

Nonlocal theoretical analysis of the dynamic behavior of two Griffith cracks in a piezoelectric strip

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THE DYNAMIC behavior of two Griffith cracks in a strip made of piezoelectric materials under anti-plane shear waves is investigated by means of the non-local theory for impermeable crack surface conditions. A one-dimensional non-local kernel is used instead of a two-dimensional one for the anti-plane dynamic problem to obtain the stress and the electric displacement near the crack tips. By utilizing the Fourier transform, the problem can be solved by means of two pairs of triple integral equations. These equations are solved using the Schmidt method. Contrary to the classical solution, it is found that no stress and electric displacement singularity are present at the crack tip. This is shown to be consistent with the physical nature.

Key words: elastic waves, piezoelectric materials, non-local theory, Fourier integral transform, collinear cracks, Schmidt method.

1. Introduction

IN THE THEORETICAL studies of crack problems in the piezoelectric materials, numerous researchers have proposed several different electric boundary conditions at the crack surfaces. For example, for the sake of analytical simplification, the assumption that the crack surfaces are impermeable to electric fields was adopted by many researchers [1–4]. In this model, the assumption of impermeable cracks refers to the fact that the crack surfaces are free of surface charge and thus the electric displacements vanish inside the crack. In fact, cracks in piezoelectric materials consist of vacuum, air or some other gases. This requires that the electric fields can propagate through the crack, so the electric displacement component perpendicular to the crack surfaces should be continuous across the crack surfaces. Along this line, the crack problems in piezoelectric materials were analyzed in [5–7]. Due to a much simpler treatment from the mathematical point of view, the impermeable crack and the conducting crack are still employed extensively in the study of the crack problems of piezoelectric materials. However, these solutions contain the stress and electric displacement singularity. This is not reasonable according to the physical nature.

As it is commonly known, one of the principal postulates of the traditional mechanics of continuous media is the principle of local action. This principle excludes the action at a distance, and attributes changes occurring at a point of the medium to thermo-energetic factors acting at the point. Thus the classical theory, by restricting the response of a continuum to strictly local actions, constitutes a so-called local theory. A familiar example is provided by the conventional theory of elasticity, in which, when determining the stress at a point, one disregards the deformation and the temperature fields outside an arbitrarily small neighborhood of the point. However, the application of classical elasticity to micro-mechanics leads to some physically unreasonable results. A singularity appearing in a stress field is a typical one; the existence of stress singularities also leads to difficulties in development of experiments in micro-mechanics. In fact, the stress at the crack tip is finite. As a result of this, beginning from Griffith, all fracture criteria in practice today are based on other considerations, e.g. the energy, and the J -integral [8]. In contrast to this local approach of zero-range internal interactions, the modern non-local continuum mechanics, originated and developed in the last four decades, postulates that the local state at a point is influenced by the action of all particles of the body. This was done primarily by EDELEN [9], ERINGEN [10], GREEN and RIVLIN [11]. According to the non-local theory, the stress at a point X in a body depends not only on the strain at point X but also on those at all other points of the body. This is different from the classical theory. In the classical theory, the stress at a point X in a body depends only on the strain at point X . In the Reference [12], the basic theory of the non-local elasticity was stated with emphasis on the difference between the non-local theory and the classical continuum mechanics. The basic idea of non-local elasticity is to establish a relationship between the macroscopic mechanical quantities and the microscopic physical quantities within the framework of continuum mechanics. The constitutive theory of non-local elasticity has been developed widely [9], where the microstructures of the material have effect on the elastic modulus. It has been found that the microstructures of the material have their effect not only on the constitutive equation but also on the basic balance laws and boundary conditions [13–14].

Other advances have been made by the application of non-local elasticity to such fields as the dislocation theory [15–16], solid defects [17–18] and fracture mechanics [19–20]. While the literature on the fundamental aspects of non-local continuum mechanics is relatively extensive, applications of the theory are not too numerous. The results, however, of those concrete problems that were solved display a rather remarkable agreement with experimental evidence. This can be used to predict the cohesive stress for various materials close to that obtained in atomic lattice dynamics [21–22]. Likewise, a non-local study of the secondary flow of viscous fluid in a pipe furnishes a streamline pattern similar to that

obtained experimentally by NIKURADZE [23]. Other examples of the effectiveness of the non-local approach are: (i) prediction of the dispersive character of elastic waves demonstrated experimentally (and lacking in the classical theory) [24] and (ii) calculation of the velocity of short Love waves whose non-local estimates agree better with seismological observations than the local ones [25]. Various non-local theories have been formulated to address the strain-gradient and size effects (see, for example, FOREST [26]).

To avoid the stress singularity in the classical elastic theory, the non-local theory was used to discuss the state of stress near the tip of a sharp line crack in an elastic plane subject to uniform tension, shear and anti-plane shear [27–30]. Recently, the non-local theory was used to analyze the crack problems in the piezoelectric material [31–32]. These solutions obtained do not contain any stress singularity, thus resolving a fundamental problem that existed over many years. This enables us to employ the maximum stress hypothesis to deal with fracture problems in a natural way. To our knowledge, the dynamic electro-elastic behavior of the piezoelectric materials strip with two cracks subjected to anti-plane shear and in-plane electric loading has not been studied by the non-local theory for impermeable crack surface boundary conditions.

In the present paper, scattering of the harmonic elastic anti-plane shear waves by two Griffith impermeable cracks in a piezoelectric strip is investigated by use of the non-local theory. The traditional concept of linear elastic fracture mechanics and the non-local theory are extended to include the piezoelectric effects. To overcome the mathematical difficulties, one-dimensional non-local kernel function is used instead of the two-dimensional kernel function for the anti-plane dynamic problem to obtain the stress and electric displacement occurring at the crack tips. To obtain the theoretical solution and discuss the probability of using the non-local theory to solve the dynamic fracture problem in the piezoelectric materials strip, one has to accept some assumptions, such as in NOWINSKI's works [25, 33]. Certainly, the assumption should be further investigated to satisfy the realistic conditions. Fourier transform is applied and a mixed boundary value problem is reduced to two pairs of triple integral equations. In solving the triple integral equations, the crack surface displacement and electric potential are expanded in a series of Jacobi polynomials. This process is quite different from that adopted in previous works [1–30]. As expected, the solution in this paper does not contain the stress and electric displacement singularity at the crack tip, thus clearly indicating the physical nature of the problem.

2. Basic equations of non-local piezoelectric materials

According to non-local theory, the stress at a point X in a body depends not only on the strain at point X , but also on those at all other points of the body.

This observation is made in accordance with atomic theory of lattice dynamics and experimental observation of phonon dispersion [34]. For the anti-plane shear problem, the basic equations of linear, homogeneous, transversely isotropic, non-local piezoelectric materials, with vanishing body force are [30–32, 35–36]

$$(2.1) \quad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = \rho \frac{\partial^2 w}{\partial t^2},$$

$$(2.2) \quad \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = 0,$$

$$(2.3) \quad \tau_{kz}(X, t) = \int_V \left[c'_{44}(|X' - X|) w_{,k}(X', t), \right. \\ \left. + e'_{15}(|X' - X|) \phi_{,k}(X', t) \right] dV(X') \quad (k = x, y),$$

$$(2.4) \quad D_k(X, t) = \int_V \left[e'_{15}(|X' - X|) w_{,k}(X', t), \right. \\ \left. - \varepsilon'_{11}(|X' - X|) \phi_{,k}(X', t) \right] dV(X') \quad (k = x, y),$$

where the only difference with the classical elastic theory and the piezoelectric theory is in the stress and the electric displacement constitutive equations (2.3)–(2.4), in which the stress $\tau_{zk}(X, t)$ and the electric displacement $D_k(X, t)$ at a point X depends on $w_{,k}(X, t)$ and $\phi_{,k}(X, t)$, at all points of the body. w and ϕ are the mechanical displacement and the electric potential. For homogeneous and isotropic piezoelectric materials there exist only three material parameters, $c'_{44}(|X' - X|)$, $e'_{15}(|X' - X|)$ and $\varepsilon'_{11}(|X' - X|)$ which are functions of the distance $|X' - X|$. ρ is the density of the piezoelectric materials. The integrals in (2.3)–(2.4) are extended over the volume V of the body enclosed within the surface ∂V .

As discussed in the papers [23, 29–30], it can be assumed in the form of $c'_{44}(|X' - X|)$, $e'_{15}(|X' - X|)$ and $\varepsilon'_{11}(|X' - X|)$ for which the dispersion curves of plane elastic waves coincide with those known in lattice dynamics. Among several possible curves, the following one has been found to be very useful

$$(2.5) \quad (c'_{44}, e'_{15}, \varepsilon'_{11}) = (c_{44}, e_{15}, \varepsilon_{11}) \alpha(|X' - X|).$$

$\alpha(|X' - X|)$ is known as the influence function, and is a function of the distance $|X' - X|$. $c_{44}, e_{15}, \varepsilon_{11}$ are the shear modulus, piezoelectric coefficient and dielectric parameter, respectively.

Substitution of Eq. (2.5) into Eqs. (2.3)–(2.4) yields

$$(2.6) \quad \tau_{kz}(X, t) = \int_V \alpha(|X' - X|) \sigma_{kz}(X', t) dV(X') \quad (k = x, y),$$

$$(2.7) \quad D_k(X, t) = \int_V \alpha(|X' - X|) D_k^c(X', t) dV(X') \quad (k = x, y),$$

where

$$(2.8) \quad \sigma_{kz} = c_{44} w_{,k} + e_{15} \phi_{,k},$$

$$(2.9) \quad D_k^c = e_{15} w_{,k} - \varepsilon_{11} \phi_{,k}.$$

The expressions (2.8)–(2.9) are the classical constitutive equations of piezoelectric materials.

3. The crack model

Consider an infinitely long piezoelectric strip of width $2h$, containing two collinear Griffith cracks parallel to the edges of the strip. Cracks occupy the region $b \leq |x| \leq 1$, $y = 0$, b is the distance between two cracks. The geometry of the problem is shown in Fig. 1. Let ω be the circular frequency of the incident wave. $-\tau_0$ is the magnitude of the incident wave. In what follows, the time-dependence of all field quantities is assumed to be of the form $e^{-i\omega t}$. It was further supposed that the two faces of the crack do not come in contact during vibrations. The piezoelectric boundary-value problem for anti-plane shear will be considerably simplified if we consider only the out-of-plane displacement and the in-plane electric fields. When the crack is subjected to harmonic elastic waves and a constant electric displacement $D_y = -D_0$, as discussed in [30, 38], the boundary conditions on the crack faces at $y = 0$ are

$$(3.1) \quad \tau_{yz}(x, 0, t) = -\tau_0, \quad D_y(x, 0, t) = -D_0, \quad b \leq |x| \leq 1,$$

$$(3.2) \quad \tau_{yz}(x, \pm h) = D_y(x, \pm h) = 0, \quad |x| \leq \infty,$$

$$(3.3) \quad w(x, 0, t) = \phi(x, 0, t) = 0, \quad |x| < b, \quad 1 < |x|,$$

$$(3.4) \quad w(x, y, t) = \phi(x, y, t) = 0, \quad \text{for} \quad (x^2 + y^2)^{1/2} \rightarrow \infty.$$

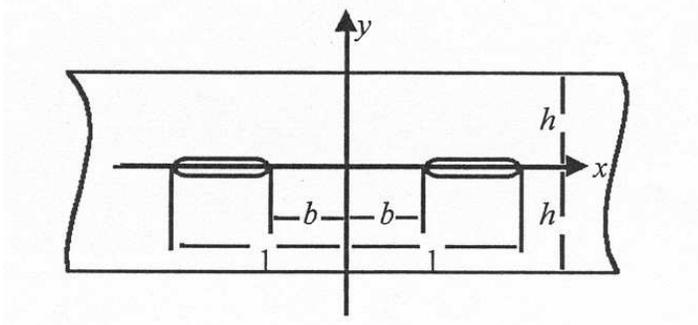


FIG. 1. Cracks in a piezoelectric strip.

Substituting Eqs. (2.6)–(2.7) into Eqs. (2.1)–(2.2), respectively, using Green–Gauss theorem, it can be obtained [30]:

$$(3.5) \quad \iint_V \alpha(|x' - x|, |y' - y|) [c_{44} \nabla^2 w(x', y', t) + e_{15} \nabla^2 \phi(x', y', t)] dx' dy' \\ - \int_{-l}^l \alpha(|x' - x|, 0) [\sigma_{yz}(x', 0, t)] dx' = \rho \frac{\partial^2 w}{\partial t^2},$$

$$(3.6) \quad \iint_V \alpha(|x' - x|, |y' - y|) [e_{15} \nabla^2 w(x', y', t) - \varepsilon_{11} \nabla^2 \phi(x', y', t)] dx' dy' \\ - \int_{-l}^l \alpha(|x' - x|, 0) [D_y^c(x', 0, t)] dx' = 0,$$

where the boldface bracket indicates a jump at the crack line. $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two-dimensional Laplace operator. Because of the assumed symmetry in geometry and loading, it is sufficient to consider the problem for $0 \leq x \leq \infty$, $0 \leq y \leq h$ only. Under the applied anti-plane shear load on the closed surfaces of the crack, the displacement field and the electric potential possess the following symmetry regulations

$$(3.7) \quad w(x, -y, t) = -w(x, y, t), \quad \phi(x, -y, t) = -\phi(x, y, t).$$

Using the Eq. (3.7), we find that

$$(3.8) \quad [\sigma_{yz}(x, 0, t)] = 0,$$

$$(3.9) \quad [D_y^c(x, 0, t)] = 0.$$

Hence the line integrals in (3.5)–(3.6) vanish. By taking the Fourier transform of (3.5)–(3.6) with respect to x' , it can be shown that

$$(3.10) \quad \int_0^{\infty} \bar{\alpha}(|s|, |y' - y|) \left\{ c_{44} \left[\frac{d^2 \bar{w}(s, y', t)}{dy^2} - s^2 \bar{w}(s, y', t) \right] + e_{15} \left[\frac{d^2 \bar{\phi}(s, y', t)}{dy^2} - s^2 \bar{\phi}(s, y', t) \right] \right\} dy' = -\rho \omega^2 \bar{w},$$

$$(3.11) \quad \int_0^{\infty} \bar{\alpha}(|s|, |y' - y|) \left\{ e_{15} \left[\frac{d^2 \bar{w}(s, y', t)}{dy^2} - s^2 \bar{w}(s, y', t) \right] - \varepsilon_{11} \left[\frac{d^2 \bar{\phi}(s, y', t)}{dy^2} - s^2 \bar{\phi}(s, y', t) \right] \right\} dy' = 0.$$

Here a superposed bar indicates the Fourier transform, e.g.

$$\bar{f}(s, y) = \int_0^{\infty} f(x, y) \exp(isx) dx.$$

What now remains is to solve the integrodifferential Eqs. (3.10)–(3.11) for the functions w and ϕ . It seems to be obvious that a rigorous solution of such a problem encounters serious but not unsurmountable mathematical difficulties, and one has to resort to an approximate procedure. In the given problem, according to the assumptions shown in (3.11)–(3.12), the non-local interaction in y -direction can be ignored. It can be assumed that

$$(3.12) \quad \bar{\alpha}(|s|, |y - y|) = \bar{\alpha}_0(s) \delta(y - y).$$

From Eqs.(3.10)–(3.11), it can be shown that

$$(3.13) \quad \bar{\alpha}_0(s) \left\{ c_{44} \left[\frac{d^2 \bar{w}(s, y, t)}{dy^2} - s^2 \bar{w}(s, y, t) \right] + e_{15} \left[\frac{d^2 \bar{\phi}(s, y, t)}{dy^2} - s^2 \bar{\phi}(s, y, t) \right] \right\} = -\rho \omega^2 \bar{w},$$

$$(3.14) \quad e_{15} \left[\frac{d^2 \bar{w}(s, y, t)}{dy^2} - s^2 \bar{w}(s, y, t) \right] - \varepsilon_{11} \left[\frac{d^2 \bar{\phi}(s, y, t)}{dy^2} - s^2 \bar{\phi}(s, y, t) \right] = 0.$$

The solution of the Eqs. (3.13)–(3.14) does not present any difficulties, it can be written for $y \geq 0$ as follows:

$$(3.15) \quad \begin{aligned} w(x, y, t) &= \frac{2}{\pi} \int_0^{\infty} [A_1(s)e^{-\gamma y} + A_2(s)e^{\gamma y}] \cos(xs) ds, \\ \phi(x, y, t) &= \frac{e_{15}}{\varepsilon_{11}} w(x, y, t) + \frac{2}{\pi} \int_0^{\infty} [B_1(s)e^{-\gamma y} + B_2(s)e^{\gamma y}] \cos(xs) ds, \end{aligned}$$

where $\gamma^2 = s^2 - \omega^2/c^2 \bar{\alpha}_0(s)$, $c^2 = \mu/\rho$, $\mu = c_{44} + e_{15}^2/\varepsilon_{11}$. $A_1(s)$, $A_2(s)$, $B_1(s)$ and $B_2(s)$ are to be determined from the boundary conditions.

Because of the symmetry, it suffices to consider the problem in the first quadrant only. According to the boundary conditions (3.1)–(3.3), it can be obtained that

$$(3.16) \quad \frac{2}{\pi} \int_0^{\infty} \bar{\alpha}_0(s) \gamma \frac{1 - \exp(-2\gamma h)}{1 + \exp(-2\gamma h)} A(s) \cos(sx) ds = \frac{1}{\mu} \left(\tau_0 + \frac{e_{15} D_0}{\varepsilon_{11}} \right) \quad b \leq |x| \leq 1,$$

$$(3.17) \quad \frac{2}{\pi} \int_0^{\infty} A(s) \cos(sx) ds = 0, \quad |x| > 1, |x| < b$$

and

$$(3.18) \quad \frac{2}{\pi} \int_0^{\infty} \bar{\alpha}_0(s) s \frac{1 - \exp(-2sh)}{1 + \exp(-2sh)} B(s) \cos(sx) ds = -\frac{D_0}{\varepsilon_{11}}, \quad b \leq |x| \leq 1,$$

$$(3.19) \quad \frac{2}{\pi} \int_0^{\infty} B(s) \cos(sx) ds = 0, \quad |x| > 1, |x| < b,$$

where $\bar{\alpha}_0(s) = 1$ for the limit $a \rightarrow 0$. The relationships between the functions $A(s)$, $B(s)$, $A_1(s)$, $A_2(s)$, $B_1(s)$ and $B_2(s)$ are obtained by applying a Fourier sine transform [39] to Eq. (3.2):

$$\begin{aligned} A(s) &= [1 + e^{-2\gamma h}] A_1(s), & A_2(s) &= e^{-2\gamma h} A_1(s), \\ B(s) &= [1 + e^{-2sh}] B_1(s), & B_2(s) &= e^{-2sh} B_1(s). \end{aligned}$$

To determine the unknown functions $A(s)$ and $B(s)$, the above two pairs of triple-integral equations (3.16)–(3.19) must be solved.

4. Solution of the triple integral equation

The triple integral equations (3.16)–(3.19) can not be transformed into the Fredholm integral equation of the second kind because the kernel of the second-kind Fredholm integral equation in the paper [30] is divergent. The kernel of the second-kind Fredholm integral equation in [30] can be written as follows:

$$L(x, u) = (xu)^{1/2} \int_0^{\infty} tk(\varepsilon't)J_0(xt)J_0(ut) dt, \quad 0 \leq x, u \leq 1,$$

where $J_n(x)$ is the Bessel function of order n .

$$k(\varepsilon t) = -\Phi(\varepsilon't), \quad \lim_{t \rightarrow \infty} k(\varepsilon't) \neq 0 \quad \text{for} \quad \varepsilon' = \frac{a}{2\beta l} \neq 0,$$

(l is the length of the crack),

$$J_0(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{1}{4}\pi\right) \quad \text{for} \quad x \gg 0.$$

The limit of $tk(\varepsilon't)J_0(xt)J_0(ut)$ is not equal to zero for $t \rightarrow \infty$. So the kernel $L(x, u)$ in ERINGEN's paper [30] is divergent. Of course, the triple integral Eqs. (3.16)–(3.19) can be considered to be a single integral equation of the first kind with a discontinuous kernel [28]. It is well known in the literature that integral equations of the first kind are generally ill-posed in the sense of Hadamard, e.g. small perturbations of the data can yield arbitrarily large changes in the solution. This makes the numerical solution of such equations quite difficult. In this paper, The Schmidt method [40] was used to overcome the difficulty. As discussed in [25, 27–32], it was taken

$$(4.1) \quad \alpha_0 = \chi_0 \exp\left[-(\beta/a)^2(x' - x)^2\right],$$

$$(4.2) \quad \chi_0 = \frac{\beta}{a\sqrt{\pi}},$$

where β is a constant (here β is a constant appropriate to each material). a is the lattice parameter. So it can be obtained that:

$$(4.3) \quad \bar{\alpha}_0(s) = \exp(-(sa)^2/(2\beta)^2)$$

and $\bar{\alpha}_0(s) = 1$ for the limit $a \rightarrow 0$, so that Eqs. (2.5)–(2.8) reduces to the well-known equation of the classical theory. Here the Schmidt method can be used

to solve the triple integral Eqs.(2.5)–(2.8). The displacement w and the electric potential ϕ can be represented by the following series:

$$(4.4) \quad w(x, 0, t) = \sum_{n=0}^{\infty} a_n P_n^{(\frac{1}{2}, \frac{1}{2})} \left(\frac{x - \frac{1+b}{2}}{\frac{1-b}{2}} \right) \left(1 - \frac{\left(x - \frac{1+b}{2}\right)^2}{\left(\frac{1-b}{2}\right)^2} \right)^{1/2},$$

for $b \leq x \leq 1, \quad y = 0,$

$$(4.5) \quad w(x, 0, t) = 0, \quad \text{for } x > 1, \quad x < b, \quad y = 0,$$

$$(4.6) \quad \phi(x, 0, t) = \sum_{n=0}^{\infty} b_n P_n^{(\frac{1}{2}, \frac{1}{2})} \left(\frac{x - \frac{1+b}{2}}{\frac{1-b}{2}} \right) \left(1 - \frac{\left(x - \frac{1+b}{2}\right)^2}{\left(\frac{1-b}{2}\right)^2} \right)^{1/2},$$

for $b \leq x \leq 1, \quad y = 0,$

$$(4.7) \quad \phi(x, 0, t) = 0, \quad \text{for } x > 1, \quad x < b, \quad y = 0,$$

where a_n and b_n are unknown coefficients to be determined and $P_n^{(1/2, 1/2)}(x)$ is a Jacobi polynomial [39]. The Fourier transformations of Eqs. (4.4)–(4.7) and (34) are

$$(4.8) \quad A(s) = \bar{w}(s, 0, t) = \sum_{n=0}^{\infty} a_n Q_n G_n(s) \frac{1}{s} J_{n+1} \left(s \frac{1-b}{2} \right),$$

$$(4.9) \quad B(s) = \bar{\phi}(s, 0, t) - \frac{e_{15}}{\varepsilon_{11}} \bar{w}(s, 0, t) \\ = \sum_{n=0}^{\infty} \left(b_n - \frac{e_{15}}{\varepsilon_{11}} a_n \right) Q_n G_n(s) \frac{1}{s} J_{n+1} \left(s \frac{1-b}{2} \right),$$

$$(4.10) \quad Q_n = 2\sqrt{\pi} \frac{\Gamma\left(n + 1 + \frac{1}{2}\right)}{n!},$$

$$G_n(s) = \begin{cases} (-1)^{n/2} \cos\left(s \frac{1+b}{2}\right), & n = 0, 2, 4, 6, \dots \\ (-1)^{(n+1)/2} \sin\left(s \frac{1+b}{2}\right), & n = 1, 3, 5, 7, \dots \end{cases}$$

where $\Gamma(x)$ and $J_n(x)$ are the Gamma and Bessel functions, respectively.

Substituting Eqs. (4.8) and (4.9) into Eqs. (3.16)–(3.19), respectively, the Eqs. (3.17) and (3.19) can be automatically satisfied. Then the remaining equations (3.16) and (3.18) reduce to the form:

$$(4.11) \quad \sum_{n=0}^{\infty} a_n Q_n \int_0^{\infty} \bar{\alpha}_0(s) G_n(s) \frac{\gamma[1 - e^{-2\gamma h}]}{s[1 + e^{-2\gamma h}]} J_{n+1}(sl) \cos(sx) ds = \frac{\pi}{2\mu} \tau_0(1 + \lambda),$$

$$(4.12) \quad \sum_{n=0}^{\infty} \left(b_n - \frac{\epsilon_{15}}{\epsilon_{11}} a_n \right) Q_n \int_0^{\infty} \bar{\alpha}_0(s) G_n(s) \frac{1 - e^{-2sh}}{1 + e^{-2sh}} J_{n+1}(sl) \cos(sx) ds = -\frac{\pi D_0}{2\epsilon_{11}},$$

where $\lambda = \frac{\epsilon_{15} D_0}{\epsilon_{11} \tau_0}$.

The semi-infinite integral in Eq. (4.11) can be evaluated numerically, except for the singularities in the integrands of the integrals in Eq. (4.11). These singularities are poles that occur in the complex plane at the zero of the function $1 + \exp(-2\gamma h)$, such as $2\gamma h = i\pi, 3i\pi, 5i\pi, \dots$. All poles depend on the material, the incident wave frequency ω and the lattice parameter. It may be noted that the integral of Eq. (4.11) is not convergent at these poles. However, there is no pole for $\omega/c < \pi/2h$. So the integral of Eq. (4.11) is convergent at these poles for $\omega/c < \pi/2h$. In this paper, we have only discussed the case of $\omega/c < \pi/2h$. From the reference [38], this case may be consistent with the statement that only the shear waves with $\omega/c < \pi/2h$ can be propagated in an elastic strip of width $2h$. This fact is in agreement with the well-known results that the waves can not propagate with frequencies less than a parameter depending on the width of the strip. For $\omega/c > \pi/2h$, it should be further investigated. For larges, the integrands of Eqs. (4.11) and (4.12) decrease almost exponentially. So, they can be evaluated numerically by Filon's method. Equations (4.11) and (4.12) can now be solved for the coefficients a_n and b_n by means of the Schmidt method [40] for $\omega/c < \pi/2h$. For brevity, the Eq. (4.11) can be rewritten (Eq. (4.12) can be solved using a similar method) as follows:

$$(4.13) \quad \sum_{n=0}^{\infty} a_n E_n(x) = U(x), \quad b < x < 1,$$

where $E_n(x)$ and $U(x)$ are known functions and coefficients a_n are to be determined. A set of functions $P_n(x)$ which satisfy the orthogonality condition

the entire dynamic stress field and the electric displacement from coefficients a_n and b_n , it is of importance in fracture mechanics to determine the stress τ_{yz} and the electric displacement D_y in the vicinity of the crack tips. τ_{yz} and D_y along the crack line can be expressed respectively as

$$(5.1) \quad \tau_{yz}(x, 0, t) = -\frac{2}{\pi} \sum_{n=0}^{\infty} \left[\mu a_n Q_n \int_0^{\infty} \bar{\alpha}_0(s) G_n(s) \frac{\gamma[1-e^{-2\gamma h}]}{s[1+e^{-2\gamma h}]} J_{n+1}(sl) \cos(xs) ds \right. \\ \left. + e_{15} \left(b_n - \frac{e_{15}}{\varepsilon_{11}} a_n \right) Q_n \int_0^{\infty} \bar{\alpha}_0(s) G_n(s) \frac{1-e^{-2sh}}{1+e^{-2sh}} J_{n+1}(sl) \cos(xs) ds \right],$$

$$(5.2) \quad D_y(x, 0, t) \\ = -\frac{2}{\pi} \sum_{n=0}^{\infty} (e_{15} a_n - \varepsilon_{11} b_n) Q_n \int_0^{\infty} \bar{\alpha}_0(s) G_n(s) \frac{1-e^{-2sh}}{1+e^{-2sh}} J_{n+1}(sl) \cos(xs) ds.$$

For $a = 0$ at $x = b, 1$, we have the classical stress and electric displacement singularity. However, as long as $a \neq 0$, the semi-infinite integration and the series in the Eqs. (5.1) and (5.2) are convergent for any value of variable x . Equations (5.1) and (5.2) give a finite stress along $y = 0$, so there is no stress and electric displacement singularity at the crack tips. At $b < x < 1$, τ_{yz}/τ_0 is very close to unity, and for $x > 1$, τ_{yz}/τ_0 and D_y/D_0 possess finite values decreasing from a finite value at $x = 1$ to zero at $x \rightarrow \infty$. Since $a/[2\beta(1-b)] > 1/100$ represents a crack length of less than 100 atomic distances as stated in [30], and other serious questions arise regarding the interatomic arrangements and force laws, we do not pursue solutions valid at such small crack sizes. The semi-infinite numerical integrals, which occur, are evaluated easily by Filon's method because of the rapid decrease of the integrands. In all computations, the material constants are not considered, we consider the incident wave frequency, the wave velocity, the crack length, the distance between two cracks, the thickness of the strip, the distance between two cracks and the lattice parameter in this paper, because the stress fields do not depend on the material constants. Since the integrands of Eqs. (5.1) and (5.2) are complex, the stress along the crack face exhibits a slight variation. The results are plotted in Figs. 2–11.

The following observations are very important:

(i) The dynamic stress and the electric displacement at the crack tip become infinite as the atomic distance $a \rightarrow 0$ ($\bar{\alpha}_0(s) = 1$ for the limit $a \rightarrow 0$). This is the classical continuum limit of square root singularity. This can be obtained from two pairs of triple-integral equations (3.16)–(3.19). For the local theory,

we can only obtain the stress and electric displacement intensity factors for the variation of $\omega l/c$.

(ii) For the $a/\beta = \text{constant}$, the atomic distance does not change, the value of the stress and the electric displacement concentrations (at the crack tip) become higher with the increase of the crack length as shown in Figs. 2 to 3. Not this fact, the experiments indicate that the piezoelectric materials with smaller cracks are more resistant to fracture than those with larger cracks [30]. The stress and the electric displacement fields near the left-hand tip are greater than the ones near the right-hand tip for the crack, as shown in Figs. 2, 3.

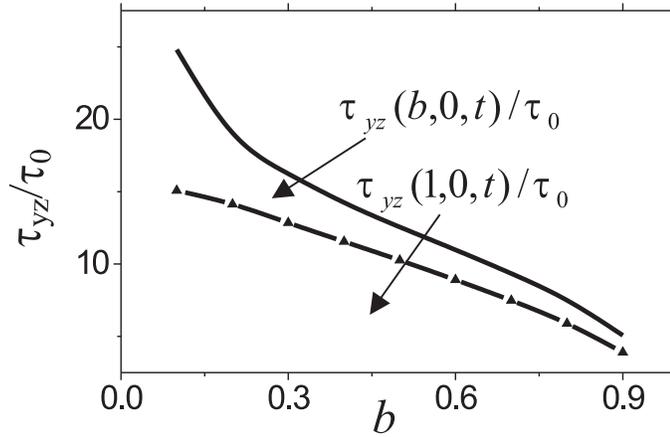


FIG. 2. The stress at the crack tip versus b for $h = 1.0$, $a/(2\beta) = 0.0005$, $\omega/c = 0.4$, $\lambda = 0.4$. (PZT-5H).

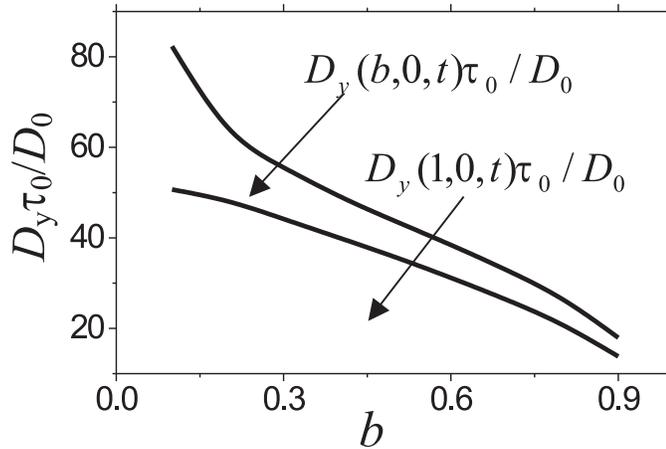


FIG. 3. The electric displacement at the crack tip versus b for $h = 1.0$, $a/(2\beta) = 0.0005$, $\omega/c = 0.4$, $\lambda = 0.4$. (PZT-5H).

(iii) The significance of this result is that the fracture criteria are unified at both the macroscopic and microscopic scales, and we can solve the problem of any crack scales.

(iv) The present results will change to the classical ones when we introduce the function $\alpha(|X' - X|) = \delta(|X' - X|)$.

(v) The dynamic stress concentration occurs at the crack tip as defined in [29–30], and this is given by

$$(5.3) \quad \tau_{yz}(b, 0, t)/\tau_0 = c_1/\sqrt{a/[2\beta(1-b)]},$$

$$(5.4) \quad \tau_{yz}(1, 0, t)/\tau_0 = c_2/\sqrt{a/[2\beta(1-b)]},$$

where c_1 and c_2 represent the stress concentration value at the tip of the crack. c_1 is almost equal to $c_1 \approx 0.533$. c_2 is almost equal to $c_2 \approx 0.533$. It is larger than the static stress concentration of the static non-local problem [30].

(vi) The dimensionless stress is found to be independent of the electric loads and the material parameters. It just depends on the length of the crack, the lattice parameter, the thickness of the strip, the circular frequency of the incident wave and the wave velocity. However, the electric field is found to be independent of the material parameters and the circular frequency of the incident wave and the wave velocity. It just depends on the length of the crack, the thickness of the strip and the lattice parameter. The dynamic stress at the crack tips tends to increase with the frequency for $\omega/c < 1.2$ as shown in Figs. 4, 5.

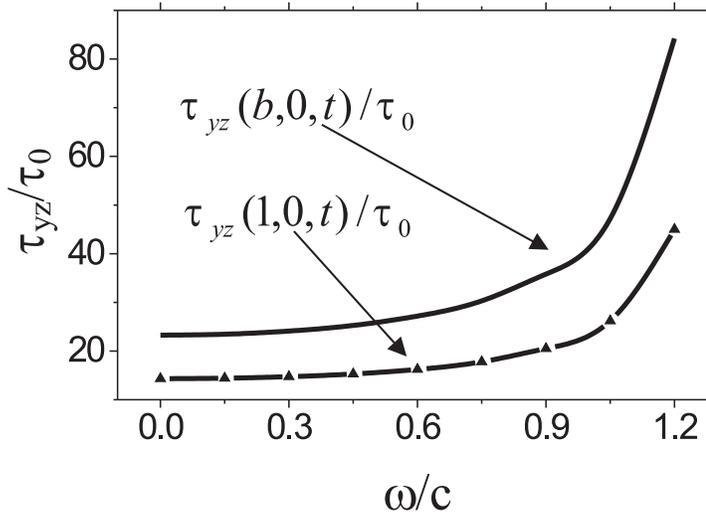


FIG. 4. The stress at the crack tip versus ω for $h = 1.0$, $a/(2\beta) = 0.0005$, $b = 0.1$, $\lambda = 0.4$ (PZT-5H).

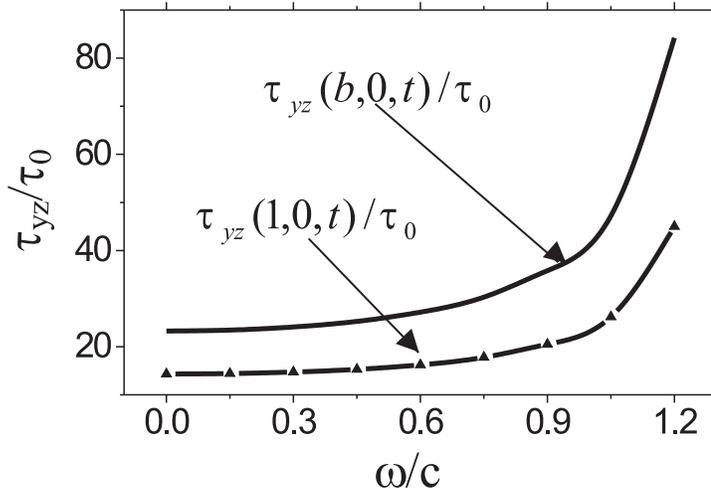


FIG. 5. The stress at the crack tip versus ω for $h = 1.0$, $a/(2\beta) = 0.0005$, $b = 0.1$, $\lambda = 0.4$. (PZT-4).

(vii) The dynamic stress and the dynamic electric displacement at the crack tips tend to decrease with the thickness of the strip, they reach a minimum and then they increase in magnitude, as shown in Figs. 6, 7.

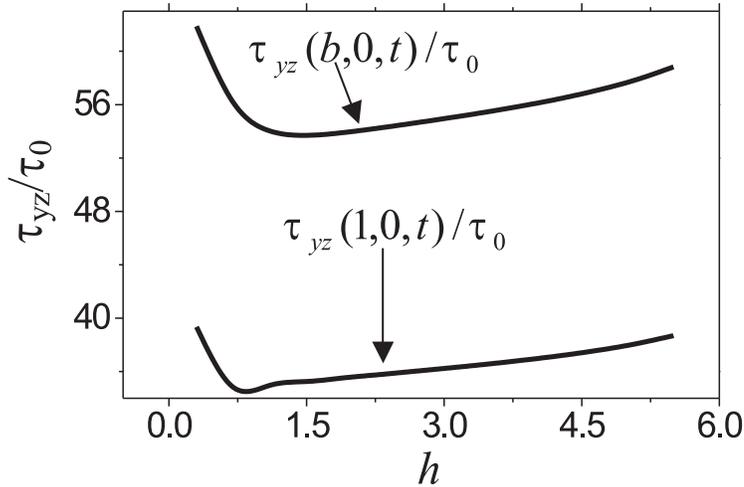


FIG. 6. The stress at the crack tip versus h for $b = 0.1$, $a/(2\beta) = 0.0005$, $\omega/c = 0.4$, $\lambda = 0.4$. (PZT-5H).

(viii) The maximum stress does not occur at the crack tip, but slightly away from it, as shown in Figs. 8, 9. This phenomenon has been thoroughly analysed

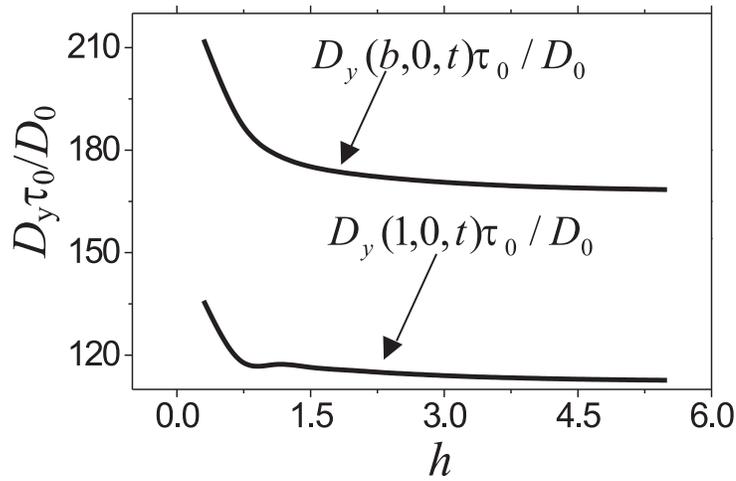


FIG. 7. The electric displacement at the crack tip versus h for $b = 0.1$, $a/(2\beta) = 0.0005$, $\omega/c = 0.4$, $\lambda = 0.4$. (PZT-5H).

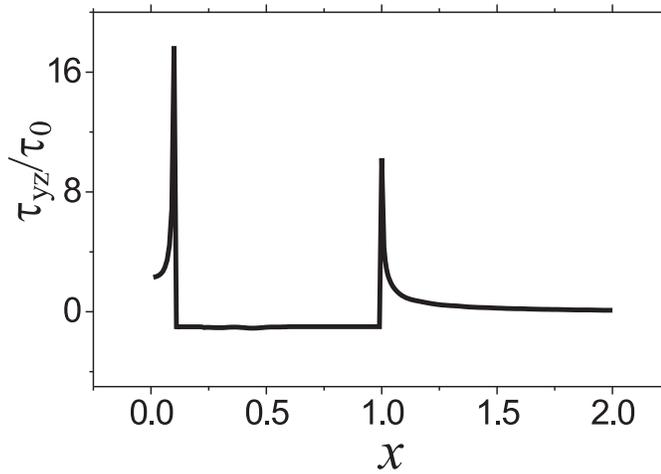


FIG. 8. The stress along crack line versus x for $h = 0.3$, $b = 0.1$, $a/(2\beta) = 0.0001$, $\omega/c = 0.4$, $\lambda = 0.4$. (PZT-4).

in [43]. The maximum stress is finite. The distance between the crack tip and the maximum stress point is very small, and it depends on the crack length and the lattice parameter. Contrary to the classical piezoelectric theory solution, it is found that no stress and electric displacement singularity is present at the crack tip, and also the present results converge to the classical ones far away from the crack tip.

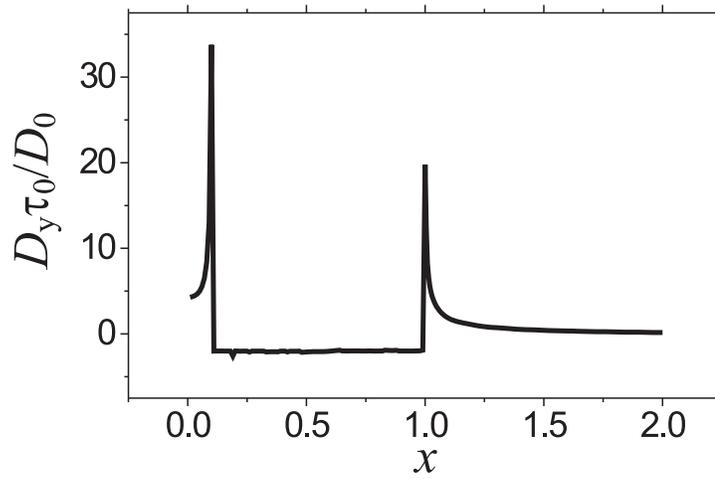


FIG. 9. The electric displacement along crack line versus x for $b = 0.1$, $h = 0.3$, $a/(2\beta) = 0.0001$, $\omega/c = 0.4$, $\lambda = 0.4$. (PZT-4).

(ix) The dynamic stress and the dynamic electric displacement at the crack tips tend to decrease with increasing lattice parameter $a/(2\beta)$ as shown in Figs. 10, 11.

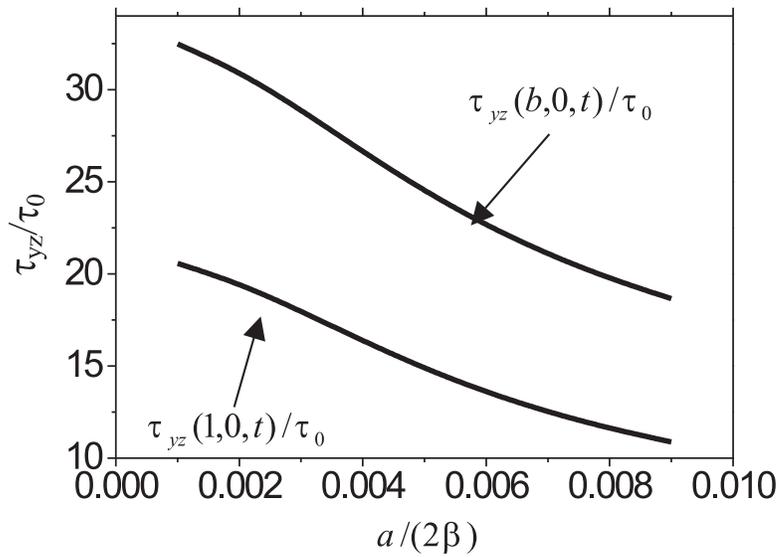


FIG. 10. The stress at the crack tip versus $a/(2\beta)$ for $h = 1.0$, $b = 0.1$, $\omega/c = 0.4$, $\lambda = 0.4$. (PZT-5H).

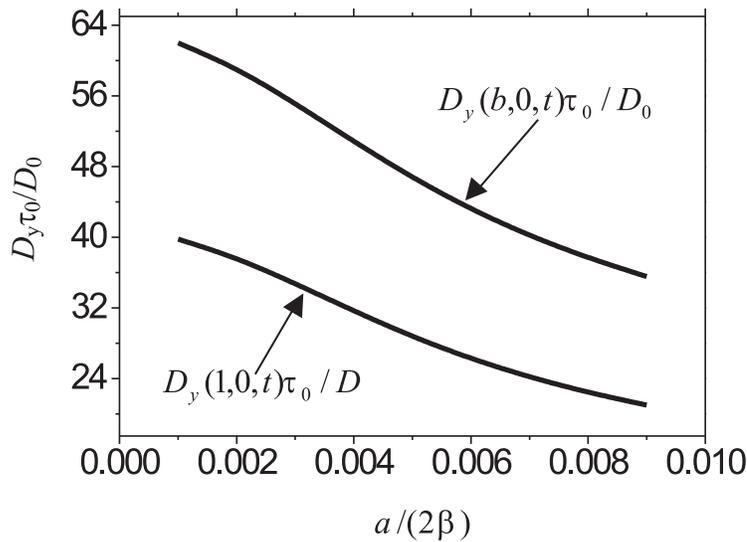


FIG. 11. The electric displacement at the crack tip versus $a/(2\beta)$ for $h = 1.0$, $b = 0.1$, $\omega/c = 0.4$, $\lambda = 0.4$. (PZT-5H).

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