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Formulas for the Rayleigh wave speed in orthotropic elastic solids

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Formulas for the speed of Rayleigh waves in orthotropic compressible elastic materials are obtained in explicit form by using the theory of cubic equations. Different formulas are obtained by using different forms of the (cubic) secular equation. Each formula is expressed as a continuous function of three dimensionless material parameters, which are the ratios of certain elastic constants. It is interesting to note that one of the formulas includes as a special case the formula obtained recently by Malischewsky for isotropic materials.

Key words: Rayleigh waves, wave speed, orthotropy.

1. Introduction

Rayleigh waves were first studied by Rayleigh [1] for compressible isotropic elastic materials. It was not until recently, however, that explicit formulas were obtained for the Rayleigh wave speed. The first such formula was given by Rahman and Barber [2] using the theory of cubic equations. The next contribution was due to Nkemzi [3], who used the theory of the Riemann problem to derive a formula for the Rayleigh wave speed expressed as a continuous function of the material parameter $\epsilon = \mu/(\lambda + 2\mu)$, where λ and μ are the Lamé moduli. As pointed out by Destrade [4] this formula is rather cumbersome and, as noted by Malischewsky [5], the final result is also incorrect. However, it was shown by Romeo [6] that the integral representation of Nkemzi is indeed correct and he generalized it to the case of a viscoelastic orthorhombic half-space. Malischewsky [5] obtained a formula for the Rayleigh wave speed by using Cardan's formula from the theory of cubic equations together with the trigonometric formulas for the roots of the cubic equation and MATHEMATICA, while, using a different method, Royer [7] also obtained explicit Rayleigh wave speeds for isotropic materials. An alternative formula has been given recently by PHAM and OGDEN [8] together with a detailed derivation of Malischewsky's formula, again based on the theory of cubic equations.

Turning now to consideration of Rayleigh waves in anisotropic elastic solids, we note that for some special cases of compressible monoclinic materials with symmetry plane $x_3 = 0$, formulas for the squared wave speed were found by TING [9] and DESTRADE [4] as the roots of quadratic equations. For *incompressible* orthotropic elastic solids an explicit formula for the Rayleigh wave speed has been obtained by OGDEN and PHAM [10], and a link can be made to results for surface waves in certain incompressible anisotropic elastic solids obtained recently by DESTRADE *et al.* [11].

The aim of the present paper is to use the theory of cubic equations to obtain formulas for the Rayleigh wave speed for *compressible* orthotropic elastic solids expressed as continuous functions of three dimensionless material parameters, which are defined in Sec. 3.

First, in Sec. 2, we obtain the secular equation for an orthotropic elastic half-space whose boundary is a plane of symmetry of the material. In Sec. 3. the secular equation is transformed into a cubic equation, which is solved explicitly to give a formula for the Rayleigh wave speed. Next, in Sec. 4., another cubic equation representation for the secular equation (with a different variable) is obtained by transformation, squaring and rearrangement of the original equation. This cubic equation is then solved explicitly to provide an alternative formula for the Rayleigh wave speed. In each case it is shown how specialization to isotropy yields the various formulas obtained previously for this case.

2. Secular equation

Consider a compressible elastic body possessing a stress-free configuration of semi-infinite extent in which the material exhibits orthotropic symmetry. The boundary of this configuration is assumed to be parallel to the (001) mirror plane of the material and, accordingly, rectangular Cartesian axes (x_1, x_2, x_3) are chosen such that the x_3 direction is normal to the boundary and the body occupies the region $x_3 \leq 0$.

The equations for time-harmonic waves propagating parallel to the boundary of the half-space in the direction of the x_1 (or x_2) axis decouple into a plane motion, in the plane defined by the half-space normal and the direction of propagation, and a motion normal to that plane (see, for example, CHADWICK [12] and ROYER and DIEULESAINT [13]). It therefore suffices to consider the plane motion in the (x_1, x_3) plane with displacement components (u_1, u_2, u_3) such that

$$u_i = u_i(x_1, x_3, t), \qquad i = 1, 3, \qquad u_2 \equiv 0, \tag{2.1}$$

where t is time.

For small deformations from the reference configuration, the constitutive equations relating the stress components σ_{ij} and the components of displacement gradient $u_{i,j}$ $(= \partial u_i/\partial x_j)$ are (see, for example, CHADWICK [12])

$$\sigma_{11} = c_{11}u_{1,1} + c_{13}u_{3,3}, \quad \sigma_{33} = c_{13}u_{1,1} + c_{33}u_{3,3}, \quad \sigma_{13} = c_{55}(u_{1,3} + u_{3,1}), \tag{2.2}$$

where the elastic constants c_{11}, c_{33}, c_{55}, c_{13} satisfy the inequalities

$$c_{ii} > 0, \quad i = 1, 3, 5, \qquad c_{11}c_{33} - c_{13}^2 > 0, \tag{2.3}$$

which are necessary and sufficient conditions for the strain energy of the material to be positive definite. Note that because of the restriction to plane strain and a plane of symmetry, the usual nine constants of orthotropy reduce to the four considered here.

The equations governing infinitesimal motion, expressed in terms of the displacement components u_i, are

$$\begin{aligned} c_{11}u_{1,11} + c_{55}u_{1,33} + (c_{13} + c_{55})u_{3,31} &= \rho\ddot{u}_1, \\ c_{55}u_{3,11} + c_{33}u_{3,33} + (c_{13} + c_{55})u_{1,13} &= \rho\ddot{u}_3, \end{aligned} \tag{2.4}$$

where ρ is the mass density of the material and a superposed dot signifies differentiation with respect to t. These equations are taken together with the boundary conditions of zero traction, which are expressed as

$$\sigma_{3i} = 0, \qquad i = 1, 3, \quad \text{on} \quad x_3 = 0. \tag{2.5}$$

We also impose the usual requirement that the displacement and stress components decay away from the boundary, so that

$$u_i \to 0 \qquad (i = 1, 3), \qquad \sigma_{ij} \to 0 \qquad (i, j = 1, 3) \quad \text{as } x_3 \to -\infty. \tag{2.6}$$

We now consider harmonic waves propagating in the x_1 direction, and we write

$$u_i = \phi_i(y) \exp[\mathrm{i}k(x_1 - ct)], \qquad i = 1, 3, \tag{2.7}$$

where k is the wave number, c is the wave speed, $y = kx_3$ and the functions ϕ_i, $i = 1, 3$, are to be determined. Substitution of (2.7) into the equations (2.4) yields

$$\begin{aligned} (c_{11} - \rho c^2)\phi_1 - c_{55}\phi_1'' - \mathrm{i}(c_{13} + c_{55})\phi_3' &= 0, \\ (c_{55} - \rho c^2)\phi_3 - c_{33}\phi_3'' - \mathrm{i}(c_{13} + c_{55})\phi_1' &= 0, \end{aligned} \tag{2.8}$$

where, in (2.8) and the following, a prime on ϕ_i indicates differentiation with respect to y.

In terms of ϕ_i, $i = 1, 3$, after taking account of (2.2) and (2.7), the boundary conditions (2.5) become

$$\mathrm{i}c_{13}\phi_1 + c_{33}\phi_3' = 0, \qquad \phi_1' + \mathrm{i}\phi_3 = 0 \qquad \text{on } y = 0, \tag{2.9}$$

while from (2.6) we obtain

$$\phi_i,\ \phi_i' \to 0 \quad \text{as } y \to -\infty, \qquad i = 1, 3. \tag{2.10}$$

It is then easy to verify that the solution of (2.8) satisfying (2.10) is

$$\begin{aligned} \phi_1 &= A_1 \exp(s_1 y) + A_2 \exp(s_2 y), \\ \phi_3 &= A_1\alpha_1 \exp(s_1 y) + A_2\alpha_2 \exp(s_2 y), \end{aligned} \tag{2.11}$$

where s_1, s_2 are the solutions of the equation

$$\begin{aligned} c_{33}c_{55}s^4 + \big[(c_{13} + c_{55})^2 + c_{33}(\rho c^2 - c_{11}) + c_{55}(\rho c^2 - c_{55})\big]s^2 \\ + (c_{11} - \rho c^2)(c_{55} - \rho c^2) = 0 \end{aligned} \tag{2.12}$$

having positive real parts, α_j $(j = 1, 2)$ is determined from

$$\mathrm{i}(c_{13} + c_{55})\alpha_j s_j = (c_{11} - \rho c^2 - c_{55}s_j^2), \tag{2.13}$$

and A_i, $i = 1, 2$, are constants to be determined from the boundary conditions (2.9).

From (2.12) we have

$$\begin{aligned} s_1^2 + s_2^2 &= -\big[(c_{13} + c_{55})^2 + c_{33}(\rho c^2 - c_{11}) + c_{55}(\rho c^2 - c_{55})\big]/c_{33}c_{55}, \\ s_1^2 s_2^2 &= (c_{11} - \rho c^2)(c_{55} - \rho c^2)/c_{33}c_{55}. \end{aligned} \tag{2.14}$$

If the roots s_1^2 and s_2^2 of the quadratic equation (2.12) for s^2 are real then they must be positive to ensure that s_1 and s_2 can have positive real parts. If they are complex then they are conjugate. In either case the product $s_1^2 s_2^2$ must be positive. Hence, from (2.3) and $(2.14)_2$ we have

$$(c_{11} - \rho c^2)(c_{55} - \rho c^2) > 0, \tag{2.15}$$

and it follows that either $0 < \rho c^2 < \min\{c_{11}, c_{55}\}$ or $\rho c^2 > \max\{c_{11}, c_{55}\}$. However, if the latter inequality holds then it is easy to verify that the discriminant of Eq. (2.12) is non-negative, and hence, since the right-hand side of $(2.14)_1$ is negative in this case, equation (2.12) has two *negative real* roots s_1^2, s_2^2. This contradicts the requirement that s_1, s_2 should have positive real parts. Hence, the Rayleigh wave speed must satisfy the inequality

$$0 < \rho c^2 < \min\{c_{11}, c_{55}\}. \tag{2.16}$$

Note that (2.16) is a necessary condition for the existence of a surface wave but may not be sufficient because of the possible presence of a limiting wave speed (see, for example, CHADWICK and WILSON [14]).

Substitution of (2.11) in (2.9) leads to a homogeneous linear system of algebraic equations for A_1, A_2. For non-trivial solutions the determinant of coefficients of this system must vanish. This condition yields the secular equation. After removing the factor $(s_1 - s_2)$ and using the equalities (2.13) and (2.14), the secular equation of the Rayleigh waves propagating in orthotropic compressible elastic materials is obtained in the form

$$(c_{55} - \rho c^2)[c_{13}^2 - c_{33}(c_{11} - \rho c^2)] + \rho c^2 \sqrt{c_{33}c_{55}}\sqrt{(c_{11} - \rho c^2)(c_{55} - \rho c^2)} = 0 \tag{2.17}$$

(CHADWICK [12]).

We note that CHADWICK [12] has proved that for c_{11}, c_{33}, c_{55}, c_{13} satisfying (2.3), Eq. (2.17) has a unique (real) solution satisfying (2.16) and ensuring that (2.12) has two distinct roots with positive real part. It was also shown in that paper that the case $s_1 = s_2$ does not yield a surface wave.

On use of (2.16), Eq. (2.17) can be transformed into

$$\sqrt{1 - \frac{\rho c^2}{c_{55}}}\left(1 - \frac{c_{13}^2}{c_{11}c_{33}} - \frac{\rho c^2}{c_{11}}\right) = \sqrt{\frac{c_{11}}{c_{33}}}\,\frac{\rho c^2}{c_{11}}\sqrt{1 - \frac{\rho c^2}{c_{11}}}, \tag{2.18}$$

as obtained by STONELEY [15], or

$$\psi(\zeta) \equiv \zeta - \sqrt{\frac{c_{33}}{c_{55}}\,\frac{c_{55} - \zeta}{c_{11} - \zeta}}(c^* - \zeta) = 0, \tag{2.19}$$

as given by ROYER and DIEULESAINT [13], where

$$\zeta \equiv \rho c^2, \qquad 0 < c^* \equiv c_{11} - c_{13}^2/c_{33} < c_{11}. \tag{2.20}$$

In [13], with the assumption $c_{55} < c_{11}$, the authors used Eq. (2.19) to deduce that $\rho c^2 = \zeta$ does not belong to the interval (c_{11}, ∞) by showing that $\psi(\zeta) > 0$ for all $\zeta \in (c_{11}, \infty)$. Unfortunately, when $\zeta \in (c_{11}, \infty)$ the relevant rearrangement of (2.17) is not (2.19) but

$$\psi(\zeta) \equiv \zeta + \sqrt{\frac{c_{33}}{c_{55}}\,\frac{c_{55} - \zeta}{c_{11} - \zeta}}(c^* - \zeta) = 0 \tag{2.21}$$

since $c_{55} - \rho c^2 = -\sqrt{(c_{55} - \rho c^2)^2}$ when $c_{55} < \rho c^2$.

3. Formulas for the Rayleigh wave speed

In order to proceed it is convenient to introduce three dimensionless material parameters defined by

$$\gamma = c_{55}/c_{11}, \qquad \alpha = c_{33}/c_{11}, \qquad \delta = 1 - c_{13}^2/c_{11}c_{33}, \tag{3.1}$$

such that

$$\gamma > 0, \qquad \alpha > 0, \qquad 0 < \delta < 1. \tag{3.2}$$

We also define the variable x by

$$x = \rho c^2/c_{55}. \tag{3.3}$$

From (2.16) we then have

$$0 < x < 1 \leq \frac{1}{\gamma} \quad \text{if} \;\; 0 < c_{55} \leq c_{11}, \qquad 0 < x < \frac{1}{\gamma} < 1 \quad \text{if} \;\; 0 < c_{11} < c_{55}. \tag{3.4}$$

Equation (2.17) may now be written

$$\sqrt{\alpha}\,(1-x)(x-\sigma\delta) + x\sqrt{1-x}\sqrt{1-\gamma x} = 0, \tag{3.5}$$

where $\sigma \equiv 1/\gamma$ and x satisfies (3.4). Since $1-x \neq 0$ equation (3.5) is equivalent to

$$\sqrt{\alpha}\,(x-\sigma\delta) + x\sqrt{\frac{1-\gamma x}{1-x}} = 0. \tag{3.6}$$

CASE 1. $\gamma \neq 1$

On introducing the variable t defined by

$$t = \sqrt{\frac{1-\gamma x}{1-x}}, \qquad x = \frac{1-t^2}{\gamma - t^2} \tag{3.7}$$

Eq. (3.6) becomes

$$f(t) \equiv t^3 + a_2 t^2 - t + a_0 = 0, \tag{3.8}$$

where

$$a_0 = -\sqrt{\alpha}\,(1-\delta), \qquad a_2 = \sqrt{\alpha}\,(1-\sigma\delta) \tag{3.9}$$

and

$$1 < t < \infty \quad \text{if} \;\; 0 < \gamma < 1, \qquad 0 < t < 1 \quad \text{if} \;\; \gamma > 1. \tag{3.10}$$

We remark that for $\gamma = 1$ the transformation (3.7) is not one-to-one. This case will be considered separately below.

If $0 < \gamma < 1\,(\sigma > 1)$ then, from (3.8) and (3.9) we have

$$f(1) = -\sqrt{\alpha}\,(\sigma - 1)\delta < 0 \tag{3.11}$$

and $f(t) \to \infty$ as $t \to \infty$. The existence of a solution of Eq. (3.8) in the interval $(1, \infty)$ is therefore assured.

If $\gamma > 1\,(0 < \sigma < 1)$ we obtain

$$f(0) = -\sqrt{\alpha}\,(1 - \delta) < 0, \qquad f(1) = \sqrt{\alpha}\,(1 - \sigma)\delta > 0, \tag{3.12}$$

and the existence of a solution of (3.8) in the interval $(0, 1)$ follows from these inequalities.

From (3.8) we obtain

$$f'(t) = 3t^2 + 2a_2 t - 1. \tag{3.13}$$

Since the discriminant of the equation $f'(t) = 0$ is $4(a_2^2 + 3) > 0$, it has two distinct real roots, which we denote by $t_{\min}$ and $t_{\max}$. It follows from (3.13) that $t_{\min}t_{\max} < 0$ and hence that $t_{\max} < 0 < t_{\min}$. It is now easy to verify that Eq. (3.8) has a unique real solution in the interval for t defined by (3.10). Note that if Eq. (3.8) has two or three distinct real roots, then the largest one corresponds to the Rayleigh wave and is the only solution in the required range of values of t.

We now introduce the variable z defined by

$$z = t + \frac{1}{3}a_2, \tag{3.14}$$

so that Eq. (3.8) becomes

$$z^3 - 3q^2 z + r = 0, \tag{3.15}$$

where

$$\begin{aligned} q &\equiv \frac{1}{3}\sqrt{a_2^2 + 3} = \frac{1}{2}(t_{\min} - t_{\max}), \\ r &= \frac{1}{27}(2a_2^3 + 9a_2 + 27a_0). \end{aligned} \tag{3.16}$$

We note in passing the geometrical point that if t_{N} is the value of t at the point of inflexion of the curve $y = f(t)$ then $r = f(t_{\mathrm{N}})$.

Our task is now to find the largest real root, which we denote by z_0, of Eq. (3.15). By the theory of cubic equation, the three roots of Eq. (3.15) are given by Cardan's formula (see COWLES and THOMPSON [16], for example). Accordingly, we may write

$$
\begin{aligned}
z_1 &= S + T,\\
z_2 &= -\frac{1}{2}(S+T) + \frac{1}{2}\mathrm{i}\sqrt{3}(S-T),\\
z_3 &= -\frac{1}{2}(S+T) - \frac{1}{2}\mathrm{i}\sqrt{3}(S-T),
\end{aligned}
\tag{3.17}
$$

where

$$
\begin{aligned}
&S = \sqrt[3]{R+\sqrt{D}}, \qquad T = \sqrt[3]{R-\sqrt{D}},\\
&D = R^2 + Q^3, \qquad R = -\frac{1}{2}r, \qquad Q = -q^2.
\end{aligned}
\tag{3.18}
$$

In relation to these formulas we emphasize two points: (i) the cubic root of a negative real number is taken as the negative real root; (ii) if the argument in S is complex then we take the phase angle in T as the negative of the phase angle in S such that $T = S^*$, where S^* is the complex conjugate of S.

The nature of the roots of Eq. (3.15) depends on the sign of its discriminant D. In particular, if $D > 0$ then (3.15) has one real root and two complex conjugate roots; if $D = 0$ the equation has three real roots, at least two of which are equal; if $D < 0$ then it has three distinct real roots.

We now show that in each case the largest real root z_0 of Eq. (3.15) is given by

$$
z_0 = \sqrt[3]{R+\sqrt{D}} + \sqrt[3]{R-\sqrt{D}},
\tag{3.19}
$$

within which each radical is understood as the complex root taking its principal value.

First, we consider $D > 0$. In this case it is clear that Eq. (3.15) has only one real root that, namely $z_0 = z_1$, given by the first Eq. in (3.17), in which the radicals must be understood as real roots. From the geometrical point, it is easy to show that in this case $f(t_{\min}) < 0$, $f(t_{\max}) < 0$, and hence $f(t_{\mathrm{N}}) < 0$. This leads to $r < 0$ and hence $R > 0$. Since the value of a real root of a positive real number coincides with the principal value of its corresponding complex root and since $R > 0$ and $Q < 0$, z_0 is clearly given by (3.19).

If $D = 0$ then $r = -2q^3$ and Eq. (3.15) reduces to

$$
(z+q)^2(z-2q) = 0.
\tag{3.20}
$$

Equation (3.20) has two distinct real roots, $z = -q$ (double root) and $z = 2q$. Hence, in this case $z_0 = 2q$ and from (3.18) we have $R = q^3$, and therefore (3.19) is valid.

Finally, for $D < 0$ equation (3.15) has three distinct real roots given by (3.17) and (3.18) in which complex cubic (square) roots can take one of three (two) possible values such that $T = S^*$. Here, we take their principal values and indicate that z_1 expressed by $(3.17)_1$ is the largest real root of (3.15), so that again (3.19) is valid. Throughout the remainder of this section, for simplicity, we take complex roots as their principal values.

From (3.18) we have

$$S = \sqrt[3]{R + \mathrm{i}\sqrt{-R^2 - Q^3}}, \qquad T = S^*. \tag{3.21}$$

The phase angle of the complex number $R + \mathrm{i}\sqrt{-D}$ belongs to the interval $(0, \pi)$, so that the phase angle θ corresponding to the principal value of S in (3.21) belongs to the interval $(0, \pi/3)$. By (3.21) this implies that $|S| = q$, and hence S and T can be expressed in the forms

$$S = qe^{\mathrm{i}\theta}, \qquad T = qe^{-\mathrm{i}\theta}, \qquad 0 < \theta < \pi/3, \tag{3.22}$$

where $\theta \in (0, \pi/3)$ satisfies the equation

$$\cos 3\theta = -\frac{r}{2q^3}, \tag{3.23}$$

which is obtained by substituting

$$z = S + T = 2q\cos\theta \tag{3.24}$$

into Eq. (3.15).

Note that $D < 0$ implies $|-r/2q^3| < 1$, which ensures that Eq. (3.23) has a unique solution in the interval $(0, \pi/3)$.

From (3.17) and (3.22) it is easy to verify that

$$z_1 = 2q\cos\theta, \qquad z_2 = 2q\cos(\theta + 2\pi/3), \qquad z_3 = 2q\cos(\theta + 4\pi/3). \tag{3.25}$$

Then, from (3.25), since $\theta \in (0, \pi/3)$, it is clear that $z_1 > z_3 > z_2$, i.e. z_1 is the largest real root of (3.15) and (3.19) is valid.

After some manipulations we obtain, on use of (3.9), (3.16) and (3.18),

$$\begin{aligned} R &= -\frac{1}{54}\, h(\alpha, \sigma, \delta), \\ D &= -\frac{1}{108}\Big[2\sqrt{\alpha}(1-\delta)h(\alpha, \sigma, \delta) + 27\alpha(1-\delta)^2 + \alpha(1-\sigma\delta)^2 + 4\Big], \end{aligned} \tag{3.26}$$

where

$$h(\alpha, \sigma, \delta) \equiv \sqrt{\alpha}\,[2\alpha(1-\sigma\delta)^3 + 9(3\delta - \sigma\delta - 2)]. \tag{3.27}$$

Finally, on using (3.6), (3.7), (3.9), (3.14) and (3.19), we obtain

$$\rho c^2/c_{55} = \sqrt{\alpha}\sigma\delta\left[\sqrt{\alpha}(\sigma\delta+2)/3 + \sqrt[3]{R+\sqrt{D}} + \sqrt[3]{R-\sqrt{D}}\right]^{-1}, \tag{3.28}$$

where R and D are given by (3.26) and (3.27) and the roots take their principal values. It is clear that the speed of Rayleigh waves is a continuous function of the three parameters α, γ, δ in the region $\alpha > 0$, $\gamma > 0$, $0 < \delta < 1$ with $\gamma \neq 1$.

For isotropic materials we have $c_{11} = c_{33} = \lambda + 2\mu$, $c_{55} = \mu$, $c_{13} = \lambda$, so that $\alpha = 1$, $\delta = 4\gamma(1-\gamma)$, with $\gamma = \mu/(\lambda + 2\mu)$. In this case (3.28) reduces to

$$\rho c^2/\mu = 4(1-\gamma)\left[2 - \frac{4}{3}\gamma + \sqrt[3]{R+\sqrt{D}} + \sqrt[3]{R-\sqrt{D}}\right]^{-1}, \tag{3.29}$$

where R and D are given by

$$\begin{aligned} R &= 2(27 - 90\gamma + 99\gamma^2 - 32\gamma^3)/27, \\ D &= 4(1-\gamma)^2(11 - 62\gamma + 107\gamma^2 - 64\gamma^3)/27 \end{aligned} \tag{3.30}$$

and the roots in (3.29) are understood as complex roots taking their principal values. The formula (3.29) with (3.30) was given by PHAM and OGDEN [8].

CASE 2. $\gamma = 1$

For the value $\gamma = 1$ we obtain directly from equation (3.5) the formula

$$\rho c^2/c_{55} = \frac{\sqrt{\alpha}\delta}{1+\sqrt{\alpha}} \tag{3.31}$$

for the Rayleigh wave speed. This formula may also be obtained from (3.28) on specialization to $\gamma = 1$, which requires some manipulation of the formulas (3.26) and (3.27). The formula (3.28) is therefore valid for all $\gamma > 0$.

4. Alternative formulas

In this section some other formulas for the speed of Rayleigh waves in compressible orthotropic elastic materials are derived that are different from (3.28) in form. In order to obtain these formulas we start from the secular equation rewritten as

$$F(x) \equiv (\gamma - \alpha)x^3 + (\alpha + 2\alpha\sigma\delta - 1)x^2 - \alpha\sigma\delta(\sigma\delta + 2)x + \alpha\sigma^2\delta^2 = 0, \tag{4.1}$$

which comes from Eq. (3.5) expressed in the form

$$\sqrt{a}\sqrt{1-x}(\sigma\delta - x) = x\sqrt{1-\gamma x} \tag{4.2}$$

on squaring and rearranging. Since $x \neq 1$, Eq. (4.2) is equivalent to Eq. (3.5).

From (4.2) it may be deduced that if x is its (real) solution, corresponding to the Rayleigh wave speed satisfying (3.4), then

$$0 < x < \sigma\delta. \tag{4.3}$$

Since $0 < \delta < 1$ and Eq. (4.2) is equivalent to (3.5), it is easy to see that for the values x such that

$$0 < x < 1 \quad \text{and} \quad 0 < x < \sigma\delta \tag{4.4}$$

Eq. (4.1) is equivalent to Eq. (3.5).

It should be noted that, as shown in the previous section, Eq. (3.5) has a unique (real) solution corresponding to the Rayleigh wave, which we denote by x_0. This satisfies the condition (3.4) and hence (4.3), for any values of the parameters α, $\sigma(= 1/\gamma)$, δ subject to (3.2). Equation (4.1) therefore also has a unique solution, and we have the following proposition.

PROPOSITION. For any values of the parameters α, σ, δ subject to (3.2), in the interval $(0, \sigma_m)$, where $\sigma_m = \min\{1, \sigma\delta\}$, Eq. (4.1) has a unique (real) solution (x_0), which corresponds to the Rayleigh wave speed.

We shall now indicate which real root of (4.1) is identified as x_0 in the situation when it has two or three distinct real roots so we can obtain formulas for the Rayleigh wave speed.

4.1. CASE 1. $\gamma = \alpha$

When $\gamma = \alpha$, (4.1) reduces to the quadratic equation

$$(\gamma + 2\delta - 1)x^2 - \delta(\sigma\delta + 2)x + \sigma\delta^2 = 0. \tag{4.5}$$

Keeping (3.2) in mind and taking into account the Proposition, it is not difficult to verify that Eq. (4.5) has two distinct real roots, x_0 being the smaller root when $\gamma + 2\delta - 1 > 0$ and the larger one when $\gamma + 2\delta - 1 < 0$. Thus, for the values of γ, δ such that $\gamma + 2\delta - 1 \neq 0$, the Rayleigh wave speed is given by

$$\rho c^2/c_{55} = \frac{\delta(\sigma\delta + 2) - \delta\sqrt{\sigma(\sigma\delta^2 + 4 - 4\delta)}}{2(\gamma + 2\delta - 1)}. \tag{4.6}$$

For the case $\gamma + 2\delta - 1 = 0$ ($\delta > 0 \Rightarrow \sigma > 1$), the Rayleigh wave speed is given by

$$\rho c^2/c_{55} = (\sigma - 1)/(\sigma + 3). \tag{4.7}$$

This special case has been examined by MOZHAEV [17], who also gave examples of materials for which $\gamma = \alpha$ ($c_{55} = c_{33}$).

4.2. CASE 2. $\gamma > \alpha$

In this case Eq. (4.1) is equivalent to

$$F_1(x) \equiv x^3 + a_2x^2 + a_1x + a_0 = 0, \tag{4.8}$$

where a_i, $i = 0, 1, 2$, are given by

$$a_0 = \frac{\alpha\sigma^2\delta^2}{\gamma - \alpha}, \qquad a_1 = \frac{\alpha\sigma\delta(\sigma\delta + 2)}{\alpha - \gamma}, \qquad a_2 = \frac{\alpha + 2\alpha\sigma\delta - 1}{\gamma - \alpha}. \tag{4.9}$$

It is easy to verify from (4.8) and (4.9) that

$$F_1(0) = \frac{\alpha\sigma^2\delta^2}{\gamma - \alpha}, \qquad F_1(1) = \frac{\gamma - 1}{\gamma - \alpha}, \qquad F_1(\sigma\delta) = \frac{\sigma^2\delta^2(\delta - 1)}{\gamma - \alpha}. \tag{4.10}$$

On using (4.10) and taking into account the Proposition, we can show that, for values of γ and α such that $\gamma - \alpha > 0$, Eq. (4.8) has three distinct real roots and x_0 is the intermediate one.

Analogously to the analysis in Sec. 3, in terms of the variable z given by

$$z = x + a_2/3, \tag{4.11}$$

equation (4.8) may be expressed in the form

$$z^3 - 3q^2z + r = 0, \tag{4.12}$$

where, in this case, q and r are given by

$$q^2 = (a_2^2 - 3a_1)/9, \qquad r = (2a_2^3 - 9a_1a_2 + 27a_0)/27, \tag{4.13}$$

with a_i, $i = 0, 1, 2$ defined by (4.9). Note that q and r differ from the values defined in Sec. 3; in particular, here q^2 can be negative.

Our task is now to find the intermediate real root of Eq. (4.12), which we denote by z_0. Using the theory presented in Sec. 3, it is clear that the root z_0 is

$$z_0 = \mathrm{e}^{4\pi\mathrm{i}/3}\sqrt[3]{R + \sqrt{D}} + \mathrm{e}^{-4\pi\mathrm{i}/3}\sqrt[3]{R - \sqrt{D}}, \tag{4.14}$$

where each radical is understood as the complex root taking its principal value and R and $D\,(< 0)$ are given by (3.18), (4.9) and (4.13). Using (3.18) and (4.13), after some manipulations we have

$$\begin{aligned} R &= \left(9a_1a_2 - 27a_0 - 2a_2^3\right)/54, \\ D &= \left(4a_0a_2^3 - a_1^2a_2^2 - 18a_0a_1a_2 + 27a_0^2 + 4a_1^3\right)/108, \end{aligned} \tag{4.15}$$

where a_i, $i = 0, 1, 2$ are expressible in terms of α, γ, δ through (4.9).

Taking into account $(4.9)_3$, (4.11) and (4.14), we see that the root x_0, and hence the speed of the Rayleigh wave, is given by the formula

$$\rho c^2/c_{55} = \frac{\alpha + 2\alpha\sigma\delta - 1}{3(\alpha - \gamma)} + \mathrm{e}^{4\pi\mathrm{i}/3}\sqrt[3]{R + \sqrt{D}} + \mathrm{e}^{-4\pi\mathrm{i}/3}\sqrt[3]{R - \sqrt{D}}, \tag{4.16}$$

in which the radicals are understood as complex roots taking their principal values, R and D being given by (4.9) and (4.15). This formula shows the continuous dependence of the speed c on the parameters α, γ, δ in the region $0 < \delta < 1$, $\gamma > \alpha > 0$.

4.3. CASE 3. $0 < \gamma < 1 = \alpha$

For this case we obtain another formula for the speed of Rayleigh waves. As we shall see, the formula for the speed of Rayleigh wave which was obtained recently by MALISCHEWSKY [5] for an isotropic material is a special case of this formula.

Equation (4.1) is equivalent to Eq. (4.8) with $0 < x < \sigma_m$, where the coefficients a_i, $i = 0, 1, 2$ become

$$a_0 = \frac{\sigma^2\delta^2}{\gamma - 1}, \qquad a_1 = \frac{\sigma\delta(\sigma\delta + 2)}{1 - \gamma}, \qquad a_2 = \frac{2\sigma\delta}{\gamma - 1}. \tag{4.17}$$

From (4.8) and (4.17) we have

$$F_1(0) < 0, \qquad F_1(1) > 0, \qquad F_1(\sigma d) > 0. \tag{4.18}$$

Consider the equation

$$F_1'(x) = 3x^2 + 2a_2 x + a_1 = 0. \tag{4.19}$$

If its discriminant $\Delta \leq 0$ then $F_1'(x) \geq 0$ for all x, and hence Eq. (4.8) has a unique real solution in the interval $(0, \sigma_m)$ (according to the Proposition), namely x_0. If $\Delta > 0$, Eq. (4.19) has two distinct real roots, denoted by $x_{\min}$ and $x_{\max}$, such that

$$\begin{aligned} x_{\min}x_{\max} &= \frac{a_1}{3} = \frac{\sigma\delta(\sigma\delta + 2)}{3(1 - \gamma)} > 0, \\ x_{\min} + x_{\max} &= -\frac{2a_2}{3} = \frac{4\alpha\sigma\delta}{3(1 - \gamma)} > 0. \end{aligned} \tag{4.20}$$

Hence, $0 < x_{\max} < x_{\min}$.

Bearing in mind that Eq. (4.8) has a unique (real) solution in the interval $(0, \sigma_m)$, it follows from $(4.18)_1$ that x_0 is the smallest real root in the case in which Eq. (4.8) has two or three distinct real roots.

By using the variable z related to the variable x by (4.11), Eq. (4.8) becomes (4.12), where q and r are given by (4.13) with (4.17). In order to find the smallest real root x_0 of (4.8) we now determine the smallest root z_0 of (4.12). We shall show that z_0 is given by

$$z_0 = \operatorname{sign}(-d)\sqrt[3]{\operatorname{sign}(-d)[R+\sqrt{D}]} - \sqrt[3]{-R+\sqrt{D}}, \tag{4.21}$$

where R and D are given by (4.15) and (4.17),

$$d \equiv a_2^2 - 3a_1 = 9q^2, \tag{4.22}$$

and the roots are understood as complex roots taking their principal values. In order to establish (4.21) we need the following two Lemmas.

Lemma 1. *In the* (γ, δ) *plane the set* $U = \{\gamma, \delta \colon 0 < \gamma < 1,\ 0 < \delta < 1 \colon d > 0\}$ *is a connected set.*

This can be seen immediately by reference to Fig. 1, in which the curve $d = 0$ based on (4.22) with (4.17), on recalling the definition $\sigma = 1/\gamma$, is plotted in the positive quadrant of the (γ, δ) plane. This yields the equation $\delta = 6\gamma(1-\gamma)/(3\gamma+1)$.

FIG. 1. Plot of the curve $d = 0$ in (γ, δ) plane for $\alpha = 1$. In the region enclosed by the curve and the γ axis $d < 0$. The maximum on the curve has coordinates $(1/3, 2/3)$.

Lemma 2. $R < 0$ *for the values of* γ, δ *such that* $0 < \gamma < 1$, $0 < \delta < 1$, $d > 0$, $D \geq 0$.

The proof of Lemma 2, in which the result of Lemma 1 is employed, is given in the Appendix.

We now examine the distinct cases dependent on the values of d in order to prove (4.21).

If $d < 0$ it follows from $(3.18)_{3,5}$ and (4.22) that $D > 0$ and

$$R + \sqrt{D} > 0, \qquad -R + \sqrt{D} > 0. \tag{4.23}$$

Since $D > 0$ Eq. (4.12) has a unique real solution given by $(3.17)_1$ and (3.18) in which the radicals are understood as real roots. It is clear that the inequalities (4.23) ensure that (4.21) is valid.

If $d = 0$, then $R \le 0$ (as shown below). If $R < 0$, then, by $(3.18)_{3,5}$ and (4.22), $D > 0$, so that equation (4.12) has a unique real solution and (4.21) is valid. If $R = 0$ then $D = 0$ and it is clear from (3.17) and (3.18) that Eq. (4.12) has a (triple) unique real root $z_0 = 0$ and in this case (4.21) is also valid.

Suppose that M_0 is a point with coordinates (γ_0, δ_0) in the considered region. We now show that if $d(M_0) = 0$, then $R(M_0) \le 0$, M_0 is a point on the curve OAB, where $d = 0$ (see Fig. 1), excluding the endpoints O, B (since $0 < \gamma < 1$). Indeed, if we suppose to the contrary that $R(M_0) > 0$, then $D(M_0) > 0$ by $(3.18)_3$ and (4.22). It is clear that $D(M)$ is a continuous function in the open set $\bar{U} = \{\gamma, \delta : 0 < \gamma < 1, 0 < \delta < 1\}$, where M denotes a general point with coordinates (γ, δ), and $M_0 \in \bar{U}$. Hence, there exists a sufficiently small neighborhood $U_0 = \{\gamma, \delta : (\gamma - \gamma_0)^2 + (\delta - \delta_0)^2 < \kappa^2\}$ of the point M_0, where κ is a sufficiently small positive number, such that $U_0 \subset \bar{U}$ and $D(M) > 0$ for all $M \in U_0$. Defining $\Omega = U \cap U_0$, we have $d(M) > 0$, $D(M) > 0$ for all $M \in \Omega$. Hence $R(M) < 0$ for all $M \in \Omega$ by Lemma 2. Since R is also continuous on the set $\bar{U} \supset \Omega$ and M_0 is a boundary point of Ω, we conclude that $R(M_0) \le 0$. This leads to contradiction of our assumption that $R(M_0) > 0$ and the proof is complete.

If $d > 0$ and $D > 0$, Eq. (4.12) again has a unique real solution (since $D > 0$), namely z_0, which is given by $(3.17)_1$ and (3.18). Since $R < 0$ by Lemma 2 and $d > 0$ it follows from $(3.18)_{3,5}$ and (4.22) that

$$-(R + \sqrt{D}) > 0, \qquad -R + \sqrt{D} > 0, \tag{4.24}$$

from which it is easy to see that (4.21) is valid.

If $d > 0$ and $D = 0$ then, by Lemma 2, $R < 0$. Hence, by using $(3.18)_{3-5}$ and (4.22), we have $R = -q^3$ $(q > 0)$, $r = 2q^3$, and Eq. (4.12) becomes

$$(z - q)^2(z + 2q) = 0. \tag{4.25}$$

Its solutions are q (double root) and $-2q$. Hence, in this case $z_0 = -2q$ and (4.21) is applicable.

If $d > 0$ and $D < 0$ then Eq. (4.12) has three distinct real roots and z_0 is the smallest root. Following the theory presented in the Sec. 3, the smallest real root is $2q\cos(\theta + 2\pi/3)$, where $\theta \in (0, \pi/3)$ is defined by (3.23).

To ensure that (4.21) is valid we must show that

$$-\sqrt[3]{-R+\sqrt{D}} - \sqrt[3]{-(R+\sqrt{D})} = 2d\cos(\theta + 2\pi/3), \tag{4.26}$$

where the roots are complex roots taking their principal values.

Indeed, following the theory of Sec. 3, we have $\mathrm{Arg}(R + \sqrt{D}) = 3\theta$, $\mathrm{Arg}(R - \sqrt{D}) = -3\theta$ with $3\theta \in (0, \pi)$ the solution of (3.23). Therefore, $\mathrm{Arg}[-(R+\sqrt{D})] = 3\theta - \pi$, $\mathrm{Arg}[-(R-\sqrt{D})] = -3\theta + \pi$. Since $|-(R+\sqrt{D}| = d$ it follows that

$$\sqrt[3]{-(R+\sqrt{D})} = d e^{\mathrm{i}(\theta - \pi/3)}, \quad \sqrt[3]{-(R-\sqrt{D})} = d e^{\mathrm{i}(-\theta+\pi/3)}. \tag{4.27}$$

Note that the roots in (4.27) are complex roots taking their principal values. It follows that

$$-\sqrt[3]{-R+\sqrt{D}} - \sqrt[3]{-(R+\sqrt{D})} = -2d\cos(\theta - \pi/3) = 2d\cos(\theta + 2\pi/3) \tag{4.28}$$

and (4.26) is established.

From (4.11), (4.17), (4.21) and (4.22) the root x_0 is given by the formula

$$\rho c^2/c_{55} = \frac{2\sigma\delta}{3(1-\gamma)} + \mathrm{sign}(-\bar{d})\sqrt[3]{\mathrm{sign}(-\bar{d})[R+\sqrt{D}]} - \sqrt[3]{-R+\sqrt{D}}, \tag{4.29}$$

where each radical is understood as a complex root taking its principal value, and the function d is now replaced by the function $\bar{d}$ given by

$$\bar{d} = \frac{\delta(1+3\gamma)}{12\gamma(1-\gamma)} - \frac{1}{2}. \tag{4.30}$$

It is noted that $\bar{d}$ differs from d by a positive factor. From (4.15) and (4.17), after some manipulations, we obtain

$$\begin{aligned} R &= \frac{\sigma^2\delta^2}{54(1-\gamma)^3}\Big[16\sigma\delta + 36(\gamma-1) + 18\sigma\delta(\gamma-1) + 27(\gamma-1)^2\Big], \\ D &= \frac{\sigma^3\delta^3}{108(\gamma-1)^4}\Big[27\sigma\delta(\gamma-1)^2 - 4(\gamma-1)(\sigma^3\delta^3 - 3\sigma^2\delta^2 - 6\sigma\delta + 8) \\ &\qquad - 4\sigma\delta(\sigma\delta-2)^2\Big]. \end{aligned} \tag{4.31}$$

For isotropic materials $\delta = 4\gamma(1-\gamma)$, and hence from (4.30) and (4.31) we have

$$\bar{d} = \gamma - 1/6, \quad R = 8(45\gamma - 17)/27,$$
$$D = 64(11 - 62\gamma + 107\gamma^2 - 64\gamma^3)/27. \tag{4.32}$$

We observe that the formula (4.29) reduces to a formula which was given recently by MALISCHEWSKY [5], for further discussion of which we refer to PHAM and OGDEN [8].

The situation for which $0 < \gamma < \alpha \neq 1$ has not been considered here, and it is natural to ask whether the formula (4.29) also holds in this case. The analysis required is more complicated than for the other cases and will be discussed separately elsewhere.

In conclusion, we remark that the results obtained in this paper can be extended to other types of anisotropy. Indeed, ROYER and DIEULESAINT [13] have shown that for surface (Rayleigh) waves, the results established for the orthotropic case may be applied to 16 different material symmetry classes, including cubic, tetragonal and hexagonal anisotropy.

Appendix: proof of Lemma 2

First we mention some facts required in the proof.

(i) The quantity r determined by $(4.13)_2$ is $F_1(x_{\mathrm{N}})$, where x_{N} is the point of inflexion of the cubic curve $y = F_1(x)$ in the (x, y) plane.

(ii) If $d > 0$ then the function $F_1(x)$ has maximum and minimum stationary points.

(iii) By the Proposition, Eq. (4.8) has a unique real solution in the interval $(0, \sigma\delta)$ for the values of γ, δ belonging to the set $\bar{U}$.

(iv) The quantities r, R, D are continuous functions of the independent variables γ, δ in the (open) set $\bar{U}$.

(v) It is not difficult to show that the point $M_2(3/4, 3/4)$ of the (γ, δ) plane has the properties $M_2 \in U$ and $D(M_2) < 0$.

Suppose that there exists a point M_1 in the (γ, δ) plane such that $0 < \gamma < 1$, $0 < \delta < 1$ and $d(M_1) > 0$, $D(M_1) \geq 0$, but $R(M_1) \geq 0$. If $R(M_1) = 0$ then $r(M_1) = 0$. Since $d(M_1) > 0$, by (i) and (ii) equation (4.8) has three distinct real roots at M_1. This contradicts the assumption $D(M_1) \geq 0$. Next, consider $R(M_1) > 0$, so that $r(M_1) < 0$. If $D(M_1) = 0$ then, from $d(M_1) > 0$, (4.18), (i) and $r(M_1) < 0$, since $0 < x_{\max} < x_{\min}$, it follows that equation (4.8) has two distinct real roots in the interval $(0, \sigma_m)$, but this contradicts (iii). Thus, $D(M_1) > 0$.

Since, by (v), M_1 and $M_2 \in U$, we can, by Lemma 1, connect the two points M_1 and M_2 by a simple continuous curve, which we denote by $L_{12} \in U$. By (iv), D is a continuous function on L_{12}. Since $D(M_1) > 0$, $D(M_2) < 0$ (by (v)), there must exist a point $M_0 \in L_{12}, M_0 \neq M_1, M_2$ such that $D(M_0) = 0$ and $D(M) > 0$ for all $M \in L_{10}$ (except M_0), where L_{10} is the part of L_{12} that connects M_1 and M_0. Analogously to the above arguments, it can be seen that R must not vanish at any point $M \in L_{10}$. Since R is a continuous function on L_{10} and $R(M_1) > 0$, then $R(M) > 0$ for all $M \in L_{10}$, and hence $R(M_0) > 0$, i.e. $r(M_0) < 0$. This, together with $d(M_0) > 0$, $D(M_0) = 0$, (4.18), (i), (ii) and the ordering $0 < x_{\max} < x_{\min}$, shows that Eq. (4.8) has two distinct real roots in the interval $(0, \sigma_m)$, but this contradicts (iii), and the proof is complete.

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