1]. \tag{5.3}$$

Induced velocities at time t are values of a fundamental vector field $\mathbf{U}_{12}(t)$ on $\mathbb{S}$, the velocity field of $\mathcal{F}_1$ relative to $\mathcal{F}_2$ depending on the two frames and defined by $\mathbf{U}_{12}(t)(s) = U_{12}(t).s$ (with the infinitesimal action of $\mathfrak{g}$ on $\mathbb{S}$). For three frames $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$, the induced velocities verify:

$$U_{13}(t) = U_{23}(t) + \operatorname{Ad} A_{23}(t).U_{12}(t),$$

$$\dot{U}_{13}(t) = \dot{U}_{23}(t) + \operatorname{Ad} A_{23}(t).\dot{U}_{12}(t) + [U_{23}(t), \operatorname{Ad} A_{23}(t).U_{12}(t)].$$

In the following we will use some remarks for shortening the calculations.

1) When $\mathcal{F}_1 = \mathcal{F}_3$ we obtain $U_{21}(t) = -\operatorname{Ad} A_{21}(t).U_{12}(t)$. Then U_{12} denotes an observation from the frame $\mathcal{F}_2$. Put $W_{12}(t) = \operatorname{Ad} A_{21}(t).U_{12}$, an observation from $\mathcal{F}_1$ regarding objects fixed relative to $\mathcal{F}_2$, then formulas (5.2) and (5.3) and relations for induced velocities read:

$$V_2 = \operatorname{Ad} A_{12}(t).\,(V_1 + W_{12})\,, \tag{5.4}$$

$$\dot{V}_2 = \operatorname{Ad} A_{12}(t).\,\Big(\dot{V}_1 + \dot{W}_{12} + [W_{12}, V_1]\Big)\,, \tag{5.5}$$

$$W_{13} = W_{12} + \operatorname{Ad} A_{21}.W_{23}, \tag{5.6}$$

$$\dot{W}_{13} = \dot{W}_{12} + \operatorname{Ad} A_{21}.\dot{W}_{23} + [\operatorname{Ad} A_{21}.W_{23}, W_{12},]. \tag{5.7}$$

2) When $\mathcal{F}_1$ and $\mathcal{F}_2$ are inertial frames, $A_{12}(t) = \exp(t\mathbf{u}).A$, with $\mathbf{u} \in \mathfrak{t}$ and $A \in \mathbb{G}$ independent of t. Taking into account the properties of the one-parameter subgroups of $\mathbb{G}$, we obtain

$$U_{12}(t) = \mathbf{u}, \quad U_{21}(t) = -\operatorname{Ad} A^{-1}.\mathbf{u}, \quad W_{12} = \operatorname{Ad} A^{-1}.\mathbf{u}, \quad W_{21} = -\mathbf{u}.$$

3) The previous formulas also suggest to define an operation $\odot$ on $\mathfrak{g} \times \mathfrak{g}$ by:

$$(\mathbf{u}, \mathbf{u}') \odot (\mathbf{v}, \mathbf{v}') = (\mathbf{u} + \mathbf{v}, \mathbf{u}' + \mathbf{v}' + [\mathbf{u}, \mathbf{v}]).$$

It is easy to prove that, endowed with this operation, $\mathfrak{g} \times \mathfrak{g}$ becomes a non-commutative group denoted by $\mathfrak{g}_{(2)}$ and whose identity element is $(0,0)$. The inverse of $(\mathbf{u}, \mathbf{u}')$ in $\mathfrak{g}_{(2)}$ is expressed as

$$(\mathbf{u}, \mathbf{u}')^{-1} = (-\mathbf{u}, -\mathbf{u}').$$

The adjoint representation of $\mathbb{G}$ in $\mathfrak{g}$ extends into a representation of $\mathbb{G}$ in the group $\mathfrak{g}_{(2)}$ with

$$\operatorname{Ad} g.(\mathbf{u}, \mathbf{u}') = (\operatorname{Ad} g.\mathbf{u}, \operatorname{Ad} g.\mathbf{u}') \ \text{ for } \ g \in \mathbb{G}.$$

In fact, due to the properties of the adjoint representation and the Lie bracket, $\operatorname{Ad} g$, acting to the left as in the previous definition, is an automorphism of the group $\mathfrak{g}_{(2)}$:

$$\operatorname{Ad} g.\big\{(\mathbf{u}, \mathbf{u}') \odot (\mathbf{v}, \mathbf{v}')\big\} = (\operatorname{Ad} g.(\mathbf{u}, \mathbf{u}')) \odot (\operatorname{Ad} g.(\mathbf{v}, \mathbf{v}')).$$

With the associative law $\odot$, relations (5.4), (5.5), (5.6) and (5.7) for composition of velocities and accelerations of a rigid body with respect to kinematical frames $\mathcal{F}_1$ and $\mathcal{F}_2$ may be condensed into the very compact and efficient form (similar relations exist for the induced velocities U_{ij})

$$(5.8)\qquad (V_2, \dot{V}_2) = (U_{12}, \dot{U}_{12}) \odot \mathrm{Ad}\, A_{12}.(V_1, \dot{V}_1) = \mathrm{Ad}\, A_{12}.\big\{(W_{12}, \dot{W}_{12}) \odot (V_1, \dot{V}_1)\big\},$$

$$(5.9)\qquad (W_{13}, \dot{W}_{13}) = (\mathrm{Ad}\, A_{21}.(W_{23}, \dot{W}_{23})) \odot (W_{12}, \dot{W}_{12}).$$

6. Foundations of dynamics

In Secs. 6.1 and 6.2 we propose two approaches to introduce inertial mass. They fit the two classical ways of developing mechanics namely the D'Alembert–Lagrange standpoint based on kinetic energy and the Euler standpoint based on the relation velocity-momentum and they turn out to be almost equivalent when $\mathbb{G}$ possesses a property which is verified for the Euclidean group in three dimensions. At the next stage, in Sec. 6.3, we will study the "inertial forces" acting on a body and the problem of their objectivity and of Galilean invariance of the law of dynamics. The mathematical structure of the law with respect to non-inertial frames will be studied in Sec. 6.4. In Sec. 6.6 we suggest definitions of "absolute" quantities and point out their links with the previous results.

From now on, ϑ and ∇ will denote the Maurer–Cartan form ϑ_r and connection $\nabla^{(\mathrm{r})}$.

6.1. The definition of mass in generalized rigid body dynamics

The first point of view for defining the "mass" is based on the bilinear form associated with the kinetic energy, that is on the following assumption:

(M) $\quad (\cdot \mid \cdot)$ is a left invariant Riemannian structure on $T\mathbb{S}$.

An equivalent statement is:

(M) $\quad H$ is a left invariant symmetric positive vector bundle homomorphism from $T\mathbb{S}$ to $T^*\mathbb{S}$.

These forms of (M) are related by $< H(\mathbf{x}), \mathbf{y} >= (\mathbf{x} \mid \mathbf{y})$ for $(\mathbf{x}, \mathbf{y}) \in T\mathbb{S} \times_{\mathbb{S}} T\mathbb{S}$ and $H(\mathbf{v})$ is the (generalized) momentum associated to the velocity $\mathbf{v}$. The kinetic energy is the quadratic form $\frac{1}{2}(\mathbf{v} \mid \mathbf{v}) = \frac{1}{2} < H(\mathbf{v}), \mathbf{v} >$ and conversely, this quadratic form determines the Riemannian structure.

An equivalent way to determine a symmetric left invariant vector bundle homomorphism $H\colon T\mathbb{S} \to T^*\mathbb{S}$ is to give an analytical map from $\mathbb{S}$ to $\mathcal{L}(\mathfrak{g}, \mathfrak{g}^*)$

denoted by $s \mapsto H_s$ and such that:

$$\vartheta^*\big(H(\mathbf{x})\big) = H_s\big(\vartheta(\mathbf{x})\big) \ \text{ for } \ \mathbf{x} \in T_s\mathbb{S}, \tag{6.1}$$

$$H_{g.s} = \mathrm{Ad}^* g \circ H_s \circ \mathrm{Ad}\, g^{-1} \ \text{ for } \ g \in \mathbb{G}, \ s \in \mathbb{S}, \tag{6.2}$$

$$< H_s(\mathbf{u}), \mathbf{v} > = < H_s(\mathbf{v}), \mathbf{u} > \ \text{ for } \ \mathbf{u}, \ \mathbf{v} \in \mathfrak{g}, \ s \in \mathbb{S}. \tag{6.3}$$

According to (6.1) H_s is the linear map from $T_s\mathbb{S}$ to $T_s^*\mathbb{S}$ induced by H when $T_s\mathbb{S}$ and $T_s^*\mathbb{S}$ are identified with $\mathfrak{g}$ and $\mathfrak{g}^*$; this is a generalization of the covariant inertia tensor of a rigid body in position s. Our assumption on H means only two things. First, in a fixed position s of the body, the relation velocity-momentum is linear (and symmetric positive definite here), second the "mass" is linked with the body, a natural property expressed by (6.2) (invariance property relating inertia operator to position).

Formula (6.2) may be derived from $H(g.\mathbf{x}) = g.H(\mathbf{x})$, (2.2) and (2.5):

$$\vartheta^*\big(H(g.\mathbf{x})\big) = H_{g.s}\big(\vartheta(g.\mathbf{x})\big) = H_{g.s}\big(\mathrm{Ad}\, g.\vartheta(\mathbf{x})\big),$$

$$\vartheta^*\big(g.H(\mathbf{x})\big) = \mathrm{Ad}^* g.\big(\vartheta^* H(\mathbf{x})\big) = \mathrm{Ad}^* g.H_s\big(\vartheta(\mathbf{x})\big),$$

so that $H_{g.s} \circ \mathrm{Ad}\, g = \mathrm{Ad}^* g \circ H_s$. Taking the differential of (6.2) with respect to g at $g = e$ and using (2.6) and (2.7), we obtain the formula for the derivative of $s \mapsto H_s$: for any $\mathbf{x}$ in $T_s\mathbb{S}$

$$dH_s(\mathbf{x}) = \mathrm{ad}^* \vartheta(\mathbf{x}) \circ H_s - H_s \circ \mathrm{ad}\, \vartheta(\mathbf{x}), \tag{6.4}$$

(note that both sides of (6.4) are linear operators from $\mathfrak{g}$ to $\mathfrak{g}^*$). Formula (6.4) may be interpreted in the framework of canonical connection on $\mathbb{S}$: the covariant derivative in the canonical connection of a left invariant vector bundle morphism H from $T\mathbb{S}$ to $T^*\mathbb{S}$ is expressed as:

$$(\nabla_{\mathbf{x}} H)(\mathbf{y}) = \{\mathbf{x}, H(\mathbf{y})\}_s - H\left([\mathbf{x}, \mathbf{y}]_s\right) \ \text{ for } \ \mathbf{x}, \ \mathbf{y} \in T_s\mathbb{S}, \tag{6.5}$$

(where the brackets $[\cdot,\cdot]_s$ and $\{\cdot,\cdot\}_s$ were defined in Sec. 2). Another interpretation is that, for two *left invariant* vector fields X and Y on $\mathbb{S}$:

$$\left(\nabla_X H\right)(Y) = \{X, H(Y)\} - H([\![X, Y]\!]). \tag{6.6}$$

6.2. Alternative description of inertial mass

Another statement of the basic assumption on "mass" could be

(M′) H is a left invariant vector bundle endomorphism of $T\mathbb{S}$.

That is a direct definition of a relation velocity-momentum: the momentum associated with the velocity $\mathbf{v}$ is now defined as a tangent vector $H(\mathbf{v})$ of $\mathbb{S}$ without assumption regarding symmetry or positivity of H. An equivalent definition is a map from $\mathbb{S}$ to $\mathcal{L}(\mathfrak{g})$ denoted by $s \mapsto H_s$ and such that:

$$\vartheta\big(H(\mathbf{x})\big) = H_s\big(\vartheta(\mathbf{x})\big) \quad \text{for} \quad x \in T_s\mathbb{S}, \tag{6.7}$$

$$H_{g.s} = \operatorname{Ad} g \circ H_s \circ \operatorname{Ad} g^{-1} \quad \text{for} \quad g \in \mathbb{G},\ s \in \mathbb{S}. \tag{6.8}$$

Now the operator H_s is the generalization of the mixed inertia tensor (or inertia operator) of a rigid body in position s. Taking the differential of (6.8) at $g = e$ we obtain

$$dH_s(\mathbf{x}) = \operatorname{ad} \vartheta(\mathbf{x}) \circ H_s - H_s \circ \operatorname{ad} \vartheta(\mathbf{x}). \tag{6.9}$$

In the framework of canonical connections on $\mathbb{S}$, formula (6.9) means that the covariant derivative in the canonical connection of a left invariant vector bundle endomorphism H of $T\mathbb{S}$ is expressed as:

$$(\nabla_{\mathbf{x}} H)(\mathbf{y}) = [\mathbf{x}, H(\mathbf{y})]_s - H\left([\mathbf{x}, \mathbf{y}]_s\right),\ \mathbf{x},\ \mathbf{y} \in T_s\mathbb{S}, \tag{6.10}$$

or, for two *left invariant* vector fields X and Y on $\mathbb{S}$:

$$(\nabla_X H)(Y) = [\![X, H(Y)]\!] - H([\![X, Y]\!]). \tag{6.11}$$

In general, the statements (M) or (M′) are mathematically irrelevant one to the other. Nevertheless, whenever there exists on $\mathfrak{g}$ inner product $[\cdot \mid \cdot]_{\mathfrak{g}}$ which is invariant by the operators $\operatorname{Ad} g$, then $\mathbb{G}$ and $\mathbb{S}$ may be endowed with a left and right invariant pseudo-Riemannian structure also denoted by $[\cdot \mid \cdot]$. This structure on $\mathbb{S}$ is defined by

$$[\mathbf{x} \mid \mathbf{y}] = [\vartheta(\mathbf{x}) \mid \vartheta(\mathbf{y})]_{\mathfrak{g}} \quad \text{for} \ (\mathbf{x}, \mathbf{y}) \in T\mathbb{S} \times_{\mathbb{S}} T\mathbb{S}.$$

Now it is possible to relate (M) and (M′) by

$$[H(\mathbf{x}) \mid \mathbf{y}] = (\mathbf{x} \mid \mathbf{y}). \tag{6.12}$$

When (M) is assumed, relation (6.12) defines a *symmetric and positive* left invariant endomorphism $H \in \mathcal{L}(T\mathbb{S})$. Conversely, when (M′) is assumed with additional properties of symmetry and positivity of the left-hand side, (6.12) leads to a Riemannian structure on $\mathbb{S}$. Symmetry, if it is used, means that the relation velocity-momentum is determined as soon as the kinetic energy is given. The Euclidean group in dimension 3 possesses such a structure so that, for an ordinary rigid body, the statements (M) and (M′) plus symmetry and positivity of H are equivalent.

In Sec. 6 we chiefly refer to the first point of view. Meanwhile the results proved with (M) thereafter may be transposed to the second point of view up to slight changes: keeping the notation H in both cases it is only necessary to change the bracket $\{\cdot, \cdot\}$ into $[\cdot, \cdot]$, the bracket $< \cdot, \cdot >$ into $[\cdot \mid \cdot]$ and the coadjoint representation into the adjoint representation and sometimes to specify some additional properties of symmetry and positivity when (M′) is used.

6.3. Necessary and sufficient condition for objectivity of inertial forces

Consider a motion of a rigid body observed with respect to an inertial frame and described by a function $t \mapsto s(t) \in \mathbb{S}$. In accordance with Newton–Euler mechanics, the inertial force will be defined at every time as minus the derivative of the momentum, that is the cotangent vector of $\mathbb{S}$

$$\mathbf{j}\,(t) = -\frac{\nabla}{dt}\,H(\mathbf{v}(t)) \qquad \text{where} \qquad \mathbf{v} = \frac{ds}{dt}.$$

Then the motion of a rigid body subjected to a force $\mathbf{f}$ is governed by the differential equations

$$\frac{\nabla}{dt}\,H(\mathbf{v}) = \mathbf{f}, \qquad \mathbf{v} = \frac{ds}{dt}. \tag{6.13}$$

In particular, the inertial motion of a free body with respect to an inertial frame is governed by

$$\frac{\nabla}{dt}\,H(\mathbf{v}) = 0, \qquad \mathbf{v} = \frac{ds}{dt}. \tag{6.14}$$

However, any operator H cannot lead to inertial force verifying the principle of objectivity of forces. The mathematical object playing the role of a cornerstone is a left invariant tensor field C, covariant of degree 2 and contravariant of degree 1, on $\mathbb{S}$ or what is the same, the bilinear maps $C_s\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}^*$ $(s \in \mathbb{S})$ defined by

$$C_s(\mathbf{u}, \mathbf{v}) = \{\mathbf{u}, H_s(\mathbf{v})\} + \{\mathbf{v}, H_s(\mathbf{u})\} + H_s\left([\mathbf{u}, \mathbf{v}]\right) \qquad (\mathbf{u},\ \mathbf{v}\ \in\ \mathfrak{g}).$$

Within the framework of assumption (M′) $\mathbf{j}(t)$ and $\mathbf{f}$ would be tangent vectors rather than cotangent vectors of $\mathbb{S}$ defined in the same way, the bilinear maps should be $C_s\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by

$$C_s(\mathbf{u}, \mathbf{v}) = [\mathbf{u}, H_s(\mathbf{v})] + [\mathbf{v}, H_s(\mathbf{u})] + H_s\left([\mathbf{u}, \mathbf{v}]\right).$$

The bilinear maps C_s enjoy remarkable properties and they also play an important role in the mathematical structure of the dynamic equations for multibody systems (see [8]). In Sec. 6.5 we will meet the tensor C in another way and prove that (6.14) describes geodesics of the left invariant metrics on $\mathbb{S}$: this is the general "principle of inertia" of rigid body dynamics, similar to the first Newton's law.

THEOREM 1. *For a rigid body the inertia of which is described according to assumption (M) (or (M′)), the following properties 1 and 2 are equivalent:*

1. *For all inertial frames $\mathcal{F}$, all motion of the body described by $t \mapsto s_{\mathcal{F}}(t)$ with respect to $\mathcal{F}$ and all time t, there exists an objective force represented in $\mathcal{F}$ by $\mathbf{j}_{\mathcal{F}}(t) \in T_{s(t)}\mathbb{S}$ such that:*

$$\mathbf{j}_{\mathcal{F}}(t) = -\frac{\nabla}{dt} H(\mathbf{v}_{\mathcal{F}}(t)), \qquad \mathbf{v}_{\mathcal{F}}(t) = \frac{d}{dt} s_{\mathcal{F}}(t) \tag{6.15}$$

or, what is equivalent, by the "motor" $\mathbf{J}_{\mathcal{F}}(t) = \vartheta(\mathbf{j}_{\mathcal{F}}(t)) \in \mathfrak{g}$ *such that:*

$$\mathbf{J}_{\mathcal{F}}(t) = -\frac{d}{dt} H_{s_{\mathcal{F}}(t)}(V_{\mathcal{F}}(t)), \quad V_{\mathcal{F}}(t) = \vartheta\big(\mathbf{v}_{\mathcal{F}}(t)\big). \tag{6.16}$$

2. $C_s(\mathbf{u}, \mathbf{v}) = 0$ *for all* $s \in \mathbb{S}$*, all* $\mathbf{u} \in \mathfrak{t}$ *and all* $\mathbf{v} \in \mathfrak{g}$.

In Property 2 the bilinear map and the quantification ($\forall \mathbf{u} \in \mathfrak{t}$, $\forall \mathbf{v} \in \mathfrak{g}$) are not symmetric. Due to invariance by the transitive action of $\mathbb{G}$ on $\mathbb{S}$, the statement with "there exists $s \in \mathbb{S}$" instead of "for all $s \in \mathbb{S}$" is equivalent (it is sufficient to remark that (6.2) (or (6.8)) implies that $C_{g.s}(\mathrm{Ad}\, g.\mathbf{u}, \mathrm{Ad}\, g.\mathbf{v}) = \mathrm{Ad}^* g.C_s(\mathbf{u}, \mathbf{v})$ and that $\mathfrak{t}$ is invariant under the adjoint representation).

Theorem 1 means that objectivity of inertial forces, a property *a priori* involving differential calculus, is reduced to an algebraic property of H (or of the Riemannian structure) defining the velocity-momentum relation. In fact, the result makes sense under the assumption that H is a vector-bundle morphism and needs no assumption regarding symmetry or positivity. The proof of Theorem 1 relies on two lemmas. The first lemma reduces Property 1 to a property involving inertial frames only:

LEMMA 1. *A necessary and sufficient condition for condition 1 from Theorem 1 holds is that for all motions and all pairs of* inertial *frames* $\mathcal{F}_1$ *and* $\mathcal{F}_2$*:*

$$\frac{d}{dt} H_{s_2}(V_2) = Ad^* A_{12}(t).\left(\frac{d}{dt} H_{s_1}(V_1)\right). \tag{6.17}$$

□ Since objectivity means that relation $\mathbf{J}_{\mathcal{F}_2}(t) = \mathrm{Ad}^* A_{12}(t).\mathbf{J}_{\mathcal{F}_1}(t)$ holds for all pairs of frames, condition (6.17) is necessary. Conversely when (6.17) is verified $\mathbf{J}_{\mathcal{F}}$ may be defined in every inertial frame by (6.15) and, when $\mathcal{F}_1$ et $\mathcal{F}_2$ are inertial frames and $\mathcal{F}_3$ is any kinematical frame, we have

$$\mathrm{Ad}^* A_{13}(t).\mathbf{J}_{\mathcal{F}_1}(t) = \mathrm{Ad}^* A_{23}(t).\mathrm{Ad}\, A_{12}.\mathbf{J}_{\mathcal{F}_1}(t) = \mathrm{Ad}^* A_{23}(t).\mathbf{J}_{\mathcal{F}_2}(t), \tag{$*$}$$

so that

$$\mathrm{Ad}^* A_{13}(t).\mathbf{J}_{\mathcal{F}_1}(t) = \mathrm{Ad}^* A_{23}(t).\mathbf{J}_{\mathcal{F}_2}(t).$$

If the objective force exists it is necessarily represented without ambiguity in a frame $\mathcal{F} = \mathcal{F}_3$ by

$$\mathbf{J}_{\mathcal{F}}(t) = \mathrm{Ad}^* A_{13}(t).\mathbf{J}_{\mathcal{F}_1}(t),$$

where $\mathcal{F}_1$ is any inertial frame. Now, this relation is a consistent definition of $\mathbf{J}_{\mathcal{F}}(t)$ since we have just proved that the right-hand side is independent of the choice of $\mathcal{F}_1$. Finally, the first equality ($*$), actually true when $\mathcal{F}_2$ is any frame, proves that $\mathbf{J}_{\mathcal{F}_3}(t) = \mathrm{Ad}\, A_{23}(t).\mathbf{J}_{\mathcal{F}_2}(t)$ for all pairs of kinematical frames, what is required by the definition of objective forces. ■

LEMMA 2. *When $\mathcal{F}_1$ is an* inertial *frame and $\mathcal{F}_2$ is any frame,*

$$\forall t \in \mathbb{R} : \ \mathrm{Ad}^* A_{12}(t).\mathbf{J}_{\mathcal{F}_1} = \begin{cases} -\Big(H_{s_2}\big(\dot{V}_2\big) + \{V_2, H_{s_2}(V_2)\}\Big) \\ -\, C_{s_2}(W_{21}, V_2) \\ -\Big(H_{s_2}\big(\dot{W}_{21}\big) + \{W_{21}, H_{s_2}(W_{21})\}\Big). \end{cases} \tag{6.18}$$

where, according to (6.16):

$$\mathbf{J}_{\mathcal{F}_1} = -\frac{d}{dt} H_{s_1}(V_1) = -\Big(H_{s_1}\big(\dot{V}_1\big) + \{V_1, H_{s_1}(V_1)\}\Big).$$

□ The proof of Lemma 2 is a sequence of straightforward transformations starting from (5.4) and (5.5) (where the roles of $\mathcal{F}_1$ and $\mathcal{F}_2$ are exchanged) and from property (6.2). ∎

□ For proving Theorem 1, remark that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are inertial frames, W_{12} is independent of t and belongs to $\mathfrak{t}$ so that relation (6.18) reduces to:

$$\mathrm{Ad}^* A_{12}(t).\mathbf{J}_{\mathcal{F}_1} = \mathbf{J}_{\mathcal{F}_2} - \Big(\{W_{21}, H_{s_2}(W_{21})\} + C_{s_2}(W_{21}, V_2)\Big).$$

Therefore, condition (6.17) is equivalent to the cancellation of the term inside the parenthesis for all motions of the body and and all frames, that is to say all values of W_{21} and V_2:

$$\forall s \in \mathbb{S},\ \forall \mathbf{u} \in \mathfrak{t},\ \forall \mathbf{v} \in \mathfrak{g} : \ C_s(\mathbf{u}, \mathbf{v}) + \{\mathbf{u}, H_s(\mathbf{u})\} = 0.$$

Since $C_s(\mathbf{u}, \mathbf{u}) = 2\{\mathbf{u}, H_s(\mathbf{u})\}$, the second term must be equal to zero and this condition is equivalent to property 2 from theorem. ∎

6.4. Law of dynamics in inertial and non-inertial frames

Objectivity of the inertial forces permits a direct statement of the second law of dynamics for a rigid body (needing no preliminary development of the principles of particle dynamics):

i) For every rigid body, at each time and with respect to any kinematical frame $\mathcal{F}$:

$$\mathbf{J}_{\mathcal{F}}(t) + \mathbf{F}_{\mathcal{F}}(t) = 0 \quad (\textit{equality in } \mathfrak{g}^*)$$

where $\mathbf{J}_{\mathcal{F}}$ and $\mathbf{F}_{\mathcal{F}} \in \mathfrak{g}^*$ are respectively the inertial and external forces acting on the body.

ii) With respect to an inertial *frame, the inertial force is expressed as:*

$$\mathbf{J}_{\mathcal{F}}(t) = -\frac{d}{dt} H_s(V_{\mathcal{F}}) \equiv -\Big(H_s\big(\dot{V}_{\mathcal{F}}\big) + [V_{\mathcal{F}}, H_s(V_{\mathcal{F}})]\Big),$$

so that, in an inertial *frame, the law of dynamics reads*

$$H_s\big(\dot V_{\mathcal F}\big) + [V_{\mathcal F}, H_s(V_{\mathcal F})] = \mathbf F_{\mathcal F}(t).$$

This is the "motor" (or "torsor") form of the law, the most useful to mechanicians. The law might also be expressed in $T^*\mathbb S$: it suffices to replace the preceding relations by:

$$\mathbf j_{\mathcal F}(t) + \mathbf f_{\mathcal F}(t) = 0 \quad \text{(an equality in } T^*_{s(t)}\mathbb S\text{)},$$

$$\mathbf j_{\mathcal F}(t) = -\frac{\nabla}{dt}\, H(\mathbf v) \ \text{ with respect to an inertial frame.}$$

Dynamics in non-inertial frames, that is calculation of inertial forces with respect to any frame, is based on Lemmas 2 and 3:

LEMMA 3. *When the conditions of Theorem 1 are verified, if $\mathcal F(=\mathcal F_2)$ is a kinematical frame and $W\ (=W_{21}) \in \mathfrak g$ is the relative velocity of $\mathcal F$ with respect to an inertial frame $\mathcal F_1$, then, for $s \in \mathbb S$ and $V \in \mathfrak g$, the quantities*

$$C_s(W,V) \ \text{ and } \ H_s(\dot W) + \{W, H_s(W)\}$$

are independent of the choice of $\mathcal F_1$ among inertial frames (they depend only on $\mathcal F$).

□ When the Condition 1 from Theorem 1 is verified, according to Lemmas 1 and 2, the right-hand side of (6.18) is independent of the choice of $\mathcal F_1$. Therefore, for all motion

$$C_s(W_{21},V_2) + H_s(\dot W_{21}) + \{W_{21}, H_s(W_{21})\}$$
$$= C_s(W_{20},V_2) + H_s(\dot W_{20}) + \{W_{20}, H_s(W_{20})\}.$$

Since the equality is true for any motion, choosing $V_2 = 0$ we obtain

$$H_s(\dot W_{21}) + \{W_{21}, H_s(W_{21})\} = H_s(\dot W_{20}) + \{W_{20}, H_s(W_{20})\}.$$

Equality $C_s(W_{21},V_2) = C_s(W_{20},V_2)$ follows. ■

THEOREM 2. *When the Conditions of Theorem 1 are verified, with every* kinematical *frame $\mathcal F$ are associated two functions*

$$\mathbf J^c_{\mathcal F} \colon \mathbb S \times \mathfrak g \to \mathfrak g^*, \quad \mathbf J^e_{\mathcal F} \colon \mathbb S \to \mathfrak g^*$$

such that:

i) $\mathbf J^c_{\mathcal F}$ and $\mathbf J^e_{\mathcal F}$ vanish when $\mathcal F$ is an inertial frame,

ii) the inertial force acting on the body observed with respect to $\mathcal F$ is expressed as

$$\mathbf J_{\mathcal F}(t) = \mathbf J^r_{\mathcal F}(s,V,\dot V) + \mathbf J^c_{\mathcal F}(s,V) + \mathbf J^e_{\mathcal F}(s),$$

with

$$\begin{cases} \mathbf{J}^r_{\mathcal{F}}(s, V, \dot{V}) & = -\Big(H_s(\dot{V}) + \{V, H_s(V)\}\Big) = -\dfrac{d}{dt}\, H_s(V), \\ \mathbf{J}^c_{\mathcal{F}}(s, V) & = -\, C_s(W, V), \\ \mathbf{J}^e_{\mathcal{F}}(s) & = -\Big(H_s(\dot{W}) + \{W, H_s(W)\}\Big), \end{cases}$$

where s and $V = \vartheta\,(\dot{s})$ are the position and the velocity of the body with respect to $\mathcal{F}$, W is the induced velocity with respect to any inertial frames, the choice of which is immaterial.

(In the framework of (M') the functions $\mathbf{J}^c_{\mathcal{F}}$ and $\mathbf{J}^e_{\mathcal{F}}$ should take their values in $\mathfrak{g}$ and the bracket $\{.,.\}$ should be replaced by $[.,.]$).

Theorem 2 is a straightforward consequence of Lemmas 2 and 3. The part $\mathbf{J}^r_{\mathcal{F}}(s, V, \dot{V})$ is *the relative inertial force* calculated as if $\mathcal{F}$ should be inertial. The part $\mathbf{J}^c_{\mathcal{F}}(s, V)$ is *the complementary inertial force*, in practice the Coriolis force and gyroscopic torque resulting from the rotation of the frame with respect to inertial frames and the velocity of the body with respect to $\mathcal{F}$. The part $\mathbf{J}^e_{\mathcal{F}}(s)$ is the *induced inertial force* equal to the inertial force acting on a body at rest in position s with respect to $\mathcal{F}$; it is composed of a force due to acceleration and a centrifugal force which is a quadratic function of the induced velocity. Remark that, for a symmetric H, say within assumption (M), by simple calculations:

$$< C_s(\mathbf{w}, \mathbf{v}), \mathbf{v} >= 0 \;\text{ for all } \mathbf{v} \text{ and } \mathbf{w} \text{ in } \mathfrak{g},$$

so that we find the property of the complementary inertial forces to be orthogonal to the relative velocity with respect to the non-inertial frame (and to expand no work).

6.5. Inertial motions and geodesics

As it was explained in ARNOLD [1], the motions of a free rigid body observed with respect to an inertial frame is described by the geodesics of a left invariant Riemannian structure on a Lie group. In this section we prove that, within the framework of assumption (M) of Sec. 6.1, equations (6.14) actually define geodesics and we meet the tensor C in another way. We also prove that those equations define as well the geodesics of a connection $\nabla^{(\mathrm{o})}$ with a vanishing curvature and a remarkable torsion (a fact which, to our knowledge, has not been noticed).

THEOREM 3. *The solutions of (6.14) are the geodesics of the Riemannian metrics $(\cdot \mid \cdot)$ on $\mathbb{S}$.*

□ The proof amounts to calculate the Levi–Civita connection $\nabla^{(\mathrm{m})}$ of the metrics $\gamma(\mathbf{x}, \mathbf{y}) = (\mathbf{x} \mid \mathbf{y})$ defined in (M). Due to general properties, the difference of two connections is a tensor and this connection is of the form

$$\nabla^{(\mathrm{m})}_X Y = \nabla^{(\ell)}_X Y + B(X,Y)$$

where B is a tensor we must calculate and which is covariant of degree 2 and contravariant of degree 1. By definition, the torsion of $\nabla^{(\mathrm{m})}$ vanishes and $\nabla^{(\mathrm{m})}\ \gamma = 0$. The condition on the torsion means

$$\nabla^{(\mathrm{m})}_X Y - \nabla^{(\mathrm{m})}_Y X - [\![X,Y]\!] = 0$$

for all vector fields X and $Y \in \mathfrak{X}(\mathbb{S})$. Since the left-hand side is a tensor, it is sufficient to write down the condition for left invariant vector fields, so that $\nabla^{(\mathrm{m})}_X Y = B(X,Y)$ and

$$B(X,Y) - B(Y,X) = [\![X,Y]\!] \ \text{ for left invariant } X \text{ and } Y \tag{6.19}$$

determining the skew-symmetric part of B. The condition $\nabla^{(\mathrm{m})}\gamma = 0$, again for left invariant vector fields, leads to

$$\begin{cases} \gamma(B(X,Y),Z) + \gamma(Y, B(X,Z)) &= 0, \\ \gamma(B(Y,Z),X) + \gamma(Z, B(Y,X)) &= 0, \\ \gamma(B(Z,X),Y) + \gamma(X, B(Z,Y)) &= 0, \end{cases}$$

hence, by the combination $++-$ and taking account of (6.19),

$$2\gamma(B(X,Y),Z) = \gamma([\![X,Y]\!],Z) + \gamma([\![Z,X]\!],Y) - \gamma([\![Y,Z]\!],X),$$

what is valid for all Z and determines $B(X,Y)$. Putting $\mathbf{u} = \vartheta_r(X(s))$, $\mathbf{v} = \vartheta_r(Y(s))$, $\mathbf{w} = \vartheta_r(Z(s))$, so that $\vartheta_r([\![X,Y]\!](s)) = [\mathbf{u},\mathbf{v}]$, and taking the value at the point s of $\mathbb{S}$ we easily obtain

$$2 < H_s \circ B_s(\mathbf{u},\mathbf{v}), \mathbf{w} > = < H_s([\mathbf{u},\mathbf{v}]) + \operatorname{ad}^*\mathbf{u}.H_s(\mathbf{v}) + \operatorname{ad}^*\mathbf{v}.H_s(\mathbf{u}), \mathbf{w} >,$$

hence

$$B_s(\mathbf{u},\mathbf{v}) = \frac{1}{2}\, H_s^{-1}\big(C_s(\mathbf{u},\mathbf{v})\big).$$

The geodesics of $\nabla^{(\mathrm{m})}$ are the curves $t \mapsto s(t)$ verifying $\nabla^{(\mathrm{m})}_{\dot s}\dot s = 0$, that is

$$\nabla^{(\ell)}_{\dot s}\dot s + B(\dot s, \dot s) = 0. \tag{6.20}$$

For transforming this equation remark that, using (2.6):

$$\frac{d}{dt}\,\vartheta_r(\dot s) = \operatorname{Ad} g.\left(\frac{d}{dt}\,\vartheta_\ell(\dot s)\right). \tag{6.21}$$

Applying $\vartheta_r = \operatorname{Ad} g.\vartheta_\ell$ to both sides of (6.20) and taking (6.21) into account and then applying $\operatorname{Ad} g^{-1}$, we derive two forms of the equation of geodesics:

$$\frac{d}{dt}\,\vartheta_r(\dot s) + B_s(\vartheta_r(\dot s), \vartheta_r(\dot s)) \;=\; 0, \tag{6.22}$$

$$\frac{d}{dt}\,\vartheta_\ell(\dot s) + B_{s_o}(\vartheta_\ell(\dot s), \vartheta_\ell(\dot s)) \;=\; 0. \tag{6.23}$$

Equations (6.20) and (6.22) or (6.23) are equivalent to systems with dynamic and kinematic equations

$$H_s\left(\frac{dV}{dt}\right) + \{V, H_s(V)\} = 0, \qquad \vartheta_r\left(\frac{ds}{dt}\right) = V. \tag{6.24}$$

$$H_{s_o}\left(\frac{dW}{dt}\right) + \{W, H_{s_o}(W)\} = 0, \qquad \vartheta_\ell\left(\frac{ds}{dt}\right) = W. \tag{6.25}$$

A fixed operator H_{s_o} appears in Eqs. (6.25), whereas in (6.24) there were operators depending on s. These equations correspond to the *Eulerian* form (relative to "space fixed" frames) and the *Lagrangian* form (relative to body fixed frames) of the dynamic equations.

THEOREM 4. *Let $B^{(o)}$ be the tensor field such that*

$$B^{(o)}_s(\mathbf{u}, \mathbf{v}) = H_s^{-1}\left(\{\mathbf{u}, H_s(\mathbf{v})\}\right).$$

The motion of a free rigid body is also described by the geodesics of the left invariant connection $\nabla^{(o)}$ such that

$$\nabla^{(o)}_X Y = \nabla^{(\ell)}_X Y + B^{(o)}(X, Y).$$

The curvature of this connection vanishes and its torsion is defined by (6.26) below.

□ It is evident that the inertial motions are also geodesics of $\nabla^{(o)}$. The curvature is defined by

$$R(X, Y, Z) = \nabla^{(o)}_X\nabla^{(o)}_Y Z - \nabla^{(o)}_Y\nabla^{(o)}_X Z - \nabla^{(o)}_{[\![X,Y]\!]} Z,$$

and corresponds to the following trilinear maps from $\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}$ to $\mathfrak{g}$

$$R^{(o)}_s(\mathbf{u}, \mathbf{v}, \mathbf{w}) = H_s^{-1}\big(\operatorname{ad}^*\mathbf{u}.\operatorname{ad}^*\mathbf{v}.H_s(\mathbf{w}) - \operatorname{ad}^*\mathbf{v}.\operatorname{ad}^*\mathbf{u}.H_s(\mathbf{w}) - \operatorname{ad}^*[\mathbf{u}, \mathbf{v}].H_s(\mathbf{w})\big).$$

These maps vanish according to the relation $\operatorname{ad}^*[\mathbf{u}, \mathbf{v}] = \operatorname{ad}^*\mathbf{u}.\operatorname{ad}^*\mathbf{v} - \operatorname{ad}^*\mathbf{v}.\operatorname{ad}^*\mathbf{u}$ (easily derived from $\operatorname{ad}[\mathbf{u}, \mathbf{v}] = \operatorname{ad}\mathbf{u}\circ\operatorname{ad}\mathbf{v} - \operatorname{ad}\mathbf{v}\circ\operatorname{ad}\mathbf{u}$ by transposition). If we define the "coboundary" $\mathrm{d}F$ of any linear map $F\colon \mathfrak{g}\to\mathfrak{g}^*$ by $\mathrm{d}F(\mathbf{u}, \mathbf{v}) = \{\mathbf{u}, F(\mathbf{v})\} - \{\mathbf{v}, F(\mathbf{u})\} - F([\mathbf{u}, \mathbf{v}])$, then the torsion tensor of $\nabla^{(o)}$ is defined by

$$\begin{aligned} T^{(o)}_s(\mathbf{u}, \mathbf{v}) &= H_s^{-1}\left(\{\mathbf{u}, H_s(\mathbf{v})\}\right) - H_s^{-1}\left(\{\mathbf{v}, H_s(\mathbf{u})\}\right) - [\mathbf{u}, \mathbf{v}] \\ &= H_s^{-1}\left(\mathrm{d}H_s(\mathbf{u}, \mathbf{v})\right). \end{aligned} \tag{6.26}$$

6.6. Algebraic aspect of Theorems 1 and 2 and "absolute" quantities

Although kinematical quantities are essentially frame-dependent, the principles of Newtonian mechanics refer to "absolute" quantities and some kind of "absolute motion" which are in some sense independent of the choice of an inertial frame of reference. For example the acceleration of a particle relative to an inertial frame or the angular velocity of a frame of reference relative to "fixed stars". The possibility of linking "absolute" velocities or accelerations to rigid motions of bodies or frames is a consequence of the existence of the class of inertial frames and some considerations on groups; the results of Secs. 6.3 and 6.4 suggest that inertial forces depend only of such absolute kinematic quantities. We will not try to state a general definition of absolute quantities in Newtonian mechanics and limit our study to a few mathematical definitions relevant to the context of the article.

First of all we consider the quotient Lie algebra $\mathfrak{r} = \mathfrak{g}/\mathfrak{t}$, also defined by the equivalence relation $X \sim Y \iff Y - X \in \mathfrak{t}$, that is isomorphic to the Lie algebra of the quotient $\mathbb{G}/\mathbb{T}$. Concretly, projections $\mathbb{G} \to \mathbb{G}/\mathbb{T}$ and $\mathfrak{t} \to \mathfrak{r}$ delete the contribution of translations to displacements and velocities. We also need a process to delete the contribution of uniform translations at the level of accelerations and in order to do this we refer to the group $\mathfrak{g}_{(2)}$ defined in Sec. 5. The subset $\mathfrak{t} \times \{0\} = \mathfrak{u}$ is a subgroup of $\mathfrak{g}_{(2)}$ isomorphic to the additive group $\mathfrak{t}$ and invariant by the action of $\mathbb{G}$. Hence, by the standard way we can define the right invariant equivalence relation $\approx$ associated to $\mathfrak{u}$ in $\mathfrak{g}_{(2)}$ by

$$(\mathbf{v}, \mathbf{v}') \approx (\mathbf{u}, \mathbf{u}') \stackrel{\Delta}{\iff} (\mathbf{v}, \mathbf{v}') \odot (\mathbf{u}, \mathbf{u}')^{-1} \in \mathfrak{u} \iff \mathbf{v} - \mathbf{u} \in \mathfrak{t} \text{and}\, \mathbf{v}' - \mathbf{u}' + [\mathbf{u}, \mathbf{v}] = 0.$$

The equivalence class of $(\mathbf{u}, \mathbf{u}')$ is the subset of $\mathfrak{g}_{(2)}$ such that

$$\mathfrak{u} \odot (\mathbf{u}, \mathbf{u}') = \{(\mathbf{u} + \mathbf{w}, \mathbf{u}' + [\mathbf{w}, \mathbf{u}]) \mid \mathbf{w} \in \mathfrak{t}\} = \overline{(\mathbf{u}, \mathbf{u}')}.$$

We define the coset space $\mathfrak{a} = \mathfrak{g} \times \mathfrak{g}/ \approx = \mathfrak{g}_{(2)} \backslash \mathfrak{u}$ and we have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{g} \times \mathfrak{g} & \longrightarrow & \mathfrak{a} \\ p_1 \downarrow & & \downarrow \text{ (canonical projection)} \\ \mathfrak{g} & \longrightarrow & \mathfrak{r} \end{array}$$

where the vertical arrow to the right maps the class $\overline{(\mathbf{u}, \mathbf{u}')}$ (according to $\approx$) onto the class $\overline{\mathbf{u}}$ (according to $\sim$). Since $\mathfrak{t}$ and $\mathfrak{u}$ are invariant by $\mathrm{Ad}\, g$, we deduce two actions of $\mathbb{G}$ on $\mathfrak{a}$ and $\mathfrak{r}$ such that

$$\mathrm{Ad}\, g.\overline{(\mathbf{u}, \mathbf{u}')} = \overline{\mathrm{Ad}\, g.(\mathbf{u}, \mathbf{u}')}, \quad \mathrm{Ad}\, g.\overline{\mathbf{u}} = \overline{\mathrm{Ad}\, g.\mathbf{u}}.$$

The reason to introduce the group $\mathfrak{u}$ is the following: when $\mathcal{F}_1$ and $\mathcal{F}_2$ are *inertial* frames $(U_{12}, \dot{U}_{12}) = (U_{12}, 0) \in \mathfrak{u}$, so that relation (5.8) shows that for any rigid body at every time

$$(V_2, \dot{V}_2) \approx \operatorname{Ad} A_{12}(t).(V_1, \dot{V}_1).$$

Let us define the "absolute" velocity and acceleration. With respect to an *inertial* frame put $\Gamma = \overline{(V, \dot{V})} \in \mathfrak{a}$ so that we deduce from (5.8) that for two *inertial* frames:

$$\Gamma_2 = \operatorname{Ad} A_{12}(t).\Gamma_1.$$

Now, if $\mathcal{F}_3$ is a *kinematical* frame we deduce from $A_{13}(t) = A_{23}(t).A_{12}(t)$ a relation

$$\Gamma_3 = \operatorname{Ad} A_{13}(t).\Gamma_1 = \operatorname{Ad} A_{23}(t).\Gamma_2$$

permitting a consistent definition of Γ_3 depending only on $\mathcal{F}_3$ and the motion of the body but not of the choice of a particular inertial frame. The just proved property means that, after the class of inertial frames was defined: *for all rigid body, at every time t, there exists a well defined* "absolute acceleration"*, which is an object of the type* $(\mathfrak{a}, \mathbb{G})$ *such that, with respect to an inertial frame* $\mathcal{F}$

$$\Gamma_{\mathcal{F}} = \overline{(V_{\mathcal{F}}, \dot{V}_{\mathcal{F}})}.$$

With respect to a kinematical frame $\mathcal{F}$, $\Gamma_{\mathcal{F}} = \operatorname{Ad} A_{\mathcal{F}_1\mathcal{F}}(t).\Gamma_{\mathcal{F}_1}$ where $\mathcal{F}_1$ is any inertial frame the choice of which is immaterial. By the projection $\mathfrak{a} \to \mathfrak{r}$ we deduce from $\Gamma_{\mathcal{F}}$ the "absolute velocity" of the body, an element $\Omega_{\mathcal{F}}$ of $\mathfrak{r}$ that may be interpreted as the angular velocity of the body relative to "fixed stars". Similar reasonings on the induced velocities with formula (5.9) in the form

$$(W_{32}, \dot{W}_{32}) = (\operatorname{Ad} A_{13}.(W_{12}, \dot{W}_{12})) \odot (W_{31}, \dot{W}_{31})$$

and, with inertial $\mathcal{F}_1$ and $\mathcal{F}_2$, lead to the following conclusion: *to every kinematical frame* $\mathcal{F}$ *is associated a well–defined induced* "absolute acceleration" $\Gamma^e_{\mathcal{F}} \in \mathfrak{a}$ *defined by:*

$$\Gamma^e_{\mathcal{F}} = \overline{(W_{\mathcal{F}\mathcal{F}_1}, \dot{W}_{\mathcal{F}\mathcal{F}_1})}, \quad \text{where } \mathcal{F}_1 \text{ is any inertial frame.}$$

Whenever $\mathcal{F}$ *is an inertial frame,* $\Gamma^e_{\mathcal{F}} = 0$ *(since we can take* $\mathcal{F}_1 = \mathcal{F}$*).* Projecting $\Gamma^e_{\mathcal{F}}$ onto $\mathfrak{r}$ we obtain the induced "absolute" velocity $\Omega^e_{\mathcal{F}} \in \mathfrak{r}$ of the frame $\mathcal{F}$, which vanish for all the inertial frames and may be interpreted as the angular velocity of the frame relative to the "fixed stars".

Theorem 5 below, where $\mathbf{H}$ may denote any operator H_s with $s \in \mathbb{S}$, clarifies the role played in dynamics by those "absolute" kinematical quantities and the algebraic meaning of Property 2 in Theorem 1:

THEOREM 5. *Let* $\mathbf{H}$ *be a linear operator from* $\mathfrak{g}$ *to* $\mathfrak{g}^*$ *and* $\mathbf{C}\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}^*$ *be the bilinear map defined by* $\mathbf{C}(\mathbf{u},\mathbf{v}) = \{\mathbf{u},\mathbf{H}(\mathbf{v})\}+\{\mathbf{v},\mathbf{H}(\mathbf{u})\}+\mathbf{H}([\mathbf{u},\mathbf{v}])$. *Then, the following properties are equivalent:*

(a) For all $\mathbf{u}\in\mathfrak{t}$ *and for all* $\mathbf{v}\in\mathfrak{g}$: $\mathbf{C}(\mathbf{u},\mathbf{v})=0$.

(b) There exists a function $\mathfrak{C}\colon \mathfrak{r}\times\mathfrak{g}\to\mathfrak{g}^*$ *such that* $\mathbf{C}(\mathbf{u},\mathbf{v})=\mathfrak{C}(\overline{\mathbf{u}},\mathbf{v})$.

(c) There exists a function $\mathfrak{J}\colon \mathfrak{a}\to\mathfrak{g}^*$ *such that* $\mathbf{H}(\mathbf{u}')+[\mathbf{u},\mathbf{H}(\mathbf{u})]=\mathfrak{J}\big(\overline{(\mathbf{u},\mathbf{u}')}\big)$.

Property (c) means that the map $(\mathbf{u},\mathbf{u}')\mapsto\mathbf{H}(\mathbf{u}')+\{\mathbf{u},\mathbf{H}(\mathbf{u})\}$, appearing in the expression of the inertial forces, splits into a composed map $\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{a}\xrightarrow{\mathfrak{J}}\mathfrak{g}^*$.

□ Equivalence between (a) and (b) is evident. Property (c) means that

$$(\mathbf{u},\mathbf{u}')\approx(\mathbf{v},\mathbf{v}')\Longrightarrow \mathbf{H}(\mathbf{u}')+\{\mathbf{u},\mathbf{H}(\mathbf{u})\}=\mathbf{H}(\mathbf{v}')+\{\mathbf{v},\mathbf{H}(\mathbf{v})\}.$$

Taking the definition of equivalence $\approx$ into account, property (c) also means

$$\forall\,\mathbf{u}\in\mathfrak{t},\ \forall\,\mathbf{v}\in\mathfrak{g}\colon \{\mathbf{u},\mathbf{H}(\mathbf{v})\}+\{\mathbf{v},\mathbf{H}(\mathbf{u})\}+\mathbf{H}([\mathbf{u},\mathbf{v}])+\{\mathbf{u},\mathbf{H}(\mathbf{u})\}=0.$$

Choosing $\mathbf{v}=0$ we see that (c) implies $\{\mathbf{u},\mathbf{H}(\mathbf{u})\}=0$ for $\mathbf{u}\in\mathfrak{t}$ and (a). Conversely, since $\mathbf{C}(\mathbf{u},\mathbf{u})=2\{\mathbf{u},\mathbf{H}(\mathbf{u})\}$, (a) implies that the left-hand side of the preceding equality cancels and (c). ■

Now the map $\mathfrak{J}_s\colon \mathfrak{a}\to\mathfrak{g}^*$ associated with H_s as in Theorem 5 (c) is well-defined for all $s\in\mathbb{S}$ and it is readily proved that, for all $g\in\mathbb{G}$ and $s\in\mathbb{S}$:

$$\operatorname{Ad}^*g\circ\mathfrak{J}_s=\mathfrak{J}_{g.s}\circ\operatorname{Ad}g$$

where to the right $\operatorname{Ad}g$ is the induced action on $\mathfrak{a}$. Property (c) implies that, when the conditions of Theorem 1 are verified, the inertial force is expressed by $\mathbf{J}_{\mathcal{F}}=\mathfrak{J}_s\Big(\overline{(V_{\mathcal{F}},\dot V_{\mathcal{F}})}\Big)=\mathfrak{J}_s(\varGamma_{\mathcal{F}})$ in an inertial frame, a function of the "absolute" acceleration only. However, objectivity of inertial forces means that if $\mathcal{F}_2$ is any frame and $\mathcal{F}_1$ is an inertial frame then at every time $\mathbf{J}_2=\operatorname{Ad}A_{12}.\mathbf{J}_1$. That is to say

$$\mathbf{J}_2=\operatorname{Ad}A_{12}.\mathfrak{J}_{s_1}\big(\varGamma_1\big)=\mathfrak{J}_{A.s_1}\big(\operatorname{Ad}A_{12}.\varGamma_1\big)=\mathfrak{J}_{s_2}\big(\varGamma_2\big).$$

Finally, in any frame, the inertial force acting on a body is a function of its "absolute" acceleration only (this is meaningful according to the definition of this acceleration as an object of the type $(\mathfrak{a},\mathbb{G})$!). Theorem 2 tells us that with respect to a non-inertial frame $\mathcal{F}$, the complementary force $\mathbf{J}^c_{\mathcal{F}}$ depends only on the "absolute velocity" of the frame and on the relative velocity of the body and the induced inertial force $\mathbf{J}^e_{\mathcal{F}}$ depends only on the "absolute acceleration" of the frame and on the position of the body. All these properties of the mathematical structure of dynamics appear as logical consequences of objectivity of inertial forces. (Remark that, since absolute kinematical quantities might be interpreted as relative to the faraway matter, the "fixed stars", they might also be used to be more specific about the links between the basic principles of dynamics in Newton's universe and Mach's principle).

7. Deduction of the mathematical form of inertia operators from the objectivity principle

7.1. Some properties of the Euclidean displacement group and its Lie algebra

In this section, we point out some connections between the Lie group theory and well-known properties in kinematics and mechanics. Let $\mathcal{E}$ be the 3-dimensional Euclidean affine space and $\mathbb{E}$ be the associated vector space (so that to every ordered pair (a, b) of points belonging to $\mathcal{E}$ is associated with a vector $\overrightarrow{ab} \in \mathbb{E}$ with usual properties). In $\mathbb{E}$ the scalar product will be denoted by $\cdot$ and the vector product by $\times$. An alternative presentation of the affine space $\mathcal{E}$ states that the additive group of the vector space $\mathbb{E}$ acts *freely* and *transitively* on $\mathcal{E}$ by an operation denoted $(\mathcal{E}, \mathbb{E}) \ni (a, \mathbf{x}) \mapsto a + \mathbf{x} \in \mathcal{E}$. In fact $b = a + \mathbf{x}$ is also the point such that $\overrightarrow{ab} = \mathbf{x}$. In the sequel, according to circumstances, we will refer to the most convenient of those equivalent points of view.

An affine transformation of $\mathcal{E}$ is a map $A\colon \mathcal{E} \to \mathcal{E}$ such that there exists $\mathbf{A} \in \mathcal{L}(\mathbb{E})$, the linear part of A denoted by the same letter in boldface, verifying the equivalent properties:

$$A(a + \mathbf{x}) = A(a) + \mathbf{A}(\mathbf{x}) \quad (\text{or } \overrightarrow{A(a)A(b)} = \mathbf{A}(\overrightarrow{ab})\,).$$

A displacement of $\mathcal{E}$ is an affine map such that $\mathbf{A} \in \mathrm{SO}(\mathbb{E})$ (the special orthogonal group of $\mathbb{E}$). The displacements make a classical Lie group $\mathbb{D}$ of transformations of $\mathcal{E}$ containing a normal Lie subgroup $\mathbb{T}$, the group of translations (translations are of the form $a \mapsto a + \mathbf{t}$ with a fixed $\mathbf{t} \in \mathbb{E}$, they are the affine maps such that $\mathbf{A} = \mathbf{1}$, so that $\mathbb{T}$ is isomorphic to the additive group of $\mathbb{E}$). For each point $c \in \mathcal{E}$ let $\mathbf{R}_c$ be the subgroup of rotations about c. For every $c \in \mathcal{E}$, $\mathbb{D} = \mathbb{T} \times \mathbf{R}_c$ (semi-direct product).

We will now describe the Lie algebra of $\mathbb{D}$. First of all, due to the affine structure, the tangent space of the manifold $\mathcal{E}$ may be identified with $\mathcal{E} \times \mathbb{E}$ so that a vector field on this manifold "is" merely a map: $\mathcal{E} \to \mathbb{E}$. A standard result in "torsor" theory states that the following properties of a vector field X on $\mathcal{E}$ are equivalent:

(i) For all a and $b \in \mathcal{E}$: $X(a) \cdot \overrightarrow{ab} = X(b) \cdot \overrightarrow{ab}$,

(ii) there exists $\boldsymbol{\omega} \in \mathbb{E}$ such that for all a and $b \in \mathcal{E}$: $X(b) = X(a) + \boldsymbol{\omega} \times \overrightarrow{ab}$,

(the uniquely defined vector $\boldsymbol{\omega}$ will be denoted by $\boldsymbol{\omega}_X$ if necessary). Property (i) characterizes *skew-symmetric vector fields* and appears for instance in calculation of the velocities of the particles of a rigid body: it is a straightforward consequence of preservation of the distance under displacements. Property (ii) characterizes the *moment fields*. Equivalence between (i) and (ii) points up the angular velocity and the instantaneous axis of rotation and translation (existing when $\boldsymbol{\omega}_X \neq 0$).

The vector fields X with properties (i) and (ii) make a vector space $\mathfrak{D}(\mathcal{E})$ over $\mathbb{R}$. If X and Y are in $\mathfrak{D}(\mathcal{E})$, the field $[X,Y] = U$ such that

$$U(a) = \boldsymbol{\omega}_X \times Y(a) - \boldsymbol{\omega}_Y \times X(a), \quad a \in \mathcal{E} \tag{7.1}$$

is also in $\mathfrak{D}(\mathcal{E})$ with $\boldsymbol{\omega}_{[X,Y]} = \boldsymbol{\omega}_X \times \boldsymbol{\omega}_Y$ and, endowed with the bracket $[\cdot,\cdot]$, $\mathfrak{D}(\mathcal{E})$ is a Lie algebra. Moreover, the natural image A_*X of a field $X \in \mathfrak{D}(\mathcal{E})$ under displacement $A \in \mathbb{D}$ is the vector field defined by

$$A_*X(a) = \mathbf{A}\big(X(A^{-1}(a))\big), \quad a \in \mathcal{E} \tag{7.2}$$

and it is readily proved that $A_*X \in \mathfrak{D}(\mathcal{E})$ and that $A \mapsto A_*$ is a linear representation of the group $\mathbb{D}$ in the Lie algebra $\mathfrak{D}(\mathcal{E})$ (in particular $[A_*X, A_*Y] = A_*[X,Y]$).

The Lie algebra $\mathfrak{D}(\mathcal{E})$ has remarkable properties. First, there exists on $\mathfrak{D}(\mathcal{E})$ an inner product $[\cdot \mid \cdot]$ (non-degenerate bilinear symmetric form), the Klein form, that is invariant under the action of $\mathbb{D}$ on $\mathfrak{D}(\mathcal{E})$ ($[A_*X \mid A_*Y] = [X \mid Y]$) and defined by

$$[X \mid Y] = \boldsymbol{\omega}_X \cdot Y(a) + \boldsymbol{\omega}_Y \cdot X(a)$$

where the choice of the point a in $\mathcal{E}$ is immaterial. The following theorem specifies the relations between the general Lie group theory and the familiar mathematical objects in kinematics:

THEOREM 6. *$\mathbb{D}$ is a Lie group and*

1. *The Lie algebra $\mathfrak{d}$ of $\mathbb{D}$ (according to the general theory) is isomorphic to the Lie algebra $\mathfrak{D}(\mathcal{E})$ endowed with the bracket defined as in (7.1);*
2. *The adjoint representation of $\mathbb{D}$ is equivalent to the representation $A \mapsto A_*$ defined in (7.2).*

The theorem follows from a general property of an effective action of a Lie group on a manifold: there exists a one–to–one relation between the Lie algebra of the group and the Lie algebra of fundamental vector fields and the adjoint representation is equivalent to the transformation of fundamental vector fields by ordinary image (see [13], T. I, Chap. 1, Proposition 4.1). Here, the fundamental vector fields of the action of $\mathbb{D}$ on $\mathcal{E}$ are the skew-symmetric vector fields.

At the level of Lie algebras the splitting of the group into a semi-direct product implies:

1. The Lie algebra $\mathfrak{t}$ of the subgroup $\mathbb{T}$ (according to the general theory) is commutative and isomorphic to the ideal of $\mathfrak{D}(\mathcal{E})$ whose elements are the constant vector fields such that $\boldsymbol{\omega}_X = \mathbf{0}$;
2. The Lie algebra $\mathfrak{z}_c$ of the subgroup $\mathbf{R}_c$ is isomorphic to the Lie subalgebra of $\mathfrak{D}(\mathcal{E})$ whose elements are the skew-symmetric fields such that $X(c) = \mathbf{0}$;
3. For every c, $\mathfrak{d} = \mathfrak{t} \times \mathfrak{z}_c$ (semi-direct product of Lie algebras corresponding to the semi-direct product $\mathbb{T} \times \mathbf{R}_c$).

From now on we will identify $\mathfrak{d}$ with $\mathfrak{D}(\mathcal{E})$, $\mathfrak{z}_c$ and $\mathfrak{t}$ with the isomorphic subalgebra or ideal of $\mathfrak{D}(\mathcal{E})$. As a Lie algebra $\mathfrak{z}_c$ is isomorphic with the Lie algebra $(\mathbb{E}, \times)$ (the isomorphism is $X \mapsto \boldsymbol{\omega}_X$). As a vector space $\mathfrak{t}$ is isomorphic with $\mathbb{E}$. When c is a fixed point in $\mathcal{E}$, any member of $\mathfrak{d}$ may be described by the pair of vectors $(X(c), \boldsymbol{\omega}_X) \in \mathbb{E} \times \mathbb{E}$ (Plücker vectors at c corresponding to the components in $\mathfrak{t}$ and $\mathfrak{z}_c$).

Another important property of $\mathfrak{d}$ we will use in what follows is to be an algebra over the dual number ring (this structure is the base of the algebra developed by Kotelnikov and Dimentberg to express the "screw theory" for the needs of kinematics, see [7] for an exposition in accordance with the standpoint of this article). Recall that the dual numbers are the "numbers" of the form $z = x + \boldsymbol{\epsilon} y$ where x and $y \in \mathbb{R}$ and $\boldsymbol{\epsilon}$ verifies $\boldsymbol{\epsilon}^2 = 0$ and they make a commutative ring Δ where addition and multiplication are defined in the natural way. The product of a dual number z by in element X of $\mathfrak{d}$ is such that $\boldsymbol{\epsilon} X \in \mathfrak{t}$ is the constant vector field equal to $\boldsymbol{\omega}_X$ so that $\boldsymbol{\omega}_{\boldsymbol{\epsilon} X} = 0$, $(\boldsymbol{\epsilon}^2)X = \boldsymbol{\epsilon}(\boldsymbol{\epsilon} X) = 0$ for all X and $\mathfrak{t}$ is the set of elements such that $\boldsymbol{\epsilon} X = 0$. In general,

$$zX = xX + y\boldsymbol{\omega}_X \quad \text{(vector field } a \mapsto xX(a) + y\boldsymbol{\omega}_x).$$

It is readily verified that $\boldsymbol{\epsilon}[X, Y] = [\boldsymbol{\epsilon} X, Y] = [X, \boldsymbol{\epsilon} Y]$ and, more generally, $z[X, Y] = [zX, Y] = [X, zY]$ for all $z \in \Delta$, X and $Y \in \mathfrak{d}$. Endowed with these operations $\mathfrak{d}$ is a module and a Lie algebra over Δ.

In the following we will use this property:

Lemma 4. *A necessary and sufficient condition for a $\mathbb{R}$-linear function $f \in \mathcal{L}(\mathfrak{d})$ to verify $[X, f(X)] = 0$ for all $X \in \mathfrak{d}$ is that there exists a dual number μ such that $f(X) = \mu X$ for all $X \in \mathfrak{d}$.*

We will not explain all the details of the proof needing a lot of elementary calculations in the Lie algebra $\mathfrak{d}$ over Δ. Say that this result may be derived from the following remark:

$$\text{If } X \neq \mathfrak{t} \text{ then } [X, Y] = 0 \iff \text{there exists } \mu \in \Delta \text{ such that } Y = \mu X,$$

(meaning in screw theory that Y has the same axis as X if $Y \notin \mathfrak{t}$ and is directed as the axis of X if $Y \in \mathfrak{t}$). After it is easy to prove that if $f(X) = \mu_X X$ for all $X \notin \mathfrak{t}$ then $\mu_X = \mu_Y$ for linearly independent pairs (X, Y) *over* Δ and to conclude.

7.2. Inertia operators

According to condition 2 of Theorem 1, under assumption (M'), we will say that a linear operator $\mathbf{H} \in \mathcal{L}(\mathfrak{d})$ is an *inertia operator* if

$$\text{for all } U \in \mathfrak{t} \text{ and for all } V \in \mathfrak{d}\text{: } [U, \mathbf{H}(V)] + [V, \mathbf{H}(U)] + \mathbf{H}([U, V]) = 0.$$

A necessary and sufficient condition for the statement of the law of dynamics of Sec. 6.4 which agrees with the principle of objectivity is that, for a reference position $s_o \in \mathbb{S}$ of a rigid body, H_{s_o} should be an inertia operator $\mathbf{H}$ within the preceding meaning. Then

$$H_s = \mathrm{Ad}\, A \circ \mathbf{H} \circ \mathrm{Ad}\, A^{-1} \text{ when } s = A.s_o,\ A \in \mathbb{D}$$

will be an inertia operator for all s. Since a canonical Ad-invariant inner product, for instance the Klein form, is defined on the Lie algebra $\mathfrak{d}$ of $\mathbb{D}$, the left invariant Riemannian structure on $\mathbb{S}$ is defined by the linear operator $\mathbf{H} \in \mathcal{L}(\mathfrak{d})$ such that $(\mathbf{x} \mid \mathbf{y}) = [\mathbf{H}(\vartheta_\ell(\mathbf{x})) \mid \vartheta_\ell(\mathbf{y})]$ and $\mathbf{H}$ must be symmetric and positive definite according to the Klein form on $\mathfrak{d}$, that is to say for all X and Y in $\mathfrak{d}$:

$$[X \mid \mathbf{H}(Y)] = [\mathbf{H}(X) \mid Y], \quad [X \mid \mathbf{H}(X)] > 0 \text{ for } X \neq 0.$$

The Riemannian structures on $\mathbb{S}$ leading to a law agreeing with the objectivity principle are defined by inertia operators verifying this condition (see Sec. 6.2).

7.3. Structure of inertia operators

THEOREM 7. *A linear operator $\mathbf{H}$ in $\mathfrak{d}$ is an inertia operator if and only if there exist real numbers m and q and maps $P \colon \mathcal{E} \to \mathbb{E}$, $\mathbf{I} \colon \mathcal{E} \to \mathcal{L}(\mathbb{E})$ such that if $L = \mathbf{H}(V)$, the Plücker vectors of V and L at a are related by:*

$$\begin{bmatrix} \boldsymbol{\omega}_L \\ L(a) \end{bmatrix} = \begin{bmatrix} \widetilde{P}_a & m\mathbf{1} \\ \mathbf{I}_a & q\mathbf{1} - \widetilde{P}_a \end{bmatrix} \times \begin{bmatrix} \boldsymbol{\omega}_V \\ V(a) \end{bmatrix} \tag{7.3}$$

where $\widetilde{P}_a$ is the operator "vector product by P_a" in $\mathbb{E}$ and, for a and b in $\mathcal{E}$: $P_b = P_a + m\overrightarrow{ab}$,

$$\mathbf{I}_b(\mathbf{x}) = \mathbf{I}_a(\mathbf{x}) + m\overrightarrow{ab} \times \left(\mathbf{x} \times \overrightarrow{ab}\right) + q\left(\overrightarrow{ab} \times \mathbf{x}\right) + \left(\overrightarrow{ab} \times \mathbf{x}\right) \times P_a + (P_a \times \mathbf{x}) \times \overrightarrow{ab}.$$

In (7.3) $\boldsymbol{\omega}_L$ is the *linear momentum* (mass$\times$velocity of the center of mass for non-singular operators, see below) and $L(a)$ is the *angular momentum at a*. The form (7.3) of the inertia operators is coordinate-free, however we can consider $\mathbb{E}$ to be $\mathbb{R}^3$, $\widetilde{P}_a$ and $\mathbf{I}_a$ to be 3×3 matrices, $\boldsymbol{\omega}$ and $V(a)$ to be 3×1 matrices, what comes down to chose coordinates in $\mathbb{E}$. In consequence of (7.3), there exist several kinds of inertia operators according to m equal zero or not.

1) Singular inertia operators (such that $m = 0$). Then the vector P_a is independent of a and

$$\mathbf{I}_b(\mathbf{x}) = \mathbf{I}_a(\mathbf{x}) + q\left(\overrightarrow{ab} \times \mathbf{x}\right) + \left(\overrightarrow{ab} \times \mathbf{x}\right) \times P + (P \times \mathbf{x}) \times \overrightarrow{ab},$$

$$\begin{bmatrix} \boldsymbol{\omega}_L \\ L(a) \end{bmatrix} = \begin{bmatrix} \widetilde{P} & \mathbf{0} \\ \mathbf{I}_a & q\mathbf{1} - \widetilde{P} \end{bmatrix} \times \begin{bmatrix} \boldsymbol{\omega}_V \\ V(a) \end{bmatrix}. \tag{7.4}$$

Note that such an operator is never invertible.

2) Inertia operators such that $m \neq 0$. There exists a unique point $c \in \mathcal{E}$ such that $P_c = 0$ and, putting $\overrightarrow{ac} = \mathbf{a}$, $P_a = -m\,\mathbf{a}$, $\quad \mathbf{I}_a = \mathbf{I}_c - m\,\widetilde{\mathbf{a}}^{\,2} - q\,\widetilde{\mathbf{a}}$ (so that $\mathbf{I}_a$ is known for all a as soon as $\mathbf{I}_c$ is known) and

$$\begin{bmatrix} \boldsymbol{\omega}_L \\ L(a) \end{bmatrix} = \begin{bmatrix} -m\,\widetilde{\mathbf{a}} & m\mathbf{1} \\ \mathbf{I}_c - m\,\widetilde{\mathbf{a}}^{\,2} - q\,\widetilde{\mathbf{a}} & q\mathbf{1} + m\,\widetilde{\mathbf{a}} \end{bmatrix} \times \begin{bmatrix} \boldsymbol{\omega}_V \\ V(a) \end{bmatrix} \tag{7.5}$$

In particular, the relation between the Plücker vectors of V and L at c is:

$$\begin{bmatrix} \boldsymbol{\omega}_L \\ L(c) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & m\mathbf{1} \\ \mathbf{I}_c & q\mathbf{1} \end{bmatrix} \times \begin{bmatrix} \boldsymbol{\omega}_V \\ V(c) \end{bmatrix}. \tag{7.6}$$

Note that, in all cases, $\boldsymbol{\omega}_L = m\big(-\overrightarrow{ac} \times \boldsymbol{\omega}_V + V(a)\big) = mV(c)$. Such an operator is invertible if and only if $\mathbf{I}_c \in \mathrm{Gl}(\mathbb{E})$ and we deduce that:

$$[X \mid \mathbf{H}(Y)] = mX(c) \cdot Y(c) + \boldsymbol{\omega}_X \cdot \mathbf{I}_c\,(\boldsymbol{\omega}_Y) + q\,\boldsymbol{\omega}_X \cdot Y(c).$$

3) Symmetric inertia operators such that $m \neq 0$ (according to the Klein form):

1. $\mathbf{H}$ *is symmetric if and only if* $q = 0$ *and* $\mathbf{I}_c$ *is symmetric and then:*

$$\begin{bmatrix} \boldsymbol{\omega}_L \\ L(c) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & m\mathbf{1} \\ \mathbf{I}_c & \mathbf{0} \end{bmatrix} \times \begin{bmatrix} \boldsymbol{\omega}_V \\ V(c) \end{bmatrix}. \tag{7.7}$$

1. $\mathbf{H}$ *is symmetric positive (resp. definite) if and only if* $q = 0$, $m > 0$ *and* $\mathbf{I}_c$ *is a symmetric positive (resp. definite) operator in* $\mathbb{E}$.

We recognize that the symmetric positive definite inertia operators are those of the Newton–Euler rigid body mechanics: $m > 0$ represents the total mass, $c \in \mathcal{E}$ – the center of inertia and $\mathbf{I}_c \in \mathcal{L}(\mathbb{E})$ – the central inertia operator (or "tensor", see also Sec. 7.5.). The symmetry assumption, which is necessary to find exactly the classical theory, is very natural: it means that the kinetic energy defined by the quadratic form $1/2\,[X \mid \mathbf{H}(X)]$ determines the inertia operator and conversely.

The inertia operator H_s depends on the position s of the body according to (6.8). Its matrix expression depends on the choice of the point $a \in \mathcal{E}$ where the Plücker vectors are calculated. It is convenient to use only the central tensors, formula (7.5) when $a \neq c$, and to denote the central tensor $\mathbf{I}_{c_s}$ of the body in position s by $\mathbf{I}_s$. The maps $s \mapsto c_s \in \mathcal{E}$ and $s \mapsto \mathbf{I}_s \in \mathcal{L}_s(\mathbb{E})$ are equivariant:

$$c_{D.s} = D(c_s), \quad \mathbf{I}_{D.s} = \mathbf{D} \circ \mathbf{I}_s \circ \mathbf{D}^{-1} \tag{7.8}$$

for $D \in \mathbb{D}\,(\mathbf{D} \in \mathcal{L}(\mathbb{E}),$ linear part of $D)$.

In particular we deduce with (6.8) and the above results that the invariant Riemannian structures on $(\mathbb{S};\mathbb{D})$ agreeing with the objectivity of inertial forces are necessarily of the form

$$(\mathbf{x}\mid\mathbf{y}) = mX(c_s)\cdot Y(c_s)+\boldsymbol{\omega}_X\cdot\mathbf{I}_s(\boldsymbol{\omega}_Y),\quad s=o(\mathbf{x})=o(\mathbf{y}),\ X=\vartheta(\mathbf{x}),\ Y=\vartheta(\mathbf{y}).$$

In Case 2, the Plücker vectors of $C_s(W,V)=G$ at c is expressed by

$$\begin{bmatrix}\boldsymbol{\omega}_G\\ G(c)\end{bmatrix}=\begin{bmatrix}2m\,\boldsymbol{\omega}_W\times V(c)\\ \boldsymbol{\omega}_W\times\mathbf{I}_c(\boldsymbol{\omega}_V)+\boldsymbol{\omega}_V\times\mathbf{I}_c(\boldsymbol{\omega}_W)+\mathbf{I}_c(\boldsymbol{\omega}_W\times\boldsymbol{\omega}_V)+2q\,\boldsymbol{\omega}_W\times V(c)\end{bmatrix}$$

with $q=0$ in the symmetric case 3. It is evident that $C_s(W,V)=0$ whenever $W\in\mathfrak{t}$.

7.4. Proof of Theorem 7

The theorem results from several lemmas pointing up successively the total mass m of the body, the number q (vanishing for symmetric inertia operators) and the center of mass (when $m\neq 0$).

LEMMA 5. *In order to make* $\mathbf{H}$ *an inertia operator, it is necessary and sufficient that the following two properties should be verified: (a) There exists a dual number* μ *such that* $\mathbf{H}(\boldsymbol{\epsilon}X)+\boldsymbol{\epsilon}\mathbf{H}(X)=\mu X$ *for all* $X\in\mathfrak{d}$.

(b) $[\mathbf{H}(X),Y]+[X,\mathbf{H}(Y)]-\mathbf{H}([X,Y])\in\mathfrak{t}$ *for all* X *and* $Y\in\mathfrak{d}$.

Remark that, if $\mu=m+\boldsymbol{\epsilon}q$, condition (a) implies in particular that $\boldsymbol{\epsilon}H(U)=mU$ when $U\in\mathfrak{t}$. Condition (b) means that $\mathrm{d}\mathbf{H}(X,Y)\in\mathfrak{t}$ for all X and $Y\in\mathfrak{d}$.

□ First of all, since $\mathfrak{t}=\{\boldsymbol{\epsilon}X\mid X\in\mathfrak{d}\}$, the condition for $\mathbf{H}$ to be an inertia operator reads

$$\text{(7.9)}\qquad \text{for all } X \text{ and all } Y \text{ in } \mathfrak{d}\text{:}\ \ [\boldsymbol{\epsilon}X,\mathbf{H}(Y)]+[Y,\mathbf{H}(\boldsymbol{\epsilon}X)]+\mathbf{H}(\boldsymbol{\epsilon}[X,Y])=0.$$

(We have used $[\boldsymbol{\epsilon}X,Y]=\boldsymbol{\epsilon}[X,Y]$). Condition (7.9) for $Y=X$ implies that:

$$[\boldsymbol{\epsilon}X,\mathbf{H}(X)]+[X,\mathbf{H}(\boldsymbol{\epsilon}X)]=[X,\mathbf{H}(\boldsymbol{\epsilon}X)+\boldsymbol{\epsilon}\mathbf{H}(X)]=0.$$

Lemma 4 applied with $f\colon X\mapsto\mathbf{H}(\boldsymbol{\epsilon}X)+\boldsymbol{\epsilon}\mathbf{H}(X)$ leads to (a). Substituting (a), namely $\mathbf{H}(\boldsymbol{\epsilon}X)=\mu X-\boldsymbol{\epsilon}\mathbf{H}(X)$, into (7.9) and using the bilinearity of the Lie bracket over Δ leads to:

$$\boldsymbol{\epsilon}\big\{[\mathbf{H}(X),Y]+[X,\mathbf{H}(Y)]-\mathbf{H}([X,Y])\big\}=0,\qquad (*)$$

what is equivalent to the statement that the element inside the braces is in $\mathfrak{t}$, that is (b). Conversely, if conditions (a) and (b) hold, we can reverse the calculations to prove that (7.9) is verified. ■

If a and $b \in \mathcal{E}$ and $U_{ab} \in \mathfrak{t}$ is the constant vector field equal to $\overrightarrow{ab}$ then the map $\mathbf{1} + \operatorname{ad} U_{ab}$ is an isomorphism of Lie algebras from $\mathfrak{z}_a$ onto $\mathfrak{z}_b$ (the reciprocal isomorphism being $\mathbf{1} - \operatorname{ad} U_{ab}$). This property is readily proved by direct calculations, in particular, if $Z \in \mathfrak{d}$ and $Z(a) = 0$ we have

$$(\mathbf{1} + \operatorname{ad} U_{ab}).Z(b) = Z(b) - \boldsymbol{\omega}_Z \times \overrightarrow{ab} = Z(a) = 0.$$

(This is linked with the general theory of Lie groups: the translation $T \in \mathbb{T}$ mapping a on b is $\exp U_{ab} = T$ and $\operatorname{Ad} T = \operatorname{Ad} \exp U_{ab} = \exp \operatorname{ad} U_{ab} = \mathbf{1} + \operatorname{ad} U_{ab}$ because $\operatorname{ad} U_{ab}$ is nilpotent of order 2.) If $a \in \mathcal{E}$, the map $Z \in \mathfrak{z}_a \mapsto \boldsymbol{\epsilon} Z \in \mathfrak{t}$ is a linear isomorphism (equivalent to $Z \mapsto \boldsymbol{\omega}_Z$). We will note $N_a \colon \mathfrak{t} \to \mathfrak{z}_a$ the reciprocal isomorphism, so that $\boldsymbol{\epsilon} N_a(U) = U$ for $U \in \mathfrak{t}$ and $\pi_a \colon \mathfrak{d} \to \mathfrak{z}_a$ the projection of $\mathfrak{d}$ onto $\mathfrak{z}_a$ according to the direct sum $\mathfrak{d} = \mathfrak{t} \oplus \mathfrak{z}_a$. It is readily proved that, for $Z \in \mathfrak{z}_a$, $N_a([U, Z]) = [N_a(U), Z]$ (since $\boldsymbol{\epsilon}[N_a(U), Z] = [\boldsymbol{\epsilon} N_a(U), Z] = [U, Z]$) and that

Lemma 6. *Let $\Delta_a \in \mathcal{L}(\mathfrak{z}_a)$ be the restriction of $\pi_a \circ \mathbf{H}$ to $\mathfrak{z}_a$ ($\Delta_a(Z) = \pi_a(\mathbf{H}(Z))$ for $Z \in \mathfrak{z}_a$). If condition (a) holds then the following conditions (b$'$) or (b$''$) are equivalent to condition (b)*

(b$'$) There exists a in $\mathcal{E}$ such that Δ_a is a derivation of the Lie algebra $\mathfrak{z}_a$.

(b$''$) For all a in $\mathcal{E}$, Δ_a is a derivation of the Lie algebra $\mathfrak{z}_a$.

Then, for a and $b \in \mathcal{E}$, the following property is verified:

$$\Delta_b - (\mathbf{1} + \operatorname{ad} U_{ab}) \circ \Delta_a \circ (\mathbf{1} - \operatorname{ad} U_{ab}) = m \operatorname{ad} N_b(U_{ab}).$$

Remark that Δ_a and Δ_b operate in different spaces and to compare them it is necessary to transmute Δ_a with the natural isomorphism from $\mathfrak{z}_a$ onto $\mathfrak{z}_b$.

□ We shall prove that (b)⇒(b$''$)⇒(b$'$)⇒(b). Relation ($*$) also reads

$$\boldsymbol{\epsilon}\big\{[\pi_a \mathbf{H}(X), Y] + [X, \pi_a \mathbf{H}(Y)] - \pi_a \mathbf{H}([X, Y])\big\} = 0$$

(the contributions of the projections on $\mathfrak{t}$ are killed by the product by $\boldsymbol{\epsilon}$ as $\boldsymbol{\epsilon} X = \boldsymbol{\epsilon} \pi_a X$ for all X) so that, taking X and Y in $\mathfrak{z}_a$, we obtain

$$\boldsymbol{\epsilon}\big\{[\Delta_a(X), Y] + [X, \Delta_a(Y)] - \Delta_a([X, Y])\big\} = 0$$

and the content of the braces, that is the member of $\mathfrak{z}_a$, must vanish, proving (b$''$) and (b$'$) that is an evident consequence. Let us prove (b$'$)⇒(b). The proof relies on the following property:

If condition (a) *holds, then* $\mathrm{d}\mathbf{H}(X, Y) \in \mathfrak{t}$ *when at least one of the elements X and Y is in* $\mathfrak{t}$.

It suffices to prove that $\mathrm{d}\mathbf{H}(U, Y) \in \mathfrak{t}$ if for instance $U \in \mathfrak{t}$. Now

$$\begin{aligned} \boldsymbol{\epsilon} \mathrm{d}\mathbf{H}(U, Y) &= \boldsymbol{\epsilon}\big\{[\mathbf{H}(U), Y] + [U, \mathbf{H}(Y)] - \mathbf{H}([U, Y]\big\} \\ &= [\boldsymbol{\epsilon} \mathbf{H}(U), Y] - \boldsymbol{\epsilon} \mathbf{H}([U, Y] = [mU, Y] - m[U, Y] = 0. \end{aligned}$$

If (b') holds, when U and V are in $\mathfrak{t}$, X and Y are in $\mathfrak{z}_a$:

$$\boldsymbol{\epsilon}\mathrm{d}\mathbf{H}(U+X, V+Y) = \boldsymbol{\epsilon}\mathrm{d}\mathbf{H}(X,Y) = \boldsymbol{\epsilon}\mathrm{d}\Delta_a(X,Y) = 0$$

and, as $\mathfrak{d} = \mathfrak{t} \oplus \mathfrak{z}_a$, condition (b) is verified. Let $Z \in \mathfrak{z}_b$, put

$$Z' = \{\Delta_b - (\mathbf{1} + \mathrm{ad}\, U_{ab}) \circ \Delta_a \circ (\mathbf{1} - \mathrm{ad}\, U_{ab})\}\,(Z).$$

Taking into account $\boldsymbol{\epsilon}\pi_a X = \boldsymbol{\epsilon} X$ and part (a) of Lemma 5, we deduce that

$$\begin{aligned}\boldsymbol{\epsilon} Z' = \boldsymbol{\epsilon}\,\{\pi_b \mathbf{H}(Z) - \pi_a \mathbf{H}(Z) + \pi_a \mathbf{H}([U,Z])\}& \\ = \boldsymbol{\epsilon}\pi_a \mathbf{H}([U,Z]) = \boldsymbol{\epsilon}\mathbf{H}([U,Z]) = m[U,Z].&\end{aligned}$$

This relation is equivalent to $Z' = mN_b([U,Z]) = m[N_b(U), Z]$, proving the last part of the lemma. ∎

LEMMA 7. *If condition (a) holds then for all $a \in \mathcal{E}$ there exists a unique $\boldsymbol{\xi}_a \in \mathfrak{z}_a$ such that $\Delta_a = \mathrm{ad}\,\boldsymbol{\xi}_a$ restricted to $\mathfrak{z}_a$, put $P_a = \boldsymbol{\omega}_{\boldsymbol{\xi}_a}$. For a and $b \in \mathcal{E}$ and if $U_{ab} \in \mathfrak{t}$ is the constant field equal to $\overrightarrow{ab}$ then*

$$\boldsymbol{\xi}_b - (\mathbf{1} + ad\, U_{ab}).\boldsymbol{\xi}_a = mN_b(U_{ab}), \tag{7.10}$$

$$P_b - P_a = m\overrightarrow{ab}. \tag{7.11}$$

Whenever $m \neq 0$ there exists a unique point c such that $P_c = 0$ (and $\Delta_c = 0$) and then $P_a = m\overrightarrow{ca}$. Whenever $m = 0$ then $P_a = P$ is a vector independent of a.

In other words, if $\mathfrak{z}_a$ is identified with $\mathbb{E}$, when $m \neq 0$ the derivation Δ_a is nothing else but $\boldsymbol{\omega} \mapsto -m\overrightarrow{ac} \times \boldsymbol{\omega}$ and when $m = 0$ it is $\boldsymbol{\omega} \mapsto P \times \boldsymbol{\omega}$ with a constant $P \in \mathbb{E}$. Expressed with the $\boldsymbol{\xi}_a$ rather than P_a the results or the demonstrations should be more complicated because the $\boldsymbol{\xi}_a$ are not in the same space. For instance, whenever $m = 0$ for all a and b we only find $\boldsymbol{\xi}_b = (\mathbf{1} + \mathrm{ad}\, U_{ab}).\boldsymbol{\xi}_a$.

□ The existence of $\boldsymbol{\xi}_a$ results from the fact that the Lie algebra $\mathfrak{z}_a$ is isomorphic with $(\mathbb{E}, \times)$ and all the derivations of $\mathfrak{z}_a$ are inner derivations. With lemma 6 we also have

$$\mathrm{ad}\,\boldsymbol{\xi}_b - (\mathbf{1} + \mathrm{ad}\, U_{ab}) \circ \mathrm{ad}\,\boldsymbol{\xi}_a \circ (\mathbf{1} - \mathrm{ad}\, U_{ab}) = m\,\mathrm{ad}\, N_b(U_{ab})$$

and, transforming the left-hand side into $\mathrm{ad}\big(\boldsymbol{\xi}_b - (\mathbf{1} + \mathrm{ad}\, U_{ab}).\boldsymbol{\xi}_a\big) = \mathrm{ad}\big(mN_b(U_{ab})\big)$, equality (7.10) follows since the center of $\mathfrak{z}_a \simeq (\mathbb{E}, \times)$ is reduced to 0. Translated into $\mathbb{E}$, relation (7.10) leads to equality (7.11). Now, it is evident that when $m \neq 0$, the point c exists and $P_a = m\overrightarrow{ca}$. ∎

Lemma 7 relies on a profound property: $\mathbb{D}/\mathbb{T}$ is a semi-simple group, in fact a simple group here. The Lie algebra $\mathfrak{d}/\mathfrak{t}$ is semi-simple and so for the Lie algebras $\mathfrak{z}_a$. Therefore all the derivations in $\mathfrak{z}_a$ are inner derivations, the center of $\mathfrak{z}_a$ is reduced to 0 and $\operatorname{ad}\boldsymbol{\xi} = 0 \Leftrightarrow \boldsymbol{\xi} = 0$ for $\boldsymbol{\xi} \in \mathfrak{z}_a$.

The results we have obtained may be summarized by the following formulas, where a is a fixed point and we refer to the direct sum $\mathfrak{d} = \mathfrak{z}_a \oplus \mathfrak{t}$: if $Z \in \mathfrak{z}$ and $U \in \mathfrak{t}$ then

$$\begin{aligned}
\pi_a \mathbf{H}(Z) &= [\boldsymbol{\xi}_a, Z] \quad (\Leftrightarrow \boldsymbol{\omega}_{\mathbf{H}(Z)} = P_a \times \boldsymbol{\omega}_Z),\\
\mathbf{H}(U) &= mN_a(U) + qU - \boldsymbol{\epsilon}[\boldsymbol{\xi}_a, U]\\
&\qquad\qquad (\Leftrightarrow \boldsymbol{\omega}_{\mathbf{H}(U)} = mU(a), \;\; \mathbf{H}(U)(a) = qU(a) - P_a \times U(a)),
\end{aligned}$$

(having in mind that to give the projections of X on $\mathfrak{z}_a$ and $\mathfrak{t}$ is equivalent to give $\boldsymbol{\omega}_X$ and $X(a)$). The formula for $\mathbf{H}(U)$ is a consequence of (a) applied to $N_a(U) \in \mathfrak{z}$. In order to complete the determination of $\mathbf{H}$ we must know the projection of $\mathbf{H}(Z)$ on $\mathfrak{t}$ or, what is equivalent, the relation between $\mathbf{H}(Z)$ and $\boldsymbol{\omega}_Z$ and we put:

$$\mathbf{H}(Z)(a) = \mathbf{I}_a(\boldsymbol{\omega}_Z) \;\text{ for }\; Z \in \mathfrak{z}_a \;\text{ where }\; \mathbf{I}_a \in \mathcal{L}(\mathbb{E}).$$

Now Theorem 7, in particular formula (7.3), is proved except the relation between $\mathbf{I}_a$ and $\mathbf{I}_b$.

□ Consider $\mathbf{x} \in \mathbb{E}$ and Z_a and Z_b such that $\boldsymbol{\omega}_{Z_a} = \boldsymbol{\omega}_{Z_b} = \mathbf{x}$ so that $Z_b - Z_a \in \mathfrak{t}$.

$$\mathbf{I}_b(\mathbf{x}) - \mathbf{I}_a(\mathbf{x}) = \mathbf{H}(Z_b)(b) - \mathbf{H}(Z_a)(a) = \mathbf{H}(Z_b - Z_a)(b) + \mathbf{H}(Z_a)(b) - \mathbf{H}(Z_a)(a)$$

With the already proved results

$$\begin{aligned}
\mathbf{I}_b(\mathbf{x}) - \mathbf{I}_a(\mathbf{x}) &= -q\mathbf{x} \times \overrightarrow{ab} + P_b \times (\mathbf{x} \times \overrightarrow{ab}) + \boldsymbol{\omega}_{\mathbf{H}(Z_a)} \times \overrightarrow{ab}\\
&= q\overrightarrow{ab} \times \mathbf{x} + m\overrightarrow{ab} \times (\mathbf{x} \times \overrightarrow{ab}) + P_a \times (\mathbf{x} \times \overrightarrow{ab}) + (P_a \times \mathbf{x}) \times \overrightarrow{ab}.
\end{aligned}$$

Taking $a = c$ and $b = a$ we obtain $\mathbf{I}_a(\mathbf{x}) = \mathbf{I}_c(\mathbf{x}) - (m\, \widetilde{\mathbf{a}}^2 + q\, \widetilde{\mathbf{a}})(\mathbf{x})$ and (7.5). To prove (7.8) consider $\mathbf{H} = H_s$ and $\mathbf{H}' = H_{D.s} = \operatorname{Ad} D \circ \mathbf{H} \circ \operatorname{Ad} D^{-1}$. The associated derivations of Lemma 6 verify $\Delta'_a = \operatorname{Ad} D \circ \Delta_{D^{-1}(a)} \circ \operatorname{Ad} D^{-1}$ (on $\mathfrak{z}_a$) and $a = c_{D.s}$ means that Δ'_a vanishes, what is equivalent to $D^{-1}(a) = c_s$. Now the relation for $\mathbf{I}_s$ is derived from $\mathbf{I}_{D.s}(\boldsymbol{\omega}_Z) = \mathbf{H}'(Z)(D(c_s)) = \mathbf{D}\left(\mathbf{I}_s(\mathbf{D}^{-1}(\boldsymbol{\omega}_Z))\right)$. ■

7.5. Relation with the standard form of dynamic equations

In this section we point out the links between the framework of Secs. 2 to 6 and the usual form of kinematics and dynamics, however we will not give detailed calculations. Some relations were outlined in Sec. 7.1, in particular the elements of $\mathfrak{d}$ correspond to fundamental vector fields on $\mathcal{E}$ that are the classical moment fields.

First of all we have to calculate the vector fields V and $W \in \mathfrak{d}$ corresponding to $\vartheta_r(\mathbf{v})$ and $\vartheta_\ell(\mathbf{v})$ with $\mathbf{v} = \dot{s}$ for a motion defined by $s(t) = D.s_o$ where $D \in \mathbb{D}$ depends on t. We may express D as the composition of a rotation $Q \in \mathbf{R}_c$ about c and a translation $\mathbf{r} \in \mathbb{E}$, then $D(p) = Q(p) + \mathbf{r}$ for $p \in \mathcal{E}$,

$$\frac{d}{dt} D(p) = \dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{Q}(\overrightarrow{cp}) = \begin{cases} V(D(p)) \\ \mathbf{D}(W(p)) \end{cases}$$

($\mathbf{D} = \mathbf{Q} \in \mathrm{SO}(\mathbb{E})$ is the linear part of D and Q and $\boldsymbol{\omega}$ is defined by $\dfrac{d}{dt}\mathbf{Q}(\mathbf{x}) = \boldsymbol{\omega} \times \mathbf{Q}(\mathbf{x})$, a well-known property of the derivative of a time-dependent orthogonal operator). The Plücker vectors of V and W are:

$$\begin{cases} \boldsymbol{\omega}_V = \boldsymbol{\omega}, & V(D(c)) = \dot{\mathbf{r}}, \\ \boldsymbol{\omega}_W = \mathbf{Q}^{-1}(\boldsymbol{\omega}) = \boldsymbol{\Omega}, & W(c) = \mathbf{Q}^{-1}(\dot{\mathbf{r}}) = \mathbf{u}. \end{cases}$$

$\boldsymbol{\omega}$ and $\dot{\mathbf{r}}$ are the angular velocity of the body and the velocity of the particle that is at c in the configuration s_o with respect to an inertial frame ("space fixed frame"), $\boldsymbol{\Omega}$ and $\mathbf{u}$ are the same quantities observed in a body fixed frame. Note that

$$\frac{d\mathbf{u}}{dt} + \boldsymbol{\Omega} \times \mathbf{u} = \mathbf{Q}^{-1}(\ddot{\mathbf{r}}).$$

For developing the dynamic equations we consider the case of symmetric inertia operators and take the Plüker vectors at the point c defined in Sec. 7.3 (so that $D(c) = c_s$).

$$H(\dot{V}) \equiv \begin{bmatrix} \mathbf{0} & m\mathbf{1} \\ \mathbf{I}_s & \mathbf{0} \end{bmatrix} \times \begin{bmatrix} \dot{\boldsymbol{\omega}}_V \\ \dot{V}(c_s) \end{bmatrix} = \begin{bmatrix} m\dot{V}(c_s) \\ \mathbf{I}_s(\dot{\boldsymbol{\omega}}_V) \end{bmatrix},$$

$$[V, H(V)] \equiv \begin{bmatrix} \boldsymbol{\omega}_V \times \boldsymbol{\omega}_L \\ \boldsymbol{\omega}_V \times L(c) - \boldsymbol{\omega}_L \times V(c) \end{bmatrix} = \begin{bmatrix} m\boldsymbol{\omega}_V \times V(c) \\ \boldsymbol{\omega}_V \times \mathbf{I}_s(\boldsymbol{\omega}_V) \end{bmatrix}.$$

Note that $\dot{V}$ is the derivative $\dfrac{dV}{dt}$ in $\mathfrak{d}$ and $\dot{V}(c_s)$ is not the derivative of $V(c_s)$. In fact

$$\frac{d}{dt}\left(V(c_s)\right) = \dot{V}(c_s) + \boldsymbol{\omega}_V \times \dot{V}(c_s) = \ddot{\mathbf{r}},$$

so that the dynamic equations with respect to a space-fixed frame are

$$m\ddot{\mathbf{r}} = \boldsymbol{\Phi}, \qquad \mathbf{I}_s\left(\frac{d\boldsymbol{\omega}}{dt}\right) + \boldsymbol{\omega} \times \mathbf{I}_s(\boldsymbol{\omega}) = \mathbf{M}, \tag{7.12}$$

where $\mathbf{\Phi}$ and $\mathbf{M}$ are the Plüker vectors at c_s of the “force” $\mathbf{F}$ interpreted as the ordinary force and torque applied to the body. With respect to a body-fixed frame, the previous equations read

$$m\left(\frac{d\mathbf{u}}{dt}+\mathbf{\Omega}\times\mathbf{u}\right)=\boldsymbol{\varphi},\qquad \mathbf{I}\left(\frac{d\mathbf{\Omega}}{dt}\right)+\mathbf{\Omega}\times\mathbf{I}(\mathbf{\Omega})=\boldsymbol{\mu}, \tag{7.13}$$

where $\mathbf{I}=\mathbf{I}_{s_o}$, $\boldsymbol{\varphi}=\mathbf{Q}^{-1}(\mathbf{\Phi})$ and $\boldsymbol{\mu}=\mathbf{Q}^{-1}(\mathbf{M})$ are the same quantities observed in a body-fixed frame. Finally we recognize in (7.12) and (7.13) the classical dynamic equations for the linear and angular momentum about the center of inertia, so that we can conclude that dynamics of the ordinary rigid body may be derived from:

1. A very general *a priori* assumption on the mathematical form of the relation velocity-momentum;
2. Condition for the objectivity of the inertial forces;
3. Algebraic properties of the Lie algebra $\mathfrak{d}$ of the Euclidean group permitting to determine precisely the mathematical form of the inertia operators verifying the previous condition;
4. The criteria of symmetry and positivity of the inertia operator.

Of course, at the present stage, it would be natural, and perhaps quite interesting, to drop the last assumption and to study also “strange” rigid bodies with $q \neq 0$ or with a singular inertia operator.

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