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Brief Notes

Shape of dispersion curves in the Rayleigh–Lamb spectrum

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THE FOLLOWING FEATURES of the symmetric Lamb modes in an elastic waveguide are well known:

1. There exists no mode with phase speed less than c_R.
2. There is only one mode whose speed asymptotically approaches c_R.
3. A horizontal line above $c = c_T$ (including the line $c = c_L$) *cannot* be an asymptote to *any* of the modes.
4. Phase speed of all modes, except the lowest mode, approaches c_T as the frequency becomes very large.

The above features characterize the spectrum which is obtained *numerically* or *experimentally* but are not *fully* understood *analytically*. We analyze the Rayleigh–Lamb equation and provide *analytical* explanation for the above features.

1. Introduction

CONSIDER AN INFINITE isotropic plate of thickness $2h$ characterized by the phase speeds c_T and c_L, respectively, of the transverse and longitudinal bulk waves. Let ω and k denote the frequency and the wave number of a wave which propagates in a direction parallel to the plate surfaces which are assumed to be free of traction. The dispersion relation for the *symmetric* modes is given by [1]

$$\frac{\tan(qh)}{\tan(ph)} = \frac{-4pqk^2}{(q^2 - k^2)^2}, \tag{1.1}$$

where

$$p = \sqrt{\frac{\omega^2}{c_L^2} - k^2}, \tag{1.2}$$

$$q = \sqrt{\frac{\omega^2}{c_T^2} - k^2}. \tag{1.3}$$

The corresponding dispersion relation for the antisymmetric modes is obtained from Eq. (1.1) by interchanging p and q on the left-hand side of the equation.

Equation (1.1) is known as the Rayleigh–Lamb equation [2, 3]. Dispersion curves, expressing the circular frequency ω or the phase speed c in terms of the wave number k, are obtained *numerically*. Sketching of these curves is facilitated, to a great extent, by Mindlin's method of bounds [4].

In 1973, UBERALL, while discussing the spectrum for a free aluminum plate, made the following observation [5]: "As $h\omega$ tends to infinity (which includes the case of the elastic half-space bounded by vacuum), the lowest modes s_0 and a_0 are *seen* to converge, and to coincide with the Rayleigh velocity c_R on the free elastic surface, while all the higher modes (both s_i and a_i) *seem* to approach the transverse bulk speed c_T. If c_T is *actually* the limit of all these dispersion curves, then the longitudinal bulk speed c_L does not seem to be similarly represented as a limiting value of any Lamb wave speeds. This point remains to be investigated". Although a discussion of the Lamb spectrum can be found in practically every graduate text-book on elastic waves (see, for example, [1, 6, 7]) and hundreds of other places in the literature, it appears that main features of the spectrum, alluded to by Uberall nearly three decades ago, still await an *analytical* explanation.

In this paper we shall analyze Eq. (1.1) for the symmetric modes. Analysis of the equation dealing with the antisymmetric modes runs along similar lines. We shall attempt to find *analytical* answers to the following questions:

1. Why there is no mode with velocity smaller than c_R?
2. Why does the velocity of the lowest mode, and only that of the lowest mode, asymptotically approach c_R?
3. Why no mode has c_L as the limiting speed?
4. Why do the dispersion curves for all modes, except the lowest one, approach the line $c = c_T$ as the frequency becomes very large?

2. Dispersion curves

Defining

$$c = \frac{\omega}{k}, \qquad u = hk, \tag{2.1}$$

we rewrite Eq. (1.1) in the form

$$\frac{\tan\left[u\sqrt{\dfrac{c^2}{c_T^2} - 1}\right]}{\tan\left[u\sqrt{\dfrac{c^2}{c_L^2} - 1}\right]} = \frac{-4\sqrt{\dfrac{c^2}{c_L^2} - 1}\sqrt{\dfrac{c^2}{c_T^2} - 1}}{\left(\dfrac{c^2}{c_T^2} - 2\right)^2}. \tag{2.2}$$

In order to find the dispersion curves, it is usual to fix the frequency ω in Eq. (1.1) and scan the equation for the values of k which satisfy the equation. We shall adopt an alternative approach in which we fix a value of the phase velocity, c, in Eq. (2.2) and look for the values of u. The task is facilitated by the fact that the right-hand side of Eq. (2.2) is a function of c only.

When $0 < c < c_T$, both functions under the square root sign are negative. Defining

$$\alpha(c) = \sqrt{1 - \frac{c^2}{c_T^2}}, \qquad \beta(c) = \sqrt{1 - \frac{c^2}{c_L^2}}, \qquad \gamma(c) = \frac{4\alpha(c)\,\beta(c)}{(1 + \alpha^2\,(c))^2}, \tag{2.3}$$

we can write Eq. (2.2) in the form

$$\frac{\tanh(u\alpha)}{\tanh(u\beta)} = \gamma, \tag{2.4}$$

the dependence of α, β and γ on c being suppressed for simplicity.

The function $\tanh x$ is an increasing function of its argument. Since

$$\beta(c) > \alpha(c), \qquad 0 < c < c_T, \tag{2.5}$$

and $u > 0$, we conclude that

(2.6) *the left-hand side of Eq. (2.4) is less than unity.*

On the other hand, it is well known that the Rayleigh equation

$$4\alpha(c)\beta(c) - (1 + (\alpha(c))^2)^2 = 0, \tag{2.7}$$

has a unique solution in $(0, c_T)$. Denote the left-hand side of (2.7) by $f(c)$. A unique solution of the above equation implies that $f(c)$ changes sign only once in the interval $(0, c_T)$, at the point $c = c_R$. It is easy to verify that for $0 < c < c_R$, $f(c)$ has the positive sign while for $c_R < c < c_T$, the function is negative. Thus

$$\begin{aligned} &\gamma(c) > 1, \quad 0 < c < c_R, \\ &\gamma(c) = 1, \quad c = c_R, \\ &\gamma(c) < 1, \quad c_R < c < c_T. \end{aligned} \tag{2.8}$$

It is clear from statements (2.6) and $(2.8)_1$ that Eq. (2.4) cannot be satisfied if $0 < c < c_R$. Thus no mode exists whose phase speed is less than c_R. *This answers the first question raised in the introduction.*

If $u \ll 1$, we can use the approximation $\tan u \cong u$, to obtain from Eq. (2.2)

$$\left(2 - \frac{c^2}{c_T^2}\right)^2 = 4\left(1 - \frac{c^2}{c_L^2}\right). \tag{2.9}$$

The above equation has the solution

$$c = 2c_T\sqrt{1 - \frac{1}{\kappa^2}}, \tag{2.10}$$

where $\kappa = c_L/c_T$. Equation (2.10) gives the velocity, near the cut-off point, of the only mode which exists for wavelengths large compared with the depth, $2h$, of the plate. This is the velocity given by the elementary theory for extensional motions of the plate.

When $c = c_L$, Eq. (2.2) is satisfied by

$$u = \frac{n\pi}{\sqrt{\kappa^2 - 1}}, \qquad n = 1, 2, 3, \ldots. \tag{2.11}$$

This means that there are infinitely many dispersion curves, which intersect the line $c = c_L$. For $c_T < c < c_L$, one of the radicals in Eq. (2.2) is real, while the other is imaginary. For large u,

$$\tanh\left[u\sqrt{1 - \frac{c^2}{c_L^2}}\right] \cong 1,$$

and Eq. (2.2) becomes

$$\tan\left[u\sqrt{\frac{c^2}{c_T^2} - 1}\right] = \frac{4\sqrt{\frac{c^2}{c_T^2} - 1}\sqrt{1 - \frac{c^2}{c_L^2}}}{\left(\frac{c^2}{c_T^2} - 2\right)^2}. \tag{2.12}$$

The above equation has infinitely many solutions for every c, $c_T < c < c_L$. There are infinitely many modes in this domain. From this observation it follows that the line $c = c_L$, or for that matter, any line $c = c_1$, with $c_1 > c_T$, *cannot* be an asymptote to any of the modes because, if the n-th mode did have c_1 as the limiting speed, any horizontal line between c_T and c_1 would intersect at most $(n - 1)$ modes (the dispersion curves cannot intersect), but this contradicts the fact that there are infinitely many modes in this domain. *This answers the third question.*

Now write Eq.(2.2) in the form

$$\frac{\tan\left(u\sqrt{\frac{c^2}{c_T^2}-1}\right)}{\sqrt{\frac{c^2}{c_T^2}-1}} = \frac{-4\sqrt{\frac{c^2}{c_L^2}-1}}{\left(\frac{c^2}{c_T^2}-2\right)^2}\tan\left(u\sqrt{\frac{c^2}{c_L^2}-1}\right). \tag{2.13}$$

When $c \to c_T$, the above equation reduces to

$$u = 4\sqrt{1-\frac{1}{\kappa^2}}\tanh\left(u\sqrt{1-\frac{1}{\kappa^2}}\right). \tag{2.14}$$

Equation (2.14) has the unique positive root u_0 at the point where the lowest mode crosses the line $c = c_T$. Let c approach c_T from above in such a manner that

$$\sqrt{\frac{c^2}{c_T^2}-1} = \varepsilon << 1. \tag{2.15}$$

If additionally u is large, an assumption justified *a posteriori* by Eq. (2.19), Eq. (2.2) becomes, to the first order in ε,

$$\tan(\varepsilon u) = 4\varepsilon\sqrt{1-\frac{1}{\kappa^2}}. \tag{2.16}$$

Hence

$$\varepsilon u = \tan^{-1}\left(4\varepsilon\sqrt{1-\frac{1}{\kappa^2}}\right) + n\pi, \qquad n = 0,1,2,... \tag{2.17}$$

or

$$\varepsilon u \cong 4\varepsilon\sqrt{1-\frac{1}{\kappa^2}} + n\pi, \qquad n = 0,1,2,... \tag{2.18}$$

or

$$u = u^0, \qquad u^0 + \frac{\pi}{\varepsilon}, \qquad u^0 + \frac{2\pi}{\varepsilon}, ... \tag{2.19}$$

where

$$u^0 = 4\sqrt{1-\frac{1}{\kappa^2}}.$$

Equation (2.17) was obtained under the assumption $u \gg 1$ but this will not hold when $u = u^0$. Hence the first term of the sequence on the right-side of (2.19) must be dropped. From Eq. (2.19) it is clear that all modes with the exception of the lowest one approach the line $c = c_T$ asymptotically. This happens because we can take ε arbitrarily small. *This answers the fourth question.*

When $c_R < c < c_T$, both radicals in Eq. (2.2) are imaginary and we write the equation in the form

$$\frac{\tanh(\alpha u)}{\tanh(\beta u)} - \gamma = 0, \tag{2.20}$$

where, α, β and γ have been defined above. Denote the left-hand side of (2.20) by $g(u)$. From $(2.8)_3$ $\gamma < 1$ for the range of c under consideration, also

$$g(0^+) = \frac{\alpha}{\beta} - \gamma, \tag{2.21}$$

and in the limit $u \to \infty$

$$\lim_{u \to \infty} g(u) = 1 - \gamma > 0. \tag{2.22}$$

It is easily checked that $g(0^+)$ appearing in Eq. (2.21) will be negative as long as

$$c < 2c_T\sqrt{1 - \frac{1}{\kappa^2}},$$

which is true in the present case. Since the function $g(u)$ is continuous on $[0, \infty)$ and it changes sign once in the interval, we conclude that for every c between c_T and c_R, there is a unique solution of Eq. (2.20). When $c \to c_R$ from above $\gamma(c) \to 1^-$, but from (2.20) it is clear that this can happen only if both αu and βu approach infinity. Hence when c is close to c_R, both $\exp(-\alpha u)$ as well as $\exp(-\beta u)$ will be small. In this approximation

$$\tanh(\alpha u) = \frac{1 - e^{-2\alpha u}}{1 + e^{-2\alpha u}} \cong 1 - 2e^{-2\alpha u}, \tag{2.23}$$

similarly

$$\tanh(\beta u) \cong 1 - 2e^{-2\beta u}, \tag{2.24}$$

and

$$\begin{aligned} \frac{\tanh(\alpha u)}{\tanh(\beta u)} &\cong 1 - 2e^{-2\alpha u} + 2e^{-2\beta u}, \\ &\cong 1 - 2e^{-2\alpha u}\left(1 - e^{-2(\beta - \alpha)u}\right), \\ &\cong 1 - 2e^{-2\alpha u}, \end{aligned} \tag{2.25}$$

since $\beta > \alpha$ and u is assumed to be large. From Eqs. (2.20) and (2.25) we get

$$u = -\frac{1}{2\alpha}\ln\left(\frac{1-\gamma}{2}\right). \tag{2.26}$$

Equation (2.26) gives an analytical expression for u in terms of c when c is close to c_R. In this approximation $1-\gamma$ will be a small positive number and u will be very large. In the limit $c \to c_R$, we have $u \to \infty$. This is the well-known result that the limiting velocity of the lowest mode is the speed, c_R, of the Rayleigh wave on the surface of a half-space. *This answers the last remaining question i.e. the second question raised in the introduction.*

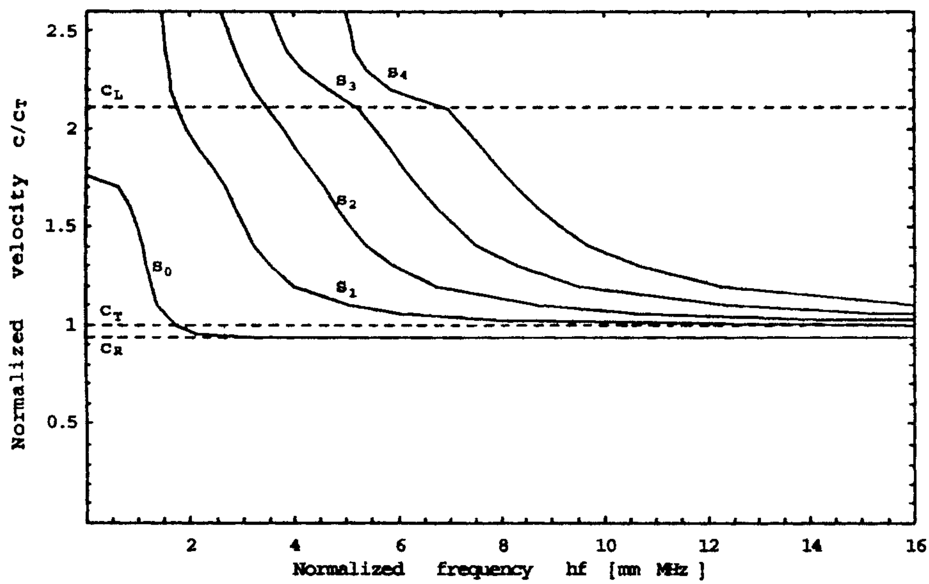

FIG. 1. Calculated dispersion curves for the first five symmetric modes of an aluminum plate, as functions of the normalized frequency.

In Fig. 1, we have plotted the dispersion curves for aluminum giving the phase velocity as a function of the normalized frequency $h\omega$ and in Fig. 2, we have plotted the same curves as functions of the dimensionless wave number hk. However, Fig. 2 has an advantage in that, in the middle part of the spectrum, say between c_T and $1.9c_T$, any horizontal line intersects the modes, except the lowest one, at *equidistant points.* This point has been discussed in some detail in [8].

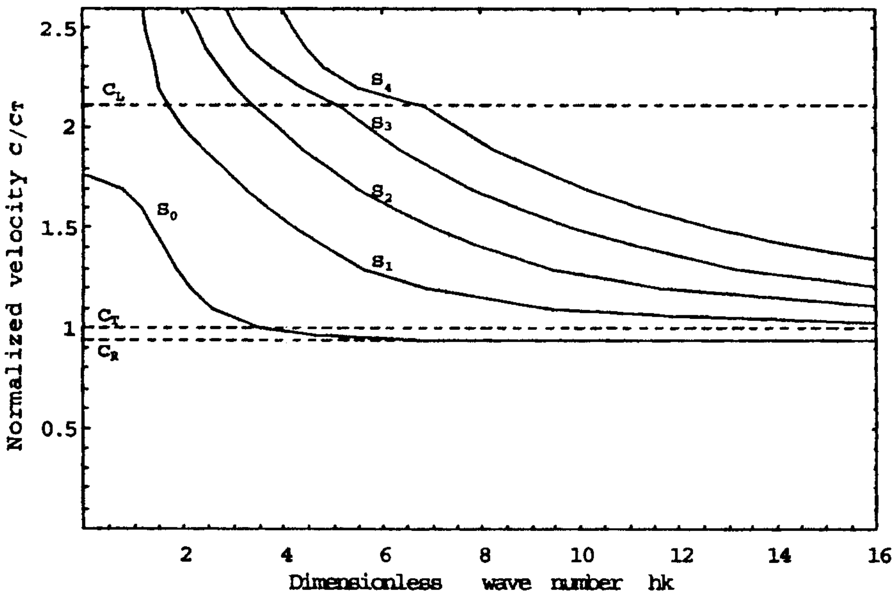

FIG. 2. Calculated dispersion curves for the first five symmetric modes of an aluminum plate, as functions of the dimensionless wave number.

3. Conclusions

We have discussed salient features of the dispersion curves of the Rayleigh-Lamb spectrum for an isotropic plate. We have found answers to some of the pertinent questions raised by UBERALL three decades ago [5].

The spectrum of a circular cylinder has many features common with the spectrum for a plate. A brief analysis of this problem has been presented elsewhere [9].

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