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Specific work of strain as a measure of material effort

Prof. M.T. HUBER (1872–1950)

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under scientific supervision of Ryszard B. Pęcherski)

I. The most general objective of the science of strength of materials can be recapitulated in an answer to the question: **What external forces cause in a given solid body (or a system of solids) a danger of fracture of a determined degree?**

It is beyond doubt that such a risk depends, above all, **on the state of stress** in a considered body, i.e. the whole of internal forces (tractions) generated by external forces. The limit of the state of stress, going beyond which must result in the body fracture, describes in the most general way the **strength** of the body. Hence, the general task of the theory of strength of materials splits into two parts:

1. To determine the stress state induced by given external forces,
2. To find the dependence of fracture upon the state of stress.

Neither of the problems has an exact general solution as yet; however, in a particular number of simple but important particular cases we can, with the help of approximate theories, obtain results that are precise enough for practical purposes. In complex cases, though, the inaccuracy of the theory comes to light and then we resort to direct and costly experiments.

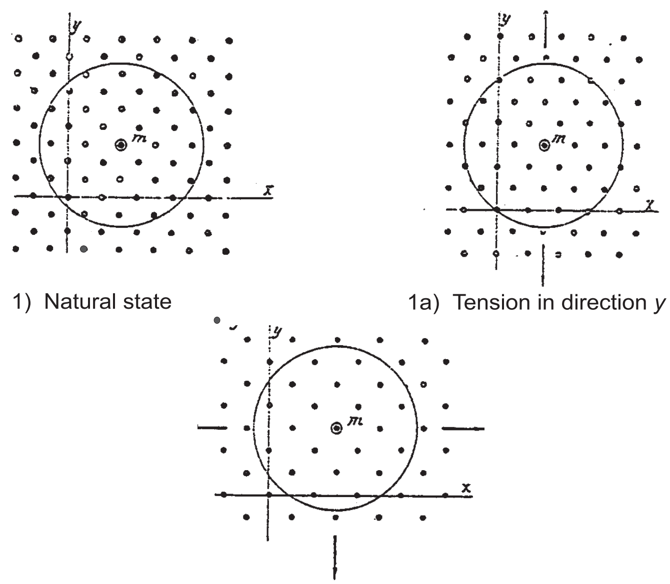
II. The first above-mentioned part of the general task of the theory of strength of materials simplifies remarkably when the **strain** accompanying a stress state in a body obeys a generalised form of HOOKE'S law, which is accepted as a basis of the mathematical theory of elasticity. Since the most significant solid bodies satisfy, within certain limits, either exactly or approximately, HOOKE's law, then the stress state determined with the help of the theory of elasticity usually solves the first part of the problem with an accuracy sufficient for practice, especially since, as a rule, we do not allow the elastic limit to be exceeded. However, there exist bodies significant in engineering practice (e.g. cast iron, stone) that show considerable deviations from Hooke's law within the limits of elasticity, as a result of which it was attempted recently to obtain greater accuracy within the theory of elasticity for those bodies by the use of empirical formulae that replace Hooke's law; but, the applicability of such formulae in the theory is, due to great mathematical difficulties, as yet rather limited.

III. Only in the simplest, though extremely important cases of **uniform tension or compression** (linear state of stress) of a perfectly homogenous and non-crystalline body *(i.e. the body that is not a monocrystal; however, ordered solids, e.g. polycrystals, have been not excluded by the Author – from translator)* do we have an exact answer to the second principal question (p. 2.) of the theory. Then the risk of fracture depends only on the magnitude of the stress, and the strength of the material can be defined by a constant that is characteristic of the material and gives the value of stress beyond which the body fractures. This constant is called, not quite properly, the coefficient of ultimate strength in tension or compression[2)].

In the beginnings of the theory of strength of materials, this constant was applied to a general state of stress due to the assumption that a risk of fracture occurs always where the stress has a maximum value. It can be justified only in certain cases of simple bending or torsion; in general, however, it leads to a contradiction since it appears even in the distinct example of uniform hydrostatic pressure acting on an isotropic body. In this very case there is no reason even for a maximum value of pressure to cause destruction of material cohesion (fracture of the body) and the sole result of such a stress state can only be a permanent volume strain. This fact has been confirmed directly or indirectly by experiments done by, among others, Föppl in München (*Mitteilungen aus d. mech.–techn. Labor. der k. techn. Hochschule in München. XXVII Heft, 1900, S.6*)[3)].

IV. Coulomb, and then Tresca, held a different opinion on the above question. According to them, a risk of fracture or **material effort** (*die Anstrengung*) is measured by the greatest change in an angle between two cross-sections of the body, produced by strain. This view does not contradict the experience of the above-mentioned case of hydrostatic pressure, for then strain changes the geometrical structure of the body into a similar one and, as a result, the angles do not undergo a change. It cannot be reconciled however, with the obviousness that a sufficiently large all-around tension may, with an arbitrarily small change of angle, cause fracture.

Poncelet and de Saint-Venant gave a third answer, claiming that material effort is measured by a unit elongation (or respectively contraction) λ, which means that a risk of fracture occurs when λ reaches a certain value characteristic of the material. This view, spread in the German technical literature by GRASHOF and WINKLER, has had the greatest number of followers until the present day, even though there have appeared new views[4)] in this regard in recent years. By all appearances it even seems difficult to doubt it, as a maximum elongation should result in a maximum separation of particles lying in the direction of this elongation, that is the greatest risk of their mutual moving apart beyond the range of the intermolecular attraction. Having given more thought to this question, however, I came to the conviction, with the help of the schematic picture of molecular distribution in a strained body (Figs. 1, 1a, 1b), that not only the separation of particles lying in the direction of the maximum elongation affects the risk of fracture but also so does a change of a distance between particles lying in all directions passing through the considered point of the body (that is elongation in all directions). A glance at the picture is sufficient to learn that with the same elongations in the y and z directions (z is perpendicular to the plane of the picture) a risk of fracture is smaller in the case depicted in Fig. 1a than in the one shown in Fig. 1b, since there are more molecules in the range of interaction of molecule m in the first case than in the second one[5)].



1) Natural state

1a) Tension in direction *y*

1b) Tension in directions *x* and *y*

FIG. 1.

V. The above elementary consideration argues against the hypothesis of PONCELET and DE SAINT-VENANT, leading at the same time to a new view on the principal question of the theory of strength of materials, which can be expressed in the following words:

The risk of fracture (material effort) at a certain point of a body is defined by the totality of the unit elongations in all directions that pass through the said point; or shorter: strain of the element of the body defines its material effort.

To determine the material effort of the element of the body it is required in general to give six independent quantities that describe a uniform strain[6)], i.e. three elongations λ_x, λ_y, λ_z and three angles of shear ϕ_x, ϕ_y, ϕ_z in the directions of Cartesian coordinate axes[7)]. If x, y, z have the same directions as principal elongations $\lambda_1, \lambda_2, \lambda_3$ then $\lambda_1 = \lambda_x$, $\lambda_2 = \lambda_y$, $\lambda_3 = \lambda_z$, $\phi_x = 0$, $\phi_y = 0$, $\phi_z = 0$, **which means that three principal elongations are sufficient to define material effort** (in uniform strain). Thus, a certain function Φ of principal elongations or elongations and angles of shear in arbitrary, mutually perpendicular directions would be a measure of material effort. Since every state of strain is related to a uniquely defined state of stress of the said element, Φ is also a function of three principal stresses ν_1, ν_2, ν_3 or respectively normal ν_x, ν_y, ν_z and shear σ_x, σ_y, σ_z components of the stress state in the considered point of the body[8)]. Moreover, we can say the following about the function Φ:

1. The function Φ increases and decreases with the respective absolute value of its arguments λ_1, λ_2, λ_3; in particular, it is equal to zero when $\lambda_1 = \lambda_2 = \lambda_3 = 0$.
2. Parameters of the function Φ depend only on the structure of matter in a natural state of the body and temperature; thus, for a homogenous structure of matter and constant temperature they are **constants** specific to the given body[9)].
3. For isotropic bodies, which I am exclusively considering in the present paper, the function Φ will of course be symmetric in its arguments.

VI. It is not difficult to notice that the above properties of the function Φ are identical to the properties of **specific work of strain** F, which, as it depends on states of stress and strain, is also a function of arguments λ_x, λ_y, λ_z; ϕ_x, ϕ_y, ϕ_z, or equivalently, ν_x, ν_y, ν_z; σ_x, σ_y, σ_z. This work is equal to the work performed by intermolecular forces acting between molecules contained in a unit volume of a body (measured in a natural state) when strain (a homogenous one) is sufficiently slow[10)]. Since, additionally, the work of intermolecular forces is the greater the larger are the relative displacements of the molecules in strain a very probable hypothesis that function Φ has the same shape as F suggests itself. In other words:

Material effort is measured by specific work of strain. Then, if in a certain point of the body work of strain goes beyond a determined value depending on the material (in a constant temperature), a permanent separation of molecules of the body, that is its fracture, must occur.

The above hypothesis is in an apparent contradiction to the discussed fact that a uniform hydrostatic pressure cannot cause fracture in a homogenous body. Then, however, in all probability the work of strain cannot exceed the said determined value at all, as it is difficult to imagine that shrinking of the body would not have natural limits beyond which even the greatest pressure would not increase the work of strain.

VII. To determine the shape of the function Φ it is, of course, required to know some general law that defines the dependence of the stress state upon strain. We do not know such a law as yet, but within the limits specific to each material HOOKE's law replaces it with better or worse accuracy. Within these limits, the function F will be then a so-called **elastic potential** given by the equation:

$$F = \frac{E}{2(1+\mu)} \left\{ \frac{1-\mu}{1-2\mu} \left(\lambda_x + \lambda_y + \lambda_z \right)^2 + \frac{1}{2} \left[\varphi_x^2 + \varphi_y^2 + \varphi_z^2 - 4\left(\lambda_x\lambda_y + \lambda_y\lambda_z + \lambda_z\lambda_x \right) \right] \right\}, \tag{1}$$

in which E is a coefficient of elasticity (the YOUNG modulus) and $\mu = \dfrac{1}{m}$ is POISSON's ratio[11].

Having expressed, with the help of the following formulas:

$$\begin{aligned} \lambda_x &= \frac{1}{E}\left[\nu_x - \mu\left(\nu_y + \nu_z\right)\right], \quad & \varphi_x &= \frac{2(1+\mu)}{E}\sigma_x, \\ \lambda_y &= \frac{1}{E}\left[\nu_y - \mu\left(\nu_z + \nu_x\right)\right], \quad & \varphi_y &= \frac{2(1+\mu)}{E}\sigma_y, \\ \lambda_z &= \frac{1}{E}\left[\nu_z - \mu\left(\nu_x + \nu_y\right)\right], \quad & \varphi_z &= \frac{2(1+\mu)}{E}\sigma_z, \end{aligned} \tag{2}$$

the strain components by the components of the stress state, we get another form of the function F,

$$F = \frac{1}{E} \left\{ \frac{1}{2} \left(\nu_x + \nu_y + \nu_z \right)^2 + (1+\mu) \left[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \left(\nu_x\nu_y + \nu_y\nu_z + \nu_z\nu_x \right) \right] \right\}, \tag{3}$$

which transforms further into

$$F = \frac{1}{E} \left\{ \frac{1}{2} \left(\nu_x + \nu_y + \nu_z \right)^2 - (1+\mu) \left(\nu_x\nu_y + \nu_y\nu_z + \nu_z\nu_x \right) \right\}, \tag{3a}$$ [12]

when x,y,z have the directions of principal stresses.

The function F in the above forms can define material effort only within the limits of elasticity (in accordance with Hooke's law). In general, then, using it for exact determination of the direct risk of fracture is out of the question. However, as a rule in engineering practice we do not allow strains to exceed the elastic limit[13)]; thus, we are not so much concerned with the risk of fracture as **with the risk of exceeding the elastic limit, which in all probability is also measured by specific work of strain, i.e. the function F.**

Having expressed a complement to the new hypothesis in this way one should give some thought to the conditions of its applicability. Namely, taking into consideration two states of stress with components equal as to their absolute value but of opposite signs, we come to the same value of the function F for both states. Therefore, for F being an exact measure of a risk of exceeding the elastic limit, the material must have the same elastic limit for both states. This condition is certainly identical to the condition that a material has equal limits of elasticity for tension and compression. In case when the condition is not satisfied for a given material, function F will define the risk of going beyond the elastic limit only with the help of additional hypotheses.

VIII. The consideration of a schematic distribution of particles in a strained body, which served above primarily to motivate the new view on the dependence of material effort upon the state of stress, has not, as I have already stressed, pretensions to scientific precision. I have expressed this fact by naming the view a 'hypothesis', and at present I am turning towards a thorough study of those of its consequences that are comparable with experiment, which is the last resort to seal the fate of every hypothesis.

In a linear state of stress, defined with only one principal stress, e.g. ν, it will be:

$$F = \frac{\nu^2}{2E}. \tag{4}$$

From the comparison of this value of F with the same magnitude in a general state of stress, there results the following relation:

$$\nu = \sqrt{(\nu_1 + \nu_2 + \nu_3)^{\,2} - 2\,(1+\mu)\,(\nu_1\nu_2 + \nu_2\nu_3 + \nu_1\nu_3)}, \tag{5}$$

which defines a simple tension or compression ν that produces the same work of strain, that is the same material effort, as the three-dimensional state of stress with components ν_1, ν_2, ν_3, or, equivalently ν_x, ν_y, ν_z, σ_x, σ_y, σ_z. The stress ν can of course serve as a convenient measure of material effort, in accordance with the presently prevailing view[14)]. An adequate name for it will be the **reduced stress** ν_{red}.

Thus,

$$\nu_{\text{red}}^2 = (\nu_1+\nu_2+\nu_3)^2 - 2\,(1+\mu)\;(\nu_1\nu_2+\nu_2\nu_3+\nu_1\nu_3) = \nu_1^2+\nu_1^2+\nu_1^2 - 2\mu\,(\nu_1\nu_2+\nu_2\nu_3+\nu_1\nu_3) \tag{6}$$

or

$$\nu_{\text{red}}^2 = (\nu_x+\nu_y+\nu_z)^2 + 2\,(1+\mu)\left[\sigma_x^2+\sigma_y^2+\sigma_z^2-(\nu_x\nu_y+\nu_y\nu_z+\nu_z\nu_x)\right] = \nu_x^2+\nu_y^2+\nu_z^2+2(1+\mu)\left(\sigma_x^2+\sigma_y^2+\sigma_z^2\right)-2\mu\,(\nu_x\nu_y+\nu_y\nu_z+\nu_z\nu_x)\,. \tag{7}$$

At present the reduced stress is calculated (according to the maximum elongation hypothesis) by the following formula:

$$\nu_{\text{red}}^* = \nu_I - \mu\,(\nu_{II}-\nu_{III})\,, \tag{a}$$

in which ν_I, ν_{II}, ν_{III} are principal stresses taken in an order such that ν_{red} is maximum. As a result of this reservation, formula (a) is less convenient in use than formula (6). This difference speaks even more clearly in favour of formula (6) when the state of stress is given by general components ν_x, ν_y, ν_z, σ_x, σ_y, σ_z. For then principal stresses are, as we know, roots of a cubic equation

$$\begin{vmatrix} \nu_x-\nu & \sigma_z & \sigma_y \\ \sigma_z & \nu_y-\nu & \sigma_x \\ \sigma_y & \sigma_x & \nu_z-\nu \end{vmatrix} = 0^{15)}, \tag{8}$$

that is

$$\nu^3 - (\nu_x+\nu_y+\nu_z)\;\nu^2 + \left(\nu_y\nu_z+\nu_z\nu_x+\nu_x\nu_y-\sigma_x^2-\sigma_y^2-\sigma_z^2\right)\;\nu - \begin{vmatrix} \nu_x & \sigma_z & \sigma_y \\ \sigma_z & \nu_y & \sigma_x \\ \sigma_y & \sigma_x & \nu_z \end{vmatrix} = 0, \tag{8a}$$

which has to be solved every time when calculating ν_{red}^*. A general solution would of course result in complicated formulas. A certain simplification is obtained by using the following relation:

$$\nu_x+\nu_y+\nu_z = \nu_I+\nu_{II}+\nu_{III},$$

due to which formula (a) assumes the following form:

$$\nu_{\text{red}}^* = (1+\mu)\,\nu_I - \mu\,(\nu_x+\nu_y+\nu_z)\,,$$

which requires calculation only of one root (ν_I)[16] that ensures the maximum absolute value of ν^*_{red}; but, even so, such a general calculation would not have any practical significance.

IX. To study the consequences of formula (6) let us assume for a while that two of the principal stresses, e.g. ν_1 and ν_3, are constant and let us calculate the effect of the third one on ν_{red} (that is on material effort).

Having expressed for this purpose formula (6) in the following form:

$$\nu^2_{\text{red}} = \left(\nu_1^2 + \nu_3^2 - 2\mu\nu_1\nu_3\right) + \left[\nu_2^2 - 2\mu\left(\nu_1 + \nu_3\right)\nu_2\right]$$

we see that

1. ν_2 **decreases** material effort when it has the same sign as $(\nu_1+\nu_3)$ and at the same time $|\nu_2| < |2\mu(\nu_1 + \nu_3)|$;
2. ν_2 **does not affect** material effort if $\nu_2 = 2\mu(\nu_1 + \nu_3)$;
3. ν_2 always **increases** material effort when it has the opposite sign to the sign of $(\nu_1+\nu_3)$ and with equal sign provided $|\nu_2| > |2\mu(\nu_1 + \nu_3)|$.

An analogous study of formula (a) is not practicable; thus, to compare the results of both hypotheses in a three-dimensional state of stress there is nothing else to do but to calculate a series of numerical examples.

In particular, if all three principal stresses are equal to ν, then $\nu_{\text{red}} = \nu\sqrt{3\left(1 - 2\mu\right)}$ and then, for $\mu = \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{\nu_{\text{red}}}{\nu} = 1, 1.22, 1.34, 1.41$ respectively. On the other hand $\nu^*_{\text{red}} = \nu\left(1 - 2\mu\right)$, and hence, for $\mu = \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$, $\frac{\nu^*_{\text{red}}}{\nu} = \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}$ respectively.

A comparison of these results speaks absolutely in favour of the new hypothesis, as it is difficult to imagine that material effort resulting from all-around uniform tension would be smaller than the one resulting from the tension in one direction, a conclusion to which de Saint-Venant's view leads.

X. The three-dimensional state of stress considered so far occurs quite rarely in the applications of the theory of strength of materials. (It occurs, e.g., in the material of a closed vessel, which bears the pressure of a liquid or gas). Far more frequent is a case of a two-dimensional state of stress with principal stresses ν_1 and ν_2. Then

$$\underline{\nu^2_{\text{red}} = \nu_1^2 + \nu_2^2 - 2\mu\nu_1\nu_2}, \tag{9}$$

$$\nu^*_{\text{red}} = \nu_I - \mu\nu_{II} \qquad \left(|\nu_1| \geq |\nu_2|\right). \tag{β}$$

Equation (9) leads to a very simple geometrical interpretation (Fig. 2). Let us draw an angle AOB, whose cosine is μ, and let us mark on one of its arms $\nu_1 = \overline{OA}$ and on the other one $\nu_2 = \overline{OB}$, then the third side of a triangle AOB will be $\overline{AB} = \nu_{\text{red}}$ if both stresses ν_1 and ν_2 have the same sign, otherwise one of the stresses, e.g. ν_2, should be marked on a prolongation of arm OB and $\nu_{\text{red}} = \overline{AB'}$. If ν_1 is constant and ν_2 variable, we see that

1. ν_2 **decreases** material effort only when, being of the same sign as ν_1, it also satisfies the condition $\nu_2 < 2\mu\nu_1$;
2. ν_2 **does not affect** material effort when $\nu_2 = 2\mu\nu_1$;
3. ν_2 **increases** material effort always when it has the opposite sign to ν_1 and in the case of equal sign provided $\nu_2 > 2\mu\nu_1$.

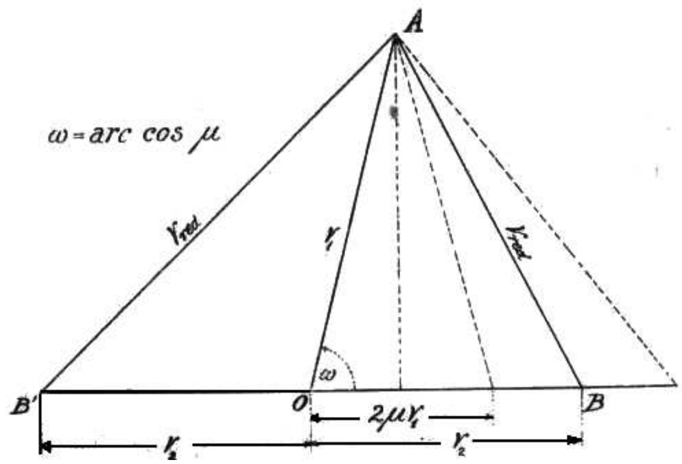


FIG. 2.

For comparison with these results formula (β) states that ν_2 always increases material effort when it is of the opposite sign to ν_1 and decreases in the case of equal signs. The latter is as equally unlikely as the result obtained before.

In particular, for $\nu_1 = -\nu_2 = \sigma$ (pure **shear**) we have $\nu_{\text{red}} = \sigma\sqrt{2\,(1+\mu)}$, $\nu^*_{\text{red}} = \sigma\,(1+\mu)$. Hence, for

$$\begin{array}{lcllll}
\mu & = & \frac{1}{3}, & \frac{1}{4}, & \frac{1}{5}, & \frac{1}{6}, \\
\frac{\nu_{\text{red}}}{\sigma} & = & 1.63, & 1.58, & 1.55, & 1.53, \\
\sigma : \nu_{\text{red}} & = & 0.61, & 0.63, & 0.65, & 0.65, \\
\frac{\nu^*_{\text{red}}}{\sigma} & = & \frac{4}{3}, & \frac{5}{4}, & \frac{6}{5}, & \frac{7}{6}\;.
\end{array}$$

Ratio $\nu_{\text{red}} : \sigma$ is at the same time a proportion of the so-called **allowable stresses** in tension (respectively: compression) to the ones in shear ($\nu_{\text{allow}} : \sigma_{\text{allow}}$) and thus the new hypothesis requires a smaller allowable stress in shear than the old one[17)].

In another particular case, that is when $\nu_1 = \nu_2$, we have $\nu_{\text{red}} = \nu\sqrt{2\,(1-\mu)}$, $\nu^*_{\text{red}} = \nu\,(1-\mu)$. Hence, for

$$\begin{aligned} \mu \quad &= \frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{5}, \quad \frac{1}{6}, \\ \frac{\nu_{\text{red}}}{\nu} &= 1.16, \ 1.22, \ 1.26, \ 1.29, \\ \nu : \nu_{\text{red}} &= 0.86, \ 0.82, \ 0.79, \ 0.78, \\ \frac{\nu^*_{\text{red}}}{\nu} &= \frac{2}{3}, \quad \frac{3}{4}, \quad \frac{4}{5}, \quad \frac{5}{6}\,. \end{aligned}$$

According to the new hypothesis we have $\nu_{\text{red}} > \nu$, while according to the older one $\nu_{\text{red}} < \nu$, which, like the preceding results, speaks in favour of the new hypothesis[18)].

XI. Considering practical needs, it seems also useful to calculate ν_{red} in a two-dimensional state of stress given by general components ν_x, ν_y, σ. Having substituted, $\nu_z = 0$, $\sigma_x = \sigma_y = 0$, $\sigma_z = \sigma$ in equation (7) we find the following formula:

$$\nu^2_{\text{red}} = \nu^2_x + \nu^2_y - 2\mu\nu_x\nu_y + 2\,(1+\mu)\,\sigma^2, \tag{10}$$

from which there also results a simple graphical scheme for determining ν_{red} (Fig.3) provided we take into account the relation: $1 + \cos\omega = 2\cos^2\dfrac{\omega}{2}$, that is

$$2\,(1+\mu) = \left(2\cos\frac{\omega}{2}\right)^2 .$$

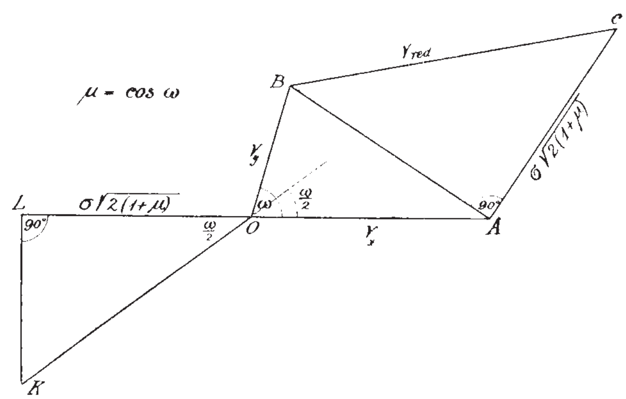


FIG. 3.

An analogous formula is derived from the older hypothesis by calculating principal stresses from equation (8), which, for the reason that $\nu_z = 0$ and $\sigma_x = \sigma_y = 0$, transforms into

$$(\nu - \nu_x)\,(\nu - \nu_y) - \sigma^2 = 0,$$

and substituting them into equation (β).

Hence

$$\nu_I = \frac{1}{2}\,(\nu_x + \nu_y) \pm \frac{1}{2}\sqrt{(\nu_x - \nu_y)^2 + 4\sigma^2},$$

$$(\gamma) \qquad \nu_{II} = \frac{1}{2}\,(\nu_x + \nu_y) \mp \frac{1}{2}\sqrt{(\nu_x - \nu_y)^2 + 4\sigma^2},$$

$$\nu^*_{\text{red}} = \frac{1}{2}\,(1 - \mu)\,(\nu_x + \nu_x) \pm \frac{1}{2}\,(1 + \mu)\sqrt{(\nu_x - \nu_y)^2 + 4\sigma^2}.$$

In accordance with the remark about equation (β) the sign of the second term is taken into account, which leads to a higher absolute value of ν^*_{red}.

If also $\nu_y = 0$ (as, for example, in the outer fibres of a shaft subjected to torsion and bending) then

$$(11) \qquad \nu^2_{\text{red}} = \nu^2 + 2\,(1 + \mu)\,\sigma^2,$$

while

$$(\delta) \qquad \nu^*_{\text{red}} = \frac{1 - \mu}{2}\nu \pm \frac{1 + \mu}{2}\sqrt{\nu^2 + 4\sigma^2}.$$

Having expressed ν in terms of the bending moment M_z and σ in terms of the torque moment M_s, we will find the equation for determining the so-called equivalent moment for a round shaft (applicable, e.g., for wrought and cast iron and steel)

$$(11') \qquad M_{\text{red}} = \sqrt{M_z^2 + \frac{1}{2}\,(1 + \mu)\,M_s^2}$$

instead of the formula

$$(\delta') \qquad M^*_{\text{red}} = \frac{1 - \mu}{2}M_z \pm \frac{1 + \mu}{2}\sqrt{M_z^2 + M_s^2}$$

used till now.

As can be seen, the new formulae are characterised by greater simplicity in application than the former ones.

XII. The concept of material effort referring so far to the homogenous strain of material in the element of the body of volume dV can be advantageously generalised by taking the work of strain of the whole body, that is:

$$\Lambda = \int F dV = \iiint F dx dy dz,$$

as a measure of its **total effort**.

The total effort will not of course describe the risk of exceeding the elastic limit (or of fracture) in general, apart from the homogenous strain; but, it will give a measure of **exploitation** of material effort through a comparison of average effort $\Lambda : V = F_s$ with maximum $F_{\max}$. The measure will thus be:

$$\eta = \frac{F_s}{F_{\max}}.$$

A homogeneous state of stress, in which $\eta = 1$, is the most favourable one, with regard to exploitation of material effort. In a non-homogenous state of stress, we have $\eta<1$, which means that material effort is not exploited completely. (In this fact lies, for instance, the superiority of truss beams compared with monolithic ones).

However, not only η is a measure of exploitation of material but also **allowable effort** F_{allow}, which was so far regarded (indirectly though) as a constant characteristic of the material. It was sufficient as long as there was no need for theoretical determination of "load carrying capacity" of bodies whose certain parts might be subjected to hydrostatic pressure (roller and ball bearings, 'wrapped' concrete). From numerous experiments it is found, however, that in these cases that material withstands safely incomparably greater stress than in a linear or two-dimensional state of stress because permanent strain beyond the elastic limit (undoubtedly exceeded in many practical applications) is relatively very small in hydrostatic pressure and, more importantly, harmless while resistance to constant pressure is unlimited.

There is no reason then not to allow in these cases the material effort to be greater than is accepted in stress states of other kinds, even though engineers still defend themselves against this possibility, rejecting exact solutions to the problems if these, to correspond with direct experience, compel them to accept "unheard-of till now" allowable stresses[19)].

Now, the question arises: how much can the allowable material effort be greater in the case when all principal stresses are pressures than the material effort in other cases? The answer was brought to my mind by a certain particular form of function F, used for the first time by Helmholtz[20)], namely:

$$F = \frac{1}{2}H\Big(\lambda_1 + \lambda_2 + \lambda_3\Big)^2 + \frac{1}{3}K\left\{(\lambda_2 - \lambda_3)^2 + (\lambda_3 - \lambda_1)^2 + (\lambda_1 - \lambda_2)^2\right\} \tag{12}$$

or

$$\begin{aligned} F = {} & \frac{1}{2}H\,(\lambda_x + \lambda_y + \lambda_z)^2 \\ & + \frac{1}{3}K\left\{(\lambda_y - \lambda_z)^2 + (\lambda_z - \lambda_x)^2 + (\lambda_x - \lambda_y)^2 + \frac{3}{2}\left(\varphi_x^2 + \varphi_y^2 + \varphi_z^2\right)\right\}, \end{aligned} \tag{12a}$$

where $H = \dfrac{E}{3\,(1-2\mu)}$ (the Helmholtz modulus), while $K = \dfrac{E}{2\,(1+\mu)}$ (the Kirchhoff modulus).

In this form, the first term represents **work of volume strain** F_ν alone, the other one being the **work of distortion** F_f. Having considered then that volume strain in compression does not affect the risk of fracture, one can in all probability consider F_f as a measure of material effort. Soon I shall return to this subject.

In the end, I should remark that after having elaborated a principal concept, developed above, I found in Beltrami's biography, published in volume VI of *"Wiadomości matematyczne"* (Warszawa 1902), among other things, a short report on the mathematician's paper entitled *"Sulle condizioni di resistenza dei corpi elastici" (Rend. Ist. Lomb. ser. vol. XVIII, 1885)* woven (as I infer from the said report) on the same idea.

A strange thing indeed, that in extensive recent literature on this subject, which I had been thoroughly studying before writing this paper, I had not found a trace of the said Beltrami's discourse. Even though since his compatriot A. Castigliano's times the concept of work of strain has played such an important role in the applications of the theory of elasticity.

(*Kraków, October 1903*).

[1)] A contribution to foundations of the strength of materials theory. The present publication contains the author's paper published in the current volume of Warsaw "Prace matematyczno-fizyczne" and entitled: "O podstawach teorii wytrzymałości" (*"On foundations of the strength of materials theory", Transactions of Mathematics and Physics, vol. XV, 1904 – the paper and the titles in Polish – from translator*).

[2)] At this point I have to bring up a misinterpretation, found in the technical literature, of tensile testing on bars cut out of the same body but having cross-sections of different shapes and sizes, for which different values of strength, measured by a quotient of force P_w, required to fracture the bar, to the area A of its cross-section, were obtained. This fact is clearly explained by a difference in behaviour of the outer layer of a body in comparison with the behaviour of its rest, resulting both from different intermolecular relations on the body surface and from inevitable changes that occur there during mechanical processing. As a result, the outer layer should generally show lower strength than the rest of the body and the quotient $\dfrac{P_w}{A}$ should be the smaller the smaller a ratio of the area of cross-section to its perimeter is. Kirkald, Bauschinger and Bach's experiments confirm this conclusion (Thullie, *„Statyka budowli" ("Structural analysis" – in Polish – from translator),* Lwów, 1902, p. 44). *(In the later works of M.T. Huber this conclusion is corrected; the outer layer should be generally stronger than the rest of the body, cf. the supplementary note on the strength of thin wires written in 1916 by M.T. Huber in his Polish translation of the book "Strength of Materials" by S. Timoshenko, published in Poland in 1922 – from translator).* The differences are in fact too small to take them into account in practice and the whole problem is only of scientific significance as yet; nevertheless, the more attention should be paid to the erroneous explanation, which can be found, e.g., in a nevertheless very useful work of Bach entitled 'Elasticität und Festigkeit' (Berlin 1902). On page 119 the author reasons in the following way:

„Den Entwicklungen der üblichen Gleichungen für die Zugelasticität und Zugfestigkeit... liegt zunächst die Voraussetzung zu Grunde, dass die Dehnungen und Spannungen in allen Punkten des Stabquerschnittes gleich gross sind, dass sich alle Fasern, aus denen der Stab bestehend gedacht werden kann, ganz gleich verhalten und nicht gegenseitig auf einender einwirken. Es ändert an jenen Gleichungen nichts ob eine Kraft P – gleichmässig verteilt – getragen wird von einem Stab, dessen Querschnitt 10 cm^2 *betragt, oder von 1000 Stäben von je 1 mm*^{2} *Querschnitt. In dem einen Fall ist* $f = 10$ *cm*^{2}*, in dem anderen* $= 1000 \cdot 0.01 = 10$ *cm*^{2}*, d.h. in beiden Fällen gleich. In Wirklichkeit aber – immer gleichmässige Verteilung der Last und gleiches Material vorausgesetzt – werden sich die 1000 Metallfäden von je 1 mm*^{2} *Querschnitt unabhängig von einender (senkrecht zur Achse) zusammenziehen können; sie werden, wenn sie sich vorher gerade berührten, die Berührung infolge der mit der Dehnung (Belastung) verknüpften Zusammenziehung aufgeben. Die einzelnen Fasern des Stabes von 10 cm*^{2} *Querschnitt jedoch besitzen eine solche Unabhängigkeit nicht; sie wirken senkrecht zur Achse aufeinander ein. Das Ergebnis dieser Einwirkung aber muss ein verschiedenes sein, je nach der Form des Querschnitts, es wir ein anderes sein bei einem kreisförmigen, als bei einem langgestreckt rechteckigen oder einem I-formigen, ... Dass aber die bezeichnete seitliche Einwirkung Dehnung und Festigkeit beeinflusst, ergibt sich aus den Betrachtungen, welche in §.7 angestellt wurden".*

(*"The development of the common equations for elasticity and strength under tension is based on the requirement that elongations and stresses are the same in all points of the cross-section of a bar and that all material fibers, of which the bar can be thought to be formed, behave in the same way and do not interact one with the other. These equations does not change if the tensile force P – uniformly distributed – is applied to a bar of the cross-section of 10 cm*^{2} *or to the 1000 bars of the cross-section of 1 mm*^{2}*. In the first case we have* $f = 10$ *cm*^{2} *and in the second one* $f = 1000 \cdot 0.01 = 10$ *cm*^{2}*, i.e. in the both cases equal. Though in reality – assuming the uniform distribution of the applied load in the same material – the 1000 metal fibers of the cross-section of 1 mm*^{2} *can contract independently (perpendicular to the axis). Accordingly, they will loose the contact due to the extension, if they were in contact before. However, the separate fibers of the bar of the cross-section of 10 cm*^{2} *have no such a freedom, they interact one with each other perpendicularly to the axis. The effect of such an interaction should be different depending on the shape of the cross-section; it will be the other for the circular cross-section than for the rectangle one or the I section.... It comes out from the discussion presented in §.7 that the mentioned above interaction affects extension and the strength of the considered bars." – from translator*).

Such reasoning contradicts of course our principal concepts of the dependence of a state of stress upon strain. According to the author's assumptions, there are no stresses at all in cross-sections of the bar that are parallel to the tensile direction; how, then, interaction of fibers should manifest itself in these cases? It occurs undoubtedly only when internal forces are not uniformly distributed over the cross-section, as it takes place, e.g., in experiments on laterally grooved bars discussed right after that.

[3)] A summary of Föppl's report can be found in *„Centralblatt der Bauverwaltung"*, 1899, p. 527, or in *„Baumaterialienkunde"*, 1900, No. 6.

[4)] Here, I have in mind the work of O. Mohr entitled *„Welche Umstände bedingen die Elasticitätsgrenze und den Bruch eines Materiales*? (Zieitschr. des Ver. deut. Ing. 1900, p. 1542), which was sharply criticised by W. Voigt in *„Annalen d. Phys."* (1901, p. 567).

[5)] This reasoning fails, as is easily noticed, in the case of compression, that is in the occurrence of negative elongations, which, provided they are not equal in all directions, can also cause fracture of the material. However, not having an intention to ground the new view exclusively on the said reasoning I neglect any required generalisations as yet, especially that, considering the ignorance of a detailed structure of matter, the above reasoning has a nature of general orientation without pretensions to scientific precision.

6) That is, what we call strain of a non-rigid body in which all plane cross-sections remain plane. Under homogenous strain a cuboid with edges a, b, c transforms into a parallelepiped with edges $a\,(1+\lambda_x)\,,\ b\,(1+\lambda_y)\,,\ c\,(1+\lambda_z)$ and dihedral angles $\angle\,(b,c) = \frac{\pi}{2} - \varphi_x,\ \ \angle\,(c,a) = \frac{\pi}{2} - \varphi_y,\ \ \angle\,(a,b) = \frac{\pi}{2} - \varphi_z$; while a sphere of a radius r maps into an ellipsoid with axes $r\,(1+\lambda_1)\,,\ r\,(1+\lambda_2)\,,\ r\,(1+\lambda_3)$. Numbers $\lambda_1,\ \lambda_2,\ \lambda_3$ are called principal elongations. In general strain, non-uniform then, plane cross-sections transform into continuously curved surfaces; hence, one can always regard strain of an element of a body as homogenous (neglecting infinitesimally small quantities of higher order than the dimensions of the element).

7) (J. N. Franke. *„Mechanika teoretyczna”*. Par. 177. $\lambda_1\,,\ \lambda_2\,,\ \lambda_3\,;\ \varphi_1\,,\ \varphi_2\,,\ \varphi_3$) – *(“Theoretical Mechanics”– in Polish – from translator).*

8) (In Franke's *„Mechanika”* $p_{11},\ p_{22},\ p_{33},\ p_{32},\ p_{31},\ p_{12}$).

9) W. Voigt (a physicist from Göttingen) and his disciples' study prove *(Annalen d. Phys. 1901, p. 567 and following)* that even the most homogenous natural bodies do not show homogeneity of structure with respect to strength, as a result of which it can be said in advance that each theory of material strength based on the assumption of perfect homogeneity in the structure of matter will be more or less inaccurate regarding real bodies.

10) In instantaneous strain a small part of the work is used to increase the kinetic energy of the molecules, i.e. to raise the temperature of the body. (J.D. Everett. *„Jednostki stałe fizyczne” – (“Physical units and constants” – in Polish – from translator).* Par. 144).

11) Franke. *“Theoret. Mech.”* p. 483.

12) To the convenience of readers not acquainted in detail with the mathematical theory of elasticity I introduce here the following elementary way of calculating specific work of strain (cf. Föppl's ‘Festigkeitlehre’ §. 12) when principal stresses $(\nu_1\,,\ \nu_2\,,\ \nu_3)$ or elongations $(\lambda_1\,,\ \lambda_2\,,\ \lambda_3)$ in the considered point of an elastic body are given:

Let us consider a cuboid element of a body with edges dx, dy, dz that are parallel to the principal directions and let us assume that during strain of this element stresses change uniformly from 0 to the above values; then, $\frac{1}{2}\nu_1 dydz$ will be the average value of the resultant of the internal forces applied to both faces dy, dz of the element (provided we neglect, due to its being very small, a change of the surface of the face) and $\lambda_1 dx$ will be a change of a distance between the two faces. Hence, work of internal forces on both faces is equal to $\frac{1}{2}\nu_1 dydz \cdot \lambda_1 dx = \frac{1}{2}\lambda_1\nu_1 dxdydz = \frac{1}{2}\lambda_1\nu_1 dV$. Analogically we find $\frac{1}{2}\lambda_2\nu_2 dV$ and $\frac{1}{2}\lambda_3\nu_3 dV$ as the works of the internal forces on the two remaining pairs of the element faces. The total work of strain in the element of volume dV will thus be

$$\frac{1}{2}\left(\lambda_1\nu_1 + \lambda_2\nu_2 + \lambda_3\nu_3\right)\,dV$$

and the specific work of strain (i.e. work related to a unit volume) reads

$$F = \frac{1}{2}\left(\lambda_1\nu_1 + \lambda_2\nu_2 + \lambda_3\nu_3\right)$$

from which, after substituting the following values

$$\lambda_1 = \frac{1}{E}\left[\nu_1 - \mu\left(\nu_2 + \nu_3\right)\right],$$

$$\lambda_2 = \frac{1}{E}\left[\nu_2 - \mu\left(\nu_3 + \nu_1\right)\right],$$
$$\lambda_3 = \frac{1}{E}\left[\nu_3 - \mu\left(\nu_1 + \nu_2\right)\right],$$

formula (3a) results.

[13)] By using the expression, ‘the elastic limit’ I am continually referring to the limit state of stress for which there begins a flow of ductile (plastic) bodies that is what Germans call ‘Streckgrenze’. For, this limit is clearly marked in experiments on tension, compression, bending and torsion of metallic bars (Bauschinger. *Mitth.* No. III). A former definition of the elastic limit as a limit of the state of stress, in which there appears permanent strain has had no significance since it turned out that as the precision of measure of strain increases the latter limit decreases more and more.

[14)] Föppl, *„Festigkeitslehre”* §. 11.

[15)] Helmholtz. *„Dynamik cont. verbr. Massen”*. Leipzig 1902, p. 84. (In Franke’s *“Theoret. Mech.”* Par. 180 there is an analogous equation for principal elongations only).

[16)] Additionally, the following simplification in the calculation of the roots of equation (8) results from its theory:

Having found the roots of the auxiliary equation

$$(\xi - \nu_x)(\xi - \nu_y) - \sigma_z^2 = 0,$$

i.e.

$$\xi_1 = \frac{1}{2}(\nu_x + \nu_y) + \frac{1}{2}\sqrt{(\nu_x - \nu_y)^2 + 4\sigma_z^2},$$

$$\xi_2 = \frac{1}{2}(\nu_x + \nu_y) - \frac{1}{2}\sqrt{(\nu_x - \nu_y)^2 + 4\sigma_z^2}$$

we learn that

$$\nu_I \rangle \xi_1, \quad \xi_1 \rangle \nu_{II} \rangle \xi_2 \quad \nu_{III} \langle \xi_2$$

(Matthiesen. *„Grundzüge der ant. u. mod. Algebra“*. Leipzig 1878, §. 196).

[17)] This is confirmed by Bauschinger’s experiments (*Mittheilungen aus dem mech.-tech. Laboratorium in München, III Heft’*) with Bessemer’s steel of a varying carbon content, the results of which are presented in the following table:

Contents of Carbon %	Stress in elastic limit under(kg/mm^2)		
	tension	compression	torsion
0.19	33	30	15
0.46	35	34	15
0.54	35	34	15
0.57	33	34	16
0.66	37	38	17
0.78	37	38	18
0.80	40	44	20
0.87	43	39	20
0.96	49	50	27

From the above numbers there results on the average $\sigma : \nu_{\text{red}} = 0.50$ instead of 0.61 according to the new, and 0.75 according to the former, hypothesis (for $\mu = \frac{1}{3}$). The value obtained from the experiments in fact still differs quite markedly from the one calculated from formula $\nu_{\mathrm{red}} = \sigma\sqrt{2\,(1+\mu)}$; however, one should not forget that in torsion of a bar of a circular cross-section the limit of elasticity is firstly exceeded in the outer layer, which, as it is known, behaves differently from the inside of the body with regard to the material strength.

[18]) Wehage studied this case experimentally (*Mitth. der techn. Versuchsanstalten zu Berlin, 6 Jahrgang, 1888*) by subjecting to bending circular iron plates supported over their whole rim; he came to the conclusion expressed in the following words:

„Wenn ein schmiedeeiserner Körper zugleich nach zwei zu einender senkrechten Richtungen gleich stark auf Zug oder Druck beansprucht wird, so wird die Elasticitätsgrenze schon bei einer Dehnung erreicht, welche kleiner ist als 0.78 von derjenigen Dehnung, welche der Elasticitätsgrenze im Falle eines einfachen Zuges entspricht“.

... Als feststehend wird nach diesen Untersuchungen anzusehen sein, dass die bisher übliche Beurteilung der Inanspruchnahme eines zugleich nach zwei Richtungen gezogenen oder gedrückten Körpers nach der grössten positiven oder negativen Dehnung allein nicht zulässig ist. Die Inanspruchnahme auf Zug, welche eine cylindrische Wand in tangentialer Richtung durch den Druck einer gepresster Flüssigkeit auf die Innenfläche erleidet, wird also durch einen gleichzeitig ausgeübten axialen Zug nicht vermindert, wie es nach der üblichen Annahme sein müsste, sondern vermehrt“.

(*„If a body of wrought iron is subjected to biaxial stress of the same amount of tension and compression, the limit of elasticity will be reached for the stretch, which is smaller than 0.78 of the stretch corresponding to the limit of elasticity in a simple tension.”*

. . . It has firm basis after these investigations to observe that the application of the commonly used assessment of material effort according to the maximum value of unit elongation or contraction for biaxially stressed bodies is not acceptable. The material effort under the tension, which is produced in the tangential direction of a cylindrical wall by the pressure of a liquid compressed inside the cylinder, will be by the applied at the same time axial tension not diminished, as it could appear according to the common assumptions, but increased.” – from translator).

This result is in distinct contradiction to the hypothesis of maximum elongation and its not too rigorous quantitative consistency with the new hypothesis can be explained, again, by a difference in behaviour of the outer layer, which plays a key role in all experiments on bending, torsion, etc. The effect of the outer layer disappears only in the homogeneous strain of bodies of not very small size such as used by W. VOIGT and L. JANUSZKIEWICZ (‘Annalen d. Physik’ vol. 53 page 43, 1894 and vol. 67 page 452, 1899). Results obtained by the two researchers do not in fact correspond quantitatively with conclusions drawn from the new hypothesis; they cannot, however, serve its refutation since the materials used in the experiments (rock-salt and a mixture of stearic and palmitic acids) do not satisfy the conditions of applicability of the new hypothesis assumed above.

Finally, one cannot pass over GUEST’s extensive paper, based on experiments, and entitled ‘On the strength of ductile materials under combined stress’ (Philosophical Magazine 1900, II), especially since the materials investigated by the author (steel, iron, copper and brass) comply with the conditions of applicability of the new hypothesis. GUEST experimented on pipes 300 mm long, with a diameter of 32 mm and wall thickness of 0.6 to 0.9 mm that were made of the materials mentioned above and could have been at the same time or in turn, stretched and twisted axially and stretched laterally with internal liquid pressure. Longitudinal compression was of course excluded on the account of risk of buckling. A series of three-dimensional states of stresses in the limits of elasticity obtained in this way comprised less than one fourth of the theoretical range of variability of the said states. The experiments proved that within this range material effort does not depend on the middle stress ν_2 but on the extreme stresses ν_1 and ν_3 only, provided we arrange the stress indices so that $\nu_1 \geq \nu_2 \geq \nu_3$ (algebraically). I have calculated, regarding the total range of variability of states of stresses and assuming that $\mu = 0.3$, the upper limit of the effect of ν_2 on material effort according to the new and to the former hypothesis ($\Delta\nu_{\text{red}}$ and Δ^*_{red}) for a comparison. It results from the calculation that $\Delta\nu_{\text{red}} \leq 0.19\nu^*_{\text{red}}$, $\Delta\nu^*_{\text{red}} \leq 0.60\nu^*_{\text{red}}$.

Then, even though Guest's experiments speak absolutely in favour of the new hypothesis, they do not yet confirm it. The inconsistency between the experiments and the theory, however, can be apparent since many a time it has been noticed that one cannot exactly regard μ as constant. It increases with the material effort; and, considering its effect on the specific work of strain, it is easy to understand that formulae (6), (7) and the following ones, which are based on the assumption that $\mu = \text{const}$, will not give precise values of the material effort in the limit of elasticity. Additionally, the different behaviour of the outer layer could have affected in all probability the results of Guest's experiments, especially if one considers that the thickness of the walls in the pipes used in the research was relatively small. Finally, one should not forget that these are the first experiments of this kind and a precision of observation in these depends upon a great many conditions.

[19)] Bach. *Elasticität u. Festigkeit.* fourth edition, Berlin 1902, p. 160-170.

[20)] *„Dynamik cont. verbr. Massen“* §. 31, equation 62.