Derivation of the normality rule for time-dependent deformation using the principle of maximal rate of entropy production

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DERIVATION OF THE normality rule for time-dependent deformation by the principle of the maximal rate of entropy production was carried out. The derivation was made within the framework of thermomechanics with internal variables. Since Ziegler did not cast his principle into an exact mathematical framework, it was done here. A condition for the multiplier in the normality rule (c.f. plasticity multiplier) was derived. If the condition gives a constant value for the multiplier, the specific (complementary) dissipation function was shown to be a homogeneous function. In the case where the value of the multiplier depends on the state variables, the dissipation potential is a non-homogeneous function.

Key words: thermodynamics, thermomechanics, internal variables, dissipation, entropy production, normality rule.

Notations

$\vec{q}$	heat flux vector,
s	specific entropy (entropy per unit mass),
$\dot{s}^i$	specific entropy production rate,
$\dot{s}^i_{\text{con}}$	specific entropy production rate (thermal part),
$\dot{s}^i_{\text{loc}}$	specific entropy production rate (mechanical part),
T	absolute temperature,
$\boldsymbol{\alpha}$	internal state variable (a second-order tensor),
$\boldsymbol{\beta}$	internal force (a second-order tensor),
$\boldsymbol{\varepsilon}$	strain tensor,
$\boldsymbol{\varepsilon}^i$	inelastic strain tensor (irreversible strain tensor),
ρ	density of the material,
$\boldsymbol{\sigma}$	stress tensor,
φ	specific dissipation function (potential),
φ_{loc}	specific dissipation function (mechanical part),
ψ	specific Helmholtz free energy,
$\text{D}/\text{D}t$	material (time) derivative operator,
$\vec{\nabla}$	vector operator del,
$(\dot{\ })$	material (time) derivative operator,
$(\vec{\ })$	quantity ( ) is a vector,
	second-order and fourth-order tensors are denoted, by bold letters.

1. Introduction

THE PRESENT PAPER combines the theory of continuum mechanics and thermodynamics and refers to the combination as thermomechanics. There are several dialects of thermodynamics. Here the theory of thermodynamics with internal variables is adopted since it provides an excellent framework within which to evaluate material models.

Basic courses in continuum mechanics already teach students learn that the body has to satisfy the equilibrium equation. They well know that if the equilibrium is not satisfied, the solution is incorrect and therefore has to be dismissed. This is because a solution that does not obey the equilibrium is in contradiction to a basic law of nature. In this case the basic law is the law of balance of momentum. The law of balance of momentum places a restriction on the values of forces and moments.

Corresponding restrictions also exist for a material model. They are the basic laws of thermodynamics, such as first law and second law. The thermodynamical restrictions for constitutive equations are dressed in the form of one inequality called the Clausius–Duhem inequality. If the material model satisfies the Clausius–Duhem inequality, it is not in contradiction to the basic laws of thermomechanics and can therefore be applied. Thermomechanical evaluation of a constitutive cannot prove that the material model is correct. This work is for micromechanical investigation of the material model and experimental work.

In order to prepare a thermomechanical investigation of a material model, the researcher has to write explicit forms for following two functions: the specific Helmholtz free energy ψ (or the specific complementary Helmholtz free energy ψ^c) and the specific dissipation function φ (or the specific complementary dissipation function φ^c or the yield function F). The material model is then obtained from these two functions using state equations and the normality rule. This paper studies the derivation of the normality for time-dependent deformation. The derivation is carried out using the principle of maximal rate of entropy production proposed by ZIEGLER [1, p. 134]. Since Ziegler did not cast his principle into an exact mathematical framework, it is done here. In this paper the normality rule is derived for time-dependent deformation. In the subsequent paper the standard approach of the principle of maximal rate of entropy production is extended for description of time-independent thermoplasticity.

Today when material models within the framework of thermomechanics are studied the normality rule belongs to the standard toolbox. However, usually the principle of maximal rate of entropy production is not used in the derivation of the normality rule. Instead of that the approach by the French school of thermodynamics is adopted. The French school of thermodynamics assumes that there exists a convex scalar-valued dissipation potential which is assumed

to obey the normality rule {see e.g. [2, p. 74]}. Furthermore, some writers have interpreted the work by Ziegler differently from what is done here. HOULSBY and PUZRIN [4, Sec. 4.1], for example, refer to Ziegler's orthogonality condition and therefore their approach deviates from the present one, since the vital part of their derivation is that the dissipation potential is a homogeneous function. SHIZAWA and ZBIB [3, Sec. V.3.1.] refer to the principle of maximal entropy production rate, but do not obtain any condition for the multiplier in the normality rule (c.f. plasticity multiplier) which plays an important role as the present work shows. On the other hand RAJAGOPAL and SRINIVASA [5] make an extensive study on "maximum rate of dissipation criterion", but they do not see the difference between the dissipation, where the terms have the form of force times flux, and the dissipation function whose variables are fluxes only. Furthermore they do not derive the normality rule, but just write it without obtaining a condition for the multiplier in normality rule. There are also writers, see e.g. [6], who just introduce the normality rule and refer to Ziegler.

2. Derivation of the Clausius–Duhem inequality

This section derives the Clausius–Duhem inequality which plays the central role in the thermomechanical verification of the material models and in the derivation of the normality rule.

Verification of a material model starts from the selection of the set of independent variables which describe the process which has to be modelled. When thermomechanics is used two kind of independent variables are present: controllable variables and internal variables. The independent variables present in the basic laws and axioms of thermomechanics are called controllable variables. The other independent variables are called internal variables.

The controllable independent variables for thermomechanical processes in deformable solids are: The strain tensor ε and the specific entropy s which is a scalar-valued quantity. The strain tensor ε is a second-order tensor describing both mechanical and thermal deformation.

The internal variables and their form are determined by the material model under consideration. Since the present paper studies time-dependent deformation in general, the inelastic strain tensor ε^i is one of the internal variables. In order to extend the scope of the this paper more internal variables are introduced. The second-order tensorial quantity α is introduced to represent any number of internal variables, which can be scalars, vectors or tensors of any order. In the theory of visco-plasticity, for example, the variables α are used for description of hardening.

The specific internal energy u is written as:

$$(2.1) \qquad u = u(\varepsilon, \varepsilon^i, \alpha, s, h(\vec{x})),$$

where the notation the notation $h(\vec{x})$ indicates that the system V^b may be thermodynamically inhomogeneous. This means that, e.g. the material properties for elastic deformation can vary from point to point.

For the sake of simplicity this paper studies material models which (also) model elastic deformation. Thus, the present formulation is for cases where the response of the system consists of elastic and inelastic deformation or pure elastic deformation. This means that the difference $\varepsilon - \varepsilon^i$ belongs to the descrption of state. Also thermal expansion can be simulated. By neglecting thermodynamically inhomogeneous systems the above assumption reduces Eq. (2.1) to the following form:

$$(2.2) \qquad u = u(\varepsilon - \varepsilon^i, \alpha, s).$$

Some writers {see e.g. MAUGIN [7, Sec. 2.3]} replace the difference $\varepsilon - \varepsilon^i$ by the elastic strain tensor ε^e. However, it is not acceptable. Variable ε is a controllable state variable whereas notation ε^i refers to an internal state variable. Thermodynamics does not define a difference between a controllable state variable and an internal state variable.

Instead of the specific internal energy u, the state of solids is usually described by the specific Helmholtz free energy ψ which is a Legendre partial transformation of the specific internal energy u. This is done because writing a material model using the specific internal energy u is very difficult, given that the specific entropy s is an argument of the specific internal energy u. It is very difficult to construct a constitutive model as a function of the specific entropy s. In the formulation of the specific Helmholtz free energy ψ the specific entropy s is replaced by the absolute temperature T. Since the absolute temperature T is a well-known quantity for a human being, writing a material model using the specific Helmholtz free energy ψ is much simpler than doing so with the specific internal energy u.

State functions are obtained as partial derivatives of the specific internal energy u with respect to the state variables. Due to the introduction of the specific Helmholtz free energy ψ state equations take the forms

$$(2.3) \qquad \sigma := \rho \frac{\partial \psi(\ldots)}{\partial(\varepsilon - \varepsilon^i)} \qquad \text{and} \qquad \beta := -\rho \frac{\partial \psi(\ldots)}{\partial \alpha}$$

and further

$$(2.4) \qquad s = -\frac{\partial \psi(\ldots)}{\partial T}, \qquad \text{where} \qquad \psi = \psi(\varepsilon - \varepsilon^i, \alpha, T).$$

In State Equations (2.2) and (2.3) the notation $\boldsymbol{\sigma}$ stands for the stress tensor, $\boldsymbol{\beta}$ is an internal force and ρ is the density.

The principle of conservation of energy, also referred to as the first law of thermodynamics, can be stated as follows: The time rate of change of the sum total of the kinetic energy K and the internal energy U in the body is equal to the sum of the rates of work done by the surface and body loads in producing the deformation (or flow) together with heat energy that may leave or enter the body at a certain rate. Thus the following is obtained:

$$(2.5) \qquad \frac{D}{Dt}(K + U) = P^{\text{ext}} + Q.$$

In Basic Law (2.5) P^{ext} is the power input of the external forces and Q is the heat input rate. The local form for the first law of thermodynamics is called the energy equation (in the non-polar case) or the equation of balance of energy. It has the following form:

$$(2.6) \qquad \rho \dot{u} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \rho r - \vec{\nabla} \cdot \vec{q},$$

where r is the heat source per unit mass and where $\vec{q}$ is the heat flux vector.

The second law of thermodynamics can be written in the form

$$(2.7) \qquad \dot{S} \geq -\oint_{\partial V} \frac{\vec{n} \cdot \vec{q}}{T} \, dA + \int_V \rho \frac{r}{T} \, dV,$$

where $\dot{S}$ is the entropy rate and $\vec{n}$ is the outward unit vector for volume V, the surface of which is denoted by ∂V. The local form of the second law of thermodynamics takes the form

$$(2.8) \qquad \rho T \dot{s} + \vec{\nabla} \cdot \vec{q} - \frac{\vec{\nabla} T}{T} \cdot \vec{q} - \rho r \geq 0.$$

The internal energy U and the entropy S are defined by

$$(2.9) \qquad U := \int_V \rho u \, dV \qquad \text{and} \qquad S := \int_V \rho s \, dV.$$

Combination of the local forms of the first and second law of thermodynamics, i.e. Eqs. (2.6) and (2.8), is called the Clausius–Duhem inequality. For the present set of state variables [see Eq. $(2.4)_2$] it takes the following form:

$$(2.10) \qquad \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{\text{i}} + \boldsymbol{\beta} : \dot{\boldsymbol{\alpha}} - \frac{\vec{\nabla} T}{T} \cdot \vec{q} \geq 0.$$

Based on the Clausius–Duhem Inequality (2.10) the specific entropy production rate $\dot{s}^{\text{i}}$ is introduced. It is defined by

$$(2.11) \qquad \rho T \dot{s}^{\text{i}} := \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{\text{i}} + \boldsymbol{\beta} : \dot{\boldsymbol{\alpha}} - \frac{\vec{\nabla} T}{T} \cdot \vec{q} \quad (\geq 0).$$

3. Principle of maximal rate of entropy production applied to time-dependent deformation

This section studies the principle of maximal rate of entropy production and derives its consequence: the normality rule. Time-dependent deformation is considered.

The principle of maximal rate of entropy production was first proposed by ZIEGLER [1, p. 134]. It should be pointed out that this principle as proposed by ZIEGLER {see e.g. [8, pp. 271 and 272]} is not (yet?) a basic law of physics, contrary to those discussed in the previous section. In Ziegler's own words, the principle of maximal rate of entropy production is quite general ZIEGLER and WEHRLI [9, p. 186]}.

According to ZIEGLER [8, p. 272] the physical foundation of this principle is as follows: From the physical point of view this principle is particularly appealing, since it may be considered as an extension of the second fundamental law. In fact, if a closed system tends towards its state of maximal entropy, it seems reasonable that the rate of entropy increase (the specific entropy production rate) under prescribed forces would take a maximum value, i.e. the system should approach its final state along the fastest (shortest) possible path.

3.1. Normality rule for time-dependent deformation when material model $\varphi = \varphi(\dot{\varepsilon}^i, \alpha, \ldots)$ is expressed by fluxes

Investigation of the expression for the specific entropy production rate $\dot{s}^i$ in Eq. (2.11) shows that $\dot{s}^i$ is dependent upon both forces and fluxes (processes). For example, in the expression $\sigma : \dot{\varepsilon}^i$ the term σ represents the force and $\dot{\varepsilon}^i$ describes the flux. Furthermore, ZIEGLER [1, p. 129] assumes the existence of the specific dissipation function {see also ZIEGLER [8, p. 76]}

$$(3.1) \qquad \varphi = \varphi(\dot{\varepsilon}^i, \dot{\alpha}, \vec{q}; \varepsilon - \varepsilon^i, \alpha, T).$$

The concept of the specific dissipation function φ is that in case of an actual process (i.e. when the maximum is present) it contains the same information (except for temperature T) about the state and the process as the specific entropy production rate $\dot{s}^i$, but the arguments of φ are only fluxes ($\dot{\varepsilon}^i, \dot{\alpha}, \vec{q}$) [and state ($\varepsilon - \varepsilon^i, \alpha, T$)], whereas the expression for $\dot{s}^i$ contains also the conjugate variables [$\sigma, \beta, -(\vec{\nabla}T)/T$]. This can be seen in Eqs. (2.11) and (3.1). The specific dissipation function φ is defined by {see ZIEGLER [1, Eq. (4.3)] and [6, Eq. (5.1)]}

$$(3.2) \qquad \text{For an actual process} \quad \varphi := T\dot{s}^i \quad \Rightarrow \quad \frac{1}{T}\varphi - \dot{s}^i = 0.$$

Ziegler did not cast his above-presented concept into an exact mathematical framework. The author therefore proposes the following formulation for the principle of maximal rate of entropy production:

The process is investigated at a certain state $(\varepsilon - \varepsilon^i, \boldsymbol{\alpha}, T)$ and the values for the fluxes $(\dot{\boldsymbol{\varepsilon}}^i, \dot{\boldsymbol{\alpha}}, \vec{q})$ have to be determined in order to maximise the specific entropy production rate $\dot{s}^i$. The state gives the values for the state functions $[\boldsymbol{\sigma}, \boldsymbol{\beta}, -(\vec{\nabla}T)/T]$ as follows: State Equations (2.13) give the values for the forces $\boldsymbol{\sigma}$ and $\boldsymbol{\beta}$. Since the state $(\varepsilon - \varepsilon^i, \boldsymbol{\alpha}, T)$ is known, the temperature T is known and therefore the state function $-(\vec{\nabla}T)/T$, is also known. Thus, the values for the state functions $[\boldsymbol{\sigma}, \boldsymbol{\beta}, -(\vec{\nabla}T)/T]$ are known.

Based on the above discussion the problem can be expressed in the following way: It is assumed that the state is known, i.e. the set $(\varepsilon - \varepsilon^i, \boldsymbol{\alpha}, T)$ is assumed to be known. This implies that the values for the forces $[\boldsymbol{\sigma}, \boldsymbol{\beta}, -(\vec{\nabla}T)/T]$ are known. The question is, what are the magnitudes of the fluxes $(\dot{\boldsymbol{\varepsilon}}^i, \dot{\boldsymbol{\alpha}}, \vec{q})$ which maximise the specific entropy production rate $\dot{s}^i$? At the same time also Definition $(3.2)_1$ has to be satisfied.

To make the evaluation shorter General Problem (2.11) is not evaluated but the concept by TRUESDELL and NOLL [10, p. 295] is followed by assuming that the specific entropy production rate $\dot{s}^i$ is separable into mechanical and thermal part as follows:

$$(3.3) \qquad \dot{s}^i = \dot{s}^i_{\text{loc}} + \dot{s}^i_{\text{con}}.$$

Based on Expression (3.3) Eq. (2.11) yields

$$(3.4) \qquad \rho\, T\, \dot{s}^i_{\text{loc}} := \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}^i + \boldsymbol{\beta}:\dot{\boldsymbol{\alpha}} \qquad (\geq 0)$$

and

$$(3.5) \qquad \rho\, T\, \dot{s}^i_{\text{con}} := -\frac{\vec{\nabla}T}{T}\cdot\vec{q} \qquad (\geq 0).$$

Quantity $\rho\, T\, \dot{s}^i_{\text{loc}}$ is often called intrinsic dissipation, whereas Quantity $\rho\, T\, \dot{s}^i_{\text{con}}$ is referred to as thermal dissipation.

Usually Mechanical Problem (3.4) and Thermal Problem (3.5) are studied separately. Since the derivation of the normality rule for a thermal problem follows the same steps that will be taken when the mechanical problem is studied, the thermal problem is not studied here.

Based on the above the principle of maximal rate of entropy production is written in the following mathematical form:

maximise with respect to the fluxes $(\dot{\boldsymbol{\varepsilon}}^i, \dot{\boldsymbol{\alpha}})$

$$(3.6) \qquad \dot{s}^i_{\text{loc}}(\dot{\boldsymbol{\varepsilon}}^i, \dot{\boldsymbol{\alpha}}, \boldsymbol{\sigma}, \boldsymbol{\beta}) = \frac{1}{\rho T}\left(\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}^i + \boldsymbol{\beta}:\dot{\boldsymbol{\alpha}}\right)$$

subject to:

$$(3.7) \qquad \tau_{\text{loc}} = \frac{1}{T}\,\varphi_{\text{loc}}(\dot{\boldsymbol{\varepsilon}}^i, \dot{\boldsymbol{\alpha}}; \varepsilon - \varepsilon^i, \boldsymbol{\alpha}, T) - \dot{s}^i_{\text{loc}}(\dot{\boldsymbol{\varepsilon}}^i, \dot{\boldsymbol{\alpha}}, \boldsymbol{\sigma}, \boldsymbol{\beta}) = 0,$$

where $\tau_{\mathrm{loc}} = 0$ is a constraint and φ_{loc} is the specific dissipation function for mechanical behaviour. It is worth noting that the specific dissipation function φ is dependent on the fluxes present in the Clausius–Duhem inequality [see Inequality (3.4)] and on the state variables [see Eq. (2.2)]. In this case this means that the function φ_{loc} is dependent on sets $(\dot{\varepsilon}^i, \dot{\alpha})$ and $(\varepsilon - \varepsilon^i, \alpha, T)$. The general maximisation problem (where the specific entropy production rate $\dot{s}^i$ is not separated) follows the above concept.

Both $\dot{s}^i_{\mathrm{loc}}$ and τ_{loc} are assumed to have at least continuous second partial derivatives with respect to the arguments $(\dot{\varepsilon}^i, \dot{\alpha})$. It should be pointed out that also Inequality (3.4) must be satisfied. Applying LUENBERGER [11, p. 225], the first-order sufficient condition for the point $(\dot{\varepsilon}^i, \dot{\alpha})$ to be a local maximum is

(3.8)
$$\frac{\partial}{\partial \dot{\varepsilon}^i} \left(\dot{s}^i_{\mathrm{loc}} + \lambda \tau_{\mathrm{loc}} \right) = \mathbf{0},$$

$$\frac{\partial}{\partial \dot{\alpha}} \left(\dot{s}^i_{\mathrm{loc}} + \lambda \tau_{\mathrm{loc}} \right) = 0 \qquad \text{and} \quad \tau_{\mathrm{loc}} = 0,$$

where λ is the Lagrange multiplier. As mentioned by ARFKEN [12, p. 946] the method based on Lagrange multipliers will fail if in Expressions $(3.8)_1$ and $(3.8)_2$ the coefficients of λ vanish at the extremum. Therefore, also special points where

(3.9)
$$\frac{\partial \tau_{\mathrm{loc}}}{\partial \dot{\varepsilon}^i} = \mathbf{0} \qquad \text{and} \qquad \frac{\partial \tau_{\mathrm{loc}}}{\partial \dot{\alpha}} = \mathbf{0}$$

must be studied. The above-mentioned coefficients of λ are

(3.10)
$$\frac{\partial}{\partial \dot{\varepsilon}^i} \tau_{\mathrm{loc}} \qquad \text{and} \qquad \frac{\partial}{\partial \dot{\alpha}} \tau_{\mathrm{loc}}.$$

The above indicates that there are two different cases for evaluation of the local maximum; utilisation of Expressions (3.8) referred to as Case A, and the special case described by Expression (3.9) referred to as Case B.

Starting with Case A:
Substitution of Eqs. (3.5) and (3.6) into Eqs. $(3.8)_1$ gives

(3.11)
$$\frac{1}{\rho T} \boldsymbol{\sigma} + \lambda \left(\frac{1}{T} \frac{\partial \varphi_{\mathrm{loc}}}{\partial \dot{\varepsilon}^i} - \frac{1}{\rho T} \boldsymbol{\sigma} \right) = \mathbf{0},$$

which yields to following result:

(3.12)
$$\boldsymbol{\sigma} = \frac{\lambda}{\lambda - 1} \rho \frac{\partial \varphi_{\mathrm{loc}}}{\partial \dot{\varepsilon}^i}.$$

Similarly Eq. $(3.8)_2$ gives

(3.13)
$$\boldsymbol{\beta} = \frac{\lambda}{\lambda - 1} \rho \frac{\partial \varphi_{\mathrm{loc}}}{\partial \dot{\alpha}}.$$

By substituting the results in Eqs. (3.12) and (3.13) into Eq. (3.8)$_3$ and re-ordering the obtained equation, the following result is obtained:

$$(3.14) \qquad \varphi_{\text{loc}}(\dot{\varepsilon}^i, \dot{\alpha}; \varepsilon - \varepsilon^i, \alpha, T) = \frac{\lambda}{\lambda - 1} \left(\frac{\partial \varphi_{\text{loc}}}{\partial \dot{\varepsilon}^i} : \dot{\varepsilon}^i + \frac{\partial \varphi_{\text{loc}}}{\partial \dot{\alpha}} : \dot{\alpha} \right).$$

It is worth noting that the value of the Lagrange multiplier λ is dependent on the set $(\varepsilon - \varepsilon^i, \alpha, T)$. This is based on the definition of the maximisation problem, which assumed that the value of the set $(\varepsilon - \varepsilon^i, \alpha, T)$ is known and that the values for the corresponding fluxes $(\dot{\varepsilon}^i, \dot{\alpha})$ have to be determined. This implies that for a certain set $(\varepsilon - \varepsilon^i, \alpha, T)$ a unique value for is obtained. Thus, the following holds: $\lambda = \lambda(\varepsilon - \varepsilon^i, \alpha, T)$.

By extending the definition for homogeneous functions given by e.g. WIDDER [13, pp. 19 and 20] the following is achieved: A function $\phi(x, y, z, u, v)$ is homogeneous of degree ω in variables x, y and z in a region R if, and only if, for x, y and z in R and for every positive value of k the following holds:

$$(3.15) \qquad \phi(k\,x, k\,y, k\,z, u, v) := k^\omega \phi(x, y, z, u, v).$$

Sometimes the definition is assumed to hold for every real k, and if the values of k are restricted to being positive, the function $\phi(x, y, z, u, v)$ is said to be a positive homogeneous function.

Euler's theorem on homogeneous functions {see original form in e.g. WIDDER [13, p. 20]} for the above extended definition reads

$$(3.16) \qquad \omega\,\phi(x, y, z, u, v) = \frac{\partial \phi}{\partial x}\,x + \frac{\partial \phi}{\partial y}\,y + \frac{\partial \phi}{\partial z}\,z.$$

In the special case that the Lagrange multiplier λ is a constant the extended Euler's theorem for homogeneous functions [Theorem (3.16)] and Result (3.14) indicate that the specific dissipation function φ is a homogeneous function of degree $(\lambda - 1)/\lambda$ in the variables $(\dot{\varepsilon}^i, \dot{\alpha})$.

The following notation is introduced:

$$(3.17) \qquad \mu := \frac{\lambda}{\lambda - 1} \qquad \text{which gives} \qquad \lambda = \frac{\mu}{\mu - 1}.$$

Equation (3.17)$_2$ shows that the multiplier $\mu = \mu(\varepsilon - \varepsilon^i, \alpha, T)$ can be any real number excluding $\mu = 1$. This means that in Case A the dissipation function φ_{loc} cannot be a homogeneous function of $1/\mu = 1$.

Continuing with Case B:

The candidates for the extremum points defined by Eq.(3.9) are investigated next. Substitution of Eq.(3.7) into Eq. (3.9)$_1$ gives

$$(3.18) \qquad \frac{1}{T} \frac{\partial \varphi_{\text{loc}}}{\partial \dot{\varepsilon}^i} - \frac{1}{\rho T}\,\sigma = 0, \qquad \text{which gives} \qquad \sigma = \rho\,\frac{\partial \varphi_{\text{loc}}}{\partial \dot{\varepsilon}^i}.$$

Correspondingly the following is obtained:

$$(3.19) \qquad\qquad \beta = \rho \, \frac{\partial \varphi_{\text{loc}}}{\partial \dot{\alpha}} \, .$$

Instead of Expression (3.14) Case B gives

$$(3.20) \qquad \varphi_{\text{loc}}(\dot{\varepsilon}^{\text{i}}, \dot{\alpha} ; \varepsilon - \varepsilon^{\text{i}}, \alpha, T) = \frac{\partial \varphi_{\text{loc}}}{\partial \dot{\varepsilon}^{\text{i}}} : \dot{\varepsilon}^{\text{i}} + \frac{\partial \varphi_{\text{loc}}}{\partial \dot{\alpha}} : \dot{\alpha}.$$

Comparison of Eqs. (3.18)$_2$, (3.19) and (3.20) with Eqs. (3.12), (3.13) and (3.14) shows that the special points defined by Case B give the same solution as Case A except that φ_{loc} is a homogeneous function of degree $1 (= 1/\mu)$.

Concluding from Cases A and B the following can be said:

At the start of this chapter the following problem was set: The state is known, which means that the set $(\varepsilon - \varepsilon^{\text{i}}, \alpha, T)$ is known. Due to State Eqs. (2.3)$_1$ and (2.3)$_2$ this implies that the values for the forces (σ, β) are known. The magnitudes of the fluxes $(\dot{\varepsilon}^{\text{i}}, \dot{\alpha})$ have to be determined in order for the specific entropy production rate $\dot{s}^{\text{i}}$ to be maximised. At the same time also Definition (3.2)$_1$ has to be satisfied. The result was as follows:

As a result the following normality rule was achieved:

$$(3.21) \qquad\qquad \sigma = \mu\,\rho \, \frac{\partial \varphi_{\text{loc}}(\dot{\varepsilon}^{\text{i}}, \dot{\alpha} ; \varepsilon - \dot{\varepsilon}^{\text{i}}, \alpha, T)}{\partial \dot{\varepsilon}^{\text{i}}}$$

and

$$(3.22) \qquad\qquad \beta = \mu\,\rho \, \frac{\partial \varphi_{\text{loc}}(\dot{\varepsilon}^{\text{i}}, \dot{\alpha} ; \varepsilon - \varepsilon^{\text{i}}, \alpha, T)}{\partial \dot{\alpha}} \, .$$

The specific dissipation function φ_{loc} has to obey the following condition:

$$(3.23) \qquad \varphi_{\text{loc}}(\dot{\varepsilon}^{\text{i}}, \dot{\alpha} ; \varepsilon - \varepsilon^{\text{i}}, \alpha, T) = \mu \left(\frac{\partial \varphi_{\text{loc}}}{\partial \dot{\varepsilon}^{\text{i}}} : \dot{\varepsilon}^{\text{i}} + \frac{\partial \varphi_{\text{loc}}}{\partial \dot{\alpha}} : \dot{\alpha} \right) .$$

The first-order sufficient condition for the point $(\dot{\varepsilon}^{\text{i}}, \dot{\alpha})$ to be a local maximum is that Eqs. (3.21), (3.22) and (3.23) hold. If φ_{loc} is a homogeneous function, according to Eq. (3.23) it is a homogeneous function of degree $1/\mu$. If φ_{loc} is not a homogeneous function, the value for $\mu = \mu(\varepsilon - \varepsilon^{\text{i}}, \alpha, T)$ is obtained from Eq. (3.23).

Equations (3.2), (3.21) and (3.22) show that the specific dissipation function φ_{loc} is a scalar potential, and it is therefore also called the specific dissipation potential.

The second-order both necessary and sufficient conditions for a local maximum lead to matrices so extensive {see LUENBERGER [11, pp. 226 and 227]}

that investigating them is very complicated and it hardly provides any practical results.

However, in practice, when constitutive models are evaluated the explicit form for the specific dissipation function φ_{loc} is assumed and the forces (σ, β) are determined by Normality Rule (3.21) and (3.22).

3.2. Normality rule for time-dependent deformation when material model $\varphi^c = \varphi^c(\sigma, \beta; \ldots)$ is expressed by forces

This section gives the normality rule for time-dependent deformation when the material model is expressed as a function of state functions, i.e. forces instead of the rates of internal variables, i.e. fluxes.

Normality Rule (3.21) and (3.22) assumes that the material model is expressed by the fluxes $(\dot{\varepsilon}^i, \dot{\alpha})$. If the conjugate forces (σ, β) are desired as the arguments of the material model, the specific complementary dissipation function φ^c_{loc} is introduced. It is a Legendre partial transformation of the specific dissipation function φ_{loc}. The transformation is defined by

$$(3.24) \qquad \rho \, \varphi^c_{loc}(\sigma, \beta \, ; \varepsilon - \varepsilon^i, \alpha, T) \; := \; \sigma : \dot{\varepsilon}^i$$
$$+ \, \beta : \dot{\alpha} - \mu \rho \, \varphi_{loc}(\dot{\varepsilon}^i, \dot{\alpha}; \varepsilon - \varepsilon^i, \alpha, T) \, ,$$

where the superscript c in notation φ^c_{loc} refers to the complementary function.

Based on Normality Rule (3.21) and (3.22) and Definition (3.24) the coefficient in Expression (A.3) of Appendix A take the values $a = \mu\rho$ and $b = \rho$. Thus, Expression (A.6) of Appendix A gives the following normality rule:

$$(3.25) \qquad \dot{\varepsilon}^{\,i} = \rho \, \frac{\partial \varphi^c_{loc}(\sigma, \beta \, ; \varepsilon - \varepsilon^i, \alpha, T)}{\partial \sigma}$$

and

$$(3.26) \qquad \dot{\alpha} = \rho \, \frac{\partial \varphi^c_{loc}(\sigma, \beta \, ; \varepsilon - \varepsilon^i, \alpha, T)}{\partial \beta} \, .$$

Substitution of Transformation (3.24) and Normality Rules (3.21), (3.22) and (3.25), (3.26) into Expression (3.23) gives

$$(3.27) \qquad \varphi^c_{loc}(\sigma, \beta \, ; \varepsilon - \varepsilon^i, \alpha, T) = (1 - \mu) \left(\frac{\partial \varphi^c_{loc}}{\partial \sigma} : \sigma + \frac{\partial \varphi^c_{loc}}{\partial \beta} : \beta \right) \, .$$

The first-order sufficient condition for the point (σ, β) to be a local maximum is that Eqs. (3.25), (3.26) and (3.27) hold. If φ^c_{loc} is a homogeneous function, according to Eq. (3.27) it is a homogeneous function of degree $1/(1 - \mu)$. If φ^c_{loc}

is not a homogeneous function, the value for $\mu = \mu(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^i, \boldsymbol{\alpha}, T)$ is obtained from Eq. (3.27).

It is worth noting that the specific complementary dissipation function $\varphi_{\text{loc}}^{\text{c}}$ cannot be a homogeneous function of degree 1 in the variables $(\boldsymbol{\sigma}, \boldsymbol{\beta})$, since according to Expression (3.27) in such a case the quantity μ would be zero and Transformation (3.24) would vanish. This means that if the specific complementary dissipation function $\varphi_{\text{loc}}^{\text{c}}$ is a homogeneous function of degree 1, Transformation (3.24) vanishes and it must be replaced by a different transformation. This new transformation is used when time-independent processes are modelled.

Equation (B.17) of Appendix B shows that if the specific dissipation function φ_{loc} is a homogeneous function of degree $1/\mu$ ($\neq 1$), the specific complementary dissipation function $\varphi_{\text{loc}}^{\text{c}}$ is a homogeneous function of degree $1/(1 - \mu)$.

4. Discussion and conclusions

The present paper refers to thermomechanics as a science which is a combination of thermodynamics and continuum mechanics. Since the field of the present work is the evaluation of constitutive equations for solid materials, the application of thermodynamics with internal variables was an obvious choice from the many dialects of thermodynamics.

The topic of this paper is the derivation of the normality rule for time-dependent deformation using the principle of maximal rate of entropy production. This principle is due to ZIEGLER [1, p. 134]. Since Ziegler did not cast his principle into an exact mathematical framework, it is done in this paper. The standard formulation of the principle of maximal rate of entropy production is shown to give a normality that is suitable for material models describing time-dependent processes.

The principle of maximal rate of entropy production is not usually used to obtain the normality rule but instead of that often the approach by the French school of thermodynamics is followed. The French school of thermodynamics assumes that the dissipation potential is a continuous and a convex scalar valued function of the flux variables. It is also non-negative function with a zero value at the origin of the space of the flux variables. According to the French school of thermodynamics, if the dissipation potential satisfies the above-mentioned properties, the normality rule is a sufficient condition for satisfaction of the Clausius–Duhem inequality. The above description is based on LEMAITRE and CHABOCHE [14, Sec. 2.4.3]. The principle of maximal rate of entropy production, however, assumes that real processes follow a path which maximises the entropy production. At the same time the Clausius–Duhem inequality is satisfied. The approach by the French school of thermodynamics is mainly a mathematical construction. Since thermomechanics describes natural events a more physical

background is needed, the author proposes to use the principle of maximal rate of entropy production.

If the state is expressed by the quantities $(\varepsilon - \varepsilon^i, \alpha, T)$ and the material model is given by the specific dissipation function whose variables are the fluxes $(\dot{\varepsilon}^i, \dot{\alpha})$ and the state $(\varepsilon - \varepsilon^i, \alpha, T)$, the mechanical part of the normality rule was shown to take the form

$$(4.1) \qquad \boldsymbol{\sigma} = \mu\rho \, \frac{\partial \varphi_{\mathrm{loc}}(\dot{\varepsilon}^i, \dot{\alpha}; \varepsilon - \varepsilon^i, \alpha, T)}{\partial \dot{\varepsilon}^i} \qquad \text{and} \qquad \boldsymbol{\beta} = \mu\rho \, \frac{\partial \varphi_{\mathrm{loc}}(\dot{\varepsilon}^i, \ldots)}{\partial \dot{\alpha}}.$$

It was also shown that the specific dissipation function φ_{loc} has to obey the following condition:

$$(4.2) \qquad \varphi_{\mathrm{loc}}(\dot{\varepsilon}^i, \dot{\alpha}; \varepsilon - \varepsilon^i, \alpha, T) = \mu \left(\frac{\partial \varphi_{\mathrm{loc}}}{\partial \dot{\varepsilon}^i} : \dot{\varepsilon}^i + \frac{\partial \varphi_{\mathrm{loc}}}{\partial \dot{\alpha}} : \dot{\alpha} \right).$$

According to Expression (4.2) if φ_{loc} is a homogeneous function, it is a homogeneous function of degree $1/\mu$. If φ_{loc} is not a homogeneous function, the value for $\mu = \mu(\varepsilon - \varepsilon^i, \alpha, T)$ is obtained from Equation (4.2).

In the case where the material model is given by the specific complementary dissipation function whose variables are the forces $(\boldsymbol{\sigma}, \boldsymbol{\beta})$ and the state $(\varepsilon - \varepsilon^i, \alpha, T)$, the specific complementary dissipation function φ_{loc}^c is a Legendre partial transformation of the specific dissipation function φ_{loc}. In this case the mechanical part of the normality rule was shown to take the form

$$(4.3) \qquad \dot{\varepsilon}^i = \rho \, \frac{\partial \varphi_{\mathrm{loc}}^c(\boldsymbol{\sigma}, \boldsymbol{\beta}; \varepsilon - \varepsilon^i, \alpha, T)}{\partial \boldsymbol{\sigma}} \qquad \text{and} \qquad \dot{\alpha} = \rho \, \frac{\partial \varphi_{\mathrm{loc}}^c(\boldsymbol{\sigma}, \ldots)}{\partial \boldsymbol{\beta}}.$$

As above a condition for the dissipation potential was obtained. It is

$$(4.4) \qquad \varphi_{\mathrm{loc}}^c(\boldsymbol{\sigma}, \boldsymbol{\beta}; \varepsilon - \varepsilon^i, \alpha, T) = (1 - \mu) \left(\frac{\partial \varphi_{\mathrm{loc}}^c}{\partial \boldsymbol{\sigma}} : \boldsymbol{\sigma} + \frac{\partial \varphi_{\mathrm{loc}}^c}{\partial \boldsymbol{\beta}} : \boldsymbol{\beta} \right).$$

According to Expression (4.4) if φ_{loc}^c is a homogeneous function, it is a homogeneous function of degree $1/(1 - \mu)$. If φ_{loc}^c is not a homogeneous function, the value for $\mu = \mu(\varepsilon - \varepsilon^i, \alpha, T)$ is obtained from Equation (4.4).

This paper does not give any examples on the application of thermomechanics for material model verification, but the reader is asked to study, for example, the lecture notes by SANTAOJA [15].

Appendix A. Legendre transformation

The investigation is started with a given scalar-valued function F, which is assumed to be a function of two independent sets of tensorial variables, which

are $\mathbf{u}^1, \ldots, \mathbf{u}^m$ and $\mathbf{w}^1, \ldots, \mathbf{w}^n$, i.e.

(A.1) $$F = F(\mathbf{u}^1, \ldots, \mathbf{u}^m, \mathbf{w}^1, \ldots, \mathbf{w}^n).$$

The new independent set of second-order tensorial variables $\boldsymbol{\gamma}^1, \ldots, \boldsymbol{\gamma}^m$ is assumed to be defined by

(A.2) $$\boldsymbol{\gamma}^i := a\, \frac{\partial F(\mathbf{u}^1, \ldots, \mathbf{u}^m, \mathbf{w}^1, \ldots, \mathbf{w}^n)}{\partial \mathbf{u}^i}, \qquad i = 1, \ldots, m,$$

where a is a coefficient independent of $\mathbf{u}^i, \mathbf{w}^j$ and $\boldsymbol{\gamma}^i$ ($i = 1, \ldots, m$ and $j = 1, \ldots, n$). The variables $\mathbf{u}^i$ are called the active variables and the variables $\mathbf{w}^j$ are called the passive variables of the transformation. A new function Ω, called the Lagende partial transformation, is introduced. It is defined by

(A.3) $$b\, \Omega(\boldsymbol{\gamma}^1, \ldots, \boldsymbol{\gamma}^m, \mathbf{w}^1, \ldots, \mathbf{w}^n)$$

$$:= \sum_{i=1}^{m} \boldsymbol{\gamma}^i : \mathbf{u}^i - a\, F(\mathbf{u}^1, \ldots, \mathbf{u}^m, \mathbf{w}^1, \ldots, \mathbf{w}^n).$$

The variables $\mathbf{w}^j$ and $\boldsymbol{\gamma}^i$ are given arbitrary variations $\delta\mathbf{w}^j$ and $\delta\boldsymbol{\gamma}^i$. Thus, Eq. (A.3) gives

(A.4) $$\delta\, b\, \Omega(\boldsymbol{\gamma}^1, \ldots, \boldsymbol{\gamma}^m, \mathbf{w}^1, \ldots, \mathbf{w}^n) = \sum_{i=1}^{m} b\, \frac{\partial \Omega}{\partial \boldsymbol{\gamma}^i} : \delta\boldsymbol{\gamma}^i + \sum_{j=1}^{n} b\, \frac{\partial \Omega}{\partial \mathbf{w}^j} : \delta\mathbf{w}^j$$

$$= \sum_{i=1}^{m} (\boldsymbol{\gamma}^i : \delta\mathbf{u}^i + \delta\boldsymbol{\gamma}^i : \mathbf{u}^i) - \sum_{i=1}^{m} a\, \frac{\partial F}{\partial \mathbf{u}^i} : \delta\mathbf{u}^i - \sum_{j=1}^{n} a\, \frac{\partial F}{\partial \mathbf{w}^j} : \delta\mathbf{w}^j,$$

which yields

(A.5) $$\sum_{i=1}^{m} b\, \frac{\partial \Omega}{\partial \boldsymbol{\gamma}^i} : \delta\boldsymbol{\gamma}^i + \sum_{j=1}^{n} b\, \frac{\partial \Omega}{\partial \mathbf{w}^j} : \delta\mathbf{w}^j$$

$$= \sum_{i=1}^{m} \left(\boldsymbol{\gamma}^i - a\, \frac{\partial F}{\partial \mathbf{u}^i}\right) : \delta\mathbf{u}^i + \sum_{i=1}^{m} \mathbf{u}^i : \delta\boldsymbol{\gamma}^i - \sum_{j=1}^{n} a\, \frac{\partial F}{\partial \mathbf{w}^j} : \delta\mathbf{w}^j.$$

According to Eq. (A.2) the first term on the right-hand side of Eq. (A.5) vanishes, giving the following equations:

(A.6) $$\mathbf{u}^i = b\, \frac{\partial \Omega(\boldsymbol{\gamma}^1, \ldots, \boldsymbol{\gamma}^m, \mathbf{w}^1, \ldots, \mathbf{w}^n)}{\partial \boldsymbol{\gamma}^i}, \qquad i = 1, \ldots, m$$

and

$$(A.7) \qquad a\,\frac{\partial F(\mathbf{u}^1,\ldots,\mathbf{u}^m,\mathbf{w}^1,\ldots,\mathbf{w}^n)}{\partial \mathbf{w}^j} = -\,b\,\frac{\partial\,\Omega(\boldsymbol{\gamma}^1,\ldots,\boldsymbol{\gamma}^m,\mathbf{w}^1,\ldots,\mathbf{w}^n)}{\partial \mathbf{w}^j},$$

$$j = 1,\ldots,n.$$

Appendix B. Legendre transformation of a homogeneous function

The Legendre transformation of a homogeneous function is investigated here. It is shown that the Legendre transformation of a homogeneous function is a homogeneous function. This does not hold if the original function is a homogeneous function of degree one.

A scalar-valued function F of m different tensorial variables $\mathbf{u}^1,\ldots,\mathbf{u}^m$ is studied. Function F is expressed as follows:

$$(B.1) \qquad F = F(\mathbf{u}^1,\ldots,\mathbf{u}^m)$$

is assumed to be a homogeneous function of degree ω and therefore it satisfies the following definition and equation:

$$(B.2) \qquad F(k\,\mathbf{u}^1,\ldots,k\,\mathbf{u}^m)\ :=\ k^\omega\,F(\mathbf{u}^1,\ldots,\mathbf{u}^m)$$

and

$$(B.3) \qquad \omega\,F(\mathbf{u}^1,\ldots,\mathbf{u}^m) = \left(\frac{\partial F}{\partial \mathbf{u}^1}:\mathbf{u}^1 + \ldots + \frac{\partial F}{\partial \mathbf{u}^m}:\mathbf{u}^m\right),$$

where k is an arbitrary positive real number {see e.g. WIDDER [13, p. 19 and 20]}.

Next, m second-order tensors $\boldsymbol{\gamma}^i,\ldots\boldsymbol{\gamma}^m$ are introduced by defining

$$(B.4) \qquad \boldsymbol{\gamma}^i\ :=\ a\,\frac{\partial F(\mathbf{u}^1,\ldots,\mathbf{u}^m)}{\partial \mathbf{u}^i}, \qquad i = 1,\ldots,m,$$

where a is an arbitrary coefficient independent of both $\mathbf{u}^i$ and $\boldsymbol{\gamma}^i$ $(i = 1,\ldots,m)$.

The Legendre transformation Ω of the function F is defined as in Appendix A, i.e.

$$(B.5) \qquad b\,\Omega(\boldsymbol{\gamma}^1,\ldots,\boldsymbol{\gamma}^m)\ :=\ \sum_{i=1}^{m}\boldsymbol{\gamma}^i:\mathbf{u}^i - a\,F(\mathbf{u}^1,\ldots,\mathbf{u}^m),$$

where the coefficient b does not depend on the tensorial variables $\mathbf{u}^i$ and $\boldsymbol{\gamma}^i$ $(i = 1,\ldots,m)$.

Substitution of Definition (B.4) into Expression (B.3) gives

(B.6)
$$\omega F(\mathbf{u}^1, \ldots, \mathbf{u}^m) = \frac{1}{a} \sum_{i=1}^{m} \boldsymbol{\gamma}^i : \mathbf{u}^i ,$$

which yields

(B.7)
$$\sum_{i=1}^{m} \boldsymbol{\gamma}^i : \mathbf{u}^i = a \, \omega \, F(\mathbf{u}^1, \ldots, \mathbf{u}^m) .$$

Substituting Eq. (B.7) into Legendre Transformation (B.5) gives

(B.8)
$$F(\mathbf{u}^1, \ldots, \mathbf{u}^m) = c \, \Omega(\boldsymbol{\gamma}^1, \ldots, \boldsymbol{\gamma}^m) ,$$

where the coefficient c is

(B.9)
$$c = \frac{b}{a \, \omega - a} .$$

Once again the definition of variables $\boldsymbol{\gamma}^i$ is used. Thus, the variables $\boldsymbol{\gamma}^i$ in Eq. (B.4) are substituted into the arguments of Ω on the right-hand side of Eq. (B.8) and the following equation is obtained:

(B.10)
$$F(\mathbf{u}^1, \ldots, \mathbf{u}^m) = c \, \Omega \left(a \, \frac{\partial F(\mathbf{u}^1, \ldots, \mathbf{u}^m)}{\partial \mathbf{u}^1} , \ldots, a \, \frac{\partial F(\mathbf{u}^1, \ldots, \mathbf{u}^m)}{\partial \mathbf{u}^m} \right) .$$

If the variables in Eq. (B.10) were changed by replacing $\mathbf{u}^i$ by $k \, \mathbf{u}^i$, Eq. (B.10) would take the following form:

(B.11)
$$F(k \, \mathbf{u}^1, \ldots, k \, \mathbf{u}^m)$$
$$= c \, \Omega \left(a \, \frac{\partial F(k \, \mathbf{u}^1, \ldots, k \, \mathbf{u}^m)}{\partial (k \, \mathbf{u}^1)} , \ldots, a \, \frac{\partial F(k \, \mathbf{u}^1, \ldots, k \, \mathbf{u}^m)}{\partial (k \, \mathbf{u}^m)} \right) .$$

The definition of a homogeneous function given by Definition (B.2) allows Eq. (B.11) to be written in the form

(B.12)
$$k^\omega \, F(\mathbf{u}^1, \ldots, \mathbf{u}^m)$$
$$= c \, \Omega \left(a \, \frac{\partial [k^\omega \, F(\mathbf{u}^1, \ldots, \mathbf{u}^m)]}{\partial (k \, \mathbf{u}^1)} , \ldots, a \, \frac{\partial [k^\omega \, F(\mathbf{u}^1, \ldots, \mathbf{u}^m)]}{\partial (k \, \mathbf{u}^m)} \right) .$$

For the partial derivatives of the arguments of the Legendre transformation Ω on the right-hand side of Eq. (B.12), the following equations hold:

(B.13)
$$k^{\omega-1} \frac{\partial F(\mathbf{u}^1, \ldots, \mathbf{u}^m)}{\partial \mathbf{u}^i} = \frac{\partial [k^\omega \, F(\mathbf{u}^1, \ldots, \mathbf{u}^m)]}{\partial (k \, \mathbf{u}^i)} , \qquad i = 1, \ldots, m .$$

Substitution of Definition (B.4) into the left-hand side of Eq. (B.13) yields

$$(B.14) \qquad \frac{1}{a} k^{\omega-1} \boldsymbol{\gamma}^i = \frac{\partial[k^\omega F(\mathbf{u}^1, \ldots, \mathbf{u}^m)]}{\partial(k \mathbf{u}^i)}, \qquad i = 1, \ldots, m.$$

Substitution of Eq. (B.8) into the left-hand side of Eq. (B.12) and Eq. (B.14) into the right-hand side of Eq. (B.12) gives the following equality:

$$(B.15) \qquad k^\omega \Omega(\boldsymbol{\gamma}^1, \ldots, \boldsymbol{\gamma}^m) = \Omega(k^{\omega-1} \boldsymbol{\gamma}^1, \ldots, k^{\omega-1} \boldsymbol{\gamma}^m).$$

Changing the variables by replacing $k^{\omega-1}$ by t allows Eq. (B.15) to be written in the form

$$(B.16) \qquad t^{\omega/(\omega-1)} \Omega(\boldsymbol{\gamma}^1, \ldots, \boldsymbol{\gamma}^m) = \Omega(t \boldsymbol{\gamma}^1, \ldots, t \boldsymbol{\gamma}^m).$$

Equation (B.16) therefore shows the Legendre transformation $\Omega(\boldsymbol{\gamma}^1, \ldots, \boldsymbol{\gamma}^m)$ to be a homogeneous function of degree $\omega/(\omega - 1)$, where ω is the degree of the original function F. This does not hold for the case $\omega = 1$, as can be seen in Eqs. (B.9) and (B.16).

If the original function F were a homogeneous function of degree $\mu = 1/\kappa$, the function Ω would be a homogeneous function of degree $1/(1 - \mu)$. As above, this does not hold for the case $\mu = 1$.

Acknowledgments

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