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The influence of internal restrictions on the elastic properties of anisotropic materials

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The influence of internal restrictions on the elastic properties of anisotropic materials described by Hooke's law is discussed. Spectral decomposition of the stiffness tensor and the compliance tensor is applied. Possible types of restrictions imposed on the deformation modes are considered. An algorithm for accounting for these restrictions in a constitutive law that minimizes stiffening of the material is proposed. As examples, the volume-isotropic materials and fiber-reinforced materials are analyzed.

1. Introduction

Majority of modern materials, that nowadays often replace traditional materials, are the materials of new generation. These materials already during their projecting are supposed to be characterized by some prescribed properties. These properties may refer to their strength, way of deformation, resistance to external forces. Therefore, it is necessary to know appropriate criteria for the assessment of these properties. It requires profound theoretical knowledge and working-out of new technological solutions. At the same time in many cases computer Finite Element Method simulations are conducted to determine elastic constants for the considered biological materials such as bones and other tissues. It is necessary to formulate the criteria of identification of symmetry type of the material.

Advanced materials usually exhibit anisotropic properties. One may control these properties in the technological process. Internal structure of the material, such as strong fibers and other reinforcements as well as the way in which these elements are tightened, imposes bounds on the admissible deformation modes. These types of restrictions will be called internal restrictions. Usually, it is accepted to describe them as strain-type internal constraints. It should be noted that also material symmetry itself introduces restrictions on the way in which the material is deformed, for example isotropic body (the body made of isotropic material) will react only by changing its volume due to hydrostatic pressure while orthotropic body will react also by changing its shape.

In the computational model, the response of the material to the prescribed stresses is determined by the constitutive law. Additional restrictions, apart from material symmetry, concerning the admissible deformation modes are traditionally taken into account by introducing additional relations that are called equations of constraints. Usually these relations are imposed on the strain tensor, for example lack of the reaction on the hydrostatic pressure is described by the equation of incompressibility.

However, in many cases one can account for the observed restrictions (or the demanded restrictions in view of the project assumptions) by modification of the material constants, in the case of linear elastic materials by the modification of the stiffness or compliance tensor. As a result, in the constitutive law one may use arbitrary strain tensor or arbitrary stress tensor.

The aim of the paper is to analyze the influence of some class of restrictions imposed on deformation modes on the anisotropic properties of the linear elastic materials using the above approach.

In order to describe the material anisotropy concept of eigen-states, the Kelvin moduli and spectral decomposition of the stiffness tensor $\mathbf{S}$ (the compliance tensor $\mathbf{C}$) [11] for different material symmetries is applied.

Imposing of restrictions considerably influences the value of Kelvin moduli and the form of orthogonal projectors. Consequently, these types of restrictions can make the material symmetry higher or lower. The influence of the possible types of restrictions on material parameters is analyzed. The obtained results are illustrated by examples for the selected material symmetry groups.

2. Description of anisotropic properties of linear elastic materials

Let us consider linear elastic materials for which the dependence between the small strain tensor $\boldsymbol{\varepsilon}$ and the stress tensor $\boldsymbol{\sigma}$ is described by Hooke's law

$$\boldsymbol{\varepsilon} = \mathbf{C} \cdot \boldsymbol{\sigma} \quad \text{or} \quad \boldsymbol{\sigma} = \mathbf{S} \cdot \boldsymbol{\varepsilon}, \qquad \mathbf{C} \circ \mathbf{S} = \mathbb{I}_S, \tag{2.1}$$

where $\mathbf{C}$ is a compliance tensor, $\mathbf{S}$ is a stiffness tensor and $\mathbb{I}_S$ is a symmetrized identity tensor[1)].

The fourth order tensors $\mathbf{C}$ and $\mathbf{S}$ are linear operators mapping the space of symmetric second order tensors S onto itself. Due to symmetry, in any Cartesian system they are described by 21 essentially different components C_{ijkl} and S_{mnrs}. These components change when the reference system in the physical space is changed, so they are not material constants.

From the theory of linear operators it is concluded that conditions

$$\mathbf{S} \cdot \boldsymbol{\omega} = \lambda \boldsymbol{\omega}, \qquad \mathbf{C} \cdot \boldsymbol{\omega} = \frac{1}{\lambda} \boldsymbol{\omega} \tag{2.2}$$

[1)] In any Cartesian system this tensor has the representation $I^S_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$.

define eigenvalues and eigenelements of these operators. Generally, the tensor **S** has at most six different real eigenvalues λ_I, λ_{II}, ..., λ_{VI}, to which six eigenstates $\boldsymbol{\omega}_I, \boldsymbol{\omega}_{II}, \ldots, \boldsymbol{\omega}_{VI}$ correspond that are mutually orthogonal, normalized and with arbitrary sign. They build an orthonormal basis in the space $\mathcal{S}$ of symmetric second order tensors

$$\boldsymbol{\omega}_K \cdot \boldsymbol{\omega}_L = \delta_{KL}, \qquad K, L = I, \ldots, VI.$$

Stiffness moduli λ_I, λ_{II}, ..., λ_{VI} that are also called Kelvin moduli [11, 12] are non-negative. These are the only restrictions imposed on the elastic constants by thermodynamics.

Knowing the Kelvin moduli λ_K and corresponding to them the elastic eigenstates $\boldsymbol{\omega}_K$, the tensors **S** and **C** could be represented in the spectral form [11, 13]:

$$\mathbf{S} = \lambda_I \mathbf{P}_I + \ldots + \lambda_{VI} \mathbf{P}_{VI}, \tag{2.3}$$

$$\mathbf{C} = \frac{1}{\lambda_I} \mathbf{P}_I + \ldots + \frac{1}{\lambda_{VI}} \mathbf{P}_{VI}, \tag{2.4}$$

where fourth order tensors $\mathbf{P}_K$ are called *orthogonal projectors* and have the form

$$\mathbf{P}_K = \boldsymbol{\omega}_K \otimes \boldsymbol{\omega}_K \quad \text{(no summation)}. \tag{2.5}$$

Consequently, the space $\mathcal{S}$ of the symmetric second order tensors has been decomposed into the direct sum of mutually orthogonal one-dimensional subspaces $\mathcal{P}_K$ of eigen-states, namely

$$\mathcal{S} = \mathcal{P}_I \oplus \mathcal{P}_{II} \oplus \ldots \oplus \mathcal{P}_K.$$

Orthogonal projectors $\mathbf{P}_K$ project any tensor $\boldsymbol{\varpi} \in \mathcal{S}$ onto the subspaces $\mathcal{P}_K$. It means that there exists an equality

$$\mathbf{P}_K \cdot \boldsymbol{\varpi} = \alpha \boldsymbol{\omega}_K \in \mathcal{P}_K$$

and

$$\mathbf{P}_I + \ldots + \mathbf{P}_{VI} = \mathbb{I}_S. \tag{2.6}$$

Each set

$$(\lambda_I, \ldots, \lambda_{VI}; \mathbf{P}_I, \ldots, \mathbf{P}_{VI}), \tag{2.7}$$

consisting of six Kelvin moduli λ_K and six elastic orthogonal projectors $\mathbf{P}_K$, describes a theoretically admissible elastic material.

In [11] and [12] a hypothesis was posed that in order to define the orthogonal projectors $\mathbf{P}_K$ it is sufficient to know 15 quantities[2)]. In [11] it has been concluded that out of these 15 parameters describing orthogonal projectors one may separate 3, that are not invariants. They fix the stiffness tensor $\mathbf{S}$ (a material sample) in the reference system (a laboratory). They, for example, correspond to three Euler angles ϕ_1, ϕ_2, ϕ_3. The remaining 12 parameters are defined by non-dimensional material constants. They characterize the distribution of stiffness along material fibers and material planes. They are identical for the stiffness tensor $\mathbf{S}$ and the compliance tensor $\mathbf{C}$ and they are called *stiffness distributors*. Proof of the Rychlewski hypothesis that for fully anisotropic material the number of irreducible parameters called *stiffness distributors* does not exceed 12 was not given. Since they are related to the invariants of projectors $\mathbf{P}_K$ (fourth order tensors), this issue is connected with the open question of irreducible basis of invariants for fourth order tensors.

Following Rychlewski, one may decompose the parameters that describe any elastic body into three groups

$$(6 + 12) + 3 = 21.$$

1. The first group consists of 6 stress-dimensional parameters $\lambda_I, \ldots, \lambda_{VI}$ – **the Kelvin moduli**.
2. The second group consists of non-dimensional 12 **stiffness distributors** $\aleph_1, \ldots, \aleph_{12}$.
3. The third group consists of 3 **Euler angles** ϕ_1, ϕ_2, ϕ_3.

Therefore, one may write instead of (2.7):

$$\langle \lambda_I, \ldots, \lambda_{VI}; \aleph_1, \ldots, \aleph_{12}; \phi_1, \phi_2, \phi_3 \rangle. \tag{2.8}$$

Two bodies are made of the same material if the values of their 18 material constants, that is $\lambda_I, \ldots, \lambda_{VI}$ and $\aleph_1, \ldots, \aleph_{12}$, are equal.

Decompositions (2.3), (2.4) and projectors $\mathbf{P}_K$ (2.5) take these forms, if the Kelvin moduli corresponding to them are single, that is if $\lambda_K \neq \lambda_L$ for $K \neq L$. Spectral decompositions (2.3), (2.4) are then unique.

If the material exhibits some external symmetries then the number of parameters defining the given material is reduced. In such a case, the set of parameters (2.8) may be rewritten as follows:

$$\langle \lambda_I, \ldots, \lambda_\rho; \aleph_1, \ldots, \aleph_t; \phi_1, \ldots, \phi_n, \rangle, \tag{2.9}$$

[2)] The hypothesis was supported by the following reasoning: on 36 components of eigenstates 21 conditions of orthonormality are imposed that reduce the number of independent components to 15. Three additional components may be reduced by appropriate basis selection.

where $\rho \leq 6$, $t \leq 12$ and $n \leq 3$. The Kelvin moduli are then multiple and the spectral theorem (2.2), instead of (2.3), (2.4), leads to

$$\mathbf{S} = \lambda_1\mathbf{P}_1 + ... + \lambda_\rho\mathbf{P}_\rho, \qquad \rho \leq 6, \tag{2.10}$$

$$\mathbf{C} = \frac{1}{\lambda_1}\mathbf{P}_1 + ... + \frac{1}{\lambda_\rho}\mathbf{P}_\rho, \qquad \rho \leq 6, \tag{2.11}$$

and

$$\mathcal{S} = \mathcal{P}_I \oplus \mathcal{P}_{II} \oplus \ldots \oplus \mathcal{P}_\rho,$$

where instead of (2.6)

$$\mathbf{P}_I + \ldots + \mathbf{P}_\rho = \mathbb{I}_S. \tag{2.12}$$

It can be shown that

$$\mathbf{P}_K \circ \mathbf{P}_L = \begin{cases} \mathbf{0} & \text{if } K \neq L, \\ \mathbf{P}_K & \text{if } K = L. \end{cases}$$

Dimension of the subspace $\mathcal{P}_\eta, \eta \in \langle I, \rho\rangle$ is equal to the multiplicity of the Kelvin modulus λ_η.

If dimensions of the eigen-subspaces $\mathcal{P}_I, \ldots, \mathcal{P}_\rho$ are denoted correspondingly by $q_I, \ldots, q_\rho$ then, according to [11], an expression

$$\langle q_I + q_{II} + \ldots + q_\rho\rangle, \quad q_I + q_{II} + \ldots + q_\rho = 6$$

is called the **first structural index** of the material, while an expression (see (2.9))

$$[\rho + t + n]$$

is called the **second structural index**. These natural numbers are material characteristics.

It should be noted that the symmetry of the tensor **S** (the tensor **C**) – the material symmetry of a linear elastic body, results from the properties of symmetric Euclidean fourth order tensors (that is from the linearity of Hooke's law (2.1) and the properties of the three-dimensional Euclidean space).

2.1. The Kelvin moduli $\lambda_I, \ldots, \lambda_{VI}$

The Kelvin moduli $\lambda_I, \ldots, \lambda_{VI}$ are roots of a characteristic polynomial of the form

$$\det(\mathbf{S} - \lambda\mathbb{I}) = \lambda^6 + a_1(\mathbf{S})\lambda^5 + \ldots + a_5(\mathbf{S})\lambda + a_6(\mathbf{S}) = 0. \tag{2.13}$$

A determinant of the fourth order tensor $\mathbf{A}$ is defined as follows

$$\det \mathbf{A} \equiv \det(A_{KL}) = \det(\boldsymbol{\nu}_K \cdot \mathbf{A} \cdot \boldsymbol{\nu}_L),$$

where $\boldsymbol{\nu}_K$, $(K = I, \ldots, VI)$ is any orthonormal basis in $\mathcal{S}$, A_{KL} is the matrix 6×6 corresponding to the tensor $\mathbf{A}$ in this basis (see Appendix). Selection of the basis $\boldsymbol{\nu}_K$ has no influence on the values of the coefficients $a_i(\mathbf{S})$ in Eq. (2.13).

For a given $\lambda^\star$, an eigen-state corresponding to it is obtained from the set of 6 linear homogeneous equations:

$$\mathbf{S} \cdot \boldsymbol{\omega}^\star = \lambda^\star \boldsymbol{\omega}^\star \Longrightarrow (\mathbf{S} - \lambda \mathbb{I}_S) \cdot \boldsymbol{\omega}^\star = \mathbf{0}$$

with constraints $\boldsymbol{\omega}^\star \cdot \boldsymbol{\omega}^\star = \operatorname{tr}(\boldsymbol{\omega}^\star)^2 = 1$. If the basis $\boldsymbol{\nu}_K = \boldsymbol{\omega}_K$ (i. e. it is the basis of eigen-states), then the matrix $S_{KL} = \boldsymbol{\omega}_K \cdot \mathbf{S} \cdot \boldsymbol{\omega}_L$ is diagonal with the Kelvin moduli on the diagonal.

2.2. Orthogonal projectors $\mathbf{P}_I, \ldots, \mathbf{P}_\rho$

Having found the Kelvin moduli λ_K from which ρ are different one may introduce some ordering rule for λ_I, ..., λ_ρ. For instance, they can be numbered according to their magnitude. After that the orthogonal projectors $\mathbf{P}_K$ corresponding to them can be derived from the following set of ρ fourth order tensorial equations [11]

$$\begin{aligned}
\mathbf{P}_I + \mathbf{P}_{II} + \ldots + \mathbf{P}_\rho &= \mathbb{I}_S, \\
\lambda_I \mathbf{P}_I + \lambda_{II} \mathbf{P}_{II} + \ldots + \lambda_\rho \mathbf{P}_\rho &= \mathbf{S}, \\
&\vdots \ddots \vdots \\
\lambda_I^{\rho-1} \mathbf{P}_I + \lambda_{II}^{\rho-1} \mathbf{P}_{II} + \ldots + \lambda_\rho^{\rho-1} \mathbf{P}_\rho &= \mathbf{S}^{\rho-1},
\end{aligned}$$

where

$$\mathbf{S}^k = \underbrace{\mathbf{S} \circ \mathbf{S} \circ \ldots \circ \mathbf{S}}_{k \quad \text{times}}.$$

Consequently, one obtains

$$\begin{bmatrix} \mathbf{P}_I \\ \mathbf{P}_{II} \\ \vdots \\ \mathbf{P}_\rho \end{bmatrix} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \lambda_I & \lambda_{II} & \ldots & \lambda_\rho \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_I^{\rho-1} & \lambda_{II}^{\rho-1} & \ldots & \lambda_\rho^{\rho-1} \end{bmatrix}^{-1} \begin{bmatrix} \mathbb{I}_S \\ \mathbf{S} \\ \vdots \\ \mathbf{S}^{\rho-1} \end{bmatrix}.$$

The inverse of the above matrix exists since the determinant of it is equal to

$$\Delta = \prod_{\rho \geq K > L \geq 1} (\lambda_K - \lambda_L)$$

and by definition $\lambda_K \neq \lambda_L$.

Distributors $\aleph_1, \ldots, \aleph_{12}$ are the parameters that should uniquely specify projectors $\mathbf{P}_K$ in the selected basis. Specific forms of these functions that allow to define uniquely projectors for the given material are not derived here and will be discussed in the next paper for the selected material symmetry classes.

From the relation (2.12) it follows that condition

$$\mathbf{1} \cdot \mathbf{P}_I \cdot \mathbf{1} + \mathbf{1} \cdot \mathbf{P}_{II} \cdot \mathbf{1} + \ldots + \mathbf{1} \cdot \mathbf{P}_\rho \cdot \mathbf{1} = \mathbf{1} \cdot \mathbb{I}_S \cdot \mathbf{1} = 3$$

is fulfilled. This equality gives the following relationship between the traces of eigen-states $\boldsymbol{\omega}_K$ for $\rho = 6$

$$(\mathrm{tr}\boldsymbol{\omega}_I)^2 + (\mathrm{tr}\boldsymbol{\omega}_{II})^2 + \ldots + (\mathrm{tr}\boldsymbol{\omega}_{VI})^2 = 3. \tag{2.14}$$

2.3. Symmetry groups of an anisotropic linear elastic material

If the material exhibits some external symmetries, then the number of the Kelvin moduli and the stiffness distributors is reduced. Symmetry group $\mathcal{Q}_S$ of the stiffness tensor $\mathbf{S}$ (the compliance tensor $\mathbf{C}$) is defined as follows:

$$\mathcal{Q}_S = \{\mathbf{Q} \in \mathcal{Q}; \mathbf{Q} \star \mathbf{S} = \mathbf{S}\},$$

where $\mathcal{Q}$ is a full orthogonal group, $\mathbf{Q}$ is the second order orthogonal tensor in the physical space, while $\star$ is the rule of rotation of the fourth order tensor. The following relationship is true

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{1}.$$

Classification of linear elastic materials in view of their symmetry corresponds to the classical set of eight classes of elastic symmetry [4, 5, 6]. Two limit classes are: full anisotropy ($\mathcal{Q}_S = \{\mathbf{1}, -\mathbf{1}\}$) and full isotropy ($\mathcal{Q}_S = \mathcal{Q}$). Sets of generators of subsequent symmetry groups are collected in Table 1. Any material that is not fully anisotropic is called a symmetric elastic material [2].

Spectral decompositions (2.10), (2.11) into subspaces $\mathcal{P}_K$, in view of material symmetry, fulfill conditions

$$\bigwedge_{\mathbf{Q} \in \mathcal{Q}_S} \mathbf{Q} \star \mathbf{P}_K = \mathbf{P}_K, \tag{2.15}$$

where $\mathcal{Q}_S$ is the symmetry group of material and consequently, the symmetry group of all projectors, so that

$$\mathcal{Q}_S = \mathcal{Q}_{P_I} \cap \mathcal{Q}_{P_{II}} \cap \ldots \cap \mathcal{Q}_{P_\rho}, \tag{2.16}$$

where $\mathcal{Q}_{P_K}$ is the symmetry group of the tensor $\mathbf{P}_K$. If the eigen-subspace $\mathcal{P}_K$ is one-dimensional, then the condition (2.15) is equivalent to

$$\bigwedge_{\mathbf{Q}\in\mathcal{Q}_S} \quad \mathbf{Q}\star(\boldsymbol{\omega}_K\otimes\boldsymbol{\omega}_K) = (\mathbf{Q}\boldsymbol{\omega}_K\mathbf{Q}^T)\otimes(\mathbf{Q}\boldsymbol{\omega}_K\mathbf{Q}^T) = \boldsymbol{\omega}_K\otimes\boldsymbol{\omega}_K.$$

It brings us to

$$\bigwedge_{\mathbf{Q}\in\mathcal{Q}_S} \quad \mathbf{Q}\boldsymbol{\omega}_K\mathbf{Q}^T = \pm\boldsymbol{\omega}_K.$$

Table 1. Generators of the symmetry groups (in every case $\mathbf{e}_1\cdot\mathbf{e}_2 = 0$).

Symmetry group	Set of generators in $\mathcal{Q}$
full anisotropy	$\mathcal{Q}^a = \{\mathbf{1}, -\mathbf{1}\}$
monoclinic symmetry (symmetry of a prism with an irregular basis)	$\mathcal{Q}^m_{\mathbf{e}_1} = \{\mathbf{1}, -\mathbf{1}, \mathbf{I}_{\mathbf{e}_1}\}$ $\mathbf{e}_1$ – normal to a prism basis (a symmetry plane)
orthotropy (symmetry of a prism with a rectangular basis)	$\mathcal{Q}^o = \{\mathbf{1}, -\mathbf{1}, \mathbf{I}_{\mathbf{e}_1}, \mathbf{I}_{\mathbf{e}_2}\}$ $\mathbf{e}_1, \mathbf{e}_2$ – normals to symmetry planes
trigonal symmetry (symmetry of a prism with a triangular basis)	$\mathcal{Q}^{3t}_{\mathbf{e}_1} = \left\{\mathbf{1}, -\mathbf{1}, \mathbf{R}^{k\frac{2}{3}\pi}_{\mathbf{e}_1}, \mathbf{I}_{\mathbf{e}_2}\right\}, \quad (k = 1, 2)$ $\mathbf{e}_1$ – a symmetry axis, $\mathbf{e}_2$ – normal to a symmetry plane perpendicular to a prism basis
tetragonal symmetry (symmetry of a prism with a quadratic basis)	$\mathcal{Q}^{4t}_{\mathbf{e}_1} = \left\{\mathbf{1}, -\mathbf{1}, \mathbf{I}_{\mathbf{e}_1}, \mathbf{I}_{\mathbf{e}_2}, \mathbf{R}^{k\frac{\pi}{2}}_{\mathbf{e}_1}\right\}, \quad (k = 1, 2, 3)$ $\mathbf{e}_1$ – symmetry axis, $\mathbf{e}_1, \mathbf{e}_2$ – normals to symmetry planes
transversal symmetry (cylindrical)	$\mathcal{Q}^{ti}_{\mathbf{e}_1} = \left\{\mathbf{1}, -\mathbf{1}, \mathbf{I}_{\mathbf{e}_1}, \mathbf{I}_{\mathbf{e}_2}, \mathbf{R}^{\phi}_{\mathbf{e}_1}\right\}, \quad (\phi \in \langle 0, 2\pi\rangle)$ $\mathbf{e}_1$ - symmetry axis, $\mathbf{e}_1, \mathbf{e}_2$ – normals to symmetry planes
cubic symmetry (symmetry of a cube)	$\mathcal{Q}^c_{\mathbf{e}_i} = \left\{\mathbf{1}, -\mathbf{1}, \mathbf{I}_{\mathbf{e}_1}, \mathbf{I}_{\mathbf{e}_2}\mathbf{R}^{k\frac{\pi}{2}}_{\mathbf{e}_1}, \mathbf{R}^{k\frac{\pi}{2}}_{\mathbf{e}_2}\right\}, \quad (k = 1, 2, 3)$ $\mathbf{e}_1, \mathbf{e}_2$ – symmetry axes, $\mathbf{e}_1, \mathbf{e}_2$ – normals to symmetry planes
isotropy	$\mathcal{Q}^I = \mathcal{Q}$

Notations: $\mathbf{I}_\mathbf{k}$ – the orthogonal tensor that describes mirror reflection with respect to the symmetry plane with normal $\mathbf{k}$, $\mathbf{R}^\alpha_\mathbf{k}$ – the orthogonal tensor that describes rotation around direction $\mathbf{k}$ about the angle α.

Using the above symmetry conditions one may derive the form of orthogonal projectors of the tensors **S** and **C** corresponding to the subsequent symmetry groups of the elastic material. In Table 2 the first and second structural indices are given for all symmetry groups. The form of the second structural index for fully anisotropic material is based on the Rychlewski conjecture. However, its form proved to be true for the symmetric materials (at least monoclinic). The forms of the corresponding orthogonal projectors one may find in the papers cited next to the symmetry name in Table 2.

Table 2. First and second structural indices for all symmetry groups of linear elastic materials.

Symmetry group	First structural index	Second structural index	number of parameters
full anisotropy [11]	$\langle 1+1+1+1+1+1 \rangle$	$[6+12+3]$	21
monoclinic symmetry [2]	$\langle (1+1+1+1)+1+1 \rangle$	$[6+6+3]$	15
orthotropy [1]	$\langle (1+1+1)+1+1+1 \rangle$	$[6+3+3]$	12
trigonal symmetry [12]	$\langle (1+1)+(2+2) \rangle$	$[4+2+3]$	9
tetragonal symmetry [8]	$\langle (1+1)+1+1+2 \rangle$	$[5+1+3]$	9
transversal symmetry [7]	$\langle (1+1)+2+2 \rangle$	$[4+1+2]$	7
cubic symmetry [9]	$\langle 1+2+3 \rangle$	$[3+0+3]$	6
isotropy [12]	$\langle 1+5 \rangle$	$[2+0+0]$	2

3. Additional restrictions

An anisotropic material, in spite of its symmetry, may exhibit some additional properties that are caused by its internal structure. Usually these properties impose some restrictions on the character of deformation [10, 14].

For an anisotropic linear elastic body with the prescribed symmetry group, that is with the corresponding form of the stiffness tensor **S** and the compliance tensor **C**, the following types of additional restrictions can be taken into account:

1. There exists a prescribed stress state $\boldsymbol{\sigma}^0$ ($\boldsymbol{\sigma}^0 \cdot \boldsymbol{\sigma}^0 = 1$) so that for any $\alpha\boldsymbol{\sigma}^0$, $\alpha \in R$ one does not observe any deformation

$$\mathbf{C} \cdot \boldsymbol{\sigma}^0 = \mathbf{0}. \tag{3.1}$$

Any stress state $\alpha\boldsymbol{\sigma}^0$ is called a *passive* state. This condition may be equivalently written in the classical form of constraints equation (see [10])

$$\bigwedge_{\boldsymbol{\varepsilon}\in\boldsymbol{\mathcal{S}}} \boldsymbol{\varepsilon}\cdot\boldsymbol{\sigma}^0 = 0.$$

2. There exists a prescribed stress state $\boldsymbol{\sigma}^\star$ ($\boldsymbol{\sigma}^\star\cdot\boldsymbol{\sigma}^\star = 1$), so that for each $\alpha\boldsymbol{\sigma}^\star$, $\alpha \in R$ one observes the following proportionality between strains and stresses

$$\mathbf{C}\cdot\boldsymbol{\sigma}^\star = \boldsymbol{\varepsilon}^\star = \frac{1}{\lambda}\boldsymbol{\sigma}^\star. \tag{3.2}$$

This condition may be rewritten in the form

$$\bigwedge_{\boldsymbol{\varepsilon}\in\boldsymbol{\mathcal{S}}} \Big(\boldsymbol{\varepsilon} - \frac{1}{\lambda}\boldsymbol{\sigma}\Big)\cdot\boldsymbol{\sigma}^\star = 0,$$

where $\boldsymbol{\sigma}$ fulfills Eq. (2.1) for the considered $\boldsymbol{\varepsilon}$.

3. There exists a prescribed stress state $\widetilde{\boldsymbol{\sigma}}$ ($\widetilde{\boldsymbol{\sigma}}\cdot\widetilde{\boldsymbol{\sigma}} = 1$) that enforces a prescribed strain state $\widetilde{\boldsymbol{\varepsilon}}$ so that any stress state $\alpha\widetilde{\boldsymbol{\sigma}}$, $\alpha \in R$ enforces some strain state $\alpha\widetilde{\boldsymbol{\varepsilon}}$ and

$$\mathbf{C}\cdot\widetilde{\boldsymbol{\sigma}} = \widetilde{\boldsymbol{\varepsilon}}. \tag{3.3}$$

This condition is equivalent to

$$\bigwedge_{\boldsymbol{\varepsilon}\in\boldsymbol{\mathcal{S}}} \boldsymbol{\varepsilon}\cdot\widetilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\cdot\widetilde{\boldsymbol{\varepsilon}} = 0,$$

where $\boldsymbol{\sigma}$ fulfills Eq. (2.1) for the considered $\boldsymbol{\varepsilon}$.

Taking into account the above types of restrictions leads to additional conditions imposed on the tensor $\mathbf{C}$ (in other words, the conditions imposed on the quantities that characterize the considered material symmetry). Therefore, spectral decomposition of the tensor $\mathbf{C}$ implies that taking into account these additional requirements has impact on the values of stiffness distributors, the Kelvin moduli and the form of eigen-subspaces. It should be noted that it is not possible to describe by linear elasticity the restrictions for which the relation between the prescribed stress state and the strain enforced by it is nonlinear.

3.1. Passive stress states

From (3.1) it follows that the passive state $\boldsymbol{\sigma}^0$ is the eigen-state of the tensor $\mathbf{C}$ and the corresponding eigenvalue is equal to zero. From spectral decomposition of the tensor $\mathbf{C}$ (2.11) one obtains that the condition (3.1) is equivalent to

$$\mathbf{C}\cdot\boldsymbol{\sigma}^0 = \frac{1}{\lambda_1}\mathbf{P}_1\cdot\boldsymbol{\sigma}^0 + \ldots + \frac{1}{\lambda_\rho}\mathbf{P}_\rho\cdot\boldsymbol{\sigma}^0 = \mathbf{0}, \qquad \rho \le 6. \tag{3.4}$$

Trivial solution of Eq. (3.4) is obtained by assuming that all Kelvin moduli fulfill the equality $1/\lambda_K \to 0$ ($K = I, \ldots, \rho$). It means that the whole space $\mathcal{S}$ is the space of passive stress states.

The form of $\boldsymbol{\sigma}^0$, as well as the type of the considered symmetry cause that $\boldsymbol{\sigma}^0$ may not be projected onto some subspaces, that is, there may exist such μ, that

$$\mathbf{P}_\mu \cdot \boldsymbol{\sigma}^0 = \boldsymbol{\sigma}^0_\mu \in \mathcal{P}_\mu, \quad \text{and} \quad \boldsymbol{\sigma}^0_\mu = \mathbf{0}. \tag{3.5}$$

Then in Eq. (3.4) all elements $(1/\lambda_\mu)\mathbf{P}_\mu \cdot \boldsymbol{\sigma}^0$ for which (3.5) is fulfilled do not appear, that is projectors that give zero-value projections of $\boldsymbol{\sigma}^0$ into the corresponding eigen-subspaces ($\lambda_\mu \neq 0$).

Equation (3.4) takes the form

$$\mathbf{C} \cdot \boldsymbol{\sigma}^0 = \frac{1}{\lambda_I}\mathbf{P}_I \cdot \boldsymbol{\sigma}^0 + \ldots + \frac{1}{\lambda_\nu}\mathbf{P}_\nu \cdot \boldsymbol{\sigma}^0 = \mathbf{0}, \quad \nu \leq \rho. \tag{3.6}$$

Among projectors $\mathbf{P}_I, \ldots, \mathbf{P}_\nu$ that are left there may exist such ones which do not depend on the distributors $\aleph_1, \ldots, \aleph_t$. For example, in the case of orthotropy this kind of projectors is represented by these $\mathbf{P}_K$ that project onto one-dimensional subspaces of pure shears in symmetry axes (see [1]).

Let the projector $\mathbf{P}_\mu$ ($\mu \in \langle I, \nu \rangle$) be independent of the distributors. Contracting Eq. (3.6) with tensor $\boldsymbol{\sigma}^0_\mu = \mathbf{P}_\mu \cdot \boldsymbol{\sigma}^0$ one obtains

$$\frac{1}{\lambda_I}\boldsymbol{\sigma}^0_I \cdot \boldsymbol{\sigma}^0_\mu + \ldots + \frac{1}{\lambda_\mu}\boldsymbol{\sigma}^0_\mu \cdot \boldsymbol{\sigma}^0_\mu + \ldots + \frac{1}{\lambda_\nu}\boldsymbol{\sigma}^0_\nu \cdot \boldsymbol{\sigma}^0_\mu = 0, \tag{3.7}$$

where

$$\boldsymbol{\sigma}^0_K = \mathbf{P}_K \cdot \boldsymbol{\sigma}^0, \qquad K = I, \ldots, \nu.$$

Because subspaces $\mathcal{P}_K$ and $\mathcal{P}_L$ are orthogonal, that is, the following equality is fulfilled

$$\boldsymbol{\sigma}^0_K \cdot \boldsymbol{\sigma}^0_L = 0, \quad \text{for} \quad K \neq L,$$

only one element remains in (3.7)

$$\frac{1}{\lambda_\mu}|\boldsymbol{\sigma}^0_\mu|^2 = 0. \tag{3.8}$$

Projection $\boldsymbol{\sigma}^0_\mu$ is not equal to zero by assumption, therefore

$$\frac{1}{\lambda_\mu} \to 0, \qquad (\lambda_\mu \to \infty) \tag{3.9}$$

and the subspace $\mathcal{P}_\mu$ becomes the subspace of passive stress states.

One may conduct the above reasoning for all projectors that do not depend on distributors. Consequently, for all projectors $\mathbf{P}_\mu$ independent of the distributors, Eq. (3.9) is fulfilled. Eigen-subspaces $\mathcal{P}_\mu$ sum up and become one eigen-subspace of passive stress states.

The above considerations refer to the case when projectors $\mathbf{P}_K$ give projections of $\boldsymbol{\sigma}^0$ onto the whole subspaces $\mathcal{P}_K$. This situation is always observed when subspaces $\mathcal{P}_K$ are one-dimensional ($\dim \mathcal{P}_K = q_K = 1$). If eigen-subspaces $\mathcal{P}_K$ have dimension $q_K \geq 2$, then there may exist such a projector $\mathbf{P}_\mu$ that

$$\mathbf{P}_\mu \cdot \boldsymbol{\sigma}^0 \in \mathcal{P}_\mu^0 \subset \mathcal{P}_\mu, \qquad \dim \mathcal{P}_\mu^0 < \dim \mathcal{P}_\mu. \tag{3.10}$$

Projector $\mathbf{P}_\mu$ projects $\boldsymbol{\sigma}^0$ onto subspace $\mathcal{P}_\mu^0$. Therefore it is possible to assume that

$$\mathbf{P}_\mu = \mathbf{P}_\mu^0 + \mathbf{P}_\mu^\perp \qquad (\mathbf{P}_\mu^0 \circ \mathbf{P}_\mu^0 = \mathbf{P}_\mu^0, \quad \mathbf{P}_\mu^0 \circ \mathbf{P}_\mu^\perp = \mathbf{0}),$$

where

$$\mathbf{P}_\mu \cdot \boldsymbol{\sigma}^0 = \mathbf{P}_\mu^0 \cdot \boldsymbol{\sigma}^0 = \boldsymbol{\sigma}_\mu^0 \in \mathcal{P}_\mu^0,$$
$$\mathbf{P}_\mu^\perp \cdot \boldsymbol{\sigma}^0 = \mathbf{0}.$$

Because the projector $\mathbf{P}_\mu^\perp$ gives zero-projection of $\boldsymbol{\sigma}^0$, in Eq. (3.6) the component with the projector $\mathbf{P}_\mu$ may be rewritten in the form

$$\frac{1}{\lambda_\mu}\mathbf{P}_\mu \cdot \boldsymbol{\sigma}^0 = \frac{1}{\lambda_\mu^0}\mathbf{P}_\mu^0 \cdot \boldsymbol{\sigma}^0 + \frac{1}{\lambda_\mu^\perp}\mathbf{P}_\mu^\perp \cdot \boldsymbol{\sigma}^0 = \frac{1}{\lambda_\mu^0}\mathbf{P}_\mu^0 \cdot \boldsymbol{\sigma}^0,$$

where without restrictions $\lambda_\mu^0 = \lambda_\mu^\perp = \lambda_\mu$. Contracting then Eq. (3.6) with $\mathbf{P}_\mu^0 \cdot \boldsymbol{\sigma}^0 = \boldsymbol{\sigma}_\mu^0$ one obtains

$$\frac{1}{\lambda_\mu^0}|\boldsymbol{\sigma}_\mu^0|^2 = 0$$

and the condition (3.9) is then equivalent to

$$\frac{1}{\lambda_\mu^0} \to 0, \qquad \left(\text{where} \quad \frac{1}{\lambda_\mu^\perp} = \frac{1}{\lambda_\mu}\right). \tag{3.11}$$

In this case the subspace $\mathcal{P}_\mu$ is decomposed into two subspaces with two distinctive Kelvin moduli. The above derivations may be applied for all projectors with property (3.10). Eigen-subspaces of the form of $\mathcal{P}_\mu^0$ sum up and constitute one subspace of passive stress states.

In Eq. (3.7) only these projectors that depend on the distributors $\aleph_1, \ldots \aleph_t$ remain. These projectors do not have to depend on all distributors. For some types of material symmetry one may divide them into groups of projectors dependent on different, independent sets of distributors. As an example, trigonal symmetry may be considered. Four projectors: $\mathbf{P}_I$, $\mathbf{P}_{II}$, $\mathbf{P}_{III}$ and $\mathbf{P}_{IV}$ dependent on 2 distributors may be divided into two groups: projectors $\mathbf{P}_{III}$ and $\mathbf{P}_{IV}$ depend only on $\aleph_1$, while $\mathbf{P}_I$ and $\mathbf{P}_{II}$ depend only on $\aleph_2$ [12].

Equation

$$\frac{1}{\lambda_1}\mathbf{P}_1 \cdot \boldsymbol{\sigma}^0 + \ldots + \frac{1}{\lambda_\gamma}\mathbf{P}_\gamma \cdot \boldsymbol{\sigma}^0 = \mathbf{0}, \qquad \gamma \le \nu, \tag{3.12}$$

may be fulfilled by imposing some restrictions on the values of distributors and Kelvin moduli. It should be stressed that Eq. (3.12) cannot be fulfilled by imposing restrictions only on the distributors.

Constraints of the form (3.1) can make material symmetry higher by reducing the number of eigen-subspaces (the number of Kelvin moduli is then lower). They can also make material symmetry lower by subdivision of subspaces in the way discussed above.

A special case of constraints (3.1) is observed, when $\boldsymbol{\sigma}^0$ is the eigen-state of the tensor $\mathbf{C}$. Then there exists eigen-subspace $\mathcal{P}_\mu$ so that $\boldsymbol{\sigma}^0 \in \mathcal{P}_\mu$. In such a case it is sufficient to assume that

$$\frac{1}{\lambda_\mu} \to 0$$

and the eigen-subspace $\mathcal{P}_\mu$ becomes the subspace of passive stress states. If the subspace $\mathcal{P}_\mu$ is not one-dimensional, it may happen that

$$\mathbf{P}_\mu \cdot \boldsymbol{\sigma}^0 \in \mathcal{P}^0_\mu \subset \mathcal{P}_\mu$$

and from the subspace $\mathcal{P}_\mu$ one may separate the subspace of passive stress states.

An example of the constraints (3.1) is obtained by assuming, as for isotropic fluids, that the material is incompressible. It means that the hydrostatic stress state (the spherical tensor) is passive, that is

$$\boldsymbol{\sigma}^0 = \frac{1}{\sqrt{3}}\mathbf{1}, \qquad \mathbf{C} \cdot \mathbf{1} = \mathbf{0}, \qquad (\Rightarrow \boldsymbol{\varepsilon} \cdot \mathbf{1} = 0). \tag{3.13}$$

Another subspace of passive stress states is obtained by assuming that the material is inextensible in some prescribed direction $\mathbf{k}$. Then, restrictions take the following form

$$\boldsymbol{\sigma}^0 = \mathbf{k} \otimes \mathbf{k}, \qquad \mathbf{C} \cdot (\mathbf{k} \otimes \mathbf{k}) = \mathbf{0}, \qquad (\Rightarrow \boldsymbol{\varepsilon} \cdot (\mathbf{k} \otimes \mathbf{k}) = 0).$$

The model of the body with forbidden deformations may be applied to describe composites reinforced by a dense family of thin fibers that are so strong and fastened to the matrix that extensions in the fiber direction can be negligible.

3.2. Proportionality between strains and stresses

Another type of restrictions is described by relation (3.2). The response of the material to the prescribed stresses is then assumed to be proportional to these stresses. This assumption is equivalent to the assumption that the stress state $\boldsymbol{\sigma}^\star$ is the eigen-state of $\mathbf{C}$.

Spectral decomposition of the tensor $\mathbf{C}$ (2.11) implies that the condition (3.2) is equivalent to

$$\left(\frac{1}{\lambda_I} - \frac{1}{\lambda}\right)\mathbf{P}_I \cdot \boldsymbol{\sigma}^\star + \ldots + \left(\frac{1}{\lambda_\rho} - \frac{1}{\lambda}\right)\mathbf{P}_\rho \cdot \boldsymbol{\sigma}^\star = \mathbf{0}, \tag{3.14}$$

where decomposition of the tensor $\boldsymbol{\sigma}^\star$ has been applied, that is (see (2.12))

$$\boldsymbol{\sigma}^\star = \mathbf{P}_I \cdot \boldsymbol{\sigma}^\star + \mathbf{P}_{II} \cdot \boldsymbol{\sigma}^\star + \ldots + \mathbf{P}_\rho \cdot \boldsymbol{\sigma}^\star.$$

After denoting

$$\frac{1}{\eta_K} \equiv \frac{1}{\lambda_K} - \frac{1}{\lambda}$$

Eq. (3.14) is reduced to the form (3.12) with $1/\lambda_K$ being here replaced by $1/\eta_K$. Therefore derivations conducted in Sec. 3.1 can be repeated. In this case conditions (3.9) or (3.11) correspond to the following ones

$$\frac{1}{\lambda_\mu} = \frac{1}{\lambda} \quad \text{or} \quad \frac{1}{\lambda^0_\mu} = \frac{1}{\lambda}.$$

Multiple Kelvin moduli appear. There exists a possibility of decomposition of eigen-subspaces.

If $\boldsymbol{\sigma}^\star$ is the eigen-state of $\mathbf{C}$ then there exists the eigen-supspace $\mathcal{P}_\mu$ so that $\boldsymbol{\sigma}^\star \in \mathcal{P}_\mu$. The subspace $\mathcal{P}_\mu$ is subjected to subdivision only if $\boldsymbol{\sigma}^\star \in \mathcal{P}^\star_\mu \subset \mathcal{P}_\mu$.

This type of restrictions may cause material symmetry to be higher or lower.

Volume-isotropic materials discussed by Burzyński in [3] are the example of materials with this type of internal restrictions. Burzyński assumed that for any material symmetry, the response of the body to the hydrostatic pressure is always restricted to the change of its volume (without the change of its shape)

$$\boldsymbol{\sigma}^\star = \frac{1}{\sqrt{3}}\mathbf{1}, \qquad \mathbf{C} \cdot \mathbf{1} = \frac{1}{\lambda}\mathbf{1}. \tag{3.15}$$

This kind of restriction is called the Burzyński restriction.

Another example could be obtained by assuming that **any** stress state given in a diagonal form in the symmetry axes for orthotropic material is the eigen-state

$$\bigwedge_{\boldsymbol{\sigma}_i \in R, i=1,2,3} \quad \boldsymbol{\sigma}^\star = \frac{1}{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} \sum_i^3 \sigma_i \mathbf{e}_i \otimes \mathbf{e}_i, \qquad \mathbf{C} \cdot \boldsymbol{\sigma}^\star = \frac{1}{\lambda}\boldsymbol{\sigma}^\star.$$

In this case one has to do with arbitrary triaxial tension or compression along the symmetry axes. It is then obtained that

$$\frac{1}{\lambda_1}=\frac{1}{\lambda_2}=\frac{1}{\lambda_3}=\frac{1}{\lambda}$$

where λ_i correspond to eigen-states with the principal directions coaxial with the orthotropy axes [1]. Number of the Kelvin moduli is reduced to 4. One of them is triple. Values of distributors are in this case not significant.

3.3. Enforced strains

The third type of restrictions is described by (3.3). Note that this relation includes also the types analyzed in previous subsections. If $\widetilde{\boldsymbol{\varepsilon}}=\mathbf{0}$ then the case analyzed in Sec 3.1 is obtained, and when $\widetilde{\boldsymbol{\varepsilon}}=(1/\lambda)\,\widetilde{\boldsymbol{\sigma}}$ then the one analyzed in Sec. 3.2 appears.

From spectral decomposition (2.11) for the restrictions (3.3) the equality follows

$$\frac{1}{\lambda_I}\mathbf{P}_I\cdot\widetilde{\boldsymbol{\sigma}}+\ldots+\frac{1}{\lambda_\rho}\mathbf{P}_\rho\cdot\widetilde{\boldsymbol{\sigma}}=\widetilde{\boldsymbol{\varepsilon}},$$

which can be rewritten in the form

$$\frac{1}{\lambda_I}\mathbf{P}_I\cdot\widetilde{\boldsymbol{\sigma}}+\ldots+\frac{1}{\lambda_\rho}\mathbf{P}_\rho\cdot\widetilde{\boldsymbol{\sigma}}=\mathbf{P}_I\cdot\widetilde{\boldsymbol{\varepsilon}}+\ldots+\mathbf{P}_\rho\cdot\widetilde{\boldsymbol{\varepsilon}}\tag{3.16}$$

due to (2.12) and $\mathcal{S}=\mathcal{P}_I\oplus\ldots\oplus\mathcal{P}_\rho$. The form of $\widetilde{\boldsymbol{\sigma}}$ as well as the type of material symmetry may cause that $\widetilde{\boldsymbol{\sigma}}$ will be not projected onto some subspaces. If

$$\mathbf{P}_K\cdot\widetilde{\boldsymbol{\sigma}}=\mathbf{0},\qquad \lambda_K>0,$$

then contracting Eq. (3.16) with the tensor $\mathbf{P}_K\cdot\widetilde{\boldsymbol{\varepsilon}}$, one obtains that

$$|\mathbf{P}_K\cdot\widetilde{\boldsymbol{\varepsilon}}|^2=0.$$

It implies that

$$\mathbf{P}_K\cdot\widetilde{\boldsymbol{\varepsilon}}=\mathbf{0}.$$

Consequently, *tensor $\widetilde{\boldsymbol{\varepsilon}}$ cannot be projected onto the subspaces $\mathcal{P}_K$ onto which $\widetilde{\boldsymbol{\sigma}}$ is not projected.*

On the other hand, if

$$\mathbf{P}_J\cdot\widetilde{\boldsymbol{\varepsilon}}=\mathbf{0},$$

then contracting Eq. (3.16) with the tensor $\mathbf{P}_J\cdot\widetilde{\boldsymbol{\sigma}}$ one obtains that

$$\frac{1}{\lambda_J}|\mathbf{P}_J\cdot\widetilde{\boldsymbol{\sigma}}|^2=0$$

and it is not necessary for $\mathbf{P}_J \cdot \widetilde{\boldsymbol{\sigma}}$ to be equal to zero (it is sufficient to assume $1/\lambda_J \to 0$). One may therefore omit in Eq. (3.16) projectors $\mathbf{P}_K$ that give zero-projections of $\widetilde{\boldsymbol{\sigma}}$ onto corresponding subspaces $\mathcal{P}_K$.

The following equality is then obtained

$$\frac{1}{\lambda_I}\mathbf{P}_I \cdot \widetilde{\boldsymbol{\sigma}} + \ldots + \frac{1}{\lambda_\nu}\mathbf{P}_\nu \cdot \widetilde{\boldsymbol{\sigma}} = \mathbf{P}_I \cdot \widetilde{\boldsymbol{\varepsilon}} + \ldots + \mathbf{P}_\nu \cdot \widetilde{\boldsymbol{\varepsilon}}, \quad (\nu \leq \rho).$$

At the right-hand side of the above equality, the elements that are equal to zero may appear. Let the $\chi < \nu$ elements be not equal to zero at the right-hand side of this equation. One may group them in the following way:

$$\frac{1}{\lambda_I}\mathbf{P}_I \cdot \widetilde{\boldsymbol{\sigma}} - \mathbf{P}_I \cdot \widetilde{\boldsymbol{\varepsilon}} + \ldots + \frac{1}{\lambda_\chi}\mathbf{P}_\chi \cdot \widetilde{\boldsymbol{\sigma}} - \mathbf{P}_\chi \cdot \widetilde{\boldsymbol{\varepsilon}} + \frac{1}{\lambda_{\chi+1}}\mathbf{P}_{\chi+1} \cdot \widetilde{\boldsymbol{\sigma}} + \ldots + \frac{1}{\lambda_\nu}\mathbf{P}_\nu \cdot \widetilde{\boldsymbol{\sigma}} = \mathbf{0}. \tag{3.17}$$

For $K > \chi$, similarly to (3.8), it is obtained that

$$\frac{1}{\lambda_K}|\mathbf{P}_K \cdot \widetilde{\boldsymbol{\sigma}}|^2 = \frac{1}{\lambda_K}|\widetilde{\boldsymbol{\sigma}}_K|^2 = 0.$$

Trivial solution is one of the form (3.9). It is the only solution for the subspaces $\mathcal{P}_K$ that are one-dimensional and do not depend on the distributors. In such a case these subspaces sum up and become one subspace of the passive stress states. If the projectors $\mathbf{P}_K$ depend on the distributors then fulfilling of the equality may require to adopt some restrictions imposed on the distributors and the Kelvin moduli. For the subspaces $\mathcal{P}_K$ of dimension $q_K \geq 2$ the situation considered in Sec 3.1 may take place. In this case $\widetilde{\boldsymbol{\sigma}}$ is projected onto the subspaces $\widetilde{\mathcal{P}}_K \subset \mathcal{P}_K$ and from the subspaces $\mathcal{P}_K$, the ones of passive stress states $\widetilde{\mathcal{P}}_K$ may be separated.

It remains to consider the case when $\widetilde{\boldsymbol{\sigma}}_K = \mathbf{P}_K \cdot \widetilde{\boldsymbol{\sigma}} \neq \mathbf{0}$ and $\widetilde{\boldsymbol{\varepsilon}}_K = \mathbf{P}_K \cdot \widetilde{\boldsymbol{\varepsilon}} \neq \mathbf{0}$.

Let the tensors $\widetilde{\boldsymbol{\sigma}}_\mu$ and $\widetilde{\boldsymbol{\varepsilon}}_\mu$ be the elements of the same eigen-subspace $\mathcal{P}_\mu$ ($\mu \in \langle I, \chi \rangle$). If the subspace dimension $\dim \mathcal{P}_\mu \geq 2$, then these tensors do not have to be proportional. Generally $\widetilde{\boldsymbol{\sigma}}_\mu \neq \eta_\mu \widetilde{\boldsymbol{\varepsilon}}_\mu$.

Projecting $\widetilde{\boldsymbol{\sigma}}_\mu$ onto the direction of $\widetilde{\boldsymbol{\varepsilon}}_\mu$ it is obtained that

$$\widetilde{\boldsymbol{\sigma}}_\mu = \widetilde{\boldsymbol{\sigma}}_\mu^{(1)} + \widetilde{\boldsymbol{\sigma}}_\mu^{(2)},$$

where $\widetilde{\boldsymbol{\sigma}}_\mu^{(1)}, \widetilde{\boldsymbol{\sigma}}_\mu^{(2)} \in \mathcal{P}_\mu$ and

$$\widetilde{\boldsymbol{\sigma}}_\mu^{(1)} = |\widetilde{\boldsymbol{\sigma}}_\mu| \cos\varphi \frac{\widetilde{\boldsymbol{\varepsilon}}_\mu}{|\widetilde{\boldsymbol{\varepsilon}}_\mu|}, \tag{3.18}$$

$$\widetilde{\boldsymbol{\sigma}}_\mu^{(2)} = \widetilde{\boldsymbol{\sigma}}_\mu - |\widetilde{\boldsymbol{\sigma}}_\mu| \cos\varphi \frac{\widetilde{\boldsymbol{\varepsilon}}_\mu}{|\widetilde{\boldsymbol{\varepsilon}}_\mu|} \tag{3.19}$$

while φ is the angle between the tensors $\widetilde{\boldsymbol{\varepsilon}}_\mu$ and $\widetilde{\boldsymbol{\sigma}}_\mu$ in the space $\mathcal{S}$ (here the summation convention does not apply). From Eq. (3.18) it is obtained, that between tensors $\widetilde{\boldsymbol{\sigma}}_\mu^{(1)}$ and $\widetilde{\boldsymbol{\varepsilon}}_\mu$ proportionality is observed

$$\widetilde{\boldsymbol{\sigma}}_\mu^{(1)} = \eta_\mu \widetilde{\boldsymbol{\varepsilon}}_\mu, \quad \text{where} \quad \eta_\mu = \frac{|\widetilde{\boldsymbol{\sigma}}_\mu|}{|\widetilde{\boldsymbol{\varepsilon}}_\mu|}\cos\varphi. \tag{3.20}$$

Consequently, Eq. (3.17) may be rewritten as

$$\left(\frac{1}{\lambda_I} - \frac{1}{\eta_I}\right)\widetilde{\boldsymbol{\sigma}}_I^{(1)} + \frac{1}{\lambda_I}\widetilde{\boldsymbol{\sigma}}_I^{(2)} + \ldots + \left(\frac{1}{\lambda_\chi} - \frac{1}{\eta_\chi}\right)\widetilde{\boldsymbol{\sigma}}_\chi^{(1)} + \frac{1}{\lambda_\chi}\widetilde{\boldsymbol{\sigma}}_\chi^{(2)} = \mathbf{0}. \tag{3.21}$$

If there exists such a subspace $\mathcal{P}_\nu$ that

$$\widetilde{\boldsymbol{\sigma}}_\nu = \eta_\nu \widetilde{\boldsymbol{\varepsilon}}_\nu, \tag{3.22}$$

then $\widetilde{\boldsymbol{\sigma}}_\nu^{(1)} = \widetilde{\boldsymbol{\sigma}}_\nu$ and in Eq. (3.21) the element $(1/\lambda_\nu)\,\widetilde{\boldsymbol{\sigma}}_\nu^{(2)}$ disappears.

In such a case from the orthogonality of the eigen-subspaces it can be concluded that

$$\frac{1}{\lambda_\nu} = \frac{1}{\eta_\nu} \tag{3.23}$$

and the restrictions for the Kelvin moduli are obtained.

On the other hand, if proportionality (3.22) is not observed, then the subspace $\mathcal{P}_\nu$ is decomposed into two subspaces $\mathcal{P}_\nu^{(1)}$ and $\mathcal{P}_\nu^{(2)}$ with the corresponding Kelvin moduli equal to

$$\frac{1}{\lambda_\nu^{(1)}} = \frac{1}{\eta_\nu}, \qquad \frac{1}{\lambda_\nu^{(2)}} \to 0 \tag{3.24}$$

and $\dim \mathcal{P}_\nu^{(1)} = \dim \widetilde{\boldsymbol{\varepsilon}}_\nu$ while $\dim \mathcal{P}_\nu^{(2)} = \dim \mathcal{P}_\nu - \dim \mathcal{P}_\nu^{(1)}$. In this way, from the subspace $\mathcal{P}_\nu$ the subspace of passive stress states $\mathcal{P}_\nu^{(2)}$ is separated. It should be noted that also in this case there exists a possibility to separate from $\mathcal{P}_\nu^{(2)}$ the subspace onto which $\widetilde{\boldsymbol{\sigma}}_\nu^{(2)}$ is not projected.

If there exist l subspaces that are decomposed into two subspaces then the subspace of passive stress states is the sum of subspaces $\mathcal{P}_\nu^{(2)}$, where $\nu = 1, \ldots, l$. From ρ subspaces $\mathcal{P}_1, \ldots, \mathcal{P}_\rho$ we obtain at most $\rho + 1$ subspaces, where $\mathcal{P}_{\rho+1}$ is the subspace of the passive stress states.

Restrictions of the form (3.3) introduce conditions (3.23) imposed on the Kelvin moduli, where (see (3.20))

$$\eta_\nu = \frac{|\widetilde{\boldsymbol{\sigma}}_\nu|}{|\widetilde{\boldsymbol{\varepsilon}}_\nu|}\cos\varphi.$$

For $\varphi = 0$ the proportionality (3.14) is observed.

It seems that the conducted analysis includes all possible restrictions imposed on the deformation of linear elastic anisotropic bodies. It should be stressed that taking into account the given types of restrictions, one has to minimize the requirements imposed on the material symmetry. The restrictions considered in this section will now be discussed in examples for the selected types of material symmetry.

4. Examples

4.1. Volume-isotropic materials

Example of the restrictions analyzed in Sec. 3.2 are Burzyński's postulates (3.15). Let us consider these relations for all symmetry groups. Let us start from a fully anisotropic material. In this case the tensor $\boldsymbol{\sigma}^{\star} = 1/\sqrt{3}\,\mathbf{I}$ is projected onto all eigen-subspaces among which each subspace depends on distributors. Because these subspaces are one-dimensional, projectors are defined by Eqs. (2.5). Consequently, Eq. (3.14), after substituting $\boldsymbol{\sigma}^{\star}$ (3.15) takes the form:

$$\frac{1}{\sqrt{3}}\sum_{K=1}^{6}\left(\frac{1}{\lambda_K}-\frac{1}{\lambda}\right)(\mathrm{tr}\boldsymbol{\omega}_K)\boldsymbol{\omega}_K = \mathbf{0}.$$

Contracting the above equation with subsequent $\boldsymbol{\omega}_K$, the following six equations are obtained (no summation)

$$\frac{1}{\sqrt{3}}\left(\frac{1}{\lambda_K}-\frac{1}{\lambda}\right)\mathrm{tr}\boldsymbol{\omega}_K = 0, \quad (\mathrm{tr}\boldsymbol{\omega}_K^2 = 1), \quad (K = I,\ldots,VI), \tag{4.1}$$

where for $\mathrm{tr}\boldsymbol{\omega}_K$ the identity (2.14) should be fulfilled. Trivial solution of the above set of equations is obtained by adopting for all K that $1/\lambda_K = 1/\lambda$. However, there exists other solution that imposes restrictions on the distributors. Five out of six $\mathrm{tr}\boldsymbol{\omega}_K$ may be equal to zero and then[3)]

$$\mathrm{tr}\boldsymbol{\omega}_{VI} = \pm\sqrt{3}.$$

The last equation (4.1) will be fulfilled if

$$\frac{1}{\lambda_{VI}} = \frac{1}{\lambda}. \tag{4.2}$$

[3)] In view of the above considerations, other rule of ordering of the Kelvin moduli and orthogonal projectors than that proposed in Sec. 2.2 should be introduced, e.g. the one based on the magnitude of $\mathbf{I}\cdot\mathbf{P}_K\cdot\mathbf{I}$.

The eigen-state $\boldsymbol{\omega}_{VI}$ is then the normalized spherical tensor $\pm\boldsymbol{\sigma}^{\star}$. Further, if one assumes that the material is volumetrically incompressible, what means that constraints of the form (3.13) are prescribed, then

$$\frac{1}{\lambda_{VI}} \to 0. \tag{4.3}$$

It can be shown that the number of parameters describing a fully anisotropic elastic material with prescribed Burzyński's restrictions is reduced to 7 stiffness distributors and 6 Kelvin moduli for which Eq. (4.2) or (4.3) is fulfilled. Only $16 = 6 + 7 + 3$ out of 21 parameters are left.

In the case of monoclinic symmetry, the stress state $\boldsymbol{\sigma}^{\star}$ is not projected onto the eigen-subspaces $\mathcal{P}_I$ and $\mathcal{P}_{II}$ because two corresponding eigen-states are pure shears with common direction of shearing $\mathbf{e}_1$ [2]. By repeating the reasoning adopted for a fully anisotropic material one obtains non-trivial solution of the form

$$\mathrm{tr}\boldsymbol{\omega}_K = 0, \quad K = III, IV, V$$

and Eq. (4.2) (in the case of incompressibility constraints (4.3)). Also in this case the number of parameters that describe monoclinic material is reduced – 6 Kelvin moduli and 3 stiffness distributors are left. It should be noted that in this case, a trivial solution is obtained by adopting

$$\frac{1}{\lambda_K} = \frac{1}{\lambda} \quad \text{where} \quad K = III, IV, V, VI.$$

Considering Burzyński's restrictions for subsequent types of material symmetry one obtains the results presented in Table 3. It should be stressed that in the analyzed case, only for four types of symmetry one has to do with non-zero number of independent stiffness distributors.

Table 3. First and second structural indices for all types of symmetry of linear elastic materials with Burzyński's restrictions.

Symmetry group	First structural index	Second structural index	number of parameters
full anisotropy	$\langle 1+(1+1+1+1+1)\rangle$	$[6+7+3]$	16
monoclinic symmetry	$\langle 1+(1+1+1)+1+1\rangle$	$[6+3+3]$	12
orthotropy	$\langle 1+(1+1)+1+1+1\rangle$	$[6+1+3]$	10
trigonal symmetry	$\langle 1+1+(2+2)\rangle$	$[4+1+3]$	8
tetragonal symmetry	$\langle 1+1+1+1+2\rangle$	$[5+0+3]$	8
transversal symmetry	$\langle 1+1+2+2\rangle$	$[4+0+2]$	6
cubic symmetry	$\langle 1+2+3\rangle$	$[3+0+3]$	6
isotropy	$\langle 1+5\rangle$	$[2+0+0]$	2

4.2. Fiber-reinforced materials

Let us consider other type of restrictions (3.2), that is restrictions of the form

$$\mathbf{C}\cdot\mathbf{k}\otimes\mathbf{k}=\frac{1}{\eta}\mathbf{k}\otimes\mathbf{k}. \tag{4.4}$$

Using spectral decomposition of the compliance tensor $\mathbf{C}$ (2.4), (2.11) and contracting it with diad $\mathbf{k}\otimes\mathbf{k}$, the following equality is obtained (see (3.14))

$$\left(\frac{1}{\lambda_I}-\frac{1}{\eta}\right)\omega_I^{(kk)}\boldsymbol{\omega}_I+\ldots+\left(\frac{1}{\lambda_{VI}}-\frac{1}{\eta}\right)\omega_{VI}^{(kk)}\boldsymbol{\omega}_{VI}=\mathbf{0},$$

where $\boldsymbol{\omega}_J^{(kk)}=\mathbf{k}\cdot\boldsymbol{\omega}_J\cdot\mathbf{k}$. Contracting further this equation with subsequent $\boldsymbol{\omega}_J$, the following six scalar equations are derived

$$\left(\frac{1}{\lambda_J}-\frac{1}{\eta}\right)\omega_J^{(kk)}=0,\quad J=I,\ldots,VI$$

that could be fulfilled if

$$\omega_J^{(kk)}=0\quad\text{or}\quad\frac{1}{\lambda_J}=\frac{1}{\eta}. \tag{4.5}$$

Similarly to the identity (2.14), from (2.6) it follows that

$$(\mathbf{k}\otimes\mathbf{k})\cdot\mathbb{I}_S\cdot(\mathbf{k}\otimes\mathbf{k})=(\omega_I^{(kk)})^2+\ldots+(\omega_{VI}^{(kk)})^2=1.$$

Therefore at most 5 out of 6 $\omega_J^{(kk)}$ parameters can be equal to zero. Conditions $(4.5)_1$ reduce the number of independent components of eigen-states $\boldsymbol{\omega}_J$.

Let us look at the influence of restrictions (4.4) on the number of independent distributors for subsequent symmetry groups starting from the material of monoclinic symmetry. In each analyzed case it is assumed that the direction $\mathbf{k}$ is coaxial with $\mathbf{e}_1$ (see Table 1), and then $\omega_J^{(kk)}=\omega_{11}^J$. In such a case, the diad $\mathbf{k}\otimes\mathbf{k}$ is not projected onto the subspaces $\mathcal{P}_I$ and $\mathcal{P}_{II}$ of pure shears. Therefore one obtains four equations of the form (4.5) for $J=III,IV,V,VI$. Stiffening of the material is minimized if it is assumed that

$$\omega_{11}^{III}=\omega_{11}^{IV}=\omega_{11}^{V}=0\quad\text{and}\quad\frac{1}{\lambda_{VI}}=\frac{1}{\eta}. \tag{4.6}$$

Because for this type of material symmetry eigen-states $\boldsymbol{\omega}_J$ have one common principal direction coaxial with $\mathbf{e}_1$ ($\boldsymbol{\omega}_{11}^K$ is therefore the corresponding eigenvalue, [2]) Eqs. (4.6) cause that eigen-states $\boldsymbol{\omega}_J$ are plane states ($\det\boldsymbol{\omega}_J=0$) and $\boldsymbol{\omega}_{VI}=\pm\mathbf{k}\otimes\mathbf{k}$, so that $\mathrm{tr}\boldsymbol{\omega}_{VI}=\pm 1$. Using identity (2.14) one obtains

$$(\mathrm{tr}\boldsymbol{\omega}_{III})^2+(\mathrm{tr}\boldsymbol{\omega}_{IV})^2+(\mathrm{tr}\boldsymbol{\omega}_{V})^2=2.$$

It may be shown that in such a case only three independent distributors are left.

One can follow similar reasoning for orthotropic materials as well as materials of trigonal, tetragonal and transversal symmetry. In all these cases diad $\mathbf{k}\otimes\mathbf{k} = \mathbf{e}_1\otimes\mathbf{e}_1$ is projected only onto one-dimensional eigen-subspaces. Structural indices for the analyzed symmetry groups have the same form as those for Burzyński's restrictions.

4.3. Cubic material with the prescribed restrictions (3.3)

To illustrate restrictions of the type (3.3) let us consider the material of cubic symmetry for which the following relation between some strains and stresses is imposed:

$$\mathbf{C}\cdot(\mathbf{k}\otimes\mathbf{k}) = (a-b)(\mathbf{k}\otimes\mathbf{k}) + b\mathbf{1}. \tag{4.7}$$

The above relation ensures that an axi-symmetric cross-section of a strip subjected to tension or compression in $\mathbf{k}$ direction remains axi-symmetric for the elastic regime. The ratio between the elongation ε_{kk} in $\mathbf{k}$-direction and the change of cross-sectional diameter ε_{rr} is given by

$$\frac{\varepsilon_{rr}}{\varepsilon_{kk}} = \frac{b}{a}.$$

Note that if $b = 0$, the relation of the form (4.4) is obtained (in such a case $\varepsilon_{rr} = 0$).

If direction $\mathbf{k}$ is coaxial with one of the edges of a cubic cell, e.g. $\mathbf{e}_1$, then diad [4]

$$\mathbf{k}\otimes\mathbf{k} = \mathbf{e}_1\otimes\mathbf{e}_1 \sim \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix} \quad \text{in} \quad \{\mathbf{e}_k\}$$

is projected only onto the subspaces $\mathcal{P}_I$ and $\mathcal{P}_{II}$ (see spectral decomposition of tensor $\mathbf{C}$ for material of cubic symmetry in [9]). From spectral decomposition (2.11) and Eq. (4.7) it is obtained that

$$\left(\frac{1}{\lambda_I} - (a-b)\right)\mathbf{P}_I\cdot(\mathbf{k}\otimes\mathbf{k}) - b\mathbf{P}_I\cdot\mathbf{1} + \left(\frac{1}{\lambda_{II}} - (a-b)\right)\mathbf{P}_{II}\cdot(\mathbf{k}\otimes\mathbf{k}) = \mathbf{0}. \tag{4.8}$$

Because

$$\mathbf{P}_I\cdot(\mathbf{k}\otimes\mathbf{k}) = \frac{1}{3}\mathbf{1}, \qquad \mathbf{P}_I\cdot\mathbf{1} = \mathbf{1},$$

[4] Symbol $\sim$ denotes that the given matrix is a representation of the considered tensor in the given basis.

Eq. (4.8) takes the form

$$\frac{1}{3}\left(\frac{1}{\lambda_I}-(a+2b)\right)\mathbf{1}+\left(\frac{1}{\lambda_{II}}-(a-b)\right)\mathbf{P}_{II}\cdot(\mathbf{k}\otimes\mathbf{k})=\mathbf{0}.$$

Projectors $\mathbf{P}_I$ and $\mathbf{P}_{II}$ are independent of distributors, so one of the possible solutions is to assume that

$$\frac{1}{\lambda_I}=a+2b,\qquad \frac{1}{\lambda_{II}}=a-b. \tag{4.9}$$

Such a solution does not change the material symmetry, therefore condition (4.7) is also fulfilled for $\mathbf{k}=\mathbf{e}_2$ and $\mathbf{k}=\mathbf{e}_3$.

Subspace $\mathcal{P}_{II}$ is two-dimensional, while diad $\mathbf{k}\otimes\mathbf{k}$ is one-dimensional. Consequently, there exists a possibility to separate from $\mathcal{P}_{II}$ some subspace $\mathcal{P}_{II}^{\perp}$ onto which this diad is not projected. One obtains

$$\mathbf{P}_{II}^{\perp}=\mathbf{P}_{II}-\frac{1}{|\mathbf{P}_{II}\cdot(\mathbf{k}\otimes\mathbf{k})|^2}(\mathbf{P}_{II}\cdot(\mathbf{k}\otimes\mathbf{k}))\otimes(\mathbf{P}_{II}\cdot(\mathbf{k}\otimes\mathbf{k}))=\mathbf{P}_{II}-\mathbf{P}_{II}^{\star}$$

where

$$\mathbf{P}_{II}\cdot(\mathbf{k}\otimes\mathbf{k})=\mathbf{P}_{II}^{\star}\cdot(\mathbf{k}\otimes\mathbf{k})=\mathbf{k}\otimes\mathbf{k}-\frac{1}{3}\mathbf{1}.$$

Equation (4.8) can be rewritten in the form

$$\left(\frac{1}{\lambda_I}-(a-b)\right)\mathbf{P}_I\cdot(\mathbf{k}\otimes\mathbf{k})-b\mathbf{P}_I\cdot\mathbf{1}+\left(\frac{1}{\lambda_{II}^{\star}}-(a-b)\right)\mathbf{P}_{II}^{\star}\cdot(\mathbf{k}\otimes\mathbf{k})=\mathbf{0}. \tag{4.10}$$

After contracting the above equation subsequently with $\mathbf{P}_I\cdot(\mathbf{k}\otimes\mathbf{k})$ and $\mathbf{P}_{II}^{\star}\cdot(\mathbf{k}\otimes\mathbf{k})$, it is obtained

$$\left(\frac{1}{\lambda_I}-(a-b)\right)\frac{1}{3}-b=0,\qquad \left(\frac{1}{\lambda_{II}^{\star}}-(a-b)\right)\frac{2}{3}=0. \tag{4.11}$$

As a solution one obtains

$$\frac{1}{\lambda_I}=a+2b,\qquad \frac{1}{\lambda_{II}^{\star}}=a-b.$$

Consequently, spectral decomposition of $\mathbf{C}$ for the material of cubic symmetry with the analyzed restrictions has the form

$$\mathbf{C}^{\star}=(a+2b)\mathbf{P}_I+(a-b)\mathbf{P}_{II}^{\star}+\frac{1}{\lambda_{II}}\mathbf{P}_{II}^{\perp}+\frac{1}{\lambda_{III}}\mathbf{P}_{III}, \tag{4.12}$$

where $\dim \mathcal{P}_I = \dim \mathcal{P}^\star_{II} = \dim \mathcal{P}^\perp_{II} = 1$ and $\dim \mathcal{P}_{III} = 3$. Elements of one-dimensional space $\mathcal{P}^\perp_{II}$ are as follows:

$$\boldsymbol{\omega}^\perp_{II} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & \boldsymbol{\omega} & 0 \\ 0 & 0 & -\boldsymbol{\omega} \end{pmatrix} \quad \text{in} \quad \{\mathbf{e}_k\}.$$

It should be stressed that symmetry group of the material is not longer a cubic symmetry group, because due to condition (2.16)

$$\mathcal{Q}_{\mathrm{mat}} = \mathcal{Q}_{\mathbf{P}_I} \cap \mathcal{Q}_{\mathbf{P}^\star_{II}} \cap \mathcal{Q}_{\mathbf{P}^\perp_{II}} \cap \mathcal{Q}_{\mathbf{P}_{III}} = \mathcal{Q} \cap \mathcal{Q}^{ti}_{\mathbf{e}_1} \cap \mathcal{Q}^{4t}_{\mathbf{e}_1} \cap \mathcal{Q}^{c}_{\mathbf{e}_i} = \mathcal{Q}^{4t}_{\mathbf{e}_1}$$

and the material possesses only tetragonal symmetry where the privileged direction is $\mathbf{e}_1 = \mathbf{k}$. The independent distributor $\aleph_1$ distinctive for tetragonal symmetry [8] is in this case equal to zero. This type of material symmetry is determined by the least symmetric projector $\mathbf{P}^\perp_{II}$. From Eq. (4.12) it can be concluded that if $b = 0$, subspaces $\mathcal{P}_I$ and $\mathcal{P}^\star_{II}$ sum up, while if $a = b$, subspace $\mathcal{P}^\star_{II}$ becomes the subspace of passive stress states.

An interesting solution is also obtained if one assumes that direction $\mathbf{k}$ is coaxial with one of the main diagonals of a cubic cell, for example

$$\mathbf{k} = \frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \Rightarrow \mathbf{k} \otimes \mathbf{k} \sim \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{in} \quad \{\mathbf{e}_k\}.$$

This is, as well as the direction $\mathbf{e}_1$, a direction of extremal Young modulus [9]. In this case diad $\mathbf{k} \otimes \mathbf{k}$ is projected only onto subspaces $\mathcal{P}_I$ and $\mathcal{P}_{III}$. For restrictions (4.7) from spectral decomposition of $\mathbf{C}$ one obtains that

$$\left(\frac{1}{\lambda_I} - (a-b)\right) \mathbf{P}_I \cdot (\mathbf{k} \otimes \mathbf{k}) - b\mathbf{P}_I \cdot \mathbf{1} + \left(\frac{1}{\lambda_{III}} - (a-b)\right) \mathbf{P}_{III} \cdot (\mathbf{k} \otimes \mathbf{k}) = \mathbf{0}. \tag{4.13}$$

One of the possible solutions of this equation is obtained, similarly to (4.9), by assuming that

$$\frac{1}{\lambda_I} = a + 2b, \qquad \frac{1}{\lambda_{III}} = a - b.$$

Such a solution does not change material symmetry, therefore conditions (4.7) are fulfilled also for $\mathbf{k}$ coaxial with any main diagonal of a cubic cell.

Another solution is obtained by decomposition of the three-dimensional subspace $\mathcal{P}_{III}$. Diad $\mathbf{k} \otimes \mathbf{k}$ is not projected onto a two-dimensional subspace defined by the projector

$$\mathbf{P}^\perp_{III} = \mathbf{P}_{III} - \frac{1}{|\mathbf{P}_{III} \cdot (\mathbf{k} \otimes \mathbf{k})|^2} (\mathbf{P}_{III} \cdot (\mathbf{k} \otimes \mathbf{k})) \otimes (\mathbf{P}_{III} \cdot (\mathbf{k} \otimes \mathbf{k})) = \mathbf{P}_{III} - \mathbf{P}^\star_{III}$$

where

$$\mathbf{P}_{III} \cdot (\mathbf{k} \otimes \mathbf{k}) = \mathbf{P}^{\star}_{III} \cdot (\mathbf{k} \otimes \mathbf{k}) = \mathbf{k} \otimes \mathbf{k} - \frac{1}{3}\mathbf{1}.$$

Equation (4.13) can be written in the form

$$\left(\frac{1}{\lambda_I} - (a-b)\right) \mathbf{P}_I \cdot (\mathbf{k} \otimes \mathbf{k}) - b\mathbf{P}_I \cdot \mathbf{1} + \left(\frac{1}{\lambda^{\star}_{III}} - (a-b)\right) \mathbf{P}^{\star}_{III} \cdot (\mathbf{k} \otimes \mathbf{k}) = \mathbf{0}.$$

By finding a solution in the same way as for (4.10), it is obtained

$$\frac{1}{\lambda_I} = a + 2b, \qquad \frac{1}{\lambda^{\star}_{III}} = a - b.$$

Spectral decomposition of the tensor $\mathbf{C}$ with the analyzed restrictions takes the form

$$\mathbf{C}^{\star} = (a+2b)\mathbf{P}_I + \frac{1}{\lambda_{II}}\mathbf{P}_{II} + (a-b)\mathbf{P}^{\star}_{III} + \frac{1}{\lambda_{III}}\mathbf{P}^{\perp}_{III},$$

where $\dim \mathcal{P}_I = \dim \mathcal{P}^{\star}_{III} = 1$ and $\dim \mathcal{P}_{II} = \dim \mathcal{P}^{\perp}_{III} = 2$. Elements of the subspace $\mathcal{P}^{\perp}_{III}$ are as follows:

$$\boldsymbol{\omega}^{\perp}_{III} \sim \begin{pmatrix} 0 & u & v \\ u & 0 & -(u+v) \\ v & -(u+v) & 0 \end{pmatrix} \quad \text{in} \quad \{\mathbf{e}_k\}.$$

Also in this case, the material symmetry group has changed. Note that

$$\mathcal{Q}_{\text{mat}} = \mathcal{Q}_{\mathbf{P}_I} \cap \mathcal{Q}_{\mathbf{P}_{II}} \cap \mathcal{Q}_{\mathbf{P}^{\star}_{III}} \cap \mathcal{Q}_{\mathbf{P}^{\perp}_{III}} = \mathcal{Q} \cap \mathcal{Q}^{\mathbf{e}_i}_k \cap \mathcal{Q}^{\mathbf{k}}_t \cap \mathcal{Q}^{\mathbf{k}}_{3t} = \mathcal{Q}^{\mathbf{k}}_{3t},$$

so the material has trigonal symmetry and the symmetry axis is coaxial with $\mathbf{k}$. Here, two independent distributors $\aleph_1$ and $\aleph_2$ distinctive for this type of symmetry take the values specified by (compare page 157 in [12])

$$\tan\kappa = \sqrt{2}, \qquad \tan\rho = \sqrt{2}.$$

Again, the type of material symmetry is determined by the least symmetric projector $\mathbf{P}^{\perp}_{III}$.

When $\mathbf{k}$ will be coaxial with any other direction in a cubic cell, the diad $\mathbf{k} \otimes \mathbf{k}$ will be projected on all subspaces $\mathcal{P}_K$ and it may further change the material symmetry.

5. Summary

Thorough analysis conducted in the paper allows to conclude about the important role played by the theorem on spectral decomposition of elasticity tensors. Especially, it is visible during examining the influence of internal restrictions on the properties of anisotropic linear elastic bodies. Prescribing some deformation modes imposes additional conditions on the values of the stiffness distributors and the form of eigen-subspaces.

Three types of internal restrictions imposed on the form of deformation of the body were considered. Algorithms, that optimize the influence of these restrictions on the elastic properties of the material for calculating the Kelvin moduli, stiffness distributors and eigen-subspaces, were proposed.

It was shown that equations of restrictions could be also fulfilled by imposing the conditions on Kelvin moduli only. These solutions were called trivial, because they lead to over-stiffening of the body. Therefore, one should look for the solutions by imposing conditions on the stiffness distributors and the form of eigen-subspaces, sometimes by decomposing some subspaces.

Bounds imposed on the deformation modes may not change the material symmetry (then they are only imposed on the material constants) or they may make it higher or lower.

The proposed algorithms were applied to analyze the influence of Burzyński's restriction (volumetric deformation is then enforced only by hydrostatic pressure – volume isotropic materials notation may be used in this case) on the elastic properties of the material and to describe the behaviour of fiber-reinforced materials.

Appendix A

The space $\mathcal{S}$ of symmetric second order tensors possesses all the properties of the six-dimensional Euclidean space with the scalar product defined as follows:

$$\bigwedge_{\mathbf{a},\mathbf{b}\in\mathcal{S}} \qquad \mathbf{a}\cdot\mathbf{b} = \mathrm{tr}(\mathbf{ab}) = a_{ij}b_{ij},$$

where a_{ij}, b_{ij}, $i,j = 1,2,3$ are components of tensors $\mathbf{a}$ and $\mathbf{b}$ in some orthonormal basis $\{\mathbf{e}_i\}$ in the three-dimensional physical space. Therefore, any second-order tensor has all the properties of the vector in the six-dimensional Euclidean space.

Due to this property of $\mathcal{S}$ it is possible to select in $\mathcal{S}$ a subset of six mutually orthogonal and normalized tensors $\{\mathbf{a}_K\}$, $K = I,\ldots,VI$ which constitute the basis. One of the possible bases is the following orthonormal subset of basis diads $\{\mathbf{e}_i\otimes\mathbf{e}_j\}$ of the form:

$$\mathbf{a}_I = \mathbf{e}_1 \otimes \mathbf{e}_1 \qquad \mathbf{a}_{IV} = \frac{1}{\sqrt{2}}(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2),$$

$$\mathbf{a}_{II} = \mathbf{e}_2 \otimes \mathbf{e}_2, \qquad \mathbf{a}_{V} = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1),$$

$$\mathbf{a}_{III} = \mathbf{e}_3 \otimes \mathbf{e}_3, \qquad \mathbf{a}_{VI} = \frac{1}{\sqrt{2}}(\mathbf{e}_2 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_2).$$

A basis in the six-dimensional space is called a polybasis. In the above polybasis, any symmetrical tensor of the second order is described in the following way

$$\mathbf{a} = a_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = a_K \mathbf{a}_K, \qquad K = I, \ldots, VI, \qquad \text{where} \qquad \mathbf{a} \cdot \mathbf{b} = a_K b_K$$

and interdependence between representations a_{ij} and a_K is given by

$$\begin{aligned} &a_I = a_{11}, &&a_{II} = a_{22}, &&a_{III} = a_{33},\\ &a_{IV} = \sqrt{2}a_{23}, &&a_V = \sqrt{2}a_{13}, &&a_{VI} = \sqrt{2}a_{12}. \end{aligned} \tag{A.1}$$

Consequently, linear projection from the space $\mathcal{S}$ onto $\mathcal{S}$ treated as the six-dimensional Euclidean space will be described by the second-order tensor belonging to tensorial product $\mathcal{S} \otimes \mathcal{S}$. This reasoning brings us to the conclusion that the fourth-order tensor $\mathbf{A}$ that describes projection from the space of second-order symmetric tensors to the space of the second-order symmetric tensors in the three-dimensional physical space possesses all the properties of the second-order tensor in the six-dimensional Euclidean space. One may therefore write

$$\mathbf{A} = A_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l = A_{KL}\mathbf{a}_K \otimes \mathbf{a}_L.$$

The set of all basis diads $\{\mathbf{a}_I \otimes \mathbf{a}_J\}$ is the basis in the space $\mathcal{S} \otimes \mathcal{S}$. Components A_{KL} depend on components A_{ijkl} of the fourth order tensor $\mathbf{A}$ in the basis $\{\mathbf{e}_i\}$ in the physical space, in the following way:

$$[A_{KL}] = \begin{pmatrix} A_{1111} & A_{1122} & A_{1133} & \sqrt{2}A_{1123} & \sqrt{2}A_{1113} & \sqrt{2}A_{1112} \\ A_{2211} & A_{2222} & A_{2233} & \sqrt{2}A_{2223} & \sqrt{2}A_{2213} & \sqrt{2}A_{2212} \\ A_{3311} & A_{3322} & A_{3333} & \sqrt{2}A_{3323} & \sqrt{2}A_{3313} & \sqrt{2}A_{3312} \\ \sqrt{2}A_{2311} & \sqrt{2}A_{2322} & \sqrt{2}A_{2333} & 2A_{2323} & 2A_{2313} & 2A_{2312} \\ \sqrt{2}A_{1311} & \sqrt{2}A_{1322} & \sqrt{2}A_{1333} & 2A_{1323} & 2A_{1313} & 2A_{1312} \\ \sqrt{2}A_{1211} & \sqrt{2}A_{1222} & \sqrt{2}A_{1233} & 2A_{1223} & 2A_{1213} & 2A_{1212} \end{pmatrix}. \tag{A.2}$$

The following products can be obtained in two alternative but fully corresponding ways ($\mathbf{a},\mathbf{b} \in \mathcal{S}$; $\mathbf{A},\mathbf{B},\mathbf{C} \in \mathcal{S}\otimes\mathcal{S}$):

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= a_{ij}b_{ij} = a_K b_K, \\ \mathbf{b} = \mathbf{A}\cdot\mathbf{a} &\Leftrightarrow b_{ij} = A_{ijkl}a_{kl} \quad \text{or} \quad b_K = A_{KL}a_L, \\ \mathbf{D} = \mathbf{A}\circ\mathbf{B} &\Leftrightarrow D_{ijkl} = A_{ijmn}B_{mnkl} \quad \text{or} \quad D_{KL} = A_{KM}B_{ML}, \end{aligned}$$

where $a_{ij}, b_{ij}, A_{ijkl}, B_{ijkl}, D_{ijlk}$ and $a_K, b_K, A_{KL}, B_{KL}, D_{KL}$ are related by Eqs. (A.1) and (A.2).

It should be stressed that, due to the fact that the tensor **A** describes linear projection between spaces of the symmetric second-order tensors, one obtains $A_{ijkl} = A_{jikl} = A_{ijlk}$. Note that in the case of the stiffness tensor **S** and the compliance tensor **C**, additionally one has to do with the situation when $A_{KL} = A_{LK}$ ($A_{ijkl} = A_{klij}$). Matrix of the components $[A_{KL}]_{6\times 6}$ is then symmetric.

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