

Influence of asymptotic corrector on the description of the dynamic behaviour of a circular elastic plate^{*)}

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THE AIM of the present paper is to study an initial-boundary value problem for an isotropic circular plate. General theory was developed by the authors in [12]. The novelty consists in the study of influence of second-order terms (*correctors*), obtained by variational-asymptotic analysis, on the dynamic solution of plates.

1. Introduction

THEORY OF PLATES has already a long history and is still an extensive field of research, cf. [5, 6] and the references therein. Refined theory of plates can be derived either with the use of asymptotic expansions [1, 2, 4, 9–12] or by assuming suitable displacements and/or stress hypotheses [3, 5–8]. By using the asymptotic method combined with variational approach, a new dynamic model of elastic orthotropic plates, clamped at the whole boundary, was derived in [12]. This model accounts for rotational inertia and involves the second order terms $\mathbf{U}^{(2)}$ and $\boldsymbol{\sigma}^{(2)}$ of the asymptotic expansions of displacements and stresses. These terms are called (*first*) *correctors*. They improve the 2-D classical solutions of plate problems. The solution that takes into account *asymptotic correctors* is a better approximation of a 3-D solution than the classical one [9, 12].

The main aim of the present contribution is to apply the general results achieved in our paper [12] to a specific case of an initial-boundary value problem for a circular isotropic plate. An influence of second-order terms on dynamic response of such a plate is revealed.

To facilitate the reading of the paper, in Appendix are gathered the basic results pertaining to the general isotropic case.

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2. Formulation of the problem

We shall analyse an initial-boundary value problem for an elastic clamped circular plate in the case of isotropy and rotational symmetry. Let $\Omega \subset \mathcal{R}^2$ be the mid-plane of the plate with the boundary Γ . By $2h$ we denote the thickness of the plate, R is its radius, ϱ is the density, E is the Young modulus and ν is the Poisson ratio of the material of the plate. We shall carry out the analysis in the polar coordinate system (r, θ) , $r \in (0, R)$, $\theta \in (0, 2\pi)$. Because of the rotational symmetry of the considered dynamic problem, the displacements and stresses will depend on r and time $t \in (0, T)$ only, where T is prescribed.

We assume that uniformly distributed loading of the plate, $g_3^+ = g_3$, is suddenly imposed on the upper face (see the Appendix). In this case of loading the r.h.s. of Eq. (A.19) simplifies considerably. For the mid-plane deflection of the real plate, $u_3(r, t)$, we have the following formula, accounting for an *asymptotic corrector* [9, 12],

$$(2.1) \quad u_3(r, t) = u_3^0(r, t) + h^2 u_3^{(2)}(r, t).$$

By using the two general problems (\mathcal{P}^0) and $(\mathcal{P}^2 f)$, given in the Appendix, we formulate (P^h) that is the initial-boundary value problem accounting for *asymptotic correctors*, for the midplane of the circular plate with radius R (and thickness $2h$):

$$(2.2) \quad 2\varrho h u_3'' + \frac{2}{3} \frac{E}{1-\nu^2} h^3 \Delta^2 u_3 = -2h g \varrho + g_3^+ + \varrho h^3 \frac{34-14\nu}{15(1-\nu)} \Delta u_3^{0''},$$

$$(2.3) \quad u_3 = -h^2 \frac{\nu}{10(1-\nu)} \Delta u_3^0 \quad \text{on } \Gamma \times (0, T), \quad (P^h)$$

$$(2.4) \quad \partial_n u_3 = -h^2 \frac{8+\nu}{10(1-\nu)} \partial_n \Delta u_3^0 \quad \text{on } \Gamma \times (0, T),$$

$$(2.5) \quad u_3(0, r) = \tilde{U}_3^0(r) + h^2 \tilde{u}_3^{(2)}(r), \quad u_3'(0, r) = 0.$$

Here $u_3^0 \in L^\infty(0, T; H_0^2(\Omega))$ is a solution to

$$(2.6) \quad 2h \varrho u_3^{0''} + \frac{2}{3} \frac{E}{1-\nu^2} h^3 \Delta^2 u_3^0 = -2h g \varrho + g_3^+ \quad \text{in } \Omega \times (0, T),$$

$$(2.7) \quad u_3^0(0, r) = \tilde{U}_3^0(r), \quad u_3^{0'}(0, r) = 0 \quad \text{in } \Omega,$$

$$(2.8) \quad u_3^0(R, t) = 0 \quad \text{for } t \in (0, T), \quad \partial_n u_3^0(R, t) = 0 \quad \text{for } t \in (0, T),$$

where g is the acceleration of gravity, $(\cdot)' = \frac{\partial(\cdot)}{\partial t}$, $t \in (0, T)$; ∂_n denotes the normal derivative, whereas \mathbf{n} is the external unit normal to Γ , and Δ is the Laplacian in the polar coordinates: $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$.

REMARK 1. Equation (2.2) follows from (A.19) if $u_3^{(2)}$ denoted by Eq. (2.1) is substituted to it. Similarly, in this way we get Eqs. (2.3)–(2.5) using (A.20)–(A.21).

In this simple case of loading the in-plane components of the displacement in the plate considered vanish. Referring to the Appendix, we see that in such a case we have $u_\alpha^0 = 0$, and since $[K \mathbf{u}^{(2)}, \mathbf{v}] = 0$, we conclude that $u_\alpha^{(2)} = 0$, $\alpha = 1, 2$.

3. Problem (\mathcal{P}^0)

According to the Appendix we solve problem (2.2)–(2.8) in two steps. First, problem (\mathcal{P}^0) described by Eqs. (2.6)–(2.8) is solved. To find a solution to Eqs. (2.6)–(2.8) we first consider the homogeneous differential equation [7, 8, 12]:

$$(3.1) \quad D \Delta^2 u_3^0 + 2 \varrho h u_3^{0''} = 0, \quad D = \frac{2}{3} \frac{Eh^3}{(1-\nu^2)},$$

that describes free vibrations of the plate. For the harmonic motion we set:

$$(3.2) \quad u_3^0(r, t) = \mathcal{Z}_3^0(r) \sin(pt).$$

Equation (3.1) can be written as follows:

$$(3.3) \quad (\Delta + \beta^2)(\Delta - \beta^2)\mathcal{Z}_3^0 = 0,$$

where

$$(3.4) \quad \beta^2 = p \sqrt{\frac{2\varrho h}{D}}.$$

The solution of Eq. (3.3) for the case of axisymmetric vibrations takes the form, cf. [7, 8],

$$(3.5) \quad \mathcal{Z}_0(r) = A_0 J_0(\beta r) + B_0 I_0(\beta r),$$

where J_0, I_0 , are Bessel functions of zero-order and modified Bessel functions of zero-order, respectively. Taking into account boundary conditions (2.8), one obtains the frequency equation in the form:

$$(3.6) \quad J_0(\beta R) I_1(\beta R) + I_0(\beta R) J_1(\beta R) = 0.$$

The first three roots of this transcendental equation, i.e. the eigenvalues of the considered problem, described by (3.1) and (2.7)–(2.8), are

$$(3.7) \quad \beta_0 R = 3.196; \quad \beta_1 R = 6.306; \quad \beta_2 R = 9.439; \dots$$

For frequencies p_i we have the formula, cf. Eq. (3.4),

$$(3.8) \quad p_i = \frac{(\beta_i R)^2}{R^2} \sqrt{\frac{D}{2\rho h}}$$

and the corresponding eigenfunctions $\mathcal{Z}_i(r)$. Performing the orthonormalization with respect to a mass density distribution (what follows from the definition of the kinetic energy of the deformable body)

$$(3.9) \quad \iint_{\Omega} 2 h \rho \mathcal{Z}_i(r) \mathcal{Z}_j(r) d\Omega = \delta_{ij},$$

one gets the expression for the eigenfunctions in the form:

$$(3.10) \quad \mathcal{Z}_i(r) = \frac{1}{R} \sqrt{\frac{1}{4\pi\rho h}} \left[\frac{J_0(\beta_i r)}{J_0(\beta_i R)} - \frac{I_0(\beta_i r)}{I_0(\beta_i R)} \right].$$

Being the orthonormal system in the Hilbert space of square-integrable functions, defined on the middle plane of the plate, functions $\mathcal{Z}_i(r)$, $i = 1, 2, \dots$, form a complete system. Indeed, one can notice that the only function orthogonal to every $\mathcal{Z}_i(r)$ is the function assuming only zero value. So, coming back to the problem of forced vibrations (2.6)–(2.8) we express its solution by means of the modal series

$$(3.11) \quad u_3^0 = \sum_i \mathcal{Z}_i(r) \cdot \eta_i(t).$$

For the function $\eta_i(t)$ (using the orthonormality condition) we have the equation

$$(3.12) \quad \eta_i''(t) + p_i^2 \eta_i(t) = F_i(t),$$

where

$$(3.13) \quad F_j(t) = \iint_{\Omega} g_3(r, t) \mathcal{Z}_j(r) d\Omega,$$

whereas $d\Omega = r dr d\theta$.

For the conditions given by Eq. (2.7) in the case when $u_3^0(0, r) = 0$ we get

$$(3.14) \quad \eta_i(t) = \frac{1}{p_i} \int_0^t F_i(\tau) \sin p_i(t - \tau) d\tau.$$

For our subsequent analysis the dynamic loading is assumed:

$$(3.15) \quad g_3(r, t) = H(t) \cdot g_3(r); \quad g_3(r) = g_3 = \text{const},$$

where $H(t)$ denotes the Heaviside function.

Then the generalized force $F_j(t)$ is given by

$$(3.16) \quad F_j(t) = \frac{2\pi R H(t) g_3}{\beta_j R} \sqrt{\frac{1}{4\pi \rho h}} \left[\frac{J_1(\beta_j R)}{J_0(\beta_j R)} - \frac{I_1(\beta_j R)}{I_0(\beta_j R)} \right].$$

It is now possible to derive the expression for $\eta_j(t)$. Finally, returning to Eqs. (3.10), (3.11) and (3.14) we get the formula describing the dynamic deflection $u_3^0(r, t)$ of the plate:

$$(3.17) \quad u_3^0(r, t) = \frac{g_3 R^4}{D} \sum_{i=1}^{\infty} \frac{1}{(\beta_i R)^5} \left[\frac{J_1(\beta_i R)}{J_0(\beta_i R)} - \frac{I_1(\beta_i R)}{I_0(\beta_i R)} \right] \cdot \left[\frac{J_0(\beta_i r)}{J_0(\beta_i R)} - \frac{I_0(\beta_i r)}{I_0(\beta_i R)} \right] (1 - \cos(p_i t)).$$

REMARK 2. Figure 1 shows very rapid convergence of the series in the r.h.s. of Eq. (3.17). Consequently, in our subsequent analysis only the first term of that series is taken into account.

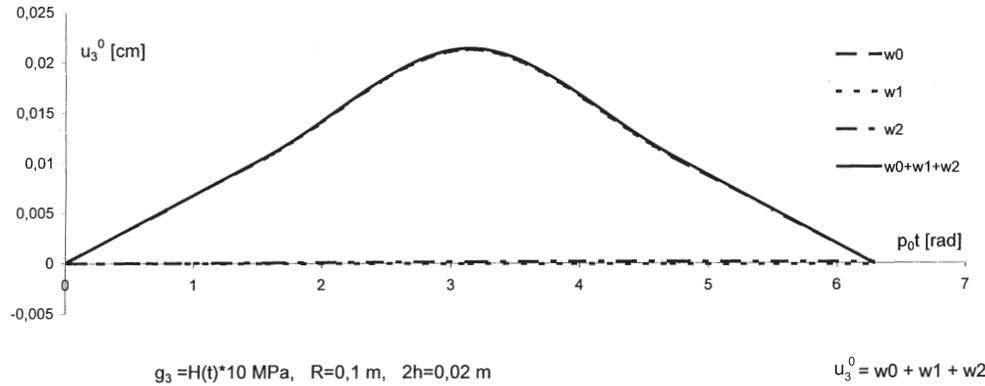


FIG. 1. The deflection of the circular plate middle-point not accounting for the *corrector*.

4. Problem (\mathcal{P}^h)

In our paper [12] we have formulated the dynamic problem for real plate with the thickness $2h$ (after rescaling) accounting for the first *asymptotic corrector*. In this paper it is given by Eqs. (2.2)–(2.8).

Having solved problem (\mathcal{P}^0) we can proceed to the second step of our analysis, i.e. to problem (2.2)–(2.5). Further we will neglect the weight of the plate. We observe that this problem can be transformed to the problem with homogeneous boundary conditions. Indeed, setting

$$(4.1) \quad \begin{aligned} \widehat{u}_3(r, t) &= u_3(r, t) - w_s(r, t), \\ w_s(r, t) &= u_3(R, t) - \frac{1}{2R} \partial_n u_3(R, t) (R^2 - r^2), \end{aligned}$$

problem (2.2)–(2.5) is transformed to the following initial-boundary value problem:

$$(4.2) \quad 2 \varrho h \widehat{u}_3'' + D \Delta^2 \widehat{u}_3 = g_3 + \varrho h^3 c(\nu) \Delta u_3^{0''} - 2 \varrho h w_s'' \quad \text{in } \Omega \times (0, T),$$

$$(4.3) \quad \widehat{u}_3 = 0 \quad \text{for } r = R, \quad t \in (0, T),$$

$$(4.4) \quad \partial_n \widehat{u}_3 = 0 \quad \text{for } r = R, \quad t \in (0, T),$$

$$(4.5) \quad \widehat{u}_3(0) = 0 \quad \text{and} \quad \widehat{u}_3'(0) = 0 \quad \text{in } \Omega,$$

where

$$(4.6) \quad c(\nu) = \frac{34 - 14\nu}{15(1 - \nu)}.$$

Now we can solve problem (4.2)–(4.5) performing the analysis similar to the one performed in Sec. 3. Note that in the r.h.s. of Eq. (4.2) we have a sum of three terms. Consequently, our solution u_3 will be the sum of solutions for these three terms. Having solved problem (\mathcal{P}^0) we derive the formula:

$$(4.7) \quad \begin{aligned} \Delta u_3^{0''}(r, t) &= -\frac{g_3 R^2}{D} \frac{p_0^2}{(\beta_0 R)^3} \left[\frac{J_1(\beta_0 R)}{J_0(\beta_0 R)} - \frac{I_1(\beta_0 R)}{I_0(\beta_0 R)} \right] \\ &\quad \cdot \left[\frac{J_0(\beta_0 r)}{J_0(\beta_0 R)} - \frac{I_0(\beta_0 r)}{I_0(\beta_0 R)} \right] \cos(p_0 t). \end{aligned}$$

Now we can find the quantities $u_3(R, t)$, $\partial_n u_3(R, t)$ and $w_s(r, t)$. In order to derive the formula for η_i (see (3.14)), for every term of the r.h.s. of Eq. (4.2), we calculate

$$(4.8) \quad \frac{1}{p_0} \int_0^t \cos(p_0 \tau) \sin p_0(t - \tau) d\tau = \frac{1}{2} \frac{t}{p_0} \sin(p_0 t).$$

REMARK 3. It is easy to notice here that the influence of *asymptotic corrector* on the plate deflection will increase with the time of the dynamic process. We can observe this fact in Figs. 2 and 3.

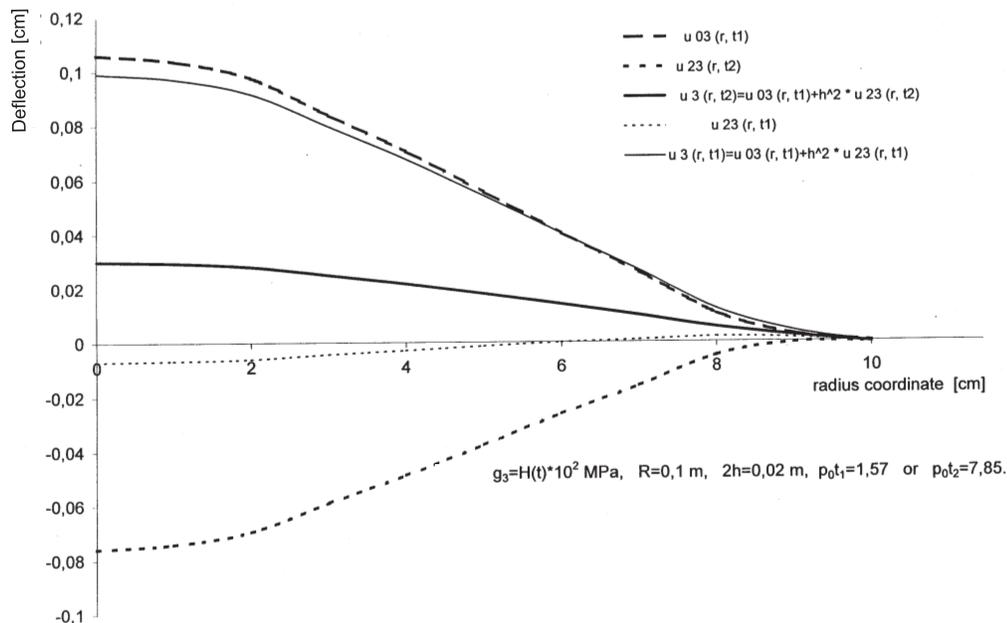


FIG. 2. The deflection of the circular plate accounting for the *asymptotic corrector* versus the radius.

Some numerical results are presented in Figs. 1 and 3. The calculations were performed for the following data: $E = 2.1 \cdot 10^5$ MPa, $\nu = 0.3$, $\rho = 7.8 \cdot 10^3$ kg/m³, $g_3 = H(t) \cdot 10$ MPa or $g_3 = H(t) \cdot 10^2$ MPa, $2h = 0.02$ m, $R = 0.2$ m or $R = 0.1$ m. Figure 1 clearly shows a dominating role of the first term in series (3.17) that represents u_3^0 , the solution of problem $(P^0 f)$. The convergence of this series is very rapid.

The notations appearing in the figures should be interpreted as follows:

$$u3 \equiv u_3, \quad u03 \equiv u_3^0, \quad u23 \equiv u_3^{(2)}, \quad u03 + h^2 u23 \equiv u_3^0 + h^2 u_3^{(2)}.$$

Figure 2 illustrates the influence of the corrector on the deflection u_3 (of the real plate with the thickness $2h$) in the case of two different time instants; the thin line refers to $p_0 t_1 = 1.57$ whilst the thick line to $p_0 t_2 = 7.85$. We observe the (increasing in time) influence of the *corrector*. It is also seen that the boundary conditions, calculated in problem (P^h) with the use of the assumptions of the X space, cf. (A.3), are represented by very small quantities:

$u_3(R, t) = -0.2815 \cdot 10^{-3}$ [cm]; $\partial_n u_3(R, t) = -0.2053 \cdot 10^{-2}$ [rad]. These numbers concern the case where $R = 0.1$ m and $g_3 = H(t) \cdot 10^2$ MPa.

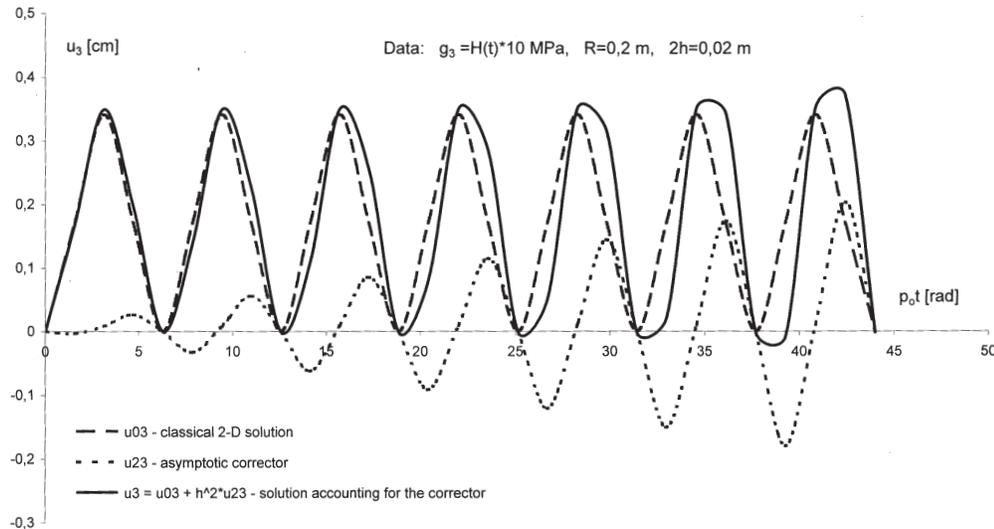


FIG. 3. The influence of the *asymptotic corrector* on the deflection of the plate middle point.

Figure 3 depicts the oscillating process of the deflection of the plate middle-point versus time. We observe that the harmonic vibrations (dashed line) are perturbed by the influence of the *corrector* (dotted line). The continuous line represents u_3 , the solution of problem (P^h) , i.e. the sum of these two graphs. We see that the influence of the *corrector* increases with time. Figure 3 shows the dynamic behaviour of the plate for which $2h/R = 1/10$; for this case $p_0 = 0.794 \cdot 10^4$ [rad/sec]. In Figs. 1 and 2 the ratio $2h/R = 1/5$; then $p_0 = 3.176 \cdot 10^4$ [rad/sec]. Let us notice that these graphs represent the first terms of the suitable series.

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5. Concluding remarks

Plate theories are two-dimensional ones and present only an approximation to exact three-dimensional solutions. In order to approximate the two-dimensional solution, one can include higher-order terms (*correctors*) of displacements and stresses. A general procedure for isotropic and orthotropic dynamic linear elastic plates involving *first-order correctors* was developed in [9, 12]. In the present paper we showed the influence of the displacement *corrector* on the dynamic

response of a circular plate. The analysis performed does not reveal the effect of the boundary layer. The last one was studied in [2] where the asymptotic expansion of displacements included also specific odd terms, responsible for the boundary layer phenomenon.

Appendix

To facilitate the reading of the paper, in the Appendix we gather basic results pertaining to the dynamic asymptotic model of a thin linear elastic plate made of an isotropic material. For details the reader is referred to [9, 12]. In our paper [12] the following notations are used: $\overline{\mathcal{B}}^\varepsilon = \overline{\Omega} \times [-\varepsilon, \varepsilon]$ is the region occupied by the undeformed plate and 2ε is the plate thickness. The boundary of the plate $\partial\mathcal{B}^\varepsilon = \Gamma_0^\varepsilon \cup \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon$, where $\Gamma_0^\varepsilon = \Gamma \times [-\varepsilon, \varepsilon]$, $\Gamma_\pm^\varepsilon = \Omega \times \{\pm\varepsilon\}$, $\Gamma = \partial\Omega$; Ω denotes the midplane of the plate. E, ν are isotropic material constants. The quantities σ_{ij} , \mathbf{U} and $\gamma_{ij}(\mathbf{U})$ denote the stress tensor, the displacement vector and the linearized strain tensor, respectively. Obviously, these quantities depend on the space coordinate \mathbf{x} and time t . Here $(\cdot)' = \partial(\cdot)/\partial t$, etc. The initial data ${}^\varepsilon\tilde{\mathbf{U}}, {}^\varepsilon\tilde{\mathbf{V}}$ are prescribed. ${}^\varepsilon\mathbf{f}, {}^\varepsilon\mathbf{g}$ are the body forces and the external surface forces.

Using the method of asymptotic expansion [1, 9, 11–12] it is convenient to work with the fixed domain, say $\mathcal{B} = \Omega \times (-1, 1)$. To this end, for $\varepsilon > 0$ we define the mapping, cf. [12],

$$(A.1) \quad F^\varepsilon : \mathbf{x} = (x^1, x^2, x^3) \in \mathcal{B} \longrightarrow F^\varepsilon(\mathbf{x}) = (x^1, x^2, \varepsilon x^3) = \mathbf{x}^\varepsilon \in \mathcal{B}^\varepsilon.$$

So, for the scaled displacement field \mathbf{U}^ε we have the relationship:

$$(A.2) \quad \mathbf{U}^\varepsilon = (U_1^\varepsilon, U_2^\varepsilon, U_3^\varepsilon) = (U_1, U_2, \varepsilon U_3).$$

Obviously, also other fields, i.e. stresses and loadings are scaled, cf. [9, 11–12]. The scaled stresses, body forces and loadings are called $\boldsymbol{\sigma}^\varepsilon$, \mathbf{f}^ε and \mathbf{g}^ε respectively.

Passing to the variational formulation of the problem and the asymptotic analysis, cf. [1, 9, 12], we introduce the spaces of stresses and displacements $\Sigma = L^2(\mathcal{B}, \mathbb{E}_s^3)$, $X = X_{12} \times X_3$, where

$$(A.3) \quad \begin{aligned} X_{12} &= \left\{ \mathbf{V} \in H^1(\mathcal{B})^2 : \int_{-1}^1 V_\alpha dx_3 = 0, \int_{-1}^1 x_3 V_\alpha n_\alpha dx_3 = 0 \text{ on } \Gamma \right\}, \\ X_3 &= \left\{ V_3 \in H^1(\mathcal{B}) : \int_{-1}^1 (1 - x_3^2) V_3 dx_3 = 0 \text{ on } \Gamma \right\}, \quad \alpha = 1, 2. \end{aligned}$$

The variational formulation of the isotropic plate dynamics (after scaling) has the form:

Problem $(\mathcal{P}^\varepsilon)$

Find $(\boldsymbol{\sigma}^\varepsilon, \mathbf{U}^\varepsilon) \in L^\infty(0, T; \Sigma \times X)$, such that $\mathbf{U}^{\varepsilon'} \in L^\infty(0, T; L^2(\mathcal{B})^3)$ and

$$(A.4) \quad \forall \boldsymbol{\tau} \in \Sigma, \quad A^\varepsilon(\boldsymbol{\sigma}^\varepsilon, \boldsymbol{\tau}) + B(\boldsymbol{\tau}, \mathbf{U}^\varepsilon) = 0,$$

$$(A.5) \quad \forall \mathbf{V} \in X, \quad -\varrho(U_3^{\varepsilon''}, V_3) - \varrho\varepsilon^2(U_\alpha^{\varepsilon''}, V_\alpha) + B(\boldsymbol{\sigma}^\varepsilon, \mathbf{V}) = F^0(\mathbf{V}), \quad (\mathcal{P}^\varepsilon)$$

$$(A.6) \quad \mathbf{U}^\varepsilon(\mathbf{x}, 0) = \tilde{\mathbf{U}}^\varepsilon(\mathbf{x}), \quad \mathbf{U}^{\varepsilon'}(\mathbf{x}, 0) = \tilde{\mathbf{V}}^\varepsilon(\mathbf{x}), \quad \mathbf{x} \in \mathcal{B}.$$

Here $(U_\alpha^{\varepsilon''}, V_\alpha) = \int_{\mathcal{B}} U_\alpha^{\varepsilon''} V_\alpha d\mathbf{x}$. The forms A^ε , B and F^0 are defined as follows:

$$\forall \{\boldsymbol{\sigma}, \boldsymbol{\tau}\} \in \Sigma \times \Sigma,$$

$$(A.7) \quad \begin{aligned} A^\varepsilon(\boldsymbol{\sigma}, \boldsymbol{\tau}) &= A^0(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \varepsilon^2 A^{(2)}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \varepsilon^4 A^{(4)}(\boldsymbol{\sigma}, \boldsymbol{\tau}), \\ A^0(\boldsymbol{\sigma}, \boldsymbol{\tau}) &= \frac{1+\nu}{E} (\sigma_{\alpha\beta}, \tau_{\alpha\beta}) - \frac{\nu}{E} (\sigma_{\mu\mu}, \tau_{\rho\rho}), \\ A^{(2)}(\boldsymbol{\sigma}, \boldsymbol{\tau}) &= 2\frac{1+\nu}{E} (\sigma_{\alpha 3}, \tau_{\alpha 3}) - \frac{\nu}{E} \left((\sigma_{33}, \tau_{\rho\rho}) + (\sigma_{\mu\mu}, \tau_{33}) \right), \end{aligned}$$

where (\cdot, \cdot) denotes the scalar product in $L^2(\mathcal{B})$;

$$(A.8) \quad B(\boldsymbol{\sigma}, \mathbf{V}) = - \int_{\mathcal{B}} \gamma_{ij}(\mathbf{V}) \sigma_{ij} d\mathbf{x}, \quad F^0(\mathbf{V}) = - \int_{\mathcal{B}} f_i^0 V^i d\mathbf{x} - \int_{\Gamma_\pm} g_i^0 V^i d\Gamma,$$

$$\forall \boldsymbol{\sigma} \in \Sigma, \quad \forall \mathbf{V} \in H^1(\mathcal{B})^3.$$

We assume the asymptotic expansions of $\{\boldsymbol{\sigma}^\varepsilon, \mathbf{U}^\varepsilon\}$ as follows:

$$(A.9) \quad \begin{aligned} \boldsymbol{\sigma}^\varepsilon &= \boldsymbol{\sigma}^0 + \varepsilon^2 \boldsymbol{\sigma}^{(2)} + \dots, \\ \mathbf{U}^\varepsilon &= \mathbf{U}^0 + \varepsilon^2 \mathbf{U}^{(2)} + \dots. \end{aligned}$$

Performing now the asymptotic analysis, i.e. substituting (A.9) into (A.4) and (A.5), we arrive at problem (\mathcal{P}^0) linked with $\{\boldsymbol{\sigma}^0, \mathbf{U}^0\}$ and problem (\mathcal{P}^2) , linked with $\{\boldsymbol{\sigma}^{(2)}, \mathbf{U}^{(2)}\}$, cf. [9, 12].

The solution $\{\boldsymbol{\sigma}^{(2)}, \mathbf{U}^{(2)}\}$ of problem (\mathcal{P}^2) yields the *first corrector* to $\{\boldsymbol{\sigma}^0, \mathbf{U}^0\}$. We limit ourselves to these two asymptotic problems. It is well-known that

$\mathbf{U}^{(2)}$ does not satisfy, in general, the homogeneous boundary condition on Γ_0^1 , [9, 11–12]. We observe that the boundary conditions involved in the X space (A.3) are satisfied only in an averaged sense. To proceed further we introduce two spaces

$$(A.10) \quad \begin{aligned} X_{KL} &= \{ \mathbf{U} \mid \gamma_{\alpha 3}(\mathbf{U}) = 0, \quad \gamma_{33}(\mathbf{U}) = 0 \}, \\ S &= \{ \boldsymbol{\sigma} \in \Sigma \mid \tau_{\alpha 3} = 0, \quad \tau_{33} = 0 \}. \end{aligned}$$

Problem ($\mathcal{P}^0 f$)

(I) For $U_3^0 \in L^\infty(0, T; H_0^2(\Omega))$, $U_3^{0'} \in L^\infty(0, T; L^2(\Omega))$ we get

$$(A.11) \quad 2\rho U_3^{0''} + \frac{2}{3} \frac{E}{1-\nu^2} \Delta^2 U_3^0 = \int_{-1}^1 f_3^0 dx_3 + g_3^{0+} + g_3^{0-} + \partial_\alpha (g_\alpha^{0+} - g_\alpha^{0-});$$

$$(A.12) \quad U_\alpha^0 = u_\alpha^0 - x_3 \partial_\alpha U_3^0,$$

$$(A.13) \quad U_3^0(0) = \tilde{U}_3^0, \quad U_3^{0'}(0) = \tilde{V}_3^0.$$

(II) For $\mathbf{u}^0 \in L^\infty(0, T; H_0^1(\Omega)^2)$, we have

$$(A.14) \quad [K \mathbf{u}^0, \mathbf{v}] = F^0(-\mathbf{v}, 0), \quad \forall \mathbf{v} \in L^\infty(0, T; H_0^1(\Omega)^2),$$

$$(A.15) \quad [K \mathbf{u}, \mathbf{v}] := \frac{2E}{1-\nu^2} \left[(1-\nu) \gamma_{\alpha\beta}(\mathbf{u}) + \nu \gamma_{\mu\mu}(\mathbf{u}) \delta_{\alpha\beta} + \gamma_{\alpha\beta}(\mathbf{v}) \right],$$

$$\alpha, \beta, \lambda = 1, 2,$$

where $[\cdot, \cdot]$ denotes the scalar product in $L^2(\Omega)$.

Problem ($\mathcal{P}^2 f$)

From the general formulation of ($\mathcal{P}^2 f$), cf. [12], one can derive expressions for the displacement vector $\mathbf{U}^{(2)} \in L^\infty(0, T; X)$:

$$(A.16) \quad \begin{aligned} U_\alpha^{(2)} &= u_\alpha^{(2)} - x_3 \partial_\alpha u_3^{(2)} + \frac{1}{2} \frac{\nu}{1-\nu} x_3^2 \partial_\alpha \gamma_{\mu\mu}(\mathbf{u}^0) \\ &\quad - \frac{1}{1-\nu} \partial_\alpha \Delta u_3^0 \left[x_3 + \frac{1}{3} x_3^3 \left(\frac{\nu}{2} - 1 \right) \right], \quad u_3^0 \equiv U_3^0, \end{aligned}$$

$$(A.17) \quad U_3^{(2)} = u_3^{(2)} - \frac{\nu}{1-\nu} x_3 \gamma_{\mu\mu}(\mathbf{u}^0) + \frac{1}{2} \frac{\nu}{1-\nu} x_3^2 \Delta u_3^0.$$

(III) For $\mathbf{u}^{(2)} \in L^\infty(0, t; H^1(\Omega))^2$ we obtain the in-plane equation (in $\Omega \times (0, T)$) with boundary condition

$$(A.18) \quad K\mathbf{u}^{(2)} = -2\rho\mathbf{u}^{0''} + \frac{1}{3} \frac{E\nu}{(1-\nu^2)(1-\nu)} \text{grad } \Delta\gamma_{\mu\mu}(\mathbf{u}^0) + \frac{\nu}{1-\nu} \text{grad} \int_{-1}^1 \sigma_{33}^0 d\zeta,$$

$$u_\alpha^{(2)} = -\frac{\nu}{6(1-\nu)} \partial_\alpha \gamma_{\mu\mu}(\mathbf{u}^0) \quad \text{on } \Gamma \times (0, T).$$

(IV) For $u_3^{(2)} \in L^\infty(0, T; H^2(\Omega))$ we derive the plate bending equation (in $\Omega \times (0, T)$) with boundary and initial conditions:

$$(A.19) \quad 2\rho u_3^{(2)''} + \frac{2}{3} \frac{E}{1-\nu^2} \Delta^2 u_3^{(2)} = \frac{34-14\nu}{15(1-\nu)} \rho \Delta u_3^{0''} - \frac{\nu}{1-\nu} \Delta \int_{-1}^1 (x_3 \int_{-1}^{x_3} f_3 dx_3) + \frac{3\nu-8}{10(1-\nu)} \Delta \left[\int_{-1}^1 f_3^o dx_3 + g_3^{o+} + g_3^{o-} + \partial_\alpha (g_\alpha^{o+} - g_\alpha^{o-}) \right],$$

$$(A.20) \quad u_3^{(2)} = -\frac{\nu}{10(1-\nu)} \Delta u_3^0 \quad \text{on } \Gamma \times (0, T),$$

$$(A.21) \quad \partial_n u_3^{(2)} = -\frac{8+\nu}{10(1-\nu)} \partial_n \Delta u_3^0 \quad \text{on } \Gamma \times (0, T),$$

$$(A.22) \quad u_3^{(2)}(0) = \tilde{u}_3^{(2)}, \quad u_3^{(2)'}(0) = \tilde{v}_3^{(2)}, \quad \mathbf{x} \in \Omega.$$

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