Arch. Mech., **56**, 2, pp. 103–115, Warszawa 2004

Imperfect transmission conditions for a thin weakly compressible interface. 2D problems

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Imperfect transmission conditions are evaluated in the case of weakly compressible elastic interphase between different bonded elastic materials. It is shown that the corresponding conditions can be different in comparison with those for the elastic compressible interface.

1. Introduction

Composite materials are usually considered as piecewise homogeneous solids with perfect bonding between different phases of the composites (e.g. [2, 3]). On the other hand, such structures contain, in fact, thin intermediate layers matching together the materials of the phases. Depending on the intermediate zone features (for example, soft or stiff interfaces), the respective transmission conditions evaluated by asymptotical methods may take essentially different forms [1, 4, 5, 8]. These transmission conditions have been evaluated, in fact, under the assumption that elastic constants of the intermediate layer are comparable in values: $\mu \sim \lambda$. However, if the interphase material is weakly compressible then $\mu/\lambda << 1$ and the earlier asymptotic analysis can fail. In [6], it has been shown by FEM-analysis that the known imperfect transmission conditions for the compressible interface cannot be always applied for the weakly compressible one. The aim of this paper is to evaluate the respective transmission conditions for the weakly compressible inhomogeneous elastic interphase. We restrict ourselves to the 2D plane problems. It is important to note that for the so-called Mode III problem, the corresponding transmission conditions have the same forms [1, 5] in the cases of the compressible and weakly compressible interfaces, because this problem does not depend on the value of Poisson's coefficient ν of the interphase.

2. Asymptotic evaluation of the transmission conditions

Let us consider a model plane problem for bimaterial elastic solid with an intermediate layer of constant thickness $\Omega_h = \Omega_+ \cup \Omega_- \cup \Omega$, where $\Omega_\pm = \{(x, y), \pm y \geq h\}$, $\Omega = \{(x, y), |y| \leq h\}$ (see Fig. 1). We assume throughout the paper that layer Ω is inhomogeneous and isotropic, while the bonded materials are isotropic and homogeneous.

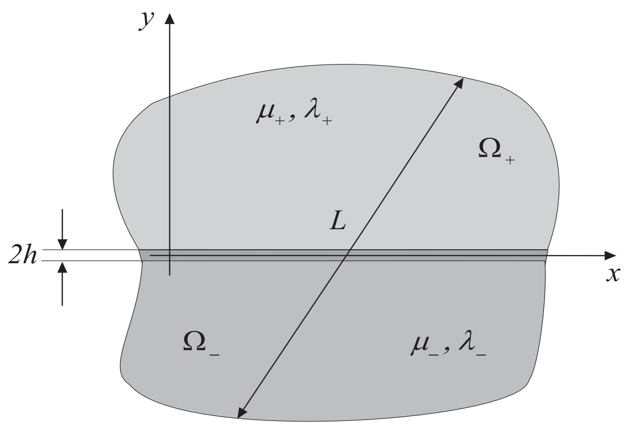

FIG. 1. Bimaterial solid with a thin intermediate zone.

Let $\mathbf{u}_\pm(x, y)$ and $\mathbf{u}(x, y)$ be the vectors of displacements: $\mathbf{u}_\pm = [u_x^\pm, u_y^\pm]^\top$, $\mathbf{u} = [u_x, u_y]^\top$. They satisfy the Lamé equations in the corresponding domains :

$$\mathcal{L}_\pm \mathbf{u}_\pm = \mathbf{0}, \quad (x, y) \in \Omega_\pm, \qquad \mathcal{L}\mathbf{u} = \mathbf{0}, \quad (x, y) \in \Omega, \tag{2.1}$$

where differential operators $\mathcal{L}_\pm$ and $\mathcal{L}$ are defined in the following manner:

$$\mathcal{L}_\pm = \begin{pmatrix} \lambda_\pm + 2\mu_\pm)D_x^2 + \mu_\pm D_y^2 & (\lambda_\pm + \mu_\pm)D_x D_y \\ (\lambda_\pm + \mu_\pm)D_x D_y & (\lambda_\pm + 2\mu_\pm)D_y^2 + \mu_\pm D_x^2, \end{pmatrix}, \tag{2.2}$$

$$\mathcal{L} = \begin{pmatrix} D_x(\lambda + 2\mu)D_x + D_y \mu D_y & D_x \lambda D_y + D_y \mu D_x \\ D_y \lambda + D_x + D_x \mu D_y & (\lambda + 2\mu)D_y + D_x \mu D_x \end{pmatrix}; \tag{2.3}$$

here by D_x, D_y are denoted the respective partial derivatives.

On the exterior boundary $\partial\Omega_h$ some boundary conditions are assumed to be satisfied:

$$\mathcal{B}_\pm \mathbf{u}_\pm = \mathbf{0}, \quad (x, y) \in \partial\Omega_h \cap \partial\Omega_\pm, \qquad \mathcal{B}\mathbf{u} = \mathbf{0}, \quad (x, y) \in \partial\Omega_h \cap \partial\Omega. \tag{2.4}$$

We do not state precisely here the forms of boundary operators $\mathcal{B}_\pm$ and $\mathcal{B}$, because they will not play any role in the formal asymptotic procedure. However, such information is absolutely necessary to prove the final estimate for the asymptotic solution obtained.

Along the interior boundaries $y = \pm h$ the perfect transmission conditions should be satisfied:

$$\mathbf{u}_\pm(x, \pm h) = \mathbf{u}(x, \pm h), \qquad \boldsymbol{\sigma}_\pm^{(y)}(x, \pm h) = \boldsymbol{\sigma}^{(y)}(x, \pm h), \tag{2.5}$$

where

$$\boldsymbol{\sigma}_\pm^{(y)}(x, y) = \mathcal{M}_\pm \mathbf{u}_\pm(x, y), \qquad \boldsymbol{\sigma}^{(y)}(x, y) = \mathcal{M}\mathbf{u}(x, y), \tag{2.6}$$

$$\mathcal{M}_\pm = \begin{pmatrix} \mu_\pm D_y & \mu_\pm D_x \\ \lambda_\pm D_x & (\lambda_\pm + 2\mu_\pm)D_y, \end{pmatrix} \quad \mathcal{M}_\pm = \begin{pmatrix} \mu D_y & \mu D_x \\ \lambda D_x & (\lambda + 2\mu)D_y, \end{pmatrix}. \tag{2.7}$$

We assume that the intermediate layer is essentially thinner in comparison with the characteristic size of the body: $h \ll L$. This allows us to introduce into the problem a small dimensionless parameter $\epsilon \ll 1$ rescaling the variable y within the interphase $(x, y) \in \Omega$:

$$y = \epsilon\xi, \qquad \xi \in [-h_0, h_0], \qquad h_0 \sim L. \tag{2.8}$$

In this paper we consider a practically incompressible layer that can be defined in terms of the Poison coefficient in the following manner:

$$\nu(x, y) = 0.5 - \epsilon\nu_0(x, y), \qquad \nu_0 > 0. \tag{2.9}$$

Then in terms of the Lamé coefficients of the interphase one can conclude that:

$$\lambda(x, y) \sim \epsilon^{-1}\mu(x, y). \tag{2.10}$$

2.1. Soft weakly compressible interface

When the anisotropic homogeneous interphase layer is essentially softer than the matched materials and $\mu \sim \lambda$, the corresponding transmission conditions have been evaluated in [1]. Here we analyze the soft inhomogeneous weakly compressible isotropic interphase:

$$\mu(x, \epsilon\xi) = \epsilon\mu_0(x, \xi), \qquad \mu_0 \sim \mu_\pm. \tag{2.11}$$

Because the interphase under consideration is weakly compressible (2.9), one can easily conclude from (2.10) and (2.11) that:

$$\lambda(x, \epsilon\xi) = \lambda_0(x, \xi) \sim \mu_0(x, \xi). \tag{2.12}$$

Let us denote by $\mathbf{w}(x,\xi) = \mathbf{u}(x,\epsilon\xi)$ a solution within the rescaling layer $\Omega_0 = \{(x,\xi),\, |\xi| \le h_0\}$. In these new notations all operators can be rewritten as follows:

$$\mathcal{L} = \epsilon^{-2}\mathcal{L}_0 + \epsilon^{-1}\mathcal{L}_1 + \mathcal{L}_2 + \epsilon\mathcal{L}_3, \qquad \mathcal{M} = \epsilon^{-1}\mathcal{M}_0 + \mathcal{M}_1 + \epsilon\mathcal{M}_2, \tag{2.13}$$

where $\mathcal{L}_0 = D_\xi \mathbf{A}_0 \lambda_0 D_\xi$, $\mathcal{L}_3 = D_x \mathbf{A}_3 \mu_0 D_x$, $\mathcal{M}_0 = \mathbf{A}_0 \lambda_0 D_\xi$, $\mathcal{M}_2 = \mathbf{A}_2 \mu_0 D_x$, and

$$\begin{gathered}\mathbf{A}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{L}_1 = \begin{pmatrix} D_\xi \mu_0 D_\xi & D_x \lambda_0 D_\xi \\ D_\xi \lambda_0 D_x & 2D_\xi \mu_0 D_\xi \end{pmatrix}, \\ \mathcal{M}_1 = \begin{pmatrix} \mu_0 D_\xi & 0 \\ \lambda_0 D_x & 2\mu_0 D_\xi, \end{pmatrix},\end{gathered} \tag{2.14}$$

$$\mathcal{L}_2 = \begin{pmatrix} D_x \lambda_0 D_x & D_\xi \mu_0 D_x \\ D_x \mu_0 D_\xi & 0 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{A}_3 = \mathbf{I} + \mathbf{A}_0. \tag{2.15}$$

To evaluate the appropriate transmission conditions it is necessary to solve within the intermediate domain Ω_0 the following equation:

$$\Big(\mathcal{L}_0 + \epsilon\mathcal{L}_1 + \epsilon^2\mathcal{L}_2 + \epsilon^3\mathcal{L}_3\Big)\mathbf{w} = \mathbf{0}, \qquad (x,\xi) \in \Omega_0, \tag{2.16}$$

with the boundary conditions:

$$\mathbf{u}_\pm(x, \pm\epsilon h_0) = \mathbf{w}(x, \pm h_0), \tag{2.17}$$

$$\boldsymbol{\sigma}_\pm^{(y)}(x, \pm\epsilon h_0) = \Big(\epsilon^{-1}\mathcal{M}_0 + \mathcal{M}_1 + \epsilon\mathcal{M}_2\Big)\mathbf{w}|_{\xi=\pm h_0}. \tag{2.18}$$

According to the standard procedure [8], solution within the corresponding domains will be sought in the form of asymptotic series:

$$\mathbf{w}(x,\xi) = \sum_{j=0}^{\infty} \epsilon^j \mathbf{w}_j(x,\xi), \qquad \mathbf{u}_\pm(x,y) = \sum_{j=0}^{\infty} \epsilon^j \mathbf{u}_j^\pm(x,y). \tag{2.19}$$

As a result, a sequence of the BVPs determining the respective terms in asymptotic expansions (2.19) will be found. Thus for the first term $\mathbf{w}_0$ one can obtain:

$$D_\xi \mathbf{A}_0 \lambda_0 D_\xi \mathbf{w}_0 = \mathbf{0}, \quad (x,\xi) \in \Omega_0, \tag{2.20}$$

(2.21) $$\mathbf{u}_0^{\pm}(x,\pm 0) = \mathbf{w}_0(x,\pm h_0),$$

(2.22) $$\mathbf{0} = \mathbf{A}_0\lambda_0 D_\xi \mathbf{w}_0|_{\xi=\pm h_0}.$$

From (2.20) and (2.22) it immediately follows that the only second component of the vector $\mathbf{w}_0$ can be calculated at this step:

(2.23) $$w_{02}(x,\xi) = a_0(x), \qquad a_0(x) = \mathbf{A}_0\mathbf{u}_0^{\pm}(x,\pm 0),$$

where additionally the following necessary condition has to be satisfied

(2.24) $$\mathbf{A}_0\Big(\mathbf{u}_0^{+}(x,+0) - \mathbf{u}_0^{-}(x,-0)\Big) = \mathbf{0},$$

which substitutes a part of the transmission conditions for the soft, weakly compressible interface. The first component $w_{01}(x,\xi)$ will be found in the next step only. Then the still unused part of the boundary condition (2.20) has to be taken into account:

(2.25) $$\mathbf{A}_1\Big(\mathbf{u}_0^{\pm}(x,\pm 0) - \mathbf{w}_0(x,\pm h_0)\Big) = \mathbf{0},$$

where $\mathbf{A}_1 = \mathbf{I} - \mathbf{A}_0$.

To obtain the next term of the asymptotic expansion (2.19) we need to solve the following BVP:

(2.26) $$D_\xi \mathbf{A}_0\lambda_0 D_\xi \mathbf{w}_1 + \mathcal{L}_1\mathbf{w}_0 = \mathbf{0}, \qquad (x,\xi)\in\Omega_0,$$

(2.27) $$\mathbf{u}_1^{\pm}(x,\pm 0) \pm h_0 D_y \mathbf{u}_0^{\pm}(x,\pm 0) = \mathbf{w}_1(x,\pm h_0),$$

(2.28) $$\boldsymbol{\sigma}_0^{(y)\pm}(x,\pm 0) = \mathcal{M}_0\mathbf{w}_1|_{\xi=\pm h_0} + \mathcal{M}_1\mathbf{w}_0|_{\xi=\pm h_0},$$

together with condition (2.25). In turn, Eq. (2.26) can be rewritten in the form:

(2.29) $$D_\xi\mathcal{L}_4\begin{pmatrix} w_{01} \\ w_{12}\end{pmatrix} \equiv D_\xi\begin{pmatrix} \mu_0 D_\xi & 0 \\ \lambda_0 D_x & \lambda_0 D_\xi\end{pmatrix}\begin{pmatrix} w_{01} \\ w_{12}\end{pmatrix} = \mathbf{0}.$$

As a result, one can find the first component of the vector $\mathbf{w}_0$ and the second component of the term $\mathbf{w}_1$:

(2.30) $$w_{01}(x,\xi) = a_1(x) + b_1(x)\int_0^{\xi}\mu_0^{-1}(x,t)dt,$$

(2.31)
$$w_{12}(x,\xi) = c_1(x) + d_1(x)\int\limits_0^\xi \frac{dt}{\lambda_0(x,t)} - \xi D_x a_1(x) - D_x\left(b_1(x)\int\limits_0^\xi \frac{(\xi-t)dt}{\mu_0(x,t)}\right).$$

Substituting (2.30) and (2.31) into (2.25) and (2.27), one respectively obtains:

(2.32)
$$u_{01}^{\pm}(x,\pm 0) = a_1(x) + b_1(x)\int\limits_0^{\pm h_0}\frac{dt}{\mu_0(x,t)},$$

(2.33)
$$\boldsymbol{\sigma}_0^{(y)\pm}(x,\pm 0) = \mathcal{L}_4[w_{01},w_{12}]^\top|_{\xi=\pm h_0} = [b_1(x),d_1(x)]^\top.$$

Conditions (2.25), (2.33) and the condition for the second component in (2.27) allow us to calculate the unknown functions $a_1(s)-d_1(s)$ in (2.30), (2.31). However, it is possible only under additional conditions with respect to the first term of the external solution:

(2.34)
$$\boldsymbol{\sigma}_{0+}^{(y)}(x,+0) = \boldsymbol{\sigma}_{0-}^{(y)}(x,-0),$$

(2.35)
$$\mathbf{u}_0^+(x,+0) - \mathbf{u}_0^-(x,-0) = \tau_1(x)\mathbf{A}_1\boldsymbol{\sigma}_0^{(y)}(x,0),$$

where, in order to evaluate the last conditions, we have taken into account Eq. (2.24) and

(2.36)
$$\tau_1(x) = \int\limits_{-h_0}^{h_0}\frac{dt}{\mu_0(x,t)} = \int\limits_{-h}^{h}\frac{dy}{\mu(x,y)}.$$

The iterative process can be continued to construct the solution with an arbitrary degree of accuracy.

On the other hand, if we restricts ourselves to the first term of the asymptotic expansion (2.19), Eqs. (2.34), (2.35) will substitute the sought for transmission conditions for the weakly compressible interface.

2.2. Weakly compressible interface

Let us assume now that the shear modulus of the interface is comparable in value with those of the bonded materials:

(2.37)
$$\mu(x,\epsilon\xi) = \mu_0(x,\xi) \sim \mu_\pm.$$

In the case when additionally $\lambda \sim \mu$, the corresponding transmission conditions [8] coincide with the classical ones (perfect or ideal interface). Here we extend the results to the weakly compressible interface. Then from (2.10) and (2.37) it follows that:

$$\lambda(x, \epsilon\xi) = \epsilon^{-1}\lambda_0(x, \xi), \qquad \lambda_0 \sim \mu_0. \tag{2.38}$$

Taking these new estimates into account instead of (2.11) and (2.12), Eq. (2.16) will be still valid as well as condition (2.17), while condition (2.18) will take other form:

$$\boldsymbol{\sigma}_{\pm}^{(y)}(x, \pm\epsilon h_0) = \Big(\epsilon^{-2}\mathcal{M}_0 + \epsilon^{-1}\mathcal{M}_1 + \mathcal{M}_2\Big)\mathbf{w}|_{\xi=\pm h_0}. \tag{2.39}$$

Solution to the problem will be still sought for in form of the asymptotic expansion (2.19). For the first term, the problem (2.20)–(2.22) is still actual and the corresponding solution and part of the transmission conditions have been found in (2.23) and (2.24), respectively. For the second term, the problem (2.26), (2.27) and

$$\mathbf{0} = \mathcal{M}_0\mathbf{w}_1|_{\xi=\pm h_0} + \mathcal{M}_1\mathbf{w}_0|_{\xi=\pm h_0} = \mathcal{L}_4[w_{01}, w_{12}]^\top|_{\xi=\pm h_0}, \tag{2.40}$$

instead of (2.28) holds true. Corresponding solutions (the first component of the vector $\mathbf{w}_0$ and the second component of the term $\mathbf{w}_1$) are easily obtainable from (2.30)–(2.35):

$$w_{01}(x, \xi) = a_1(x), \qquad w_{12}(x, \xi) = c_1(x) - \xi D_x a_1(x). \tag{2.41}$$

Unknown functions $a_1(x)$, $c_1(x)$ ($b_1(x) = d_1(x) = 0$) should be found from conditions (2.27), (2.32) and (2.33). Then additionally, one more necessary condition has to be satisfied among others:

$$\mathbf{A}_1\Big(\mathbf{u}_0^+(x, +0) - \mathbf{u}_0^-(x, -0)\Big) = \mathbf{0}, \tag{2.42}$$

which together with (2.24) gives the transmission condition:

$$\mathbf{u}_0^+(x, +0) - \mathbf{u}_0^-(x, -0) = \mathbf{0}. \tag{2.43}$$

For the third term, the following BVP has to be solved:

$$D_\xi \mathbf{A}_0 \lambda_0 D_\xi \mathbf{w}_2 + \mathcal{L}_1\mathbf{w}_1 + \mathcal{L}_2\mathbf{w}_0 = \mathbf{0}, \qquad (x, \xi) \in \Omega_0, \tag{2.44}$$

$$\mathbf{u}_2^{\pm}(x, \pm 0) \pm h_0 D_y \mathbf{u}_1^{\pm}(x, \pm 0) + \frac{h_0^2}{2} D_y^2 \mathbf{u}_0^{\pm}(x, \pm 0) = \mathbf{w}_2(x, \pm h_0), \tag{2.45}$$

$$\boldsymbol{\sigma}_0^{(y)\pm}(x,\pm 0) = \mathcal{M}_0\mathbf{w}_2|_{\xi=\pm h_0} + \mathcal{M}_1\mathbf{w}_1|_{\xi=\pm h_0} + \mathcal{M}_2\mathbf{w}_0|_{\xi=\pm h_0}. \tag{2.46}$$

Equation (2.44) can be rewritten in the form

$$D_\xi \mathcal{L}_4[w_{11}, w_{22}]^\top = -D_\xi \mu_0 [D_x w_{02}, 2D_\xi w_{12}]^\top. \tag{2.47}$$

Corresponding solution is then of the form:

$$w_{11}(x,\xi) = a_2(x) + b_2(x)\int_0^\xi \mu_0^{-1}(x,t)dt - \xi D_x w_{02}(x), \tag{2.48}$$

$$w_{22}(x,\xi) = c_2(x) + d_2(x)\int_0^\xi \frac{dt}{\lambda_0(x,t)} - \xi D_x a_2(x) \tag{2.49}$$

$$- D_x\left(b_2(x)\int_0^\xi \frac{(\xi-t)dt}{\mu_0(x,t)}\right) + \frac{\xi^2}{2}D_x^2 w_{02}(x) + 2D_x a_1(x)\cdot\int_0^\xi \frac{\mu_0(x,t)dt}{\lambda_0(x,t)}.$$

Substituting this solution in (2.46) one can obtain:

$$\boldsymbol{\sigma}_0^{(y)\pm}(x,\pm 0) = [b_2(x), d_2(x)], \tag{2.50}$$

or, that it follows immediately:

$$\boldsymbol{\sigma}_0^{(y)+}(x,+0) = \boldsymbol{\sigma}_0^{(y)-}(x,-0). \tag{2.51}$$

Unknown functions $a_2(x) - d_2(x)$ should be found from the equations for the first and the second components of (2.27) and (2.45), respectively and from (2.50).

Condition (2.51) together with (2.43) substitute the classical transmission conditions that completely coincide with those for compressible interphase in the case when the Lamé moduli of all materials are comparable in values [8].

2.3. Stiff weakly compressible interface

The last case may happen if the shear modulus of the intermediate thin layer μ is essentially greater than $\mu_\pm$:

$$\mu(x,\epsilon\xi) = \epsilon^{-1}\mu_0(x,\xi), \qquad \mu_0 \sim \mu_\pm. \tag{2.52}$$

Then from (2.10) one can conclude that:

$$\lambda(x,\epsilon\xi) = \epsilon^{-2}\lambda_0(x,\xi), \qquad \lambda_0 \sim \mu_0. \tag{2.53}$$

As a result, the BVP (2.16), (2.17) and

$$\boldsymbol{\sigma}^{(y)}_{\pm}(x, \pm\epsilon h_0) = \left(\epsilon^{-3}\mathcal{M}_0 + \epsilon^{-2}\mathcal{M}_1 + \epsilon^{-1}\mathcal{M}_2\right)\mathbf{w}|_{\xi=\pm h_0}, \tag{2.54}$$

must be considered. For the first two terms, absolutely the same line of reasoning should be repeated as in the previous subsection. Thus, solutions (2.23), (2.41) are still valid and the transmission conditions (2.43) are satisfied. For the third term, the problem (2.47) with boundary conditions:

$$\mathbf{0} = \mathcal{M}_0\mathbf{w}_2|_{\xi=\pm h_0} + \mathcal{M}_1\mathbf{w}_1|_{\xi=\pm h_0} + \mathcal{M}_2\mathbf{w}_0|_{\xi=\pm h_0}, \tag{2.55}$$

instead of (2.46) should be solved. Then, corresponding solution can be easily obtained from (2.48), (2.49) where $b_2(x) = d_2(x) = 0$, as it follows from (2.46), (2.50) and (2.55).

The fourth term has to be found from the following problem:

$$D_\xi \mathbf{A}_0 \lambda_0 D_\xi \mathbf{w}_3 + \mathcal{L}_1\mathbf{w}_2 + \mathcal{L}_2\mathbf{w}_1 + \mathcal{L}_3\mathbf{w}_0 = \mathbf{0}, \quad (x,\xi) \in \Omega_0, \tag{2.56}$$

$$\mathbf{u}_3^{\pm}(x, \pm 0) \pm h_0 D_y \mathbf{u}_2^{\pm}(x, \pm 0) + \frac{h_0^2}{2} D_y^2 \mathbf{u}_1^{\pm}(x, \pm 0) \pm \frac{h_0^3}{6} D_y^3 \mathbf{u}_0^{\pm}(x, \pm 0) = \mathbf{w}_3(x, \pm h_0), \tag{2.57}$$

$$\boldsymbol{\sigma}^{(y)}_{0\pm}(x, \pm 0) = \mathcal{M}_0\mathbf{w}_3|_{\xi=\pm h_0} + \mathcal{M}_1\mathbf{w}_2|_{\xi=\pm h_0} + \mathcal{M}_2\mathbf{w}_1|_{\xi=\pm h_0}. \tag{2.58}$$

Equation (2.56) can be eventually rewritten in the following manner:

$$D_\xi \left\{ \mathcal{L}_4 \begin{pmatrix} w_{21} \\ w_{32} \end{pmatrix} + \mu_0 \begin{pmatrix} D_x w_{12} \\ 2D_\xi w_{22} \end{pmatrix} \right\} = -4 D_x \mu_0 D_x \begin{pmatrix} w_{01} \\ 0 \end{pmatrix}, \tag{2.59}$$

while boundary condition (2.58) takes the form:

$$\boldsymbol{\sigma}^{(y)}_{0\pm}(x, \pm 0) = \left\{ \mathcal{L}_4 \begin{pmatrix} w_{21} \\ w_{32} \end{pmatrix} + \mu_0 \begin{pmatrix} D_x w_{12} \\ 2D_\xi w_{22} \end{pmatrix} \right\} \Big|_{\xi=\pm h_0}. \tag{2.60}$$

Equation (2.59) is integrated to give:

$$
\begin{aligned}
w_{21}(x,\xi) &= a_3(x) + \int\limits_0^{\xi} \frac{b_3(x)}{\mu_0(x,t)}dt \\
&\quad - \int\limits_0^{\xi} D_x w_{12}(x,t)dt - \int\limits_0^{\xi} \frac{4dt}{\mu_0(x,t)} \int\limits_0^{t} D_x\mu_0(x,s)D_x w_{01}(x,s)ds, \\
w_{32}(x,\xi) &= c_3(x) + \int\limits_0^{\xi} \frac{d_3(x)}{\mu_0(x,t)}dt \\
&\quad - \int\limits_0^{\xi} D_x w_{21}(x,t)dt - 2\int\limits_0^{\xi} \frac{\mu_0(x,t)}{\lambda_0(x,t)} D_t w_{22}(x,t)dt,
\end{aligned}
\tag{2.61}
$$

where functions $a_3(x) - d_3(x)$ should be calculated from Eqs. (2.45) and (2.57) for the first and second components, respectively and from equation (2.58). Additional solvability condition is then of the form:

$$
\boldsymbol{\sigma}^{(y)}_{0+}(x,+0) - \boldsymbol{\sigma}^{(y)}_{0-}(x,-0) = -D_x\eta(x)D_x\mathbf{A}_1\mathbf{u}_0(x,0), \tag{2.62}
$$

where conditions (2.20) and (2.43) have been taken into account and

$$
\eta(x) = 4\int\limits_{-h_0}^{h_0} \mu_0(x,t)dt = 4\int\limits_{-h}^{h} \mu(x,y)dy. \tag{2.63}
$$

3. Discussions and conclusions

Let us summarize all the results obtained in this paper and compare them with those known for the compressible interface.

Soft interface

In this case the transmission conditions for the weakly compressible interface take the form:

$$
[\boldsymbol{\sigma}^{(y)}] = \mathbf{0}, \qquad [\mathbf{u}] = \tau_1\mathbf{A}_1\boldsymbol{\sigma}^{(y)}, \qquad \mathbf{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.1}
$$

while those for the compressible interface can be evaluated (see [1, 6]) to give:

$$
[\boldsymbol{\sigma}^{(y)}] = \mathbf{0}, \qquad [\mathbf{u}] = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \boldsymbol{\sigma}^{(y)}, \tag{3.2}
$$

where

$$\tau_1(x) = \int_{-h}^{h} \frac{dy}{\mu(x,y)}, \qquad \tau_2(x) = \int_{-h}^{h} \frac{(1-2\nu(x,y))dy}{2(1-\nu(x,y))\mu(x,y)}. \tag{3.3}$$

Interface of comparable properties

The corresponding transmission conditions are the same for both weakly compressible and compressible interphases:

$$[\mathbf{u}] = \mathbf{0}, \qquad [\boldsymbol{\sigma}^{(y)}] = \mathbf{0}. \tag{3.4}$$

Stiff interface

In the case of weakly compressible interface, the corresponding transmission conditions are of the form:

$$[\mathbf{u}] = \mathbf{0}, \qquad [\boldsymbol{\sigma}^{(y)}] = -D_x\left(\eta_1(x) D_x \mathbf{A}_1 \mathbf{u}\right). \tag{3.5}$$

However, as it is shown in [6], the transmission conditions for the compressible interface take a slightly different form:

$$[\mathbf{u}] = \mathbf{0}, \qquad [\boldsymbol{\sigma}^{(y)}] = -D_x\left(\bar{\eta}(x) D_x \mathbf{A}_1 \mathbf{u}\right), \tag{3.6}$$

where

$$\eta(x) = 4\int_{-h}^{h} \mu(x,y)dy, \qquad \bar{\eta}(x) = \int_{-h}^{h} \frac{2\mu(x,y)dy}{1-\nu(x,y)}. \tag{3.7}$$

It is easy to observe that the transmission conditions (3.2) and (3.6), formally speaking, transform to (3.1) and (3.5), respectively, when $\nu \to 0.5$. In fact, $\tau_2 = O(\epsilon)$, $\bar{\eta} = \eta + O(\epsilon)$, when $\nu - 0.5 = O(\epsilon)$. One can think, hence, that the conditions (3.2) and (3.6) for the compressible interfaces can be simply used in the case of weakly compressible interfaces. Although this is the case for the comparable and the stiff interfaces, for the soft interface this may lead to an essential error. This fact has been shown in [6] by the accurate FEM-analysis. Moreover, application of the transmission conditions (3.2) in the case of weakly compressible interface lead to a drastic consequence when the interface crack problem with imperfect interface is under consideration. Namely, in this case asymptotic behaviour exhibits different features with each of the transmission conditions (3.2) and (3.1). For example, if the thickness of the interphase does not decrease to zero at the crack tip then the only logarithmic stress singularity exists under assumption of the transmission conditions (3.2), while using the conditions (3.1), the main stress component exhibits the square root stress singularity for the shear stress along the interface. Moreover, in any other direction just all stress components manifest the square root singularity and the only next asymptotic terms behave as a logarithm. For the general shape and/or mechanical properties of the interphase near the crack tip, the structure of the solution may have even more complex character. For details of this important feature, the reader is referred to the paper [4]. As a mathematical explanation of this phenomenon one can note that structures of the transmission conditions (3.2) and (3.1) are different, so the formal convergence from (3.2) to (3.1) leads to the singular perturbation of the interface crack problem.

Finally let us note that the transmission conditions obtained in the paper as well as all others mentioned here are valid, as it follows from the asymptotic procedure, only at a certain distance from the external boundary $\partial\Omega$ along the imperfect interface. To take into account the local edge effects, it is necessary to construct additionally the boundary layers modelling local behaviour of the accurate solution for the dissimilar body with a thin interface near the body boundary. In other words, exact forms of the boundary operators $\mathcal{B}_{\pm}$ and $\mathcal{B}$ in (2.4) have to be implemented in the analysis. The boundary layer solutions decrease exponentially, so the domains, where they influence essentially the entire solution, are very small. Decision whether such a correction is important or not in the problem under consideration should be made depending on the aim of the investigation. Some examples and advices concerning numerical estimates of the boundary layer size have been presented in [7].

Acknowledgments

This work has been supported by Grant 8 T 07A 05221 of Polish Committee for Scientific Research (KBN).

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Received May 12, 2003; revised version January 9, 2004.