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On some exponential decay estimates for porous elastic cylinders

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In this paper we introduce two new cross-sectional measures for studying the spatial behaviour of the solutions in elastostatics of the porous cylinders. This allows us to extend the range of applicability of the estimates describing the Saint–Venant's decay behaviour of enlarged classes of porous materials.

1. Introduction

In deriving the decay estimates for solutions of the models described by second-order elliptic equations defined in a finite or semi-infinite cylinder subjected to homogeneous boundary conditions on the lateral surface, it is customary to use volume energy methods. A notable exception is the study by Flavin, Knops and Payne [1] who use integrals taken over plane cross-sections of the cylinder rather than averages over partial volumes, as in previous work. In this way they are able to relax the positive definiteness of the elastic coefficients. A further study in this connection is recently made by Chiriţă [2] for linear elastodynamics. For references to recent work on Saint–Venant type decay estimates the reader is referred to the survey articles of Horgan and Knowles [3] and Horgan [4, 5].

Recently, there are prepared novel foam structures with negative Poisson's ratios and their mechanical behaviour and structure evaluated (see, e.g. Lakes [6] and Caddock and Evans [7]). Such materials are called auxetic or anti-rubber and they expand laterally when stretched, in contrast to ordinary materials. Interest in such materials lies in the fact that a negative Poisson's ratio may significantly increase many of the effective mechanical properties of a material, as for example flexural rigidity and plane-strain fracture toughness. Some anisotropic polymer foams have been prepared which exhibit Poisson's ratio exceeding 1 (see Lee and Lakes [8]). Materials of the above sorts are expected to have interesting mechanical properties, such as high energy absorbtion and fracture resistance, which may be useful in applications. Possible applications of such materials in prevention of pressure sores or ulcers are analyzed in [9].

Saint–Venant end effects for materials with negative Poisson's ratio are analyzed by Lakes [10].

In the present paper we consider the linear theory of linear elastic materials with voids, developed by Cowin and Nunziato [11]. In such a theory the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field. This representation introduces an additional degree of kinematic freedom.

In this paper we present two cross-sectional measures which allow us to establish estimates describing the spatial behaviour of the solutions under weaker assumptions than those used in the previous studies by Chiriţă [12] and Ieşan and Quintanilla [13] on the subject. One of the two cross-sectional measures is identical in form with the one used in the previous studies, but the results concerning the spatial decay are established under mild assumptions upon the elastic coefficients. It extends the class of elastic materials possessing the exponential spatial decay property, while the other cross-sectional measure covers a class of elastic materials distinct from that studied in the previous works.

2. Basic formulation

Throughout this paper we consider a prismatic cylinder of uniform cross-section D whose boundary is sufficiently smooth to allow for the application of the divergence theorem. A rectangular Cartesian coordinate system Ox_k is used. The origin of the reference system is located at an interior point of the base cross-section with the x_3-axis directed parallel to the generators of the cylinder. We shall employ the usual summation and differentiation conventions: Latin subscripts run over the integers $(1, 2, 3)$, whereas Greek subscripts are confined to the range $(1, 2)$; summation is carried out over repeated indices and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.

We consider an elastic material with voids which possesses a reference configuration in which the volume fraction is constant. We assume that the body occupies the cylinder $B = D \times [0, L]$ with the lateral surface $\sum = \partial D \times [0, L]$, where L is the length of the cylinder. We denote by $D(x_3)$ the bounded cross-section of the cylinder situated at the distance x_3 from the x_1Ox_2 plane. The boundary ∂D of the cross-section is assumed to be a piecewise smooth simple closed curve.

Let us denote by u_i the components of the displacement vector field. Then the components of the infinitesimal strain field are given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.1}$$

We denote by φ the change in volume fraction from the reference volume fraction [11, 14] (see also [15]).

The constitutive equations for the linear theory of homogeneous and isotropic materials with voids are [11]

$$t_{ij} = \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + \beta\varphi\delta_{ij}, \tag{2.2}$$

$$h_i = \alpha\varphi_{,i}, \tag{2.3}$$

$$g = -\xi\varphi - \beta e_{rr}, \tag{2.4}$$

where t_{ij} are the components of the stress tensor, h_i are the components of the equilibrated stress vector, g is the intrinsic equilibrated body force, λ, μ, α, β and ξ are constitutive constants and δ_{ij} is the Kronecker delta. The internal energy density e is defined by

$$2e = \lambda e_{rr}e_{ss} + 2\mu e_{ij}e_{ij} + 2\beta e_{rr}\varphi + \xi\varphi^2 + \alpha\varphi_{,i}\varphi_{,i}. \tag{2.5}$$

The necessary and sufficient conditions for the internal energy density to be positive definite are (cf. [11])

$$\mu > 0, \qquad \alpha > 0, \qquad \xi > 0, \qquad 3\lambda + 2\mu > 0, \qquad (3\lambda + 2\mu)\xi > 3\beta^2. \tag{2.6}$$

In this paper we will not impose all these inequalities. More precisely, the first three inequalities will be assumed, while the last two will be replaced by other ones.

In the absence of an external body force and an extrinsic equilibrated body force, the field equations governing the equilibrium of a continuum with voids are [11, 14, 15]

$$t_{ji,j} = 0, \tag{2.7}$$

$$h_{i,i} + g = 0. \tag{2.8}$$

The surface traction and the equilibrated stress acting at a point x of the surface ∂B are given by

$$t_i = t_{ji}n_j, \qquad h = h_i n_i, \tag{2.9}$$

where $n_j = \cos(n_x, x_j)$ and n_x is the unit vector of the outward normal to ∂B at x.

From (2.1)–(2.4) and (2.7), (2.8), we obtain the field of equilibrium equations in terms of the displacement and the volume fraction fields

$$\mu u_{i,jj} + (\lambda + \mu)u_{j,ji} + \beta\varphi_{,i} = 0, \tag{2.10}$$

$$\alpha\varphi_{,jj} - \xi\varphi - \beta u_{r,r} = 0. \tag{2.11}$$

Throughout this paper we will consider the following boundary conditions on the boundary of the cylinder:

$$u_i = 0 \quad \text{on} \quad (\partial D \times [0, L]) \cup D(L), \tag{2.12}$$

$$\begin{aligned} &\varphi = 0 \quad \text{on} \quad \Gamma_1 \times [0, L], \quad h = 0 \quad \text{on} \quad \Gamma_2 \times [0, L], \\ &\varphi = 0 \quad \text{or} \quad h = 0 \quad \text{on} \quad D(L), \end{aligned} \tag{2.13}$$

$$\begin{aligned} &u_i = \tilde{u}_i \quad \text{on} \quad D(0), \\ &\varphi = \tilde{\varphi} \quad \text{or} \quad h = \tilde{h} \quad \text{on} \quad D(0), \end{aligned} \tag{2.14}$$

where $\tilde{u}_i$, $\tilde{\varphi}$ and $\tilde{h}$ are prescribed functions and Γ_1 and Γ_2 are subcurves of ∂D so that $\overline{\Gamma}_1 \cup \Gamma_2 = \partial D$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Other boundary conditions can be postulated but essentially for our considerations we use the boundary condition (2.12).

3. First cross-sectional measure

Throughout this section we will assume that

$$\mu > 0, \qquad \alpha > 0, \qquad \xi > 0, \qquad 3\lambda + 4\mu > 0, \qquad (3\lambda + 4\mu)\xi > 3\beta^2. \tag{3.1}$$

In what follows we will extend some results obtained by CHIRIŢĂ [12] and IEŞAN and QUINTANILLA [13]. To this end we write the equilibrium equations (2.10) and (2.11) in the following form

$$T_{ji,j} = 0, \tag{3.2}$$

$$h_{i,i} + g = 0, \tag{3.3}$$

where

$$T_{ji} = \mu u_{i,j} + (\lambda + \mu) u_{r,r} \delta_{ij} + \beta \varphi \delta_{ij}, \tag{3.4}$$

and introduce the following cross-sectional measure:

$$\begin{aligned} M(x_3) &= \int\limits_{D(x_3)} \Big(T_{3i} u_i + h_3 \varphi \Big) da \\ &= \int\limits_{D(x_3)} \big[\mu u_i u_{i,3} + (\lambda + \mu)\, u_{r,r} u_3 + \beta \varphi u_3 + \alpha \varphi \varphi_{,3} \big] da, \quad x_3 \in [0, L]. \end{aligned} \tag{3.5}$$

By a direct differentiation and by using the equilibrium equations in the form (3.2) and (3.3), we get

$$M'(x_3) = \int\limits_{D(x_3)} w da, \tag{3.6}$$

where

$$w = w_1 + w_2 + w_3, \tag{3.7}$$

and

$$\begin{aligned} w_1 = (\lambda + 2\mu)\left(u_{1,1}^2 + u_{2,2}^2 + u_{3,3}^2\right) + \xi\varphi^2 \\ + 2(\lambda + \mu)\left(u_{1,1}u_{2,2} + u_{2,2}u_{3,3} + u_{3,3}u_{1,1}\right) \\ + 2\beta\varphi\left(u_{1,1} + u_{2,2} + u_{3,3}\right), \end{aligned} \tag{3.8}$$

$$w_2 = \mu\left(u_{1,2}^2 + u_{2,1}^2 + u_{2,3}^2 + u_{3,2}^2 + u_{3,1}^2 + u_{1,3}^2\right), \tag{3.9}$$

$$w_3 = \alpha\varphi_{,i}\varphi_{,i}. \tag{3.10}$$

Under the assumptions described by the relation (3.1) it follows that w_1 is a positive definite quadratic form. The eigenvalues of the associated linear transform are

$$\begin{aligned} &\sigma_1 = \mu > 0, \\ &\sigma_{2,3} = \frac{1}{2}\left\{\xi + 3\lambda + 4\mu \pm \sqrt{\left[\xi - (3\lambda + 4\mu)\right]^2 + 12\beta^2}\right\} > 0, \end{aligned} \tag{3.11}$$

and therefore, we can write

$$\sigma_m\left(u_{1,1}^2 + u_{2,2}^2 + u_{3,3}^2 + \varphi^2\right) \le w_1 \le \sigma_M\left(u_{1,1}^2 + u_{2,2}^2 + u_{3,3}^2 + \varphi^2\right), \tag{3.12}$$

where

$$\sigma_m = \min\{\sigma_1, \sigma_2, \sigma_3\}, \qquad \sigma_M = \max\{\sigma_1, \sigma_2, \sigma_3\}. \tag{3.13}$$

Let us consider the following bilinear form:

$$\begin{aligned} \mathcal{F}(\psi, \chi) = (\lambda + 2\mu)\left(\psi_1\chi_1 + \psi_2\chi_2 + \psi_3\chi_3\right) + \xi\psi_4\chi_4 \\ + (\lambda + \mu)\left(\psi_1\chi_2 + \psi_2\chi_1 + \psi_2\chi_3 + \psi_3\chi_2 + \psi_3\chi_1 + \psi_1\chi_3\right) \\ + \beta\left[\chi_4\left(\psi_1 + \psi_2 + \psi_3\right) + \psi_4\left(\chi_1 + \chi_2 + \chi_3\right)\right], \end{aligned} \tag{3.14}$$

(3.14) [cont.]
$$\forall\psi = \{\psi_1, \psi_2, \psi_3, \psi_4\}, \qquad \chi = \{\chi_1, \chi_2, \chi_3, \chi_4\},$$

so that we have

$$\mathcal{F}(\psi,\chi) = \mathcal{F}(\chi,\psi), \tag{3.15}$$
$$\mathcal{F}(\tilde{\psi},\tilde{\psi}) = w_1 \quad \text{for} \quad \tilde{\psi} = \{u_{1,1}, u_{2,2}, u_{3,3}, \varphi\}.$$

Then, on the basis of the relation (3.4), we deduce that

$$\begin{aligned} T_{11}^2 + T_{22}^2 + T_{33}^2 + g^2 = {} & T_{11}\left[(\lambda+2\mu)\,u_{1,1} + (\lambda+\mu)\,(u_{2,2}+u_{3,3}) + \beta\varphi\right] \\ & + T_{22}\left[(\lambda+2\mu)\,u_{2,2} + (\lambda+\mu)\,(u_{3,3}+u_{1,1}) + \beta\varphi\right] \\ & + T_{33}\left[(\lambda+2\mu)\,u_{3,3} + (\lambda+\mu)\,(u_{1,1}+u_{2,2}) + \beta\varphi\right] \\ & - g\,(\xi\varphi + \beta u_{r,r}) = \mathcal{F}(T,\tilde{\psi}), \end{aligned} \tag{3.16}$$
$$\text{for} \qquad T = \{T_{11}, T_{22}, T_{33}, -g\}, \qquad \tilde{\psi} = \{u_{1,1}, u_{2,2}, u_{3,3}, \varphi\}.$$

By means of the Schwarz inequality and by using the relations (3.12) and (3.15), we get

$$\begin{aligned} T_{11}^2 + T_{22}^2 + T_{33}^2 + g^2 & \le \left[\mathcal{F}(\tilde{\psi},\tilde{\psi})\right]^{1/2}\left[\mathcal{F}(T,T)\right]^{1/2} \\ & \le \left[\mathcal{F}(\tilde{\psi},\tilde{\psi})\right]^{1/2}\left[\sigma_M\left(T_{11}^2 + T_{22}^2 + T_{33}^2 + g^2\right)\right]^{1/2}, \end{aligned} \tag{3.17}$$

so that we obtain

$$T_{11}^2 + T_{22}^2 + T_{33}^2 + g^2 \le \sigma_M w_1. \tag{3.18}$$

Moreover, from the relations (2.3), (3.1) and (3.4) we deduce that

$$T_{31}^2 = \mu^2 u_{1,3}^2, \qquad T_{32}^2 = \mu^2 u_{2,3}^2, \qquad h_3^2 = \alpha^2\varphi_{,3}^2. \tag{3.19}$$

On the basis of the relations (3.18) and (3.19) and the Cauchy–Schwarz and the arithmetic-geometric mean inequalities, from the relation (3.5) we obtain

$$\begin{aligned} |M(x_3)| & \le \frac{1}{2}\int\limits_{D(x_3)}\left[\varepsilon_1 T_{3i}T_{3i} + \frac{1}{\varepsilon_1}u_iu_i + \varepsilon_2 h_3^2 + \frac{1}{\varepsilon_2}\varphi^2\right]da \\ & \le \frac{1}{2}\int\limits_{D(x_3)}\left\{\varepsilon_1\left[\mu^2(u_{1,3}^2 + u_{2,3}^2) + \sigma_M w_1\right] + \frac{1}{\varepsilon_1}u_iu_i \right. \\ & \qquad \left. + \varepsilon_2\alpha^2\varphi_{,3}^2 + \frac{1}{\varepsilon_2}\varphi^2\right\}da, \end{aligned} \tag{3.20}$$

for all $\varepsilon_1, \varepsilon_2 > 0$. In view of the boundary condition (2.12), we have the following inequality:

$$\int_{D(x_3)} u_{i,\varrho} u_{i,\varrho} da \geq \lambda_1 \int_{D(x_3)} u_i u_i da, \tag{3.21}$$

where λ_1 is the lowest eigenvalue of the corresponding membrane problem. Then, by using the estimate (3.21) in relation (3.20), we find

$$\begin{aligned}|M(x_3)| \leq \frac{1}{2} \int_{D(x_3)} \Big\{ & \varepsilon_1 \mu \Big[\mu \Big(u_{1,3}^2 + u_{2,3}^2 \Big) \Big] \\ & + \frac{1}{\lambda_1 \mu \varepsilon_1} \Big[\mu \Big(u_{1,2}^2 + u_{2,1}^2 + u_{3,1}^2 + u_{3,2}^2 \Big) \Big] + \frac{1}{\lambda_1 \varepsilon_1} \Big(u_{1,1}^2 + u_{2,2}^2 \Big) \\ & + \frac{1}{\varepsilon_2} \varphi^2 + \varepsilon_1 \sigma_M w_1 + \varepsilon_2 \alpha^2 \varphi_{,3}^2 \Big\} da.\end{aligned} \tag{3.22}$$

Now we set

$$\varepsilon_1 \mu = \frac{1}{\lambda_1 \mu \varepsilon_1}, \qquad \varepsilon_2 = \lambda_1 \varepsilon_1, \tag{3.23}$$

and use the relations (3.9) and (3.12) in the relation (3.22) to obtain

$$|M(x_3)| \leq \frac{1}{2} \int_{D(x_3)} \left\{ \frac{1}{\sqrt{\lambda_1}} w_2 + \left(\frac{\mu}{\sigma_m \sqrt{\lambda_1}} + \frac{\sigma_M}{\mu \sqrt{\lambda_1}} \right) w_1 + \frac{\alpha \sqrt{\lambda_1}}{\mu} w_3 \right\} da. \tag{3.24}$$

We further set

$$\frac{1}{\sigma^{*2}} = \frac{1}{2} \max \left\{ \frac{1}{\sqrt{\lambda_1}}, \left(\frac{\mu}{\sigma_m \sqrt{\lambda_1}} + \frac{\sigma_M}{\mu \sqrt{\lambda_1}} \right), \frac{\alpha \sqrt{\lambda_1}}{\mu} \right\}, \tag{3.25}$$

and note that the relations (3.6), (3.7) and (3.24) imply the following first-order differential inequality

$$\sigma^{*2} |M(x_3)| \leq M'(x_3) \quad \text{for} \quad x_3 \in [0, L]. \tag{3.26}$$

We now proceed to integrate the first-order differential inequality (3.26). To this end we first suppose that $M(0) > 0$. Since $M'(x_3) \geq 0$, it follows that $M(x_3) > 0$ for all $x_3 \geq 0$ so that we must have $M(L) > 0$. Then the differential inequality (3.26) becomes

$$M'(x_3) \geq \sigma^{*2} M(x_3), \tag{3.27}$$

which, by integration, gives

$$M(0)e^{\sigma^{*2}x_3} \leq M(x_3) \leq M(L)e^{-\sigma^{*2}(L-x_3)}, \qquad x_3 \in [0, L]. \tag{3.28}$$

Let us now suppose that $M(0) = 0$. Then either $M(L) = 0$ or $M(L) > 0$. In the first case we deduce that $M(x_3) = M'(x_3) = 0$ and therefore, the relations (3.1) and (3.6)–(3.10) imply that $u_i = \varphi = 0$ in B. In the second case it results that there exists $\hat{x}_3 = \inf\{x_3 \in [0, L] \text{ with } M(x_3) > 0\} > 0$ and then the differential inequality (3.27) gives

$$M(\hat{x}_3)e^{\sigma^{*2}(x_3-\hat{x}_3)} \leq M(x_3) \leq M(L)e^{-\sigma^{*2}(L-x_3)}, \qquad x_3 \in [\hat{x}_3, L], \tag{3.29}$$

and

$$M(x_3) = M'(x_3) = 0, \quad \text{for} \quad x_3 \in (0, \hat{x}_3). \tag{3.30}$$

Finally, we suppose that $M(0) < 0$ and then it follows that $M(L) < 0$ or $M(L) \geq 0$. In the first case it follows that $M(x_3) < 0$ for all $x_3 \in [0, L]$ and therefore the differential inequality (3.26) gives

$$-M(L)e^{\sigma^{*2}(L-x_3)} \leq -M(x_3) \leq -M(0)e^{-\sigma^{*2}x_3}, \qquad x_3 \in [0, L]. \tag{3.31}$$

In the second case we are lead to a combination of situations already discussed in the above for establishing the behaviour described by the relations (3.29) and (3.31).

We consider now the case of a cylinder of semi-infinite length. From the above analysis it follows that the cross-sectional measure has the following property

$$-M(x_3) \leq -M(0)e^{-\sigma^{*2}x_3}, \qquad x_3 \geq 0, \tag{3.32}$$

provided $M(0) < 0$; when $M(0) \geq 0$, we have

$$M(x_3) \geq M(0)e^{\sigma^{*2}x_3}, \qquad x_3 \geq 0, \tag{3.33}$$

or

$$M(x_3) \geq M(\hat{x}_3)e^{\sigma^{*2}(x_3-\hat{x}_3)}, \qquad x_3 \geq \hat{x}_3. \tag{3.34}$$

Finally, we have to note that in the case when the measure of Γ_2 is so that $meas\ \Gamma_2 = 0$ then we can estimate the term $\dfrac{1}{\varepsilon_2}\varphi^2$ in terms of w_3, by means of the inequality

$$\int_{D(x_3)} \varphi_{,\varrho}\varphi_{,\varrho}da \geq \lambda_1 \int_{D(x_3)} \varphi^2 da. \tag{3.35}$$

That means the constant σ^* is now given by the relation

$$\frac{1}{\sigma^{*2}} = \frac{1}{2}\max\left\{\frac{1}{\sqrt{\lambda_1}}, \frac{\mu}{\sigma_m\sqrt{\lambda_1}} + \frac{\sigma_M}{\mu\sqrt{\lambda_1}}\right\}. \tag{3.36}$$

It can be seen that the cross-sectional measure $M(x_3)$ has the same form as that used by CHIRIŢĂ [12] and IEŞAN and QUINTANILLA [13], but our hypothesis (3.1) is weaker than that used in their papers.

4. Second cross-sectional measure

Throughout this paper we shall assume the following inequalities:

$$\mu > 0, \quad \alpha > 0, \quad \xi > 0, \quad \lambda + 2\mu > 0, \quad \lambda < 0, \quad (\lambda + 2\mu)\xi > 3\beta^2. \tag{4.1}$$

In order to analyse the spatial behaviour under the above hypotheses upon the characteristic constants of the material, we write the equilibrium equations (2.10) and (2.11) in the following form

$$S_{ji,j} = 0, \tag{4.2}$$

$$h_{i,i} + g = 0, \tag{4.3}$$

where

$$S_{ji} = \mu u_{i,j} + (\lambda + \mu)u_{j,i} + \beta\varphi\delta_{ij}. \tag{4.4}$$

Further, we define the following cross-section integral:

$$N(x_3) = \int_{D(x_3)} \big[S_{3i}u_i + h_3\varphi\big]da \tag{4.5}$$

$$= \int_{D(x_3)} \big[\mu u_i u_{i,3} + (\lambda + \mu)\, u_{3,i}u_i + \beta\varphi u_3 + \alpha\varphi\varphi_{,3}\big]da.$$

By direct differentiation and by using the equilibrium equations (4.2) and (4.3) and the boundary conditions (2.12), from the relation (4.5) we deduce that

$$N'(x_3) = \int_{D(x_3)} W da, \tag{4.6}$$

where

$$W = W_1 + W_2 + W_3, \tag{4.7}$$

and

$$W_1 = (\lambda + 2\mu)\left(u_{1,1}^2 + u_{2,2}^2 + u_{3,3}^2\right) + 2\beta\varphi\left(u_{1,1} + u_{2,2} + u_{3,3}\right) + \xi\varphi^2, \tag{4.8}$$

$$W_2 = \mu\left(u_{1,2}^2 + u_{2,1}^2 + u_{2,3}^2 + u_{3,2}^2 + u_{3,1}^2 + u_{1,3}^2\right) + 2(\lambda + \mu)\left(u_{1,2}u_{2,1} + u_{2,3}u_{3,2} + u_{3,1}u_{1,3}\right), \tag{4.9}$$

$$W_3 = \alpha\varphi_{,i}\varphi_{,i}. \tag{4.10}$$

It follows then by the hypotheses described by the relation (4.1) that W_1 is a positive definite quadratic form. The associated linear transform has the following eigenvalues:

$$\begin{aligned} &\check{\sigma}_1 = \lambda + 2\mu > 0, \\ &\check{\sigma}_{2,3} = \frac{1}{2}\left\{\xi + \lambda + 2\mu \pm \sqrt{(\xi - \lambda - 2\mu)^2 + 12\beta^2}\right\} > 0, \end{aligned} \tag{4.11}$$

so that, if we set

$$\check{\sigma}_m = \min\left\{\check{\sigma}_1, \check{\sigma}_2, \check{\sigma}_3\right\}, \qquad \check{\sigma}_M = \max\left\{\check{\sigma}_1, \check{\sigma}_2, \check{\sigma}_3\right\}, \tag{4.12}$$

we have

$$\check{\sigma}_m\left(u_{1,1}^2 + u_{2,2}^2 + u_{3,3}^2 + \varphi^2\right) \le W_1 \le \check{\sigma}_M\left(u_{1,1}^2 + u_{2,2}^2 + u_{3,3}^2 + \varphi^2\right). \tag{4.13}$$

In view of the relations (4.1), (4.4), (4.8) and (4.13), we deduce that

$$S_{11}^2 + S_{22}^2 + S_{33}^2 + g^2 \le \check{\sigma}_M W_1. \tag{4.14}$$

On the other hand, the relation (4.1) shows that the quadratic form W_2 is positive definite. The eigenvalues of the associated linear transformation are

$$\hat{\sigma}_1 = -\lambda > 0, \qquad \hat{\sigma}_2 = \lambda + 2\mu > 0, \tag{4.15}$$

and therefore, we have

$$\hat{\sigma}_m\left(u_{1,2}^2 + u_{2,1}^2 + u_{2,3}^2 + u_{3,2}^2 + u_{3,1}^2 + u_{1,3}^2\right) \le W_2 \le \hat{\sigma}_M\left(u_{1,2}^2 + u_{2,1}^2 + u_{2,3}^2 + u_{3,2}^2 + u_{3,1}^2 + u_{1,3}^2\right), \tag{4.16}$$

where

$$\hat{\sigma}_m = \min\{-\lambda,\ \lambda + 2\mu\} \qquad \hat{\sigma}_M = \max\{-\lambda,\ \lambda + 2\mu\}. \tag{4.17}$$

By means of a procedure similar to that used in the above section we deduce that

$$\tag{4.18} \begin{gathered} S_{12}^2 + S_{21}^2 \le \hat{\sigma}_M^2 (u_{1,2}^2 + u_{2,1}^2), \qquad S_{23}^2 + S_{32}^2 \le \hat{\sigma}_M^2 (u_{2,3}^2 + u_{3,2}^2), \\ S_{31}^2 + S_{13}^2 \le \hat{\sigma}_M^2 (u_{3,1}^2 + u_{1,3}^2). \end{gathered}$$

From the relation (2.3) it follows that

$$\tag{4.19} h_3^2 = \alpha^2 \varphi_{,3}^2.$$

On the basis of the estimates (3.21), (4.14), (4.18) and (4.19), from the relation (4.5) we deduce

$$\tag{4.20} \begin{aligned} |N(x_3)| &\le \frac{1}{2} \int\limits_{D(x_3)} \left[\varepsilon_1 S_{3i} S_{3i} + \frac{1}{\varepsilon_1} u_i u_i + \varepsilon_2 h_3^2 + \frac{1}{\varepsilon_2} \varphi^2 \right] da \\ &\le \frac{1}{2} \int\limits_{D(x_3)} \Big\{ \varepsilon_1 \Big[\hat{\sigma}_M^2 (u_{3,1}^2 + u_{1,3}^2 + u_{2,3}^2 + u_{3,2}^2) + \check{\sigma}_M W_1 \Big] \\ &\qquad + \varepsilon_2 \alpha^2 \varphi_{,3}^2 + \frac{1}{\lambda_1 \varepsilon_1} u_{i,\varrho} u_{i,\varrho} + \frac{1}{\varepsilon_2} \varphi^2 \Big\} da, \end{aligned}$$

for all $\varepsilon_1, \varepsilon_2 > 0$. Thus, we get

$$\tag{4.21} \begin{aligned} |N(x_3)| \le \frac{1}{2} \int\limits_{D(x_3)} \Big\{ &\left(\varepsilon_1 \hat{\sigma}_M^2 + \frac{1}{\lambda_1 \varepsilon_1} \right) (u_{1,2}^2 + u_{2,1}^2 + u_{2,3}^2 + u_{3,2}^2 \\ &+ u_{3,1}^2 + u_{1,3}^2) + \left[\frac{1}{\lambda_1 \varepsilon_1} (u_{1,1}^2 + u_{2,2}^2) + \frac{1}{\varepsilon_2} \varphi^2 \right] \\ &+ \varepsilon_1 \check{\sigma}_M W_1 + \varepsilon_2 \alpha^2 \varphi_{,3}^2 \Big\} da, \end{aligned}$$

for all $\varepsilon_1, \varepsilon_2 > 0$. Further, we set

$$\tag{4.22} \varepsilon_1 = \frac{1}{\hat{\sigma}_M \sqrt{\lambda_1}}, \qquad \varepsilon_2 = \lambda_1 \varepsilon_1,$$

so that, by means of the relations (4.10), (4.13) and (4.16), we obtain

$$\tag{4.23} \begin{aligned} |N(x_3)| \le \frac{1}{2} \int\limits_{D(x_3)} \Big\{ &\frac{2\hat{\sigma}_M}{\hat{\sigma}_m \sqrt{\lambda_1}} W_2 \\ &+ \frac{1}{\sqrt{\lambda_1}} \left(\frac{\check{\sigma}_M}{\hat{\sigma}_M} + \frac{\hat{\sigma}_M}{\check{\sigma}_m} \right) W_1 + \frac{\alpha \sqrt{\lambda_1}}{\hat{\sigma}_M} W_3 \Big\} da. \end{aligned}$$

If we set

$$\frac{1}{\tilde{\sigma}^2} = \frac{1}{2}\max\left\{\frac{2\hat{\sigma}_M}{\hat{\sigma}_m\sqrt{\lambda_1}}, \frac{1}{\sqrt{\lambda_1}}\left(\frac{\check{\sigma}_M}{\hat{\sigma}_M} + \frac{\hat{\sigma}_M}{\check{\sigma}_m}\right), \frac{\alpha\sqrt{\lambda_1}}{\hat{\sigma}_M}\right\}, \tag{4.24}$$

then the relations (4.23) and (4.6) yield the following first-order differential inequality:

$$\tilde{\sigma}^2\,|N(x_3)| \le N'(x_3), \qquad x_3 \in [0, L]. \tag{4.25}$$

Such differential inequality can be solved by the same analysis as that used in the above section. Let us first discuss the case of a finite cylinder. Thus, if $N(0) > 0$ then we have $N(x_3) > 0$ for all $x_3 > 0$ and

$$N(0)e^{\tilde{\sigma}^2 x_3} \le N(x_3) \le N(L)e^{-\tilde{\sigma}^2(L-x_3)}, \qquad x_3 \in [0, L]. \tag{4.26}$$

If $N(0) = 0$ then there exists $\tilde{x}_3 = \inf\{x_3 \in [0, L]$ with $N(x_3) > 0\}$ so that we have

$$N(x_3) = N'(x_3) = 0 \quad \text{for} \quad x_3 \in (0, \tilde{x}_3), \tag{4.27}$$

and

$$N(\tilde{x}_3)e^{\tilde{\sigma}^2(x_3-\tilde{x}_3)} \le N(x_3) \le N(L)e^{-\tilde{\sigma}^2(L-x_3)}, \qquad x_3 \in [\tilde{x}_3, L]. \tag{4.28}$$

In the case when $N(x_3) < 0$ for all $x_3 \in [0, L]$, the differential inequality (4.25) gives

$$-N(L)e^{\tilde{\sigma}^2(L-x_3)} \le -N(x_3) \le -N(0)e^{-\tilde{\sigma}^2 x_3}, \qquad x_3 \in [0, L]. \tag{4.29}$$

For the case of an infinite cylinder, for $N(0) < 0$ we have

$$-N(x_3) \le -N(0)e^{-\tilde{\sigma}^2 x_3}, \qquad x_3 \ge 0, \tag{4.30}$$

while for $N(0) \ge 0$, we have

$$N(x_3) \ge N(0)e^{\tilde{\sigma}^2 x_3}, \qquad x_3 \ge 0, \tag{4.31}$$

or

$$N(x_3) \ge N(\tilde{x}_3)e^{\tilde{\sigma}^2(x_3-\tilde{x}_3)}, \qquad x_3 \ge \tilde{x}_3. \tag{4.32}$$

5. Conclusion

The main purpose of this paper was to establish some exponential decay estimates such as (3.32) and (4.30), with an estimated decay rate depending on the class of porous elastic materials to be studied. On the other hand, the foregoing analysis leading to the conclusion expressed by the relation (3.32) is valid, however, for a broader class of materials considered in [12, 13], and so (3.32) can also be regarded as an alternative result to the estimates obtained in [12, 13].

The analysis proves that slow decay of the end effects with respect to the distance can arise from the appropriate extreme eigenvalues of the porous elastic material, as well as from the geometry of the cylinder.

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