Arch. Mech., **55**, 1, pp. 91–107, Warszawa 2003

On the asymptotic partition of energy in the theory of swelling porous elastic soils

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THE CESÀRO MEANS of various parts of the total energy are introduced in the context of the linear theory of swelling porous elastic soils. Then, the relations describing the asymptotic behavior of the Cesàro means are established.

1. Introduction

IT IS ACCEPTED that the swelling of soils, drying of fibers, wood, plants, paper, etc. are problems concerning the porous media theory. Several recent articles describe the work on the subject and introduce theories for fluids infiltrating elastic media (see [1, 2] and references therein). Most research, in this area, is devoted to some modification of the classical diffusion theory [3]: solids are considered be not to deformable, fluids incompressible and inertial forces are negligible. So, the main physics are diffusion and solid transport. On the other, hand the classical mixture theory approach has been applied to derive a comprehensive macroscopic constitutive theory for swelling porous media (see [4, 5, 6, 7]). A presentation of the continuum theory of mixtures can be found in review articles by BOWEN [8], ATKIN and CRAINE [9, 10] and BEDFORD and DRUMHELLER [11]. In these works, constitutive equations and equations of motion, for mixtures consisting of arbitrary number of fluids and elastic solids, have been obtained.

In [4], ERINGEN has developed a continuum theory for a mixture consisting of three components: an elastic solid, viscous fluid and gas. The intended applications of the theory are in the field of swelling, oil exploration, slurries and consolidation problems. The theory is relevant to problems in the oil exploration industry, since oil is viscous and is usually accompanied by gas in underground rocks, porous solid in slurries and muddy river beds. Consolidation problems in the building industry, earthquake problems, swelling of plants and living tissues and a plethora of other problems fall into the domain of mixture theory considered in [4]. It is also shown that the diffusion-type theories are special cases of the present theory. We note that the theory can be extended in order to incorporate other effects, disregarded here. In this sense, Eringen pointed out: "In some cases, it may be necessary to consider additional properties of mixtures.

For example, elastic solid and/or viscous fluid may require the consideration of memory effects. This is the case for viscoelastic materials..., dislocation problems require consideration of non-local effects. In these problems, stress at a point depends on strains at all points in the body. Plastic deformations of soils and mechanics of sands are problems that require consideration of permanent deformations. These are crucial to the building industry. This divergence is made here to point out to the vast field of mixture theories that are waiting future developments."

In the present paper we continue the study of fundamental qualitative properties of Eringen's mixture theory [4], that began with the papers [12, 13, 14, 15]. Such studies are important to assess whether a given theory is mathematically acceptable for use in a given physical problem. The purpose of this work is to investigate the asymptotic partition of total energy within the context of isothermal linear theory of swelling porous elastic soils.

The question of partition of energy in the asymptotic form was first studied by LAX and PHILLIPS [16] and BRODSKY [17]. Further, this problem has been studied by GOLDSTEIN [18, 19], DUFFIN [20], LEVINE [21]. In his analysis of the abstract wave equation, Goldstein applied the semigroup theory in order to obtain an equipartition theorem stating that the difference of the kinetic energy and the potential energy vanishes as the time approaches infinity. LEVIN [21] treated an abstract version of Goldstein's approach by use of the Lagrange identity method. His result represents a simplified proof that asymptotic equipartition occurs between the Cesàro means of the kinetic and potential energies, a fact first demonstrated by GOLDSTEIN [19].

The asymptotic equipartition between the mean kinetic and strain energies within the context of linear elastodynamics was established by DAY [22]. In the classical linear theory of thermoelasticity, Chiriţă [23] proved that the mean thermal energy tends to zero as time goes to infinity and the asymptotic equipartition occurs between the Cesàro means of the kinetic and strain energies.

This article describes the temporal behavior of solutions to the initial boundary value problems associated with the isothermal linear theory of swelling porous elastic soils. Using the method developed by Chiriţă [23], we introduce the Cesàro means of the kinetic, internal and dissipation energies. Then, with the aid of some auxiliary Lagrange–Brun identities derived in [14], we establish the relations that describe the asymptotic behavior of mean energies. In fact, we prove that asymptotic equipartition occurs between the Cesàro means of the kinetic and internal energies. Therefore, the results established by Day [22] and Chiriţă [23], for elasticity and thermoelasticity, concerning Cesàro means of the energies continue to hold (with corresponding modifications) in the framework of dynamic linear theory of swelling porous elastic soils.

The method developed in [23] has also been used in [24] to study the temporal behavior of solutions in the linear thermoelasticity of materials with voids.

2. Basic equations

We refer the motion of a continuum to a fixed system of rectangular Cartesian axes $0x_k$ $(k = 1,2,3)$. We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over integers $(1,2,3)$, summation over repeated subscripts is implied, subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, and a superposed dot denotes time differentiation.

We consider a body that at time $t = 0$ occupies the bounded regular region B of Euclidean three-dimensional space whose boundary is the regular surface ∂B.

We assume that B is occupied by a mixture consisting of three components: an elastic solid, a viscous fluid and a gas. We use superscripts s, f, g to denote respectively, the elastic solid, the fluid and the gas. Let ρ_0^s, ρ_0^f and ρ_0^g denote the densities at time $t = 0$ of the three constituents, respectively. We consider the fundamental equations for mechanical behavior of the mixture in the framework of the linearized theory (see [4]). The equations of motion in the absence of the body forces are

$$
\begin{aligned}
t_{ji,j}^s + p_i^f + p_i^g &= \rho_0^s \ddot{u}_i^s, \\
t_{ji,j}^f - p_i^f &= \rho_0^f \ddot{u}_i^f, \\
t_{ji,j}^g - p_i^g &= \rho_0^g \ddot{u}_i^g,
\end{aligned} \tag{2.1}
$$

where t_{ij}^s, t_{ij}^f and t_{ij}^g are the partial stress tensors, p_i^f and p_i^g are the internal body forces and u_i^s, u_i^f and u_i^g are the displacement vector fields.

The constitutive equations for a homogeneous and isotropic mixture are

$$
\begin{aligned}
t_{ij}^s &= \Big(-\sum_{a=f,g} \sigma^a e_{rr}^a + \lambda e_{rr}^s\Big)\delta_{ij} + 2\mu e_{ij}^s, \\
t_{ij}^f &= \Big(-\sigma^f e_{rr}^s - \sum_{a=f,g} \sigma^{fa} e_{rr}^a + \lambda_\nu \dot{e}_{rr}^f\Big)\delta_{ij} + 2\mu_\nu \dot{e}_{ij}^f, \\
t_{ij}^g &= \Big(-\sigma^g e_{rr}^s - \sum_{a=f,g} \sigma^{ga} e_{rr}^a\Big)\delta_{ij}, \\
p_i^a &= \sum_{b=f,g} \xi^{ab}(\dot{u}_i^b - \dot{u}_i^s), \quad a = f, g,
\end{aligned} \tag{2.2}
$$

where σ^a $(a = f, g)$, $\lambda, \mu, \sigma^{ab}$ $(a, b = f, g)$, $\lambda_\nu, \mu_\nu, \xi^{ab}$ $(a, b = f, g)$ are constitutive constants; δ_{ij} is the Kronecker delta; and e^s_{ij}, e^f_{ij} and e^g_{ij} are defined by

$$e^s_{ij} = \frac{1}{2}(u^s_{i,j} + u^s_{j,i}) \ , \quad e^f_{ij} = \frac{1}{2}(u^f_{i,j} + u^f_{j,i}) \ , \quad e^g_{ij} = \frac{1}{2}(u^g_{i,j} + u^g_{j,i}). \tag{2.3}$$

The coefficients in relation (2.2) have the following symmetries:

$$\sigma^{ab} = \sigma^{ba}, \ \xi^{ab} = \xi^{ba}, \ a, b = f, g. \tag{2.4}$$

To the system of field equations we adjoin boundary conditions and initial conditions. Many different types of boundary conditions are suggested in applications [8]–[11], [25]. We consider the following homogeneous boundary conditions:

$$\begin{aligned} &u^s_i = 0, \qquad u^f_i = 0, \qquad u^g_i = 0 \qquad \text{on } \overline{S}_1 \times [0, \infty), \\ &(t^s_{ji} + t^f_{ji} + t^g_{ji})n_j = 0, \ u^f_i - u^s_i = 0, \ u^g_i - u^s_i = 0 \quad \text{on } S_2 \times [0, \infty), \end{aligned} \tag{2.5}$$

where S_i $(i = 1, 2)$ are subsets of ∂B such that $\partial B = \overline{S}_1 \cup S_2$, $S_1 \cap S_2 = \emptyset$. Moreover, we adjoin the following initial conditions:

$$\begin{aligned} &u^s_i(\mathbf{x}, 0) = a^s_i(\mathbf{x}) \ , \ u^f_i(\mathbf{x}, 0) = a^f_i(\mathbf{x}) \ , \ u^g_i(\mathbf{x}, 0) = a^g_i(\mathbf{x}), \\ &\dot{u}^s_i(\mathbf{x}, 0) = b^s_i(\mathbf{x}) \ , \ \dot{u}^f_i(\mathbf{x}, 0) = b^f_i(\mathbf{x}) \ , \ \dot{u}^g_i(\mathbf{x}, 0) = b^g_i(\mathbf{x}), \ \ \mathbf{x} \in B, \end{aligned} \tag{2.6}$$

where a^s_i, a^f_i, a^g_i, b^s_i, b^f_i, b^g_i are prescribed fields. We denote by $(\mathcal{P})$ the initial–boundary value problem defined by the basic equations (2.1), the constitutive equations (2.2), the geometrical equations (2.3), the boundary conditions (2.5) and the initial conditions (2.6).

As was shown by ERINGEN [4], the local form of the Clausius–Duhem inequality implies that

$$3\lambda_\nu + 2\mu_\nu \geq 0, \ \mu_\nu \geq 0, \tag{2.7}$$

and the following symmetric matrix is positive semi–definite

$$\Delta = \begin{pmatrix} \xi^{ff} & \xi^{fg} \\ \xi^{gf} & \xi^{gg} \end{pmatrix}, \tag{2.8}$$

so that the dissipation energy density Φ defined by

$$\Phi = \lambda_\nu \dot{e}^f_{ii}\dot{e}^f_{jj} + 2\mu_\nu \dot{e}^f_{ij}\dot{e}^f_{ij} + \sum_{a,b=f,g} \xi^{ab}(\dot{u}^a_i - \dot{u}^s_i)(\dot{u}^b_i - \dot{u}^s_i), \tag{2.9}$$

is non-negative.

The internal energy density $\mathcal{E}$ is defined by

$$\mathcal{E} = \frac{1}{2}\lambda e^s_{ii}e^s_{jj} + \mu e^s_{ij}e^s_{ij} - \sum_{a=f,g} \sigma^a e^a_{ii}e^s_{jj} - \frac{1}{2}\sum_{a,b=f,g} \sigma^{ab} e^a_{ii}e^b_{jj}. \tag{2.10}$$

3. Hypotheses and some preliminary results

Throughout this paper we shall assume the following:

(i) the densities ρ_0^s, ρ_0^f and ρ_0^g are strictly positive;

(ii) the following symmetric matrix is positive definite:

$$\delta = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 & -\sigma^f & -\sigma^g \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 & -\sigma^f & -\sigma^g \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 & -\sigma^f & -\sigma^g \\ 0 & 0 & 0 & 2\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu & 0 & 0 \\ -\sigma^f & -\sigma^f & -\sigma^f & 0 & 0 & 0 & -\sigma^{ff} & -\sigma^{fg} \\ -\sigma^g & -\sigma^g & -\sigma^g & 0 & 0 & 0 & -\sigma^{fg} & -\sigma^{gg} \end{pmatrix}, \tag{3.1}$$

so, the internal energy density $\mathcal{E}$ defined by (2.10) is positive;

(iii) the symmetric matrix Δ is positive definite, that is we have

$$\xi_m \sum_{a=f,g} (\dot{u}_i^a - \dot{u}_i^s)(\dot{u}_i^a - \dot{u}_i^s) \le \sum_{a=f,g} \xi^{ab}(\dot{u}_i^a - \dot{u}_i^s)(\dot{u}_i^b - \dot{u}_i^s) \le \le \xi_M \sum_{a=f,g} (\dot{u}_i^a - \dot{u}_i^s)(\dot{u}_i^a - \dot{u}_i^s), \tag{3.2}$$

for any $\dot{u}_i^a - \dot{u}_i^s$, where $\xi_m > 0$ and $\xi_M > 0$ are the minimum and the maximum eigenvalues of ξ^{ab}, respectively.

Let us introduce the following energies:
the kinetic energy

$$\mathcal{K}(t) = \frac{1}{2} \int_B \Big(\sum_{\alpha=s,f,g} \rho_0^\alpha \dot{u}_i^\alpha(t) \dot{u}_i^\alpha(t) \Big) dv, \tag{3.3}$$

the internal energy

$$\mathcal{U}(t) = \int_B \mathcal{E}(t) dv, \tag{3.4}$$

the dissipation energy

$$\mathcal{D}(t) = \int_0^t \int_B \Phi(\tau) dv\, d\tau, \tag{3.5}$$

the total energy

$$E(t) = \mathcal{K}(t) + \mathcal{U}(t) + \mathcal{D}(t), \tag{3.6}$$

and

$$I(t) = \frac{1}{2}\int_B \Big(\sum_{\alpha=s,f,g} \rho_0^\alpha u_i^\alpha(t) u_i^\alpha(t)\Big) dv + \frac{1}{2}\int_0^t\int_B \Big[\lambda_\nu e_{ii}^f(\tau) e_{jj}^f(\tau) \tag{3.7}$$
$$+ 2\mu_\nu e_{ij}^f(\tau) e_{ij}^f(\tau) + \sum_{a,b=f,g} \xi^{ab}\Big(u_i^a(\tau) - u_i^s(\tau)\Big)\Big(u_i^b(\tau) - u_i^s(\tau)\Big)\Big] dv\, d\tau.$$

Now, we recall some preliminary integral identities of Lagrange–Brun type [26], established in [14], that are essential in studying the temporal behavior of the solutions of the initial–boundary value problem $(\mathcal{P})$. For the readability of the paper we prefer to give here the proofs. Thus, in the present context, the lemmas 1,2 and 3 derived in [14] are:

LEMMA 1. (Conservation law of total energy). *For every* (u_i^s, u_i^f, u_i^g) *satisfying the equations of motion* (2.1), *the constitutive equations* (2.2) *and the geometrical equations* (2.3), *we have*

$$E(t) = E(0) + \int_0^t P(\tau,\tau) d\tau\ , \qquad t \in [0,\infty) \tag{3.8}$$

where

$$P(t,\tau) = \int_{\partial B} \Big(\sum_{\alpha=s,f,g} t_{ji}^\alpha(t) \dot{u}_i^\alpha(\tau)\Big) n_j\, da. \tag{3.9}$$

P r o o f. From the relations (2.2) and (2.10) it follows that

$$\sum_{\alpha=s,f,g} t_{ij}^\alpha \dot{e}_{ij}^\alpha = \frac{\partial \mathcal{E}}{\partial t} + \lambda_\nu \dot{e}_{ii}^f \dot{e}_{jj}^f + 2\mu_\nu \dot{e}_{ij}^f \dot{e}_{ij}^f. \tag{3.10}$$

On the other hand, in view of (2.1)–(2.3) we have

$$\sum_{\alpha=s,f,g} t_{ij}^\alpha \dot{e}_{ij}^\alpha = -\frac{1}{2}\frac{\partial}{\partial t}\Big(\sum_{\alpha=s,f,g} \rho_0^\alpha \dot{u}_i^\alpha \dot{u}_i^\alpha\Big) \tag{3.11}$$
$$- \sum_{a,b=f,g} \xi^{ab}(\dot{u}_i^a - \dot{u}_i^s)(\dot{u}_i^b - \dot{u}_i^s) + \Big(\sum_{\alpha=s,f,g} t_{ji}^\alpha \dot{u}_i^\alpha\Big)_{,j}.$$

Then from the relations (3.10) and (3.11) we get

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\sum_{\alpha=s,f,g}\rho_0^\alpha \dot{u}_i^\alpha \dot{u}_i^\alpha + \mathcal{E} + \int_0^t \Phi(\tau)d\tau\right) = \left(\sum_{\alpha=s,f,g} t_{ji}^\alpha \dot{u}_i^\alpha\right)_{,j}. \tag{3.12}$$

By an integration of the relation (3.12) over $B\times[0,t]$, and by using the divergence theorem and the relations (3.3)–(3.6) and (3.9), we obtain the identity (3.8) and the proof is complete.

LEMMA 2. *If* (u_i^s, u_i^f, u_i^g) *satisfies the relations* (2.1), (2.2) *and* (2.3), *then for every* $t\in[0,\infty)$

$$\frac{dI}{dt}(t) = \frac{dI}{dt}(0) + \int_0^t [4\mathcal{K}(\tau) + 2\mathcal{D}(\tau)]d\tau - 2E(0)t - 2\int_0^t\int_0^\tau P(r,r)dr\,d\tau + \int_0^t \mathcal{W}(\tau,\tau)d\tau. \tag{3.13}$$

where

$$\mathcal{W}(t,\tau) = \int_{\partial B}\left(\sum_{\alpha=s,f,g} t_{ji}^\alpha(t)u_i^\alpha(\tau)\right)n_j\,da\ . \tag{3.14}$$

□

P r o o f. It follows from (2.2) and (2.10) that

$$\sum_{\alpha=s,f,g} t_{ij}^\alpha e_{ij}^\alpha = 2\mathcal{E} + \lambda_\nu \dot{e}_{ii}^f e_{jj}^f + 2\mu_\nu \dot{e}_{ij}^f e_{ij}^f. \tag{3.15}$$

By taking into account the relations (2.1)–(2.3) we obtain

$$\sum_{\alpha=s,f,g} t_{ij}^\alpha e_{ij}^\alpha = -\frac{\partial}{\partial t}\left(\sum_{\alpha=s,f,g}\rho_0^\alpha \dot{u}_i^\alpha u_i^\alpha\right) - \sum_{a,b=f,g}\xi^{ab}(\dot{u}_i^a - \dot{u}_i^s)(u_i^b - u_i^s) + \sum_{\alpha=s,f,g}\rho_0^\alpha \dot{u}_i^\alpha \dot{u}_i^\alpha + \left(\sum_{\alpha=s,f,g} t_{ji}^\alpha u_i^\alpha\right)_{,j}. \tag{3.16}$$

Then the relations (3.15) and (3.16) imply

$$\frac{\partial}{\partial t}\left(\sum_{\alpha=s,f,g}\rho_0^\alpha \dot{u}_i^\alpha u_i^\alpha\right) + \lambda_\nu \dot{e}_{ii}^f e_{jj}^f + 2\mu_\nu \dot{e}_{ij}^f e_{ij}^f + \sum_{a,b=f,g}\xi^{ab}(\dot{u}_i^a - \dot{u}_i^s)(u_i^b - u_i^s) = \sum_{\alpha=s,f,g}\rho_0^\alpha \dot{u}_i^\alpha \dot{u}_i^\alpha - 2\mathcal{E} + \left(\sum_{\alpha=s,f,g} t_{ji}^\alpha u_i^\alpha\right)_{,j}. \tag{3.17}$$

If we integrate relation (3.17) over $B \times [0,t]$ and use the divergence theorem and the relations (3.3), (3.4), (3.7) and (3.14), then we get

$$\frac{dI}{dt}(t) = \frac{dI}{dt}(0) + 2\int_0^t [\mathcal{K}(\tau) - \mathcal{U}(\tau)]d\tau + \int_0^t \mathcal{W}(\tau,\tau)d\tau. \tag{3.18}$$

A combination of the relations (3.8) and (3.18) gives the identity (3.13) and the proof is complete.

LEMMA 3. *For every* (u_i^s, u_i^f, u_i^g) *satisfying* (2.1) *to* (2.3), *the following identity holds*

$$\frac{dI}{dt}(t) = L(t) + \Lambda(t) + \frac{1}{2}\int_0^t [\mathcal{W}(t-\tau, t+\tau) - \mathcal{W}(t+\tau, t-\tau)]d\tau, \quad t > 0 \tag{3.19}$$

where

$$L(t) = \frac{1}{2}\int_B \left[\sum_{\alpha=s,f,g} \rho_0^\alpha u_i^\alpha(0)\dot{u}_i^\alpha(2t) + \sum_{\alpha=s,f,g} \rho_0^\alpha \dot{u}_i^\alpha(0) u_i^\alpha(2t) \right] dv, \tag{3.20}$$

and

$$\Lambda(t) = \frac{1}{2}\int_B \Bigg[\lambda_\nu e_{ii}^f(0)e_{jj}^f(2t) + 2\mu_\nu e_{ij}^f(0)e_{ij}^f(2t) + \sum_{a,b=f,g} \xi^{ab}\Big(u_i^a(0) - u_i^s(0)\Big)\Big(u_i^b(2t) - u_i^s(2t)\Big)\Bigg] dv. \tag{3.21}$$

□

P r o o f. Let us introduce the notation

$$R(t,\tau) = \sum_{\alpha=s,f,g} t_{ij}^\alpha(t)e_{ij}^\alpha(\tau). \tag{3.22}$$

Then, by (2.2) and (2.4), we obtain

$$\begin{aligned} R(t-\tau, t+\tau) - R(t+\tau, t-\tau) &= \lambda_\nu \dot{e}_{ii}^f(t-\tau)e_{jj}^f(t+\tau) \\ &\quad + 2\mu_\nu \dot{e}_{ij}^f(t-\tau)e_{ij}^f(t+\tau) - \lambda_\nu \dot{e}_{ii}^f(t+\tau)e_{jj}^f(t-\tau) - 2\mu_\nu \dot{e}_{ij}^f(t+\tau)e_{ij}^f(t-\tau) \\ &= -\frac{\partial}{\partial\tau}\Big(\lambda_\nu e_{ii}^f(t-\tau)e_{jj}^f(t+\tau) + 2\mu_\nu e_{ij}^f(t-\tau)e_{ij}^f(t+\tau)\Big). \end{aligned} \tag{3.23}$$

On the other hand, by means of the relations (2.1)–(2.3), we get

$$(3.24)\quad R(t-\tau,t+\tau)-R(t+\tau,t-\tau)$$

$$=\frac{\partial}{\partial\tau}\Big[\sum_{\alpha=s,f,g}\Big(\rho_0^\alpha\dot u_i^\alpha(t-\tau)u_i^\alpha(t+\tau)+\rho_0^\alpha u_i^\alpha(t-\tau)\dot u_i^\alpha(t+\tau)\Big)\Big]$$

$$+\frac{\partial}{\partial\tau}\Big[\sum_{a,b=f,g}\xi^{ab}\Big(u_i^a(t-\tau)-u_i^s(t-\tau)\Big)\Big(u_i^b(t+\tau)-u_i^s(t+\tau)\Big)\Big]$$

$$+\Big[\sum_{\alpha=s,f,g}\Big(t_{ji}^\alpha(t-\tau)u_i^\alpha(t+\tau)-t_{ji}^\alpha(t+\tau)u_i^\alpha(t-\tau)\Big)\Big]_{,j}.$$

Further, from (3.23), we get

$$(3.25)\quad \int_0^t\int_B[R(t-\tau,t+\tau)-R(t+\tau,t-\tau)]dv\,d\tau$$

$$=-\int_B\Big[\lambda_\nu e_{ii}^f(0)e_{jj}^f(2t)+2\mu_\nu e_{ij}^f(0)e_{ij}^f(2t)\Big]dv$$

$$+\int_B\Big[\lambda_\nu e_{ii}^f(t)e_{jj}^f(t)+2\mu_\nu e_{ij}^f(t)e_{ij}^f(t)\Big]dv.$$

From (3.14), (3.20) and (3.24) we deduce

$$(3.26)\quad \int_0^t\int_B[R(t-\tau,t+\tau)-R(t+\tau,t-\tau)]dv\,d\tau$$

$$=2L(t)-2\int_B\Big(\sum_{\alpha=s,f,g}\rho_0^\alpha\dot u_i^\alpha(t)u_i^\alpha(t)\Big)dv$$

$$+\int_B\Big[\sum_{a,b=f,g}\xi^{ab}\Big(u_i^a(0)-u_i^s(0)\Big)\Big(u_i^b(2t)-u_i^s(2t)\Big)\Big]dv$$

$$-\int_B\Big[\sum_{a,b=f,g}\xi^{ab}\Big(u_i^a(t)-u_i^s(t)\Big)\Big(u_i^b(t)-u_i^s(t)\Big)\Big]dv$$

$$+\int_0^t[\mathcal{W}(t-\tau,t+\tau)-\mathcal{W}(t+\tau,t-\tau)]d\tau.$$

A combination of the relations (3.25) and (3.26) implies the desired result. □

4. Cesàro means and the asymptotic partition

In this section we study the time asymptotic behavior of the solutions of the problem ($\mathcal{P}$) defined by the relations (2.1) to (2.6). To this end, we introduce the Cesàro means of various parts of the total energy and then, using the identities (3.8), (3.13) and (3.19), we establish the relations that describe the asymptotic behavior of the mean energies.

If (u_i^s, u_i^f, u_i^g) is a solution for the problem ($\mathcal{P}$), then we introduce the Cesàro means

$$\mathcal{K}_C(t) := \frac{1}{t}\int_0^t \mathcal{K}(\tau)d\tau \ , \tag{4.1}$$

$$\mathcal{U}_C(t) := \frac{1}{t}\int_0^t \mathcal{U}(\tau)d\tau \ , \tag{4.2}$$

$$\mathcal{D}_C(t) := \frac{1}{t}\int_0^t \mathcal{D}(\tau)d\tau \ . \tag{4.3}$$

If meas $S_1 = 0$, where meas S represents the area $\int_S da$ of the surface S. Then there exists a family of rigid motions ($u_i^s = u_i^f = u_i^g = c_i + \varepsilon_{ijk}x_j d_k$, c_i, d_i – constants, ε_{ijk} – alternating symbol) that satisfy equations of motion (2.1), constitutive equations (2.2) and the boundary conditions (2.5). For this reason, we decompose the initial data a_i^s and b_i^s as

$$a_i^s = a_i^{\star s} + U_i^{0s} \ , \qquad b_i^s = b_i^{\star s} + V_i^{0s} \ , \tag{4.4}$$

where $a_i^{\star s}$ and $b_i^{\star s}$ are rigid displacements determined in such a way that

$$\int_B \rho_0^s U_i^{0s} dv = 0 \ , \qquad \int_B \rho_0^s \varepsilon_{ijk} x_j U_k^{0s} dv = 0 \ , \tag{4.5}$$
$$\int_B \rho_0^s V_i^{0s} dv = 0 \ , \qquad \int_B \rho_0^s \varepsilon_{ijk} x_j V_k^{0s} dv = 0 \ .$$

We consider the sets

$$\hat{\mathbf{C}}^1(B) := \{\mathbf{v} = (v_1, v_2, v_3), v_i \in C^1(\overline{B}) : v_i = 0 \text{ on } S_1 \text{ and if meas } S_1 = 0 \ ,$$
$$\text{then } \int_B \rho_0^s v_i dv = 0 \ , \qquad \int_B \rho_0^s \varepsilon_{ijk} x_j v_k dv = 0 \ \}.$$

$\hat{\mathbf{W}}_1(B)$:=the completion of $\hat{\mathbf{C}}^1$(B) by means of $||\cdot||_{\mathbf{W}_1(B)}$ where $C^1(\overline{B})$ represents the set of scalar functions that are continuous and continuously differentiable on $\overline{B}$. Moreover $\mathbf{W}_m(B) := [W_m(B)]^3$, where $W_m(B)$ is the familiar Sobolev space (see [27]).

The hypothesis (ii) assures that the following inequality [28] holds:

$$\int_B [\lambda v_{i,i}v_{j,j} + \frac{\mu}{2}(v_{i,j} + v_{j,i})(v_{i,j} + v_{j,i})]dv \geq m_1 \int_B v_i v_i dv \ , \tag{4.6}$$

$$m_1 = \text{const} > 0 \ , \qquad \forall\ \mathbf{v} \in \hat{\mathbf{W}}_\mathbf{1}(\mathbf{B}) \ .$$

If meas $S_1 = 0$, then we shall find it is a convenient practice to decompose the solution (u_i^s, u_i^f, u_i^g) in the form

$$u_i^s = a_i^{\star s} + tb_i^{\star s} + v_i^s, \quad u_i^f = a_i^{\star s} + tb_i^{\star s} + v_i^f, \quad u_i^g = a_i^{\star s} + tb_i^{\star s} + v_i^g, \tag{4.7}$$

where $(\mathbf{v^s}, \mathbf{v^f}, \mathbf{v^g}) \in \hat{\mathbf{W}}_\mathbf{1}(\mathbf{B}) \times \mathbf{W_1}(\mathbf{B}) \times \mathbf{W_1}(\mathbf{B})$ represents the solution of the initial boundary value problem $(\mathcal{P})$ in which the initial conditions are substituted by

$$\begin{aligned} &v_i^s = U_i^{0s}, \qquad v_i^f = a_i^f - a_i^{\star s}, \qquad v_i^g = a_i^g - a_i^{\star s}, \\ &\dot{v}_i^s = V_i^{0s}, \qquad \dot{v}_i^f = b_i^f - b_i^{\star s}, \qquad \dot{v}_i^g = b_i^g - b_i^{\star s}, \ \text{ on } B,\ t = 0. \end{aligned} \tag{4.8}$$

We are now ready to derive the asymptotic partition of the energies.

THEOREM 1. *Let (u_i^s, u_i^f, u_i^g) be a solution of the initial boundary value problem $(\mathcal{P})$. Then, for all choices of initial data $\mathbf{a^s}$, $\mathbf{a^f}$, $\mathbf{a^g}$, $\mathbf{b^f} \in \mathbf{W_1}(\mathbf{B})$, $\mathbf{b^s}$, $\mathbf{b^g} \in \mathbf{W_0}(\mathbf{B})$, we have:*

$1)^0$ *if* meas $S_1 \neq 0$, *then*

$$\lim_{t\to\infty} \mathcal{K}_C(t) = \lim_{t\to\infty} \mathcal{U}_C(t), \tag{4.9}$$

$$\lim_{t\to\infty} \mathcal{D}_C(t) = E(0) - 2\lim_{t\to\infty} \mathcal{K}_C(t) = E(0) - 2\lim_{t\to\infty} \mathcal{U}_C(t). \tag{4.10}$$

$2)^0$ *if* meas $S_1 = 0$, *then*

$$\lim_{t\to\infty} \mathcal{K}_C(t) = \lim_{t\to\infty} \mathcal{U}_C(t) + \frac{1}{2}\int_B (\rho_0^s b_i^{\star s} b_i^{\star s} + \sum_{a=f,g} \rho_0^a b_i^a b_i^{\star s})dv, \tag{4.11}$$

$$\begin{aligned} \lim_{t\to\infty} \mathcal{D}_C(t) &= E(0) - 2\lim_{t\to\infty} \mathcal{K}_C(t) + \frac{1}{2}\int_B (\rho_0^s b_i^{\star s} b_i^{\star s} + \sum_{a=f,g} \rho_0^a b_i^a b_i^{\star s})dv \\ &= E(0) - 2\lim_{t\to\infty} \mathcal{U}_C(t) - \frac{1}{2}\int_B (\rho_0^s b_i^{\star s} b_i^{\star s} + \sum_{a=f,g} \rho_0^a b_i^a b_i^{\star s})dv. \end{aligned} \tag{4.12}$$

P r o o f. By taking into account the fact that (u_i^s, u_i^f, u_i^g) is the solution of the problem $(\mathcal{P})$, from (3.8), we deduce

$$\mathcal{K}(t) + \mathcal{U}(t) + \mathcal{D}(t) = E(0), \qquad t \leq 0. \tag{4.13}$$

If we further use the relations (3.13) and (3.19), we get

$$\int_0^t [4\mathcal{K}(\tau) + 2\mathcal{D}(\tau)]d\tau = 2E(0)t + \Lambda(t) + \Gamma(t) - \frac{dI}{dt}(0), \qquad t \geq 0. \tag{4.14}$$

A combination of the relations (4.13) and (4.14) leads to the identity

$$\mathcal{K}_C(t) - \mathcal{U}_C(t) = \frac{1}{2t}\left[\Lambda(t) + \Gamma(t) - \frac{dI}{dt}(0)\right]. \tag{4.15}$$

By letting t tend to infinity and making use of the relations (3.20) and (3.21), we obtain

$$\begin{aligned} \lim_{t\to\infty}[\mathcal{K}_C(t) - \mathcal{U}_C(t)] = \lim_{t\to\infty} \frac{1}{4t} \int_B \Big\{ \sum_{\alpha=s,f,g} \rho_0^\alpha u_i^\alpha(0)\dot{u}_i^\alpha(2t) \\ + \sum_{\alpha=s,f,g} \rho_0^\alpha \dot{u}_i^\alpha(0)u_i^\alpha(2t) + \lambda_\nu e_{ii}^f(0)e_{jj}^f(2t) + 2\mu_\nu e_{ij}^f(0)e_{ij}^f(2t) \\ + \sum_{a,b=f,g} \xi^{ab}\Big(u_i^a(0) - u_i^s(0)\Big)\Big(u_i^b(2t) - u_i^s(2t)\Big)\Big\}dv. \end{aligned} \tag{4.16}$$

On the basis of the hypotheses (i)–(iii), relations (2.7), (3.3)–(3.5), (4.13) and Schwarz's inequality, we deduce for the terms in the right-hand side of (4.16) the following estimates:

$$\begin{aligned} \int_B \sum_{\alpha=s,f,g} \rho_0^\alpha u_i^\alpha(0)\dot{u}_i^\alpha(2t)dv \leq \Big(\int_B \sum_{\alpha=s,f,g} \rho_0^\alpha u_i^\alpha(0)u_i^\alpha(0)dv\Big)^{1/2} \\ \times \Big(\int_B \sum_{\alpha=s,f,g} \rho_0^\alpha \dot{u}_i^\alpha(2t)\dot{u}_i^\alpha(2t)dv\Big)^{1/2} \\ \leq \sqrt{2E(0)}\Big(\int_B \sum_{\alpha=s,f,g} \rho_0^\alpha u_i^\alpha(0)u_i^\alpha(0)dv\Big)^{1/2}; \end{aligned} \tag{4.17}$$

$$(4.18)\quad \int\limits_B \sum_{\alpha=s,f,g} \rho_0^\alpha \dot{u}_i^\alpha(0) u_i^\alpha(2t)\,dv = \int\limits_B \Big\{ \sum_{\alpha=s,f,g} \rho_0^\alpha \dot{u}_i^\alpha(0) u_i^s(2t)$$

$$+ \int\limits_0^{2t} \sum_{a=f,g} \rho_0^a \dot{u}_i^a(0)[\dot{u}_i^a(\tau) - \dot{u}_i^s(\tau)]\,d\tau + \sum_{a=f,g} \rho_0^a \dot{u}_i^a(0)[u_i^a(0) - u_i^s(0)]\Big\} dv$$

$$\leq \int\limits_B \sum_{\alpha=s,f,g} \rho_0^\alpha \dot{u}_i^\alpha(0) u_i^s(2t)\,dv + \Bigg(\int\limits_0^{2t}\int\limits_B \sum_{a=f,g} \frac{(\rho_0^a)^2}{\xi_m} \dot{u}_i^a(0)\dot{u}_i^a(0)\,dv\,d\tau \Bigg)^{1/2}$$

$$\times \Bigg(\int\limits_0^{2t}\int_B \sum_{a,b=f,g} \xi^{ab}[\dot{u}_i^a(\tau) - \dot{u}_i^s(\tau)][\dot{u}_i^b(\tau) - \dot{u}_i^s(\tau)]\,dv\,d\tau \Bigg)^{1/2}$$

$$+ \int\limits_B \sum_{a=f,g} \rho_0^a \dot{u}_i^a(0)[u_i^a(0) - u_i^s(0)]\,dv \leq \int\limits_B \sum_{\alpha=s,f,g} \rho_0^\alpha \dot{u}_i^\alpha(0) u_i^s(2t)\,dv$$

$$+ \sqrt{2tE(0)} \Bigg(\int\limits_B \sum_{a=f,g} \frac{(\rho_0^a)^2}{\xi_m} \dot{u}_i^a(0)\dot{u}_i^a(0)\,dv \Bigg)^{1/2}$$

$$+ \int\limits_B \sum_{a=f,g} \rho_0^a \dot{u}_i^a(0)[u_i^a(0) - u_i^s(0)]\,dv;$$

$$(4.19)\quad \int\limits_B [\lambda_\nu e_{ii}^f(0) e_{jj}^f(2t) + 2\mu_\nu e_{ij}^f(0) e_{ij}^f(2t)]\,dv = \int\limits_0^{2t}\int\limits_B [\lambda_\nu e_{ii}^f(0) \dot{e}_{jj}^f(\tau)$$

$$+ 2\mu_\nu e_{ij}^f(0)\dot{e}_{ij}^f(\tau)]\,dv\,d\tau + \int\limits_B [\lambda_\nu e_{ii}^f(0) e_{jj}^f(0) + 2\mu_\nu e_{ij}^f(0) e_{ij}^f(0)]\,dv$$

$$\leq \sqrt{2tE(0)} \Bigg(\int\limits_B [\lambda_\nu e_{ii}^f(0) e_{jj}^f(0) + 2\mu_\nu e_{ij}^f(0) e_{ij}^f(0)]\,dv \Bigg)^{1/2}$$

$$+ \int\limits_B [\lambda_\nu e_{ii}^f(0) e_{jj}^f(0) + 2\mu_\nu e_{ij}^f(0) e_{ij}^f(0)]\,dv;$$

$$(4.20)\quad \int_B \sum_{a,b=f,g} \xi^{ab}\Big(u_i^a(0)-u_i^s(0)\Big)\Big(u_i^b(2t)-u_i^s(2t)\Big)dv$$

$$= \int_0^{2t}\int_B \sum_{a,b=f,g} \xi^{ab}\Big(u_i^a(0)-u_i^s(0)\Big)\Big(\dot{u}_i^b(\tau)-\dot{u}_i^s(\tau)\Big)dv\, d\tau$$

$$+\int_B \sum_{a,b=f,g} \xi^{ab}\Big(u_i^a(0)-u_i^s(0)\Big)\Big(u_i^b(0)-u_i^s(0)\Big)dv$$

$$\leq \sqrt{2tE(0)}\Big(\int_B \sum_{a,b=f,g} \xi^{ab}\Big(u_i^a(0)-u_i^s(0)\Big)\Big(u_i^b(0)-u_i^s(0)\Big)dv\Big)^{1/2}$$

$$+\int_B \sum_{a,b=f,g} \xi^{ab}\Big(u_i^a(0)-u_i^s(0)\Big)\Big(u_i^b(0)-u_i^s(0)\Big)dv.$$

Using the estimates (4.17)–(4.20) in (4.16) we obtain

$$(4.21)\qquad \lim_{t\to\infty}[\mathcal{K}_C(t)-\mathcal{U}_C(t)] = \lim_{t\to\infty}\frac{1}{4t}\int_B \sum_{\alpha=s,f,g} \rho_0^\alpha \dot{u}_i^\alpha(0)u_i^s(2t)dv.$$

Let us first consider $1)^0$. Since meas $S_1 \neq 0$ and $\mathbf{u}^s \in \hat{\mathbf{W}}_1(\mathbf{B})$, from (2.10), (3.4), (3.6), (4.6) and (4.13), we deduce

$$(4.22)\qquad \int_B u_i^s(\tau)u_i^s(\tau)dv \leq \frac{1}{m_1}\int_B 2\mathcal{E}(\tau)dv \leq \frac{2}{m_1}E(0)\ ,$$

so, by means of the Schwarz inequality, we get

$$(4.23)\qquad \lim_{t\to\infty}\frac{1}{4t}\int_B \sum_{\alpha=s,f,g} \rho_0^\alpha \dot{u}_i^\alpha(0)u_i^s(2t)dv = 0.$$

Thus, the relations (4.21) and (4.23) give the relation (4.9). A combination of the relations (4.9) and (4.13) give the relation (4.10).

Let us consider $2)^0$. Using the decomposition (4.4) and (4.7), we have

$$\frac{1}{4t}\int_B \sum_{\alpha=s,f,g} \rho_0^\alpha \dot{u}_i^\alpha(0) u_i^s(2t)dv \tag{4.24}$$

$$= \frac{1}{4t}\int_B \Big[\rho_0^s(b_i^{\star s} + V_i^{0s}) + \sum_{a=f,g} \rho_0^a b_i^a\Big]\Big[a_i^{\star s} + 2tb_i^{\star s} + v_i^s(2t)\Big] dv$$

$$= \frac{1}{4t}\int_B \Big[\rho_0^s b_i^{\star s} a_i^{\star s} + \sum_{a=f,g} \rho_0^a b_i^a a_i^{\star s}\Big] dv + \frac{1}{4t}\int_B \sum_{\alpha=s,f,g} \rho_0^\alpha b_i^\alpha v_i^s(2t)dv$$

$$+ \frac{1}{2}\int_B \Big[\rho_0^s b_i^{\star s} b_i^{\star s} + \sum_{a=f,g} \rho_0^a b_i^a b_i^{\star s}\Big] dv.$$

The Korn inequality (4.6) and the relations (2.10), (3.4) and (4.13) imply

$$\int_B v_i^s(\tau) v_i^s(\tau) dv \le \frac{1}{m_1}\int_B 2\mathcal{E}(\tau)dv \le \frac{2}{m_1}E(0)\ , \tag{4.25}$$

so, by means of the Schwarz inequality, we deduce

$$\lim_{t\to\infty}\frac{1}{4t}\int_B \sum_{\alpha=s,f,g} \rho_0^\alpha \dot{u}_i^\alpha(0) u_i^s(2t)dv = \frac{1}{2}\int_B \Big[\rho_0^s b_i^{\star s} b_i^{\star s} + \sum_{a=f,g} \rho_0^a b_i^a b_i^{\star s}\Big] dv\ . \tag{4.26}$$

Thus, using (4.26) from (4.21) we obtain (4.11). The relation (4.12) follows then by coupling the relations (4.11) and (4.13). The proof is complete. □

REMARK 1. Relations (4.9) and (4.11) (restricted to the class of initial data for which $b_i^{\star s} = 0$) prove the asymptotic equipartition of the mean kinetic and internal energies.

REMARK 2. Similarly to the previous papers concerning asymptotic partition of energy, we supposed that body forces are absent ($f_i^s = f_i^f = f_i^g = 0$) and this assumption is essential in our analysis. Generally, the asymptotic partition of energy will be modified by the presence of body forces. For a solution (u_i^s, u_i^f, u_i^g) of the problem $(\mathcal{P})$ the relation (4.13) becomes

$$\mathcal{K}(t) + \mathcal{U}(t) + \mathcal{D}(t) = E(0) + \int_0^t\int_B \sum_{\alpha=s,f,g} \rho_0^\alpha f_i^\alpha(\tau)\dot{u}_i^\alpha(\tau)dv d\tau, \quad t \le 0 \tag{4.13'}$$

and it leads to estimate terms like that in the right-hand side of (4.16). Our analysis in the above can be applied under appropriate assumptions concerning the behavior of the forces at infinity, but the calculation becomes more complicated.

References

1. M. A. MURAD and J.H. CUSHMAN, *Thermomechanical theories for swelling porous media with microstructure*, Int. J. Engng Sci. **38**, 517–564, 2000.

2. L.E. PAYNE, J.F. RODRIGUES and B. STRAUGHAN, *Effect of anisotropic permeability on Darcy's law*, Math. Methods Appl. Sci. **24**, 427–438, 2001.

3. J.R. PHILIP, *Hydrostatics and hydrodynamics in swelling soils*, Water Resour. Res. **5**, 1070–1077, 1969.

4. A.C. ERINGEN, *A continuum theory of swelling porous elastic soils*, Int. J. Engng Sci. **32**, 1337–1349, 1994.

5. T. HUECKEL, *Water mineral interaction in hydromechanics of clays exposed to environmental loads: a mixture theory approach*, Can. Geotech. J. **29**, 1071–1086, 1992.

6. T. HUECKEL, *Effects of inter-phase mass transfer in heated clays: A mixture theory*, Int. J. Engng Sci. **30**, 1567–1582, 1992.

7. T.K. KARALIS, *Water flow in non-saturated swelling soil*, Int. J. Engng Sci. **31**, 751–774, 1993.

8. R.M. BOWEN, *Theory of mixtures*, In Continuum Physics, A.C. ERINGEN [Ed.], Vol. III. Academic Press, New York 1976.

9. R.J. ATKIN and R.E. CRAINE, *Continuum theories of mixtures: basic theory and historical development*, Q.J. Mech. Appl. Math., **29**, 209–245, 1976.

10. R.J. ATKIN and R.E. CRAINE, *Continuum theories of mixtures: applications*, J. Inst. Math. Appl., **17**, 153–207, 1976.

11. A. BEDFORD and D.S. DRUMHELLER, *Theory of immiscible and structured mixtures*, Int.J. Engng Sci., **21**, 863–960, 1983.

12. R. QUINTANILLA, *On the linear problem of swelling porous elastic soils*, J. Math. Anal. Appl., **269**, 50–72, 2002.

13. R. QUINTANILLA, *On the linear problem of swelling porous elastic soils with incompressible fluid*, Int.J. Engng Sci., **40**, 1485–1494, 2002.

14. C. GALEŞ, *Some uniqueness and continuous dependence results in the theory of swelling porous elastic soils*, Int. J. Engng Sci. **40**, 1211–1231, 2002.

15. C. GALEŞ, *On the spatial behavior in the theory of swelling porous elastic soils*, Int. J. Solids Structures, **39**, 4151–4165.

16. P.D. LAX and R.S. PHILLIPS, *Scattering theory*, Academic Press, New York 1967.

17. A.R. BRODSKY, *On the asymptotic behavior of solutions of the wave equation*, Proc. Amer. Math. Soc., **18**, 207–208, 1967.

18. J.A. GOLDSTEIN, *An asymptotic property of solutions of wave equations*, Proc. Amer. Math. Soc., **23**, 359–363, 1969.

19. J.A. GOLDSTEIN, *An asymptotic property of solutions of wave equations II*, J. Math. Anal. Appl., **32**, 392–399, 1970.

20. R.J. DUFFIN, *Equipartition of energy in wave motion*, J. Math. Anal. Appl., **32**, 386–391, 1970.

21. H.A. LEVINE, *An equipartition of energy theorem for weak solutions of evolutionary equations in Hilbert space: the Lagrange identity method*, J. Differential Equations, **24**, 197–210, 1977.

22. W.A. DAY, *Means and autocorrelations in elastodynamics*, Arch. Rational Mech. Anal., **73**, 243–256, 1980.

23. S. CHIRIŢĂ, *On the asymptotic partition of energy in linear thermoelasticity*, Quart. Appl. Math., vol. XLV, 327–340, 1987.

24. S. CHIRIŢĂ, *On the spatial and temporal behavior in linear thermoelasticity of materials with voids*, J. Thermal Stresses, **24**, 433–455, 2001.

25. D. IEŞAN, *On the theory of mixtures of thermoelastic solids*, J. Thermal Stresses, **14**, 389–408, 1991.

26. L. BRUN, *Méthodes énergétiques dans les systèmes évolutifs linéaires*, J. Mécanique, **8**, 125–192, 1969.

27. R. A. ADAMS, *Sobolev Spaces*, Academic Press, New York 1975.

28. I. HLAVACEK and J. NECAS, *On inequalities of Korn's type*, Arch. Rational Mech. Anal., **36**, 305–334, 1970.

Received April 16, 2002; revised version October 15, 2002.