Non-local coupling of viscoplasticity and anisotropic viscodamage for impact problems using the gradient theory

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Dedicated to Professor Piotr Perzyna
on the occasion of his 70th birthday

A GENERAL FRAMEWORK for the analysis of heterogeneous media that assesses a strong coupling between viscoplasticity and anisotropic viscodamage evolution is formulated for-impact related problems within the framework of thermodynamic laws and nonlinear continuum mechanics. The proposed formulations include thermo-elasto-viscoplasticity with anisotropic thermo–elasto–viscodamage, a dynamic yield criterion of a von Mises type and a dynamic viscodamage criterion, the associated flow rules, non-linear strain hardening, strain-rate hardening, and temperature softening. The constitutive equations for the damaged material are written according to the principle of strain energy equivalence between the virgin material and the damaged material. That is, the damaged material is modeled using the constitutive laws of the effective undamaged material in which the nominal stresses are replaced by the effective stresses. The evolution laws are impeded in a finite deformation framework based on the multiplicative decomposition of the deformation gradient into elastic, viscoplastic, and viscodamage parts. Since the material macroscopic thermomechanical response under high-impact loading is governed by different physical mechanisms on the macroscale level, the proposed three-dimensional kinematical model is introduced with manifold structure accounting for discontinuous fields of dislocation interactions (plastic hardening), and crack and void interactions (damage hardening). The non-local theory of viscoplasticity and viscodamage that incorporates macroscale interstate variables and their higher-order gradients is used here to describe the change in the internal structure and in order to investigate the size effect of statistical inhomogeneity of the evolution-related viscoplasticity and viscodamage hardening variables. The gradients are introduced here in the hardening internal state variables and are considered to be independent of their local counterparts. It also incorporates the thermomechanical coupling effects as well as the internal dissipative effects through the rate-type covariance constitutive structure with a finite set of internal state variables. The model presented in this paper can be considered as a framework, which enables one to derive various non-local and gradient viscoplasticity and viscodamage theories by introducing simplifying assumptions.

Key words: non-local theory; anisotropic viscodamage; viscoplasticity; gradient theory; heterogeneous media.

1. Introduction

THE INELASTIC MATERIAL behavior of engineering materials may be attributed
to two distinct material mechanical processes: viscoplasticity (i.e. dislocations
along crystal slip planes) and/or damage mechanics (cracks, voids nucleation
and coalescence, decohesions, grain boundary cracks, and cleavage in the regions
of high stress concentration). Plasticity/viscoplasticity theories, by themselves,
are insufficient for modeling the material behavior since both damage defects
(cracks and voids) and dislocation densities (viscoplastic flow) are present in
their inelastic response. A constitutive model should address equally the two
distinct physical modes of irreversible changes and should satisfy the basic pos-
tulates of mechanics and thermodynamics. A multi-dissipative model that ac-
counts for both the material decohesions (discontinuities within a material) and
the dislocations along slip planes is necessary. This is accomplished by adopting
two loading surfaces and two potential functions, one for the viscoplasticity and
another for the damage.

Experimental observations show that in general the processes of cold-working,
forming, machining of mechanical parts, etc. can cause an initial evolution of de-
fects in the virgin material state at localized zones, such as the nucleation of cer-
tain amount of cracks, voids, dislocation densities, and shear bands. Those local-
ized defects of viscoplasticity (rate-dependent plasticity) and viscodamage (rate-
dependent damage) induced in the material structure along with the subsequent
defects that occur during deformation process lead to a heterogeneous (non-
uniform) material behavior. Further loading of materials of this type will cause
failure mechanisms to occur at localized zones of viscoplasticity and viscodam-
age. In those localized zones, a lot of defects may undergo irreversible growth; co-
alescence of pre-existing cracks and voids may occur; propagation of dislocations
may proceed; and new defects may nucleate and their ultimate coalescence results
in failure. Moreover, intensive interaction mechanisms of the evolved defects may
take place at those localized zones; such as dislocation-dislocation interaction,
microdamages-microdamages interaction, crack dominated-dislocation interac-
tion, dislocation-dominated crack interaction, dislocation-crack grain boundary
interaction, etc.

As the viscoplasticity and viscodamage defects localize over narrow regions of
the continuum, the characteristic length-scale governing the variations of those
defects and their average interactions over multiple length-scales falls far below
the scale of the local state variables of viscoplasticity and viscodamage used to
describe the response of the continuum. This leads to the loss of the statistical
homogeneity in the representative volume element (RVE), in such a way that all
the macroscopic response functions of interest (e.g. the Helmholtz free energy, Ψ;
the dissipation potential, Π; the Cauchy stress tensor, $\boldsymbol{\sigma}$; the small strain tensor,

ε; the stiffness tensor; $\mathbf{C}$; etc.) are sensitive to the distribution, size, and orientation of the mesostructural and macrostructural defects within the RVE. The viscoplasticity and viscodamage evolution processes are, therefore, statistically inhomogeneous at the macroscale level (at the RVE scale). This suggests that the macroscopic inelastic deformations and failure are governed by mechanisms at different scale levels (nonlocality). For example, dislocation interactions are observed on a mesolevel with length-scale $0.1 - 10\mu m$ GAO et al [36] affecting strongly the material behavior on the macrolevel with length-scale $\geq 100\mu m$. Thus, different methodologies rather than the local theories are necessary to adequately capture the decrease in the length-scale from the macroscale to the mesoscale level. The non-local theories are expanding steadily in order to appropriately overcome this problem, which take into account the influence of the nth nearest neighbor of the material points or the long-range microstructural interaction.

Moreover, it is a well-known fact that the use of classical rate-independent plasticity theory or local theory do not possess an intrinsic length-scale, which leads to numerical stability problems, such as mesh size and mesh alignment sensitivities, particularly, in problems exhibiting strain localization phenomena. However, several regularization approaches have been proposed in the constitutive modeling to accommodate this problem. They include: viscoplastic models (e.g. PERZYNA [73], NEEDLEMAN [70], WANG et al. [106] DORNOWSKI and PERZYNA [30], GLEMA et al., [43]), thermal dissipation models (e.g. LE MONDS and NEEDLEMAN [57], non-local models (e.g. AIFANTIS [2], PIJAUDIER-CABOT and BAZANT [85], BAZANT and PIJAUDIER-CABOT [14], VOYIADJIS and DE-LIKTAS [94], VOYIADJIS and DORGAN [97], and gradient models (AIFANTIS [3], ZBIB and AIFANTIS [107], DE BORST and SLUYS [27], FLECK and HUTCHINSON [35], BAMMANN et al., [10], BUSSO et al. [19], VOYIADJIS et al. [96], BASSANI [11], GURTIN [44], VOYIADJIS and ABU AL-RUB [93]).

In the literature, many non-local plasticity/visco plasticity and damage/viscodamage models were proposed to introduce intrinsic length-scale measures in the constitutive equations, which can be grouped into two classes: integral models and gradient models.

KRÖNER [53] and ERINGEN and EDELEN [33] incorporated nonlocal terms through integral equations of elasticity. PIJAUDIER-CABOT and BAZANT [85], extended this concept to continuum damage mechanics. BAZANT and OZBOLT [13] also proposed a nonlocal anisotropic damage formulation, which is based on nonlocal tensorial variable. However, integration in the nonlocal integral models requires a global averaging procedure with resulting equations that can not be easily linearized (DE BORST and PAMIN [26]). This makes the nonlocal integral models computationally inefficient.

However, the integral approach generally involves an infinitely extended zone of nonlocal action and may be approxiamated by truncated Taylor series expansion, giving rise to the so-called gradient theories. Gradient approaches typically retain terms in the constitutive equations of higher-order gradients with coefficients representing length-scale measures of the deformation microstructure associated with the nonlocal continuum. Aifantis [2] was one of the first to study the gradient regularization in solid mechanics. The gradient methods suggested by Lasry and Belytschko [56] and Mühlhaus and Aifantis [69] provide an alternative approach to the nonlocal integral equations. The gradient terms in plasticity models are introduced through the yield function (e.g. Mühlhaus and Aifantis [69] de Borst and Mühlhaus [25], de Borst et al. [28], de Borst and Pamin [26], Aifantis et al. [4], Gao et al. [40], Fleck and Hutchinson [36], Voyiadjis et al. [96]). The gradient damage theory has been developed for isotropic damage (e.g. Pijaudier-Cabot and Bazant [85], Peerlings et al. [84], Lacy et al. [55], Geers et al. [42], Svedberg and Runesson [90], Zhou et al. [111], Askes and Sluys [7]) and for anisotropic damage (e.g. Kuhl et al. [54], Voyiadjis and Deliktas [94], Voyiadjis and Dorgan [97], Voyiadjis and Abu Al-Rub [93]).

The motivation of this work comes from the experimental tests of specimens made of ductile materials and heterogeneous materials loaded at low- and high-speed impacts (Belingardi and Vadori [15], Borvik et al. [18], Luo et al., [64] Espinosa et al. [34], Sierakowski [87], Zhou et al. [110], Montagnani et al. [68], Johnson and Cook [47], Albertini and Montagnani [5] etc). Generally, these kinds of laboratory tests serve to verify the constitutive concepts and material parameters. In many of those tests, the intensive nonlinearity induced in the material is attributed to the viscoplasticity and viscodamage morphologies. Furthermore, the softening behavior in those experiments mostly appears as the result of temperature rise and damage growth. Those experiments indicate that the failure mechanisms occur at localized zones of viscoplasticity and viscodamage where a lot of interactions of defects take place. In order to be able to capture such localized deformation zones and strain-softening material behavior, we aim here to introduce explicit and implicit length-scale measures in plasticity and damage governing equations through the use of the gradient-dependent and viscoplasticity theories coupled to the viscodamage theory.

It is generally assumed that the rate of deformation can be additively decomposed into an elastic (reversible) part and an inelastic (irreversible) part (e.g. Nemat-Nasser [75], Lubliner [63], Simo and Hughes [89]). "Non-instantaneously reversible" deformation is a more general description of the inelastic deformation since it is corresponding to the following set of physical phenomena: instantaneous plasticity, viscoplasticity, instantaneous damage, and viscodamage. The first type of inelastic deformation is a time-independent mechanism,

which is generally considered in the rate-independent plasticity theories. The viscoplastic deformation, which is sometimes qualified as creep, is a rate-dependent mechanism. Both of those two mechanisms or one of them is generally not sufficient to describe the set of experimental observations under high strain rates (dynamic loadings). Therefore, degradation of the mechanical properties up to complete failure should be considered in the experimental simulations, in particular, simulating the heterogeneous material response under high strain-rates. This progressive physical process is commonly referred to as damage and it can be time-independent (damage theory) and/or time-dependent process (viscodamage theory). The evolution, nucleation, and coalescence of microcracks, voids, and cavities during manufacturing processes and subsequent loading enhance the material to behave inelastically in the elastic and plastic domains. VOYIADJIS and PARK [102] tend to sum such defects as an inelastic strain called the damage strain. They tend to decompose this damage strain into an elastic-damage (recoverable) component attributed to crack closure and void contraction during unloading, and an inelastic-damage (unrecoverable) component attributed to random distribution and orientation of the cracks that make their recovery impossible. Therefore, the second underlying motivation for this study is given by the work recently proposed by VOYIADJIS and PARK [102]. They presented a framework for finite nonlinear continuum damage involving seven different deformation configurations. In accordance with their work, two irreversible strains are considered in this study: the viscoplastic and viscodamage strains. The viscodamage strain component tends to be considerable in engineering materials under impact loading processes as compared to the viscoplastic strain component. We will also use a similar approach with a more attractive physical interpretation of the viscodamage deformation mechanisms. The proposed approach is analogous to the finite elasto-plasticity (e.g. NEMAT-NASSER [76], PERZYNA [82], LUBLINER [63], SIMO and HUGHES [89]) involving the multiplicative decomposition of the deformation gradient into elastic and inelastic parts. All configurations induced by the multiplicative decomposition are, as in finite elasto-plasticity, macroscopic. The damage evolution equations, however, are based on micromechanical considerations established through the use of gradient theory (for a detailed demonstration consult LACY et al. [55]).

There are many models with weak coupling between plasticity/viscoplasticity and damage/viscodamage; hence, no consistent model realizing a strong coupling has been published yet (HESEBECK [46]), which serves as our third motivation. In this work, the strong coupling between viscoplasticity and viscodamage will be implemented by using two independent viscodamage mechanisms. One mechanism is coupled with viscoplasticity, while the other one occurs independently of viscoplastic deformation. To formulate that on the basis of the thermodynamic principles, the two viscodamage processes are represented by two additive

portions in the dissipation potentials. Because this work focuses on the development of coupled viscoplastic-viscodamage governing equations based on thermomechanical postulates, the various possibilities to describe the viscoplasticity and anisotropic viscodamage will be considered here.

To mention some of the important contributions to phenomenological damage modeling, we have to start with effective stress concept of Kachanov (1958), who was the first to introduce for the isotropic case a one-dimensional variable, which may be interpreted as the effective surface density of microdamages per unit volume (VOYIADJIS and VENSON [104], VENSON and VOYIADJIS [91], VOYIADJIS *et al.* [105]). Following Kachanov's pioneering work researchers in different fields applied continuum damage mechanics to their areas in fields like brittle materials (KRAJCINOVIC and FONESKA [52], KRAJCINOVIC [51]) and ductile materials (e.g. LEMAITRE and CHABOCHE [59], LEMAITRE [60], KACHANOV [49], MURAKAMI [71]). In the 1990's coupling of continuum damage mechanics to plasticity have appeared (e.g. VOYIADJIS and VENSON [104], VOYIADJIS and KATTAN [98, 99, 100], LUBARDA and KRAJCINOVIC [62], VOYIADJIS and ABU-LEBDEH [92], VOYIADJIS *et al.* [105]).

The objective of this paper is to derive a general thermodynamic framework for the modeling of heterogeneous media that assess a strong coupling between viscoplasticity and viscodamage evolution for impact-related problems with considering the discontinuities on the macroscale level. The essential aspects of interest here can all be examined within the context of: (1) Finite strain kinematics; (2) Rapid time variations in temperature, strain, strain rate, and other field variables; (3) Viscodamage effects on moduli and strength (hardening/softening); (4) Strong viscoplasticity and viscodamage coupling; and (5) Numerical stability through using the regularization approaches (i.e., using viscoplasticity and viscodamage gradient-dependent theories). This can be effectively characterized through a thermodynamic framework for the development of a continuum thermo-elasto-viscodamage and thermo-elasto-viscoplastic based failure model. The constitutive equations are derived from the first and second laws of thermodynamics, the expression of Helmholtz free energy, the Clausius-Duhem inequality, the maximum dissipation principle, generalized normality, and the thermomechanical heat equation. All the thermodynamic equations are expressed in the spatial configuration. The evolution laws are impeded in a finite deformation framework based on the multiplicative decomposition of the deformation gradient and the additive decomposition of the spatial rate of deformation tensor into elastic, viscoplastic, and damage parts. The nonlocality is introduced here through the viscoplasticity and viscodamage hardening variables. The first-order gradients in the gradient-dependent theory are disregarded and the second-order gradients are mainly considered in this work. The local viscoplasticity and viscodamage hardening variables and their corresponding second-order gradients

are considered to be independent of each other, allowing one to computationally introduce independently the macroscale and mesoscale levels influence.

The outline of this paper is as follows: in Sec. 2, we outline a summary of the finite deformation kinematics and some of the fundamental definitions of nonlinear continuum mechanics. In Sec. 3, we outline a general thermodynamic framework for the elasto-viscoplastic and elasto-viscodamage material behavior with thermal effects for impact-related problems using the gradient-dependent theory. In Sec.4 the rate-type constitutive relation is derived. Finally, in Sec. 5 the thermomechanical couplings equation is formulated.

2. Finite deformation kinematics and fundamental definitions

2.1. Fundamental measure of total deformation

Here we summarize some of the fundamental definitions of nonlinear continuum mechanics (SIMO and HUGHES [80], BELYTSCHKO et al. [16], DOGHRI [29]) relevant to our subsequent developments. Our notation throughout is as follows: C^o and C^t are the manifolds, where we refer to a point $\mathbf{X} \in C^o$ as a point in the reference configuration of a continuum body and a point $\mathbf{x} \in C^t$ as a point in the current configuration (at time t). Assuming that the deformation is smooth regardless of damage, one can assume a one-to-one mapping such that:

$$(2.1) \qquad x_i = x_i\,(X,\mathrm{t}) \quad \text{or} \quad X_i = X_i\,(x,\mathrm{t})\,(1)$$

which maps the reference configuration C^o onto the spatial configuration C^t at time t. The corresponding deformation gradient is expressed as follows:

$$(2.2) \qquad F_{ij} = \frac{\partial x_i}{\partial X_j}$$

which is a linear transformation for each $X \in C^o$ at time t.

Note that in this work the subscripts indicate the tensorial nature of the variables unless specifically stated otherwise.

For each $\mathbf{X} \in C^o$ there exists an orthogonal transformation $\mathbf{R}(\mathbf{X})$ called rotation such that (polar decomposition):

$$(2.3) \qquad F_{ij} = R_{ik}U_{kj} = V_{ik}R_{kj}$$

where $\mathbf{U}$ and $\mathbf{V}$ are the right and left stretch tensors, respectively.

The change in the squared length of a material filament $d\mathbf{X}$ is used as a measure of deformation such that for a Cartesian coordinate system:

$$(2.4) \qquad \begin{aligned} (dl)^2 - (dL)^2 &= dx_i\,dx_i - dX_i\,dX_i, \\ &= 2E_{ij}dX_i\,dX_j, \end{aligned}$$

or

(2.5)
$$(dl)^2 - (dL)^2 = 2e_{ij}dx_i\,dx_j,$$

where $(dL)^2$ and $(dl)^2$ are the material filaments in the reference configuration C^o and the spatial configuration C^t, respectively. $\mathbf{E}$ and $\mathbf{e}$ are the material (or Lagrangian) and the spatial (or Eulerian) strain tensors, respectively, and are given by:

(2.6)
$$E_{ij} = \frac{1}{2}\left(F_{ki}F_{kj} - \delta_{ij}\right) = \frac{1}{2}\left(C_{ij} - \delta_{ij}\right)$$

and

(2.7)
$$e_{ij} = \frac{1}{2}\left(\delta_{ij} - F_{ki}^{-1}F_{kj}^{-1}\right) = \frac{1}{2}\left(\delta_{ij} - c_{ij}\right),$$

where $\mathbf{C} = \mathbf{U}^2$ and $\mathbf{b} = \mathbf{c}^{-1} = \mathbf{V}^2$ are the right and left Cauchy-Green tensors, respectively, and δ_{ij} is the Kronecker delta. $\mathbf{C}$ is defined with respect to the reference configuration while $\mathbf{b}$ is with respect to the current configuration.

For the spatial strain tensor $\mathbf{e}$ and the material strain tensor $\mathbf{E}$ we have:

(2.8)
$$e_{ij} = F_{ki}^{-1}E_{kl}F_{lj}^{-1}.$$

The spatial velocity, denoted by $\boldsymbol{\upsilon}(\mathbf{x}, t)$, is the time derivative of the motion:

(2.9)
$$\upsilon_i = \frac{\partial x_i}{\partial t}.$$

The spatial velocity gradient $\mathbf{l}$ is defined as:

(2.10)
$$l_{ij} = \frac{\partial \upsilon_i}{\partial x_j} = \dot{F}_{ik}F_{kj}^{-1}.$$

where the dot denotes the time derivative. The symmetric part of $\mathbf{l}$, denoted by $\mathbf{d}$, is called the spatial rate of deformation tensor, and its skew-symmetric part is called the spin (or vorticity) tensor, $\boldsymbol{\omega}$. Thus one obtains:

(2.11)
$$d_{ij} = \frac{1}{2}\left(l_{ij} + l_{ji}\right)$$

and

(2.12)
$$\omega_{ij} = \frac{1}{2}\left(l_{ij} - l_{ji}\right).$$

Let us define the Lagrangian rate of deformation tensor $\mathbf{D}$ and the Eulerian rate of deformation tensor $\mathbf{d}$ as follows:

$$(2.13) \qquad D_{ij} = \dot{E}_{ij} = \frac{1}{2}\dot{C}_{ij},$$

$$(2.14) \qquad d_{ij} = \frac{1}{2}F_{ki}^{-1}\dot{C}_{kl}F_{lj}^{-1}.$$

Utilizing Eq. (2.13) into Eq. (2.14), d is then related to D by:

$$(2.15) \qquad d_{ij} = F_{ki}^{-1}D_{kl}F_{lj}^{-1}.$$

while the spatial deformation rate tensor $\mathbf{d}$ is equal to the Cotter-Rivin convected rate of the Eulerian (Almansi) strain tensor as follows:

$$(2.16) \qquad d_{ij} = L_v(e_{ij}) = \dot{e}_{ij} + e_{ik}l_{kj} + e_{kj}l_{ki}$$

where the symbol L_v denotes the well-known Lie derivative with respect to $\mathbf{v}$ (MARSDEN and HUGHES, [67]).

2.2 Rates of the stress tensors

The first Piola-Kirchhoff stress tensor $\mathbf{P}$ is a nonsymmetric nominal stress tensor obtained by performing a Piola transformation on the Cauchy stress tensor $\boldsymbol{\sigma}$, i.e.

$$(2.17) \qquad P_{ij} = JF_{ik}^{-1}\sigma_{kj}$$

where J denotes the Jacobian of the deformation and represents the ratio of the mass densities at the reference configuration and the current configuration:

$$(2.18) \qquad J = \frac{\rho^o}{\rho} = \frac{dv}{dV} = \det(\mathbf{F}),$$

where ρ^o and ρ are the mass densities of the reference and current configurations, respectively. dV and dv are the initial volume and the volume after deformation, respectively.

The symmetric (or second) Piola-Kirchhoff stress tensor $\mathbf{S}$ is defined as follows:

$$(2.19) \qquad S_{ij} = F_{ik}^{-1}P_{kj} = JF_{ik}^{-1}\sigma_{kl}F_{jl}^{-1} = F_{ik}^{-1}\tau_{kl}F_{jl}^{-1}$$

where $\boldsymbol{\tau} = \mathbf{J}\boldsymbol{\sigma}$ is called the Kirchhoff stress tensor.

The rate of the Kirchhoff stress tensor (the Lie derivative) is given as follows:

$$(2.20) \qquad L_v(\tau_{ij}) = F_{ik}\dot{S}_{kl}F_{jl} = \dot{\tau}_{ij} - l_{ik}\tau_{kj} - l_{jk}\tau_{ik}$$

where $\dot{\boldsymbol{\tau}}$ defines the material time derivative of the Kirchhoff stress tensor which is given by the following relation:

$$(2.21) \qquad \dot{\tau}_{ij} = \frac{\partial \tau_{ij}}{\partial t} + \frac{\tau_{ij}}{\partial x_k}v_k$$

The Jaumann-Zaremba stress rate of the Kirchhoff stress is defined as follows:

$$(2.22) \qquad \tau_{ij}^o = \dot{\tau}_{ij} - \omega_{ik}\tau_{kj} + \omega_{kj}\tau_{ik}$$

Using Eqs. (2.11) and (2.12) along with Eq. (2.22), we can express the Jaumann-Zaremba stress rate, $\boldsymbol{\tau}^o$, in terms of the Lie derivative of the Kirchhoff stress tensor ($L_v\boldsymbol{\tau}$) and the spatial rate of deformation (d) as follows:

$$(2.23) \qquad \tau_{ij}^o = L_v(\tau_{ij}) + d_{ik}\tau_{kj} + d_{kj}\tau_{ik}.$$

Note that both the first and second Piola-Kirchhoff stress tensors, $\mathbf{P}$ and $\mathbf{S}$, are relevant to the reference configuration C^o, while the Kirchhoff and Cauchy stress tensors, $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$, are relevant to the current configuration C^t. Also note that the elastic components of the rate of deformation (d) and the spin tensor ($\boldsymbol{\omega}$) should be substituted in the stress rate tensors $L_v\boldsymbol{\tau}$ and $\boldsymbol{\tau}^o$.

2.3. The deformation rate additive decomposition

Imagine an elastically loaded representative volume element (RVE) containing uniformly distributed (micro)-cracks of Mode I, which are triggered by the process of cold working, is deformed by a total strain ε_1. A certain part of this strain will be elastically recoverable (ε_1^e) and another part can be induced by damage (ε_1^{ed}). When the loads are released before yield limit, the body will have no permanent strains left. However, the elastic stiffness of the RVE could have been reduced by the growth of microcracks. This is clearly demonstrated in Figure 1 which shows the foregoing micromechanics of a continuum point in the RVE and the corresponding macro-stresses and strains.

Imagine now the elastically loaded RVE containing an arbitrary distribution of (micro)-voids and (micro)-cracks of mixed modes (Mode I, II, and III), which are triggered by the process of cold working, and subjected to 3-D state of stress. Generally, this situation is more likely to happen in materials than the above one. This RVE is deformed by a total strain of $\boldsymbol{\varepsilon}$ a certain part of it will be elastically recoverable ($\boldsymbol{\varepsilon}^e$) and another part is induced by damage ($\boldsymbol{\varepsilon}^d$). When the loads are released before the yield limit, the body will have, similarly to plasticity and
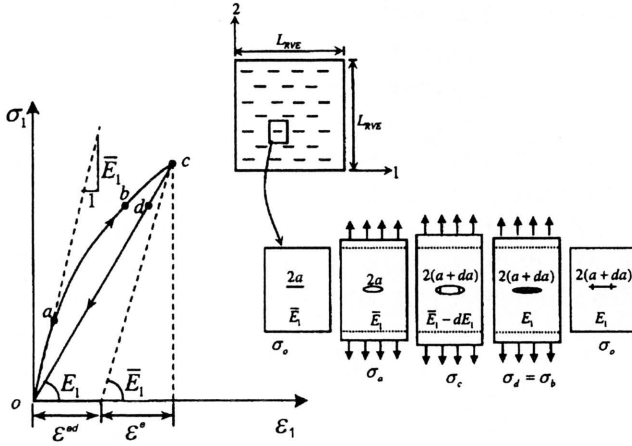
FIG. 1. Fictitious uniaxial stress-strain elastic response resulting from a growing micro-crack. All damage strain is recoverable (the crack is closed but not healed).
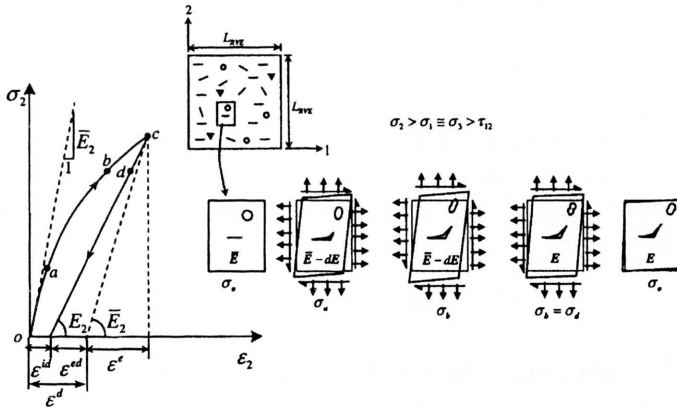

FIG. 2. Fictitious stress-strain elastic response of an RVE subjected to 3-D state of stress $\sigma_2 > \sigma_1 \cong \sigma_3 > \tau_{12}$ resulting from a growing microcrack and microvoid. Part of the damage strain is recoverable (not healed) and the other part is unrecoverable.

in contrast to the above fictitious situation, permanent deformations left (ε^{id}). Fig. 2 shows the underlying micromechanics of a continuum point in the RVE and the corresponding macro-stresses and strains in one of the directions.

Motivated by the above discussion and assuming small elastic and finite viscoplastic and viscodamage deformations under high-impact loading, we can postulate the additive decomposition of the total spatial deformation rate tensor ($\mathbf{d}$) into elastic ($\mathbf{d}^e$), viscoplastic ($\mathbf{d}^{vp}$), and viscodamage components ($\mathbf{d}^d$). Although

the damage process is an irreversible deformation thermodynamically; however, the deformation due to damage itself can be partially or completely recovered upon unloading. Thus, the damage deformation component is also decomposed into elastic-damage (reversible) and viscoinelastic-damage (irreversible) parts. The recoverable part is attributed to crack closure upon unloading (but not healing), while the unrecoverable part is attributed to the lack of crack closure and void contraction that cause permanent deformation. Both cause degradation in the material stiffness (Abu Al-Rub and Voyiadjis [1]). Hence, the total deformation rate tensor can be written as:

$$(2.24) \qquad d_{ij} = d_{ij}^e + d_{ij}^{vp} + d_{ij}^d$$

and

$$(2.25) \qquad d_{ij}^d = d_{ij}^{ed} + d_{ij}^{id},$$

where $\mathbf{d}^{ed}$ and $\mathbf{d}^{id}$ are the elastic-damage and viscoinelastic-damage parts of the damage strain, respectively. The superscripts here do not imply tensorial indices but merely indicate the corresponding deformation configuration such as "e" for elastic, "vp" for viscoplastic, "d" for damage, "ed" for elastic-damage, and "id" for inelastic-damage.

During the unloading process, two types of deformation rates are purely reversible: the ordinary elastic deformation rate $\mathbf{d}^e$ and the elastic-damage deformation rate $\mathbf{d}^{ed}$. Thus, the total reversible elastic deformation rate $\mathbf{d}^E$ upon unloading can be obtained by:

$$(2.26) \qquad d_{ij}^E = d_{ij}^e + d_{ij}^{ed}.$$

On the other hand, the total viscoinelastic rate of deformation d^I arises from the two irreversible sources: viscoinelastic damage and viscoplastic flow, such that:

$$(2.27) \qquad d_{ij}^I = d_{ij}^{id} + d_{ij}^p,$$

hence Eq. (2.24) can be rewritten as:

$$(2.28) \qquad d_{ij} = d_{ij}^E + d_{ij}^I.$$

The viscoinelastic damage may occur during only elastic deformations (in the absence of plastic deformations) under the condition that micro-cracks occur without the presence of a plastic process zone ahead of the crack tip.

Many researchers tend to adopt the traditional simple isotropic scalar damage variable, "$(1-\phi)$", to model the material micro-damage mechanism, in which

all components of the material stiffness are degraded by the same scalar damage parameter, ϕ. However, to ensure a more realistic application of the principles of the damage mechanics, anisotropic damage will be assumed. In this case different levels of damage are related to the principal directions, and thus a simple scalar damage parameter is no longer sufficient to quantify damage in all directions. Instead, the anisotropic phenomenon of the microcracks distribution in the material is interpreted using a symmetric second-order damage tensor, ϕ_{ij}.

The linear elastic constitutive equations for the damaged material are written according to the principle of elastic strain energy equivalence between the virgin material and the damaged material (SIDOROFF, [86]). That is, the damaged material is modeled using the constitutive laws of the effective undamaged material in which the Kirchhoff stress tensor τ is replaced by the effective stress tensor $\bar{\tau}$ (MURAKAMI and OHNO, [70]):

$$(2.29) \qquad \bar{\tau}_{ij} = \mathbf{M}_{ikjl}\tau_{kl}$$

where $\mathbf{M}$ is the fourth order damage-effect tensor. Many different expressions for $\mathbf{M}$ exist in the literature. A comprehensive review of the most widely used expressions are presented by VOYIADJIS and PARK [101]. The following expression for $\mathbf{M}$, which have been proposed by CORDEBOIS and SIDOROFF [24], is used here due to its attractiveness in the mathematical formulations and its symmetrization ability of the effective stress tensor $\bar{\tau}$, such that:

$$(2.30) \qquad \mathbf{M}_{ikjl} = 2\left[(\delta_{ik} - \phi_{ik})\,\delta_{jl} + \delta_{ik}\,(\delta_{jl} - \phi_{jl})\right]^{-1}$$

where δ_{ij} is the Kronecker delta.

The elastic-damage stiffness $\mathbf{C}$ in the case of finite deformation is given by VOYIADJIS and PARK [102] as follows:

$$(2.31) \qquad C_{ijkl} = N_{ikjl}\bar{C}_{klpq}N_{prqs}$$

where

$$(2.32) \qquad \begin{aligned} N_{ikjl} &= M_{ikjl}^{-1} \\ &= a_{ik}a_{jl} = \tfrac{1}{2}\left[(\delta_{ik} - \phi_{ik})\,\delta_{jl} + \delta_{ik}\,(\delta_{jl} - \phi_{jl})\right] \end{aligned}$$

and $\bar{\mathbf{C}}$ is the fourth-order elastic moduli tensor given by:

$$(2.33) \qquad \bar{C}_{ijkl} = K\delta_{ij}\delta_{kl} + 2G\left(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}\right),$$

where K is the bulk modulus and G is the shear-modulus in the effective configuration.

The rate of the Kirchhoff stress tensor is defined in terms of the elastic deformation tensor as follows:

$$(2.34) \qquad L_v(\tau_{ij}) = \bar{\mathfrak{I}}_{ijkl} d_{kl}^e$$

where $\bar{\mathfrak{I}}$ is the effective spatial elasticity tensor related to the effective fourth-order tensor of elastic constants $\bar{\mathbf{C}}$ by the (push-forward) transformation as follows:

$$(2.35) \qquad \bar{\mathfrak{I}}_{ijkl} = F_{ir} F_{js} F_{km} F_{ln} \bar{C}_{rsmn}.$$

Similarly, Eqs. (2.34) and (2.35) can be defined using the Jaumann-Zaremba stress rate tensor as follows:

$$(2.36) \qquad \tau_{ij}^o = \bar{a}_{ijkl} d_{kl}^e .$$

where

$$(2.37) \qquad \bar{a}_{ijkl} = \bar{\mathfrak{I}}_{ijkl} + \delta_{il}\tau_{jk} + \delta_{jl}\tau_{ik} .$$

2.4. Finite elasto-viscoplastic and elasto-viscodamage deformations

The processes of cold-working, forming, machining of mechanical parts, etc. can leave an initial damage. The initial damage induced in the material microstructure along with the subsequent damage that occurs during elastic loading, enhance the material to behave inelastically before viscoplasticity occurs. Therefore, if the material is elastically unloaded before forming dislocations along slip planes (viscoplasticity), permanent strains consist observed. Those strains are irreversible damage strains, while the reversible damage strains consist of two parts: elastic part and damage part. Then, if viscoplasticity occurs, both viscodamage and viscoplastic permanent deformations are anticipated.

First we motivated this basic behavior in one dimension. Consider the uni-axial tension test shown in Fig. 3. In this test, a bar of uniform cross-section is subjected to the uniaxial loading history: $O \to B \to C$, during which the length of the bar takes the following values: $L \to l^{vp} \to l^{pid} \to l^{vpd} \to l$. Stage $O \to B$ corresponds to a monotonic loading beyond the elasticity domain, and $B \to C$ to elastic-damage unloading ($C \to B$ corresponds to elastic-damage loading process). State C corresponds to a stress-free, unloaded configuration. We can write the following trivial identity:

$$(2.38) \qquad \frac{l}{L} = \frac{l}{l^{vpd}} \frac{l^{vpd}}{l^{vp}} \frac{l^{vp}}{L}$$

or

$$(2.39) \qquad \lambda = \lambda^e \lambda^d \lambda^{vp},$$

where $\lambda = l/L$ is the axial stretch at the end of $O \rightarrow B$, $\lambda^e = l/l^{vpd}$ can be viewed as the elastic stretch at the end of the elastic transformation $B \rightarrow F$, $\lambda^d = l^{vpd}/l^{vp}$ corresponds to the viscodamage stretch between D state and a viscodamage-free state between C and O, and $\lambda^{vp} = l^{vp}/L$ corresponds to viscoplastic stretch between O state and a viscoplastic-free state between C and O. Additionally, λ^d can be written as:

$$(2.40) \qquad \frac{l^{vpd}}{l^{vp}} = \frac{l^{vpd}}{l^{vpid}} \frac{l^{vpid}}{l^{vp}}$$

or

$$(2.41) \qquad \lambda^d = \lambda^{ed}\lambda^{id},$$

where $\lambda^{ed} = l^{vpd}/l^{vpid}$ is the elastic-damage stretch (recoverable viscodamage stretch) between states $D \rightarrow C$, and $\lambda^{id} = l^{vpid}/l^{vp}$ is the unrecoverable viscodamage stretch between C state and a viscodamage-free state between C and O.
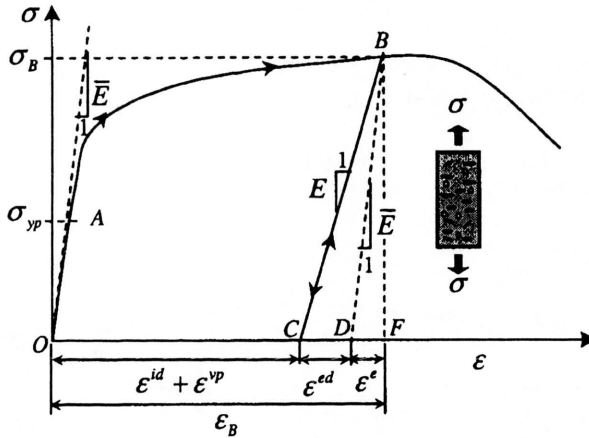

FIG. 3. Uniaxial stress-strain response of a metallic specimen.

The illustration of Fig. 4 in three dimensions is similar to that of the one-dimensional case (Fig. 3). If O designates the initial state (C^o), B the current state (C^t), and $D \rightarrow O$ (C^{dvp}, C^{vpid}, and C^{vp}) the local intermediate, stress-free, unloaded states, then the deformation gradients are: $\mathbf{F}$ for $O \rightarrow B$, $\mathbf{F}^e$ for $D \rightarrow B$, $\mathbf{F}^d$ between D state and a viscodamage-free state $D \rightarrow O$, and $\mathbf{F}^{vp}$ between state O and a viscoplastic-free state between O and C. Additionally, $\mathbf{F}^d$ can be decomposed into $\mathbf{F}^{ed}$ for $C \rightarrow B$, $\mathbf{F}^{id}$ between C state and a viscodamage-

free state between $C \to O$. Thus, the deformation gradient $\mathbf{F}(X, t)$ is split into elastic, viscoplastic, and viscodamage parts as follows:

$$(2.42) \qquad \frac{\partial x}{\partial X} = \frac{\partial x}{\partial x^{vpd}} \frac{\partial x^{vpd}}{\partial x^{vp}} \frac{\partial x^{vp}}{\partial X}$$

or

$$(2.43) \qquad \mathbf{F} = \mathbf{F}^e \cdot \mathbf{F}^{vp} \cdot \mathbf{F}^d,$$

introducing tacitly the local intermediate natural state configurations. In the sequel we suppose that elastic strains are small compared to viscoplastic and viscodamage strains. Also $\mathbf{F}^d$ is split into elastic-damage (reversible) part and inelastic-viscodamage (irreversible) part as:

$$(2.44) \qquad \frac{\partial x^{vpd}}{\partial x^{vp}} = \frac{\partial x^{vpd}}{\partial x^{vpid}} \frac{\partial x^{vpid}}{\partial x^{vp}}$$

or

$$(2.45) \qquad \mathbf{F}^d = \mathbf{F}^{ed} \cdot {}^{id}$$

Therefore, the total deformation gradient can be decomposed into total elastic part ($\mathbf{F}^E = \mathbf{F}^e . \mathbf{F}^{ed}$) and inelastic part ($\mathbf{F}^I = \mathbf{F}^{id} . \mathbf{F}^{vp}$), such that:

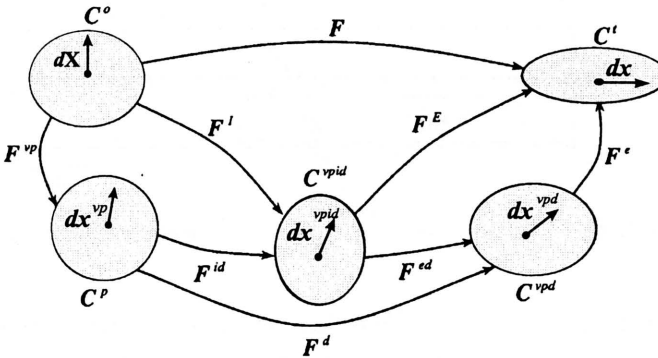$$(2.46) \qquad \mathbf{F} = \mathbf{F}^E . \mathbf{F}^I.$$

Fig. 4. Illustration of the multiplication decomposition of the deformation gradient.

The determinate of the total deformation tensor J (or the Jacobian of deformation, Eq. (2.18)) that characterizes the volumetric deformation can then be

multiplicatively decomposed into elastic, viscoplastic, and viscodamage parts as follows:

$$(2.47) \qquad J = J^e J^{vp} J^d$$

where

$$(2.48) \qquad J^e = \det\left(\mathbf{F}^e\right); \quad J^{vp} = \det\left(\mathbf{F}^{vp}\right); \quad J^d = \det\left(\mathbf{F}^d\right)$$

3. Constitutive modeling for dynamic impact loading

3.1. Thermodynamic formulation

In this section, the thermo-elasto-viscoplastic and thermo-elasto-viscodamage material behavior is considered. This means that the strain, strain rate, temperature material dependence, and the nonlinear material response are considered. The dependent constitutive variables are functions of the Eulerian total elastic strain tensor ($\mathbf{e}^E$), temperature (T), temperature gradient (∇T), the deformation gradient tensor ($\mathbf{F}$), the measure of volumetric deformation ($J = \det\left(F\right)$), and n_{int}- of phenomenological internal state variables ($\aleph_k$, $k = 1, ..., n_{int}$; $n_{int} \geq 1$). Hence, within the thermodynamic framework the thermoelastic Helmholtz free energy density at the current state of deformation can be written as (DUSZEK-PERZYNA and PERZYNA, [32]):

$$(3.1) \qquad \Psi = \tilde{\Psi}\left(e_{ij}^E, T, \nabla_i T, F_{ij}, J; \aleph_k\right).$$

However, by considering the assumption of small elastic strains (usually accepted for metals and other materials subjected to high strain-rate loading), the specific free energy function Ψ may be written as follows (MURNAGAHAM [37], GARCIA GARINO and OLIVER [41] CELENTANO [21]):

$$(3.2) \qquad \Psi = \tilde{\Psi}\left(e_{ij}^E, T, \nabla_i T, J; \aleph_k\right)$$

Moreover, under severe loading conditions the elastic strains are comparably smaller. Thus, the elastic part of the deformation gradient can be assumed to be unity (i.e., $J^e = I$). By adopting, also, the incompressibility assumption (i.e., $J^p = I$), which is an acceptable postulate for metals, we can rewrite the specific free energy function Ψ as follows:

$$(3.3) \qquad \Psi = \tilde{\Psi}\left(e_{ij}^E, T, \nabla_i T, J^d; \aleph_k\right)$$

where $J^d = \det(\mathbf{F}^d)$. This last simplified form of Ψ is used in the formulation described below.

Analogous to the additive decomposition of the deformation rate d^E into elastic and elastic-damage components (Eq. (2.26)), the Eulerian total elastic strain tensor e^E can be also decomposed as follows:

$$(3.4) \qquad e_{ij}^E = e_{ij}^e + e_{ij}^{ed}.$$

This additive decomposition of the Almansi total elastic strain can also be deduced from the multiplicative decomposition of the deformation gradient into elastic and elastic-damage parts, $\mathbf{F}^E = \mathbf{F}^e \cdot \mathbf{F}^{ed}$.

Since the main objective is to develop the rate-type constitutive equations for a viscoplastic and viscodamage material, the effects of viscoplastic strain hardening/softening, viscodamage strain hardening/softening, micro-damage mechanism, and thermomechanical coupling have to be considered. In order to compensate for such mechanisms, a finite set of internal state variables $\aleph_k$ ($k = 1, ..., n_{int}$) representing either scalars or tensorial variables are assumed, such that:

$$(3.5) \qquad \aleph_k = \tilde{\aleph}_k \left(\phi_{ij}, \Xi_n, \nabla^2 \Xi_n \right)$$

where ϕ is the average damage density, Ξ_n ($n = 1-4$) are the viscoplasticity and viscodamage hardening variables, and $\nabla^2 \Xi_n$ are the corresponding higher-order gradients (Laplacian) of Ξ_n.

We make use here of the postulate of the isotropic influence of averaging of the evolution equations of the assumed internal state variables Ξ_n over a representative volume element (RVE). The first-order gradients are disregarded and the second-order gradients are mainly considered in this work. Moreover, setting Ξ_n and $\nabla^2 \Xi_n$ as independent internal state variables allows one to computationally introduce independently the macro and meso-scales. It also allows these two different physical phenomena to be identified separately with different evolution equations. This approach is considered in this paper. The set of the macro internal variables Ξ_n is postulated as follows:

$$(3.6) \qquad \Xi_n = \tilde{\Xi}_n \left(p, \alpha_{ij}, r, \Gamma_{ij} \right)$$

where p denotes the accumulative or equivalent viscoplastic strain and α denotes the flux of the residual stress (backstress). p is associated with the isotropic hardening and α with the kinematic hardening in the viscoplastic flow process. Similarly, r denotes the accumulative viscodamage and Γ denotes the flux of the residual stress (kinematic hardening) in the damage flow process. Those viscoplasticity and viscodamage hardening variables are introduced in the Helmholtz free energy density in order to provide sufficient details of the deformation defects (cracks, voids, mobile and immobile dislocation densities) and their interactions in order to properly (i.e. physically) characterize the material microstructure behavior. These variables will provide an adequate characterization of these defects

in terms of size, orientation, distribution, spacing, interaction among defects, and so forth. In addition, in order to be able to achieve this, macroscale discontinuities influence should to be addressed and implemented properly.

For the strain-softening regime of the material behavior, the non-homogenous states of deformation can appear as localized regions with large deformations. A suitable description of the evolution of such zones can be obtained with the use of a non-local or gradient theory, examples of which are given by AIFANTIS [2], MÜHLHAUS and AIFANTIS [69], BAZANT *et al.* [12], and VOYIADJIS *et al* [96]. The gradient theory introduces in the material constitutive equations higher-order deformation gradients with coefficients that represent length-scale measures that characterize microstructural links with the non-local continuum. An attempt is made here to account for the non-uniform macroscale viscodamage and viscoplastic distribution on the overall macroscale response by assuming the thermoelastic Helmholtz free energy density Ψ to depend not only on the macroscopic response associated with the internal variables Ξ_n, but also on its macroscopic spatial higher-order gradients $\nabla^2\Xi_n$. Both Ξ_n and $\nabla^2\Xi_n$ are considered to be independent of each other. This postulate is motivated through the fact that certain internal state variables such as the mobile and immobile dislocation densities do not necessarily have the same evolution equations (BAMMANN and AIFANTIS, [8,9]). They have different physical interpretations that guide one to different evolution equations for Ξ_n and $\nabla^2\Xi_n$. Thus, the set of internal state variables $\nabla^2\Xi_n$ are postulated as follows:

$$(3.7) \qquad \nabla^2\Xi_n = \nabla^2\tilde{\Xi}_n \left(\nabla^2 p, \nabla^2\alpha_{ij}, \nabla^2 r, \nabla^2\Gamma_{ij}\right)$$

where $\nabla^2(\square)$ denotes the corresponding higher-order gradient Laplacian of $(\square)$. The assumed dependence of the Helmholtz free energy on the distinct variables $\nabla^2\Xi_n$ is also motivated by the necessity to include length-scale measures into the equations of state that link the mesoscale interactions to the macroscale viscoplasticity and viscodamage, which can not be captured by Xi_n variables only.

The viscoplastic hardening presented by the internal state variables p and α accounts for the dislocation interactions. The isotropic hardening internal state variables p and $\nabla^2 p$ are associated with the density of dislocations in the current state and characterized by statistically stored dislocations and geometrically necessary dislocations, respectively (GAO *et al.*, [40] FLECK and HUTCHINSON, [36]). The kinematic hardening variables α and $\nabla^2\alpha$ correspond to the incompatibility of viscoplastic deformation between various parts of the material (LEMAITRE and CHABOCHE,[59]). Hence, the viscoplasticity hardening variables depend on the interaction of the statistically stored or mobile dislocations and geometrical necessary or immobile dislocation densities. On the other hand, the viscodamage

hardening presented by the internal state variables r and Γ accounts for retardation of the (micro)-crack growth at higher strain rates resulting from (micro)-crack growth arrested by other (micro)-defects (VOYIADJIS and DELIKTAS, [95], ABU AL-RUB and VOYIADJIS, [1]). The gradient of the assumed internal state variable is used to describe the corresponding non-local material behavior, i.e. it is used to overcome the deficiency of the classical continua to capture the length-scale effects due to localization of viscoplasticity and viscodamage. The damage variable ϕ reflects the material degradation at a micromechanical scale due to nucleation and coalescence of voids, cavities, and microcracks in an averaged sense. The determination of the assumed internal state variables is the main challenge of the constitutive modeling.

The proposed viscoplastic and viscodamage constitutive modeling is formulated within the framework of thermodynamic principles; that is, the use of balancing laws, the conservation of mass, linear and angular momenta, and the first and second laws of thermodynamics. Those fundamental laws of continuum mechanics in the spatial representation can be written as follows (COLEMAN and GURTIN [23], LUBLINER [63], LEMAITRE and CHABOCHE [59], BELYTSCHKO et al. [16], DOGHRI [29]):

(i) Conservation of mass

$$(3.8) \qquad \dot{\rho} + \rho \, div\,(v_i) = 0 \quad \text{or} \quad \rho J = \rho^o.$$

(ii) Balance of linear momentum

$$(3.9) \qquad div\left(\frac{1}{J}\tau_{ij}\right) + \rho b_i = \rho \dot{v}_i.$$

(iii) Balance of moment of momentum

$$(3.10) \qquad \tau_{ij} = \tau_{ji}.$$

(iv) Conservation of energy (first law of the thermodynamics)

$$(3.11) \qquad \dot{u} = \frac{1}{\rho^o}\tau_{ij}d_{ij} + h - \frac{1}{\rho}q_{k,k}.$$

(v) Law of entropy (second law of the thermodynamics)

$$(3.12) \qquad \rho\dot{\eta} + div\left(\frac{q_i}{T}\right) - \rho\frac{h}{T} \geq 0.$$

and the Clausius-Duhem inequality, which can be derived from the previous laws of thermodynamics, such that:

$$(3.13) \qquad -\dot{u} + \dot{\eta}T + \frac{1}{\rho^o}\tau_{ij}d_{ij} - \frac{q_i}{\rho T}\nabla_i T \geq 0$$

where ρ^o is the reference mass density, ρ is the current mass density, v is the spatial velocity vector, b is the specific body force vector, u is the internal energy density, h is the external specific heat source, η is the specific (per unit mass) entropy, q is the heat flux vector, and $J > 0$ is the determinate of the deformation gradient tensor $\mathbf{F}$. ∇ denotes the first order gradient ($\nabla_i = \partial/\partial x_i$). Meanwhile, u, Ψ, T, and η are related by:

$$(3.14) \qquad \Psi = u - T\eta$$

Using the above equation in Eq. (3.13), one can rewrite the Clausius-Duhem inequality as follows:

$$(3.15) \qquad \tau_{ij}d_{ij} - \rho^o\left(\dot{\Psi} + \eta\dot{T}\right) - J\frac{q_i}{T}\nabla_i T \geq 0.$$

Note that the time rate of a spatial field $(\dot{f})$ is defined as the material-time derivative (Df/Dt) and given as follows:

$$(3.16) \qquad \dot{f} = \frac{Df}{Dt} = \frac{\partial f}{\partial t} + v_m\frac{\partial f}{\partial x_m}.$$

The Lie derivative of Eq. (3.3) with respect to its internal state variables in the updated configuration is given by:

$$(3.17) \qquad \dot{\psi} = \frac{\partial\psi}{\partial e_{ij}^E}d_{ij}^E + \frac{\partial\psi}{\partial T}\dot{T} + \frac{\partial\psi}{\partial\nabla_i T}\overline{\nabla_i T} + \frac{\partial\psi}{\partial J^d}\dot{j}^d + \frac{\partial\psi}{\partial\aleph_k}L_v(\aleph_k)$$

where

$$(3.18) \qquad \frac{\partial\Psi}{\partial\aleph_k}L_v(\aleph_k) = \frac{\partial\Psi}{\partial\phi_{ij}}L_v(\phi_{ij}) + \frac{\partial\Psi}{\partial\Xi_n}L_v(\Xi_n) + \frac{\partial\Psi}{\partial\nabla^2\Xi_n}L_v(\nabla^2\Xi_n).$$

The last two terms of Eq. (3.18) are given by:

$$(3.19) \qquad \frac{\partial\Psi}{\partial\Xi_n}L_v(\Xi_n) = \frac{\partial\Psi}{\partial p}\dot{p} + \frac{\partial\Psi}{\partial\alpha_{ij}}L_v(\alpha_{ij}) + \frac{\partial\Psi}{\partial r}\dot{r} + \frac{\partial\Psi}{\partial\Gamma_{ij}}L_v(\Gamma_{ij}).$$

and

$$(3.20) \qquad \frac{\partial\Psi}{\partial\nabla^2\Xi_n}L_v(\nabla^2\Xi_n) = \frac{\partial\Psi}{\partial\nabla^2 p}\overline{\nabla^2 p} + \frac{\partial\Psi}{\partial\nabla^2\alpha_{ij}}L_v(\nabla^2\alpha_{ij})$$

$$+ \frac{\partial\Psi}{\partial\nabla^2 r}\overline{\nabla^2 r} + \frac{\partial\Psi}{\partial\nabla^2\Gamma_{ij}}L_v(\nabla^2\Gamma_{ij}).$$

Note that in Eq. (3.17) the Lie derivative of the Almansi total elastic strain $(L_v e^E)$ defines the total elastic spatial rate of deformation tensor (d^E). The material-time derivative of the Jacobian of the damage deformation $(\dot{J}^d)$ is expressed as follows:

$$(3.21) \qquad \dot{J}^d = \frac{\partial J^d}{\partial F_{ij}^d} \dot{F}_{ij}^d = J^d d_{ij}^d \delta_{ij}.$$

Substituting the rate of the Helmholtz free energy density (Eq. (3.17)) into the Clausius-Duhem inequality (Eq. (3.15)), one obtains:

$$(3.22) \qquad \left(\tau_{ij} - \rho^o \frac{\partial \psi}{\partial e_{ij}^E} \right) d_{ij}^e$$

$$+ \left(\tau_{ij} - \rho^o \frac{\partial \psi}{\partial e_{ij}^E} - \rho^o \frac{\partial \psi}{\partial J^d} J^d \delta_{ij} \right) d_{ij}^{ed} + \left(\tau_{ij} - \rho^o \frac{\partial \psi}{\partial J^d} J^d \delta_{ij} \right) d_{ij}^{id}$$

$$+ \tau_{ij} d_{ij}^{vp} - \rho^o \left(\frac{\partial \psi}{\partial T} + \eta \right) \dot{T} - \rho^o \frac{\partial \psi}{\partial \nabla_i T} \overline{\nabla_i T} - \rho^o \frac{\partial \psi}{\partial \aleph_k} L_v \aleph_k - J \frac{q_i}{T} \nabla_i T \geq 0.$$

Assume that the axiom of entropy production holds, then the above inequality equation results in the following thermodynamic state laws:

$$(3.23) \qquad \begin{aligned} \tau_{ij} &= \rho^o \frac{\partial \psi}{\partial e_{ij}^E}; \quad \eta = -\frac{\partial \psi}{\partial T}; \quad \frac{q_i}{T} = \rho^o \frac{\partial \psi}{\partial \nabla_i T}; \\ P &= -\rho^o \frac{\partial \psi}{\partial J^d}; \quad \sum_k = \rho^o \frac{\partial \psi}{\partial \aleph_k}. \end{aligned}$$

The above equations describe the relation between the state variables (observable and internal) and their associated thermodynamic conjugate forces. These thermodynamic forces conjugate to their state variables are listed in Table 1, where $\Sigma_k = \{Y, R, X, K, H, R^g, X,^g, K^g, H^g\}$ are the conjugate forces corresponding to the viscoplastic and viscodamage internal variables $\Xi_k = \{\phi, p, \alpha, r, \Gamma, \nabla^2 p, \nabla^2 \alpha, \nabla^2 r, \nabla^2 \Gamma\}$, respectively. The stress τ is a measure of the elastic changes in the internal structure, while $\mathbf{Y}$ is a measure of the elastic-damage changes in the internal structure resulting from crack closure and void contraction during the unloading process. The conjugate forces R and $\mathbf{X}$ are measures of the viscoplastic changes in the internal structure, while K and $\mathbf{H}$ are measures of the viscoinelastic-damage changes in the internal structure. P is the thermodynamic pressure, where Eq. $(3.23)_4$ is consistent with the definition of thermodynamic tension (which is opposite in sign to that of pressure) of Gibbsian thermodynamics (Narasimhan [73]).

Table 1. Thermodynamic state variables and their corresponding conjugate forces.

State Variables		Associated Conjugates	
Observable	Internal		
e		τ	
T		η	
	e^E	τ	
	e^I	$-\tau$	
	J^d	P	
	∇T	q	
	$p, \quad \nabla^2 p$	$R,$	R^g
	$\alpha, \quad \nabla^2 \alpha$	$X,$	X^g
	$r, \quad \nabla^2 r$	$K,$	K^g
	$\Gamma, \quad \nabla^2 \Gamma$	$H,$	H^g
	ϕ	Y	

The additive decomposition of the rate of deformation tensor (Eq. (2.24)) implies that the Helmholtz free energy function Ψ can be written as a sum of elastic, viscoplastic, and viscodamage portions, such that:

$$(3.24) \quad \Psi \left(e_{ij}^E, T, \nabla_i T, J^d; \aleph_k \right)$$

$$= \Psi^e \left(e_{ij}^E, T, \nabla_i T, \phi_{ij} \right) + \Psi^{vp} \left(T, \nabla_i T, p, \nabla^2 p, \alpha_{ij}, \nabla^2 \alpha_{ij} \right)$$

$$+ \Psi^d \left(T, \nabla_i T, J^d, r, \nabla^2 r, \Gamma_{ij}, \nabla^2 \Gamma_{ij}, \phi_{ij} \right)$$

where Ψ^e is the thermoelastic stored energy, while Ψ^{vp} and Ψ^d are the energy stored due to material hardening.

The complexity of a model is directly determined by the form of the Helmholtz free energy Ψ and by the number of conjugate pairs of variables. It is possible to decouple the Helmholtz free energy into a potential function of each internal state variable in such a way that an analytical expression for the thermodynamic potential is given as a quadratic form of its internal state variables. However, coupling is possible in the viscoplastic potential or the viscodamage potential if they depend on more than one variable (HENRY and HASLACH [45]), which makes the evolution equations more complex. CHABOCHE [22] said that an energy Ψ with non-quadratic kinematic hardening variables leads to abnormal results, but coupling with temperature was not discussed there. In high velocity impact-related problems, very high-strain rates combined with large deformations can produce a significant temperature rise due to adiabatic heating, where most of the work done on solids is usually dissipated in the form of heat resulting in local increase

in temperature, which affects the behavior of the material during deformation. This necessitates the inclusion of the temperature in the constitutive modeling of the material. Kapoor and Nemat-Nasser [50] measured the energy converted to heat during inelastic flow using infra-red method for some metal alloys. The infra-red measurements showed that 70% of the work done is converted to heat. Bjerke et al. [17] examined experimentally the role of plastic deformation in generating heat during dynamic compression and fracture of polycarbonate, where Split Hopkinson pressure bar (SHPB) experiments and opening mode dynamic mode fracture experiments were performed to measure the thermomechanical response of polycarbonate at various loading conditions. The results indicate that plastic deformation is not the main source of heat generation during the dynamic fracture, but it only accounts for about 8% of the measured heating and the other portion is due to thermofracture coupling providing that the deformation is the only source of material heating. Hence, the thermomechanical coupling in both viscoplasticity and viscodamage mechanisms needs to be considered in the material behavior modeling for more accurate comparisons with the experiments. A necessary explicit multiplicative temperature coupling term can be introduced in the hardening state variables for more realistic description of their evolution equations and good conformity with the experimental observations that show strong dependence of such states on temperature.

The thermoelastic energy (Ψ^e) is postulated as follows:

$$(3.25) \quad \Psi^e = \frac{1}{2\rho^0} e_{ij}^E \Im_{ijkl}(\phi) e_{kl}^E$$

$$- \frac{1}{\rho^0} \beta_{ij} e_{ij}^E \Delta T - \eta_r \Delta T - \frac{1}{2} c \Delta T^2 - \frac{1}{2\rho^0 \dot{T}} k_{ij} \nabla_i T \nabla_j T$$

and the viscoplastic and viscodamage energies, Ψ^{vp} and Ψ^d, on the long term manifolds (neglecting the short term manifolds) are respectively assumed to have the following analytical forms, such that:

$$(3.26) \quad \Psi^{vp} = \frac{\vartheta}{2\rho^0} \left[a_1 p^2 + a_2 \left(\nabla^2 p \right)^2 + a_3 \alpha_{ij} \alpha_{ij} + a_4 \nabla^2 \alpha_{ij} \nabla^2 \alpha_{ij} \right]$$

$$(3.27) \quad \Psi^d = \frac{\vartheta}{2\rho^0} \left[a_5 r^2 + a_6 \left(\nabla^2 r \right)^2 + a_7 \Gamma_{ij} \Gamma_{ij} + a_8 \nabla^2 \Gamma_{ij} \nabla^2 \Gamma_{ij} \right]$$

$$+ c_v \left(T^{ig} - T_r \right)$$

where $\Im(\phi)$ is the fourth-order damage elastic tensor and is a function of ϕ, β are the thermo-mechanical coefficients, c is the coefficient of thermal expansion,

η_r is the reference entropy, $a_k(k = 1 - 8)$ are the material-dependent constants, $\mathbf{k}$ is the heat conductivity coefficients tensors, $\Delta T = T - T_r$ is the temperature difference, ϑ is the homologous temperature defined as $\vartheta = 1 - (T/T_m)^n$, where n is the temperature softening component, T_r is the reference temperature, and T_m is the melting temperature. c_v is the specific heat at constant volume (or the constant volume heat capacity), and T^{ig} is chosen to have the form of ideal gas temperature that can be expressed as follows:

$$(3.28) \quad T^{ig}$$
$$= T_r \exp\left[(\eta - \eta_r)/c_v\right] \left[1 + \varepsilon^d\right]^{(\gamma-1)} \exp\left[(\gamma - 1)\left(1\Big/\left(1 + \varepsilon^d\right) - 1\right)\right].$$

$\gamma = c_p/c_v$ is the ratio of the specific heats, where c_p is the specific heat at constant pressure. Both c_v and c_p are related to the gas constant $\Re$ by $\Re = c_p - c_v$. The above expression is postulated in order to derive an expression for the equation of state, which relates pressure to specific density. ε^d is the nominal volumetric damage strain, which can be expressed in terms of J^d as:

$$(3.29) \qquad \varepsilon^d = \frac{1}{J^d} - 1.$$

Note that in this paper the foregoing material properties are taken as independent of temperature unless specifically stated otherwise.

The proposed definition of Ψ allows the derivation of the constitutive equations and the internal dissipation described next. Moreover, the definition of the different contributions of Ψ given by Eqs. (3.25), (3.26), and (3.27) consider the density at the initial configuration ρ^o instead of its current value ρ. This simplification is consistent with most of the large strain models exists in the literature (see DOYLE and ERICKSEN [31], DUSZEK-PERZYNA and PERZYNA [32], CELENTANO [21] etc). However, any density change in the evolution equations is given by the conservation of mass law (Eq. (56)). The constitutive equations for stress and entropy, Eqs. (3.23)$_1$ and (3.23)$_2$, can be written from the thermodynamic potential equations, Eqs. (3.25), (3.26), and (3.27), by neglecting higher-order derivatives, such that:

$$(3.30) \qquad \tau_{ij} = \Im_{ijkl}e^E_{kl} - \beta_{ij}\Delta T,$$

where

$$(3.31) \qquad \Im_{ijkl} = \rho^o \frac{\partial^2 \Psi}{\partial e^E_{ij}\partial e^E_{kl}} \quad , \quad \beta_{ij} = -\rho^o \frac{\partial^2 \Psi}{\partial e^E_{ij}\partial T},$$

and

$$(3.32) \qquad \eta = \eta_e + \eta_{vp} + \eta_d,$$

where

$$(3.33) \qquad \eta_e = \eta_r + c\Delta T + \frac{1}{\rho^o}\beta_{ij}e^E_{ij},$$

$$(3.34) \qquad \eta_{vp} = \frac{1}{2\rho^o}\left[a_1 p^2 + a_2\left(\nabla^2 p\right)^2 + a_3\alpha_{ij}\alpha_{ij} + a_4\nabla^2\alpha_{ij}\nabla^2\alpha_{ij}\right]\frac{\partial\vartheta}{\partial T}$$

$$(3.35) \qquad \eta_d = \frac{1}{2\rho^o}\left[b_1 r^2 + b_2\left(\nabla^2 r\right)^2 + b_3\Gamma_{ij}\Gamma_{ij} + b_4\nabla^2\Gamma_{ij}\nabla^2\Gamma_{ij}\right]\frac{\partial\vartheta}{\partial T}$$

with

$$(3.36) \qquad \frac{\partial\vartheta}{\partial T} = \frac{n}{T_m}\left(\frac{T}{T_m}\right)^{n-1}.$$

In Eq. (3.30) the instantaneous elasticity tensor $\Im$ could be considered constant either on the spatial (updated) or in the material (reference) configuration. If it is considered constant in the material configuration, $\Im$ is obtained by "push forward" operation, while if it is considered constant on the spatial configuration, the elasticity tensor in the material configuration comes out by performing a "pull back" operation (Car et. al. [20]).

The constitutive equations for the heat flux vector $\mathbf{q}$ and the pressure stress P can be obtained from Eqs. $(3.23)_3$ and $(3.23)_4$, respectively, as follows:

$$(3.37) \qquad \mathbf{q}_i = -k_{ij}\nabla_j T$$

which is the Fourier heat conduction equation. The negative sign indicates that the heat flow is opposite to the direction of temperature increase.

The thermodynamic pressure stress P is given as follows:

$$(3.38) \qquad P = (1 - \gamma)\,c_v T^{ig}\varepsilon^d$$

which gives the equation of state necessary for high-impact loading. The equation of state accounts for compressibility effects (changes in density) and irreversible thermodynamic processes.

Coupling between elasticity and damage does exist indirectly since the elastic modulus is a function of the damage variable $\boldsymbol{\phi}$. Furthermore, coupling between viscoplasticity and viscodamage exists since the viscoplastic thermodynamic states are expressed in the current damaged configuration. The state laws of the assumed internal state variables, Eq. $(3.23)_5$, are obtained from Table 1 and the thermodynamic potential equations Eqs. (3.25), (3.26), and (3.27) are

expressed in Table 2 in terms of their associated internal state variables. The superscript 'g' in Table 2 indicates the thermodynamic conjugate force corresponding to the gradient internal variable. Other choices of the Helmholtz free energy function Ψ lead to conjugate forces which differ in algebraic form but not in the fundamental concept from the specific case considered here.

Table 2. The thermodynamic conjugate forces.

Plasticity	Isotropic Hardening	$R = \rho^o \frac{\partial \Psi^{vp}}{\partial p} = a_1 p \vartheta,$ $R^g = \rho^o \frac{\partial \Psi^{vp}}{\partial \nabla^2 p} = a_2 \left(\nabla^2 p\right) \vartheta$
	Kinematic Hardening	$X_{ij} = \rho^o \frac{\partial \Psi^{vp}}{\partial \alpha_{ij}} = a_3 \alpha_{ij} \vartheta,$ $X_{ij}^g = \rho^o \frac{\partial \Psi^{vp}}{\partial \nabla^2 \alpha_{ij}} = a_4 \left(\nabla^2 \alpha_{ij}\right) \vartheta$
Damage	Isotropic Hardening	$K = \rho^o \frac{\partial \Psi^d}{\partial r} = b_1 r \vartheta,$ $K^g = \rho^o \frac{\partial \Psi^d}{\partial \nabla^2 r} = b_2 \left(\nabla^2 r\right) \vartheta$
	Kinematic Hardening	$H_{ij} = \rho^o \frac{\partial \Psi^d}{\partial \Gamma_{ij}} = b_3 \Gamma_{ij} \vartheta, \quad H_{ij}^g = \rho^o \frac{\partial \Psi^d}{\partial \nabla^2 \Gamma_{ij}} = b_4 \left(\nabla^2 \Gamma_{ij}\right) \vartheta$
	Damage Force	$-Y_{ij} = \rho^o \frac{\partial \Psi^e}{\partial \phi_{ij}} = \frac{\partial}{\partial \phi_{ij}} \left[\frac{1}{2} e_{ij}^E \Im_{ijkl}(\phi) e_{kl}^E\right]$

Substituting of Eqs. (3.23) into Eq. (3.22) modifies the Clausius-Duhem inequality to express the fact that dissipation energy Π is necessarily positive, such that:

$$(3.39) \qquad \Pi = \tau_{ij} d_{ij}^I + P J^d d_{ij}^d \delta_{ij} - \Pi_{int} - q_i \left[\frac{\overline{\nabla_i T}}{\dot{T}} + J \frac{\nabla_i T}{T}\right] \geq 0$$

where the rate of internal dissipation Π_{int} is given by the relation:

$$(3.40) \quad \Pi_{int} = \sum_k L_v \aleph_k = -R\dot{p} - R^g \overline{\nabla^2 p} - X_{ij} L_v(\alpha_{ij}) - X_{ij}^g L_v(\nabla^2 \alpha_{ij})$$

$$- K\dot{r} - K^g \overline{\nabla^2 r} - H_{ij} L_v(\Gamma_{ij}) - H_{ij}^g L_v(\nabla^2 \Gamma_{ij}) + Y_{ij} L_v(\phi_{ij}) \geq 0.$$

Based on the previous assumption of Helmholtz free energy additive decomposition (Eq. (3.24)), the dissipation energy (Eq. (3.39)) can be rewritten as the summation of dissipation energies due to mechanical dissipation (viscoplasticity and viscodamage) and thermal dissipation, such that:

$$(3.41) \qquad \Pi = \Pi^{vp} + \Pi^d + \Pi^{th} \geq 0,$$

where

$$(3.42) \qquad \Pi^{vp} = \tau_{ij}d_{ij}^{vp} - R\dot{p} - R^g\,\overline{\nabla^2 p} - X_{ij}L_v(\alpha_{ij}) - X_{ij}^g L_v(\nabla^2\alpha_{ij}) \geq 0,$$

$$(3.43) \quad \Pi^d = \tau_{ij}d_{ij}^{id} + PJ^d d_{ij}^d \delta_{ij} - K\dot{r} - K^g\,\overline{\nabla^2 r} - H_{ij}L_v(\Gamma_{ij})$$
$$- H_{ij}^g L_v(\nabla^2\Gamma_{ij}) + Y_{ij}L_v(\phi_{ij}) \geq 0,$$

$$(3.44) \qquad \Pi^{th} = -q_i\left[\frac{\overline{\nabla_i T}}{\dot{T}} + J\frac{\nabla_i T}{T}\right] \geq 0.$$

This result requires that all inelastic work should dissipate away as heat, except for that energy which is stored because of the rearrangement of the material internal structure. Note that not dissipation occurs not only due to deviatoric stresses associated with d^{vp} and $\mathbf{d}^{id}$ but also due to the pressure stress associated with $\mathbf{d}^d$. This result suggests that viscoinelastic-damage deformation is controlled by deviatoric as well as volumetric stresses, while the elastic-damage deformation is mainly controlled by volumetric stresses. Moreover, writing the dissipation potential function Π as in the decoupled form shown in Eq. (3.41) does not mean that the corresponding physical mechanisms are decoupled. Coupling does occur in the viscoplastic potential given by Eq. (3.42) between viscoplasticity and viscodamage since the conjugate forces are expressed in the current deformed and damaged configuration of the material. Hence, two additive damage mechanisms are introduced in the dissipation function (Eq. (3.41)); one mechanism is coupled with viscoplasticity and the other occurs independently of viscoplastic deformation. Complementary laws can be related to the dissipation processes given by Eqs. (3.42) and (3.43). This implies the existence of the dissipation potential expressed as a continuous and convex scalar-valued function of the flux variables.

It is obvious that the definition of Ψ and consequently of $L_v\aleph_k$ ($k = 1 - 9$) are essential features of the thermodynamic formulation in order to describe the thermomechanical/microstructural behavior of the material involved in the deformation process. The evolution laws of $\mathbf{d}^I$ and $\boldsymbol{\phi}$ can be obtained by utilizing the calculus of several variables with Lagrange multipliers $\dot{\Lambda}^{vp}$ and $\dot{\Lambda}^d$. The dissipation function $\Pi^I = \Pi^{vp} + \Pi^d$ (Eq. (3.41)) is subjected to the two constraints, namely $f = 0$ and $g = 0$ (Voyiadjis and Kattan [98]), such that:

$$(3.45) \qquad \Omega = \Pi^{vp} + \Pi^d - \dot{\Lambda}^{vp}f - \dot{\Lambda}^d g.$$

For generality purposes, we will assume here that the time-dependent behavior of both viscoplasticity and viscodamage mechanisms are controlled by different relaxation times associated with $\dot{\Lambda}^{vp}$ and $\dot{\Lambda}^d$, which may not generally be the case. This congruous is suggested to occur only for material impact behavior above a certain material threshold of impact speed. This is not the subject of the present work, but it will be discussed thoroughly in a forthcoming paper.

Now we make use of the maximum viscoinelastic dissipation principle (Simo and Honein [88], Simo and Hughes [89]), which states that the actual state of the thermodynamic forces $(\boldsymbol{\tau}, \mathbf{Y})$ is that which maximizes the viscoinelastic dissipation function over all other possible admissible states. Thus, we maximize the objective function Ω by using the necessary conditions as follows:

$$(3.46) \qquad \frac{\partial \Omega}{\partial \tau_{ij}} = 0 \quad \text{and} \quad \frac{\partial \Omega}{\partial Y_{ij}} = 0.$$

Substitution of Eq. (3.45) into Eq. (94) along with Eq. (3.41) yields the thermodynamic laws corresponding to the evolution of the total inelastic deformation rate $(\mathbf{d}^I)$ and the viscodamage variable $(\boldsymbol{\phi})$, where Eq. $(4.46)_1$ gives the inelastic deformation rate tensor as follows:

$$(3.47) \qquad d_{ij}^I = \dot{\Lambda}^{vp} \frac{\partial f}{\partial \tau_{ij}} + \dot{\Lambda}^d \frac{\partial g}{\partial \tau_{ij}}$$

Considering the earlier postulate of the additive decomposition of the inelastic deformation rate into viscoplastic and viscodamage parts (Eq. (2.27)), the following assumption is made:

$$(3.48) \qquad d_{ij}^{vp} = \dot{\Lambda}^{vp} \frac{\partial f}{\partial \tau_{ij}} \quad \text{and} \quad d_{ij}^{id} = \dot{\Lambda}^d \frac{\partial g}{\partial \tau_{ij}}$$

while Eq. $(4.46)_2$ gives the viscodamage rate evolution law as follows:

$$(3.49) \qquad L_v(\phi_{ij}) = \dot{\Lambda}^{vp} \frac{\partial f}{\partial Y_{ij}} + \dot{\Lambda}^d \frac{\partial g}{\partial Y_{ij}}$$

where f and g are respectively the dynamic viscoplastic and viscodamage loading surfaces outlined in the following section. Eqs. (3.47) and (3.49) show that a strong coupling exists between viscoplasticity and viscodamage in such a way that $\mathbf{d}^I$ is decomposed into viscoplastic $(\mathbf{d}^{vp})$ and viscoinelastic-damage $(\mathbf{d}^{id})$ components and both $\mathbf{d}^I$ and $\boldsymbol{\phi}$ are expressed in terms of the viscoplastic and viscodamage potentials.

Now, in order to obtain non-associative rules for the viscoplasticity and viscodamage hardening variables, one can assume the existence of a viscoplastic potential F and a viscodamage potential G such that they are respectively not

equal to f and g. This postulate is essential in order to obtain nonlinear viscoplastic and viscodamage hardening rules, which give a more realistic characterzation of the material response in the deformation process. The complementary laws for the evolution of the other internal state variables can then be obtained directly from the generalized normality rule, which are summarized in Table 3. Those evolution laws show strong viscoplasticity and viscodamage coupling. This strong coupling results, on the one hand, from the fact that the viscoplasticity evolution equations are obtained in the current, deformed, and damaged state and, on the other hand, the damage evolution equations are expressed in terms of the viscoplastic and viscodamage potentials.

It is noteworthy to mention that the Lagrangian parameters $\dot\Lambda^{up}$ and $\dot\Lambda^d$, respectively, characterize the effective or equivalent viscoplastic and viscodamage deformations. Therefore, the evolution of the assumed gradient variables is associated with the gradients of $\dot\Lambda^{up}$ and $\dot\Lambda^d$. This is clearly shown by the evolution equations of $\nabla^2 p$, $L_v\alpha$, $\nabla^2 r$, and $L_v\Gamma$ in Table 3.

Table 3. The thermodynamic laws for the evolution of the internal state variables.

Plasticity	Isotropic Hardening Evolution Laws	$\dot p = -\dot\Lambda^{vp}\frac{\partial F}{\partial R}$, $\overline{\nabla^2 p} = -\nabla^2\dot\Lambda^{vp}\frac{\partial F}{\partial R^g}$
	Kinematic Hardening Evolution Laws	$L_v(\alpha_{ij}) = -\dot\Lambda^{vp}\frac{\partial F}{\partial X_{ij}}$, $L_v(\nabla^2\alpha_{ij}) = -\nabla^2\dot\Lambda^{vp}\frac{\partial F}{\partial X_{ij}^g}$
Damage	Isotropic Hardening Evolution Laws	$\dot r = -\dot\Lambda^d\frac{\partial G}{\partial K}$, $\overline{\nabla^2 r} = -\nabla^2\dot\Lambda^d\frac{\partial G}{\partial K^g}$
	Kinematic Hardening Evolution Laws	$L_v(\Gamma_{ij}) = -\dot\Lambda^d\frac{\partial G}{\partial H_{ij}}$, $L_v(\nabla^2\Gamma_{ij}) = -\nabla^2\dot\Lambda^d\frac{\partial G}{\partial H_{ij}^g}$

The non-local evolution of each of the assumed internal state variables $L_v\widehat\Xi_n$, ($n = 1 - 4$)) at position x in the current configuration, can be expressed as the weighted average of its local counterpart $L_v\Xi_n$ over a surrounding volume v at a small distance $|\zeta| \le L_C$ from the considered point, such that:

$$(3.50) \qquad L_v\widehat\Xi_n = \frac{1}{v}\int_v h(\zeta) L_v\Xi_n(x+\zeta)\, dv,$$

where L_c is an internal characteristic length and $h(\zeta)$ is a weight function that decays smoothly with distance and in this work is given by $h(\zeta) = \mathbf{I}h(\zeta)$ where $\mathbf{I}$ is an identity tensor. However, the identity tensor $\mathbf{I}$ may be suitably substituted

by another tensor in order to induce further anisotropic behavior of the material (VOYIADJIS and DORGAN [97]).The local variable $L_v \Xi_n$ in Eq. (3.50) can be approximated by a Taylor expansion at $\zeta = 0$ such that:

$$(3.51) \quad L_v\Xi_n\,(x+\zeta) = L_v\Xi_n\,(x) + \nabla L_v\Xi_n\,(x)\,\zeta + \frac{1}{2!}\nabla^2 L_v\Xi_n\,(x)\,\zeta\zeta$$

$$+ \frac{1}{3!}\nabla^3 L_v\Xi_n\,(x)\,\zeta\zeta\zeta + \cdots$$

where ∇^i denotes the i-th order gradient operator. Assuming only an isotropic influence of the averaging equation, as we stated previously, the integrals of the odd terms in Eq. (3.51) vanish. Furthermore, making use of Eqs. (3.50) and (3.51) and truncating the Taylor series after the quadratic term lead to the following expression for the nonlocal variable $L_v\widehat{\Xi}_n$:

$$(3.52) \qquad L_v\widehat{\Xi}_n = \frac{1}{v}\int\limits_v h\,(\zeta)\,L_v\Xi_n\,(x)\,dv + \frac{1}{2v}\int\limits_v h\,(\zeta)\,\nabla^2 L_v\Xi_n\,(x)\,\zeta\zeta\,dv.$$

This relation can be expressed as a partial differential equation such that:

$$(3.53) \qquad L_v\widehat{\Xi}_n = L_v\Xi_n + \left(\frac{1}{2v}\int\limits_v [h\,(\zeta)]\,\zeta\zeta\,dv\right)\nabla^2 L_v\Xi_n.$$

However, setting $\dfrac{1}{v}\displaystyle\int\limits_v [h\,(z)]\,dv = 1$, Eq. (3.53) can be re-written at constant x as follows:

$$(3.54) \qquad L_v\widehat{\Xi}_n = L_v\Xi_n + c_n L_v\nabla^2\Xi_n.$$

such that:

$$(3.55) \qquad \widehat{\dot{p}} = \dot{p} + c_1\,\overline{\nabla^2 p},$$

$$(3.56) \qquad L_v(\widehat{\alpha}_{ij}) = L_v(\alpha_{ij}) + c_2 L_v(\nabla^2\alpha_{ij}),$$

$$(3.57) \qquad \widehat{\dot{r}} = \dot{r} + c_3\,\overline{\nabla^2 r},$$

$$(3.58) \qquad L_v(\widehat{\Gamma}_{ij}) = L_v(\Gamma_{ij}) + c_4 L_v(\nabla^2\Gamma_{ij}).$$

In Eqs. (3.54)-(3.59), ($n = 1-4$) are constants proportional to length squared and weight each component of the gradient term identically, which give a rise for explicit length-scale measures. If one assumes a more general tensorial character for **h** not necessarily confined to the expression in terms of an identity tensor, then one obtains a different weighting of the individual coefficients. This will give a weighting function with a tensorial nature c_n containing several different integration constants.

Following Eq. (3.54), one can write the evolution equations of the corresponding non-local state laws as follows:

$$(3.59) \qquad L_v\widehat{\Sigma}_n = L_v\Sigma_n + L_v\Sigma_n^g$$

such that:

$$(3.60) \qquad \dot{\widehat{R}} = \dot{R} + \dot{R}^g,$$

$$(3.61) \qquad L_v(\widehat{X}_{ij}) = L_v(X_{ij}) + L_v(X_{ij}^g),$$

$$(3.62) \qquad \dot{\widehat{K}} = \dot{K} + \dot{K}^g,$$

$$(3.63) \qquad L_v(\widehat{H}_{ij}) = L_v(H_{ij}) + L_v(H_{ij}^g).$$

The next important step is the selection of the appropriate form of the viscoplastic potential function $F \geq 0$ and the viscodamage potential function $G \geq 0$ in order to establish the desired constitutive equations that describe the mechanical behavior of the material. It is clearly seen in the previous part of this work that the viscodamage evolution laws are strongly coupled with viscoplasticity. To maintain this strong coupling, two independent viscodamage mechanisms are distinguished. One mechanism is coupled with viscoplasticity, while the other one occurs independent of viscoplastic deformation. Similar argument was presented by HESEBECK [46], where he showed that the first mechanism is dominated in the case of shear stress and the second one for hydrostatic stress. In order to be consistent and satisfy the generalized normality rule of thermodynamics, a proper analytical form for the viscoplastic and the viscodamage potentials need to be postulated to obtain consistent evolution equations for the flux variables, such that:

$$(3.64) \qquad F = f + \frac{1}{2}k_1\bar{X}_{mn}\bar{X}_{mn} + \frac{1}{2}k_2\bar{X}_{mn}^g\bar{X}_{mn}^g + \frac{1}{2}k_3\bar{R}^2 + \frac{1}{2}k_4\bar{R}^{g2},$$

$$(3.65) \qquad G = g + \frac{1}{2}k_5 H_{mn}H_{mn} + \frac{1}{2}k_6 H^g_{mn}H^g_{mn} + \frac{1}{2}k_7 K^2 + \frac{1}{2}k_8 K^{g^2},$$

where k_i $(i = 1 - 8)$ are material constants used to adjust the units of the equation, which are again independent of temperature. $\bar{X}$ and $\bar{X}^g$ are the effective backstress tensors associated with the kinematic hardening and are expressed similarly to Eq. (2.29) as:

$$(3.66) \qquad \bar{X}_{ij} = M_{ikjl}X_{kl} \quad \text{and} \quad \bar{X}^g_{ij} = M_{ikjl}X^g_{kl}.$$

The isotropic hardening represents a global expansion in the size of the yield surface with no change in shape. Thus for a given yield criterion and flow rule, isotropic hardening in any process can be predicted from the knowledge of the functions $\bar{R}$ and $\bar{R}^g$, and those functions may in principle, be determined from a single test (e.g. the tension test). Therefore, the effective isotropic hardening functions $\bar{R}$ and $\bar{R}^g$ are related to the nominal isotropic hardening function R and R^g similar to Eq. (2.29) as follows:

$$(3.67) \qquad \bar{R} = \frac{R}{1 - r},$$

$$(3.68) \qquad \bar{R}^g = \frac{R^g}{1 - r},$$

where r is defined as the accumulative or equivalent damage and can be expressed as follows:

$$(3.69) \qquad r = \sqrt{\phi_{ij}\phi_{ij}}.$$

The assumed potential functions indicate the need for two loading surfaces f and g, one for viscoplasticity and another for viscodamage, respectively. Thus, the coupled anisotropic viscoplastic and viscodamage formulation is a two-surface model whereby anisotropic viscodamage is formulated in the spirit of viscoplasticity, complete with a viscodamage criterion and flow rules.

4. Viscoplasticity yield criterion and its corresponding flow rules

4.1. Viscoplasticity yield criterion

Once a material is damaged, further loading can only affect the undamaged material. Thus, the viscoplastic function f is defined in terms of the effective stresses. For the classical J_2 rate-independent plasticity, the static yield surface f_s (at negligible viscoplastic strain rate) is assumed to be of a von Mises type with isotropic and kinematic hardening, and defined as follows:

$$(4.1) \quad f_s = \sqrt{3J_2\left(\bar{\tau}'_{ij} - \bar{X}_{ij} - \bar{X}^g_{ij}\right)} - \bar{\tau}_{yp}(T) - \bar{R}(\bar{p}, T) - \bar{R}^g(\nabla^2\bar{p}, T) \leq 0$$

where $J_2 = 1/2(\bar{\tau}' - \bar{X} - \bar{X}^g) : (\bar{\tau}' - \bar{X} - \bar{X}^g)$ is the second invariant of the deviatoric stress tensor $(\bar{\tau}' - \bar{X} - \bar{X}^g)$, $\bar{\tau}_{yp}(T)$ is the initial yield strength as a function of temperature, R and R^g are the isotropic hardening functions, T is the absolute temperature, and $\bar{p}$ is the equivalent viscoplastic strain. $\bar{\tau}'$ is the effective deviatoric Kirchhoff stress tensor, $\bar{X}$ and $\bar{X}^g$ are the effective backstress tensors associated with the kinematic hardening. $\bar{\tau}'$ is expressed in terms of the viscodamage tensor M (given by Eq. (2.30)) and the corresponding damage states as follows (VOYIADJIS and KATTAN [100]):

$$(4.2) \qquad \bar{\tau}'_{ij} = M'_{ijkl}\tau_{kl} \quad \text{with} \quad M'_{ijkl} = M_{ijkl} - \frac{1}{3}M_{rrkl}\delta_{ij}.$$

The accumulative or equivalent rate of the effective viscoplastic deformation $(\dot{\bar{p}})$ is defined by:

$$(4.3) \qquad \dot{\bar{p}} = \sqrt{\frac{2}{3}\bar{d}_{ij}^{vp}\bar{d}_{ij}^{vp}}$$

where $\bar{\mathbf{d}}^{vp}$ is the viscoplastic deformation rate tensor in the effective configuration.

The extension of Eq. (4.1) to include the rate-dependent plasticity (viscoplasticity) implies that the stress state is no longer constrained to remain on the yield surface but one can have $f_s \geq 0$. Therefore, we define the dynamic yield surface f as follows:

$$(4.4) \qquad f = \sqrt{3J_2\left(\bar{\tau}'_{ij} - \bar{X}_{ij} - \bar{X}^g_{ij}\right)} - \bar{\tau}_{yp} - \bar{R} - \bar{R}^g - \bar{\tau}_{vp} \equiv 0$$

where $\bar{\tau}_{vp} = \langle f_s \rangle$ is the viscous effective stress (or the overstress, i.e. the difference between the dynamic stress and its static counterpart) and $\langle x \rangle$ denotes the MacAuley brackets defined by $\langle x \rangle = (x + |x|)/2$. Clearly, $\bar{\tau}_{vp}$ is the common notion of viscoplasticity (PERZYNA [78]), which implies that an inelastic process can only take place if, and only if, the overstress $\bar{\tau}_{vp}$ is positive. From the assumed yield function f the current effective yield stress can be defined as follows to account for high-strain rate and temperature effects:

$$(4.5) \qquad \bar{Y} = \bar{\tau}_{yp}(T) + \bar{R}(\bar{p}, T) + \bar{R}^g(\nabla^2\bar{p}, T) + \bar{\tau}_{vp}(\dot{\bar{p}}, \nabla^2\dot{\bar{p}}, \bar{p}, \nabla^2\bar{p}, T)$$

where one can define the initial yield strength $\bar{\tau}_{yp}$ as follows:

$$(4.6) \qquad \bar{\tau}_{yp}(T) = \bar{\sigma}_o\vartheta$$

where $\bar{Y}_o$ is the initial yield stress in the undamaged state, which is obtained at zero absolute temperature, zero viscoplastic strain, and static strain rate, and

$\vartheta = 1 - (T/T_m)^n$. The evolution equations of the hardening functions $\bar{R}$, $\bar{R}^g$, $\bar{X}$, and $\bar{X}^g$ will be derived in the subsequent sections.

The proposed dynamic yield criterion $f = 0$ (Eq. (4.4)) is a generalization of the classical von Mises yield criterion $f_s \leq 0$ (Eq. (4.1)) for rate-dependent materials. The latter can be simply recovered by imposing $\dot{p} = \dot{p}_o$ (rate-independent), so that one has the plasticity case $f \leq 0$, where $\dot{p}_o$ is the static strain rate (i.e. the smallest strain rate of the strain-rate range considered), and is called the control strain rate according to the notation of PERZYNA [81]. In the elastic domain, both f_s and f are equivalent since, in that case, $\dot{p} = 0$. Therefore, the admissible stress states are constrained to remain on or within the elastic domain ($f \leq 0$). The viscoplastic parameter $\dot{\Lambda}^{vp} \geq 0$, which is known as the plastic consistency parameter, is assumed to obey the following Kuhn-Tucker loading/unloading conditions (SIMO and HUGHES, 1998):

$$(4.7) \qquad f \leq 0 \quad \text{and} \quad \dot{f} \begin{cases} < 0 & \Rightarrow \dot{\Lambda}^{vp} = 0 \\ = 0 & \Rightarrow \dot{\Lambda}^{vp} = 0 \\ = 0 & \Rightarrow \dot{\Lambda}^{vp} > 0 \end{cases} \Leftrightarrow \begin{cases} \text{elastic unloading} \\ \text{neutral loading} \\ \text{plastic loading} \end{cases}$$

Thus, f still satisfies the constraint equation Eq. (3.45) and the maximum dissipation principle Eq. (94)$_1$.

4.2. Viscoplastic flow rule d^{vp}

One can substitute in Eq. (4.48)$_1$ different admissible forms for the Lagrange multiplier $\dot{\Lambda}^{vp}$ without violating the constrained maximization problem presented by Eq. (4.46)$_1$. However, the evolution equation for $\dot{\Lambda}^{vp}$ is now defined in a quiet similar way as in the classical viscoplasticity. Several evolution equations have been proposed in the literature to calculate the viscoplastic strain rate. Some of them are physically-based and others are phenomenological. The most widely used in rate-dependent plasticity (viscoplasticity) are those based on the overstress concept (ZENER and HOLLOMON [108], [109], MALVERN [65], [66], PERZYNA [77], [78], [80], [83]). One of the first and most widely used phenomenological models for rate-sensitive plastic flow is due to PERZYNA ([77], [78], [79], [80], [81]), which has been often considered in computational applications. On the other hand, ZENER and HOLLOMON ([108], [109]) proposed that the functional dependence of the magnitude of the inelastic strain rate of metals could be multiplicatively decomposed into two functions: thermal (static) and stress (dynamic). FREED and WALKER ([37], [39]) and FREED et al. [38] showed that the Zener parameter is very general and includes many viscoplastic theories as special cases. In the spirit of the Perzyna-type and Zener-type evolution equations for the viscoplastic strain rate, one can postulate the following admissible

form for the Lagrangian multiplier $\dot{\Lambda}^{vp}$, such that:

$$(4.8) \qquad \dot{\Lambda}^{vp} = \frac{1}{\eta_{vp}} \left[\frac{\langle \bar{\tau}_{vp} \rangle}{\bar{\tau}_{yp} + \bar{R} + \bar{R}^g} \right]^{m_1}$$

where m_1 is the viscoplastic rate sensitivity parameter and η_v is the viscosity or fluidity parameter, sometimes referred to as the relaxation time according to notation of Perzyna [81]. Note that the script "vp" used as a superscript or subscript does not designate tensor indices but hardly denotes the corresponding viscoplasticity.

One can now utilize the hypothesis of viscoplastic dissipation equivalence Lee *et al.* [58], Voyiadjis and Thiagarajan [103], Voyiadjis and Deliktas [95]. This hypothesis assumes that the viscoplastic energy in terms of the effective and nominal stress and strain quantities must be equal. Thus, one can write the following relation, such that:

$$(4.9) \qquad \bar{d}_{ij}^{vp} = N_{ikjl} d_{kl}^{vp}$$

where N is the inverse of the fourth-order damage tensor which is given by Eq. (2.32).

By making use of the effective stress equation (Eq. (2.29)) and the effective viscoplastic deformation rate equation (Eq. (4.9)), we can write the viscoplastic deformation rate equation (Eq. (96)$_1$) as follows:

$$(4.10) \qquad \bar{d}_{ij}^{vp} = \dot{\Lambda}^{vp} \frac{\partial f}{\partial \bar{\tau}_{ij}},$$

Since $\partial f / \partial \bar{\tau} : \partial f / \partial \bar{\tau} = 1.5$, it can be easily shown that the effective rate of the accumulative viscoplastic deformation rate $(\dot{\bar{p}})$ defined by Eq. (4.3) is related to $\dot{\nabla}^{vp}$ by:

$$(4.11) \qquad \dot{\bar{p}} = \dot{\Lambda}^{vp},$$

By making use of Eqs. (4.8) and (4.11), one can write an expression for the overstress function $\bar{\tau}_{vp}$ as follows:

$$(4.12) \qquad \bar{\tau}_{vp} = (\eta_{vp}\dot{\bar{p}})^{1/m_1} \left(\bar{\tau}_{yp} + \bar{R} + \bar{R}^g \right).$$

Note that from Eq. (4.12) the classical von Mises criterion $f_s = 0$ (Eq. (4.1)) for rate-independent materials can be simply recovered by imposing $\eta_{vp} = 0$ (no viscosity effect) on f (Eq. (4.4)). Moreover, the well-known fact that, from the relation in Eq. (4.8), it can be noted that as the viscosity parameter η_{vp} goes to zero, the consistency parameter $\dot{\nabla}^{vp}$ remains finite and positive (though indeterminate) since $\bar{\tau}_v$ also goes to zero.

Setting the dynamic yield function f in the proposed form allows one to use effectively the well-known rate-independent radial-return algorithm to integrate the viscoplasticity governing equations. Applying the consistency condition, $\dot{f} = 0$, also gives $\dot{\Lambda}^{vp}$ as proposed in Eq. (4.8).

4.3. Viscoplastic hardening rules

Now, in order to derive the evolution of the viscoplasticity isotropic hardening functions $\bar{R}$ and $\bar{R}^g$ in the effective state to be used in the dynamic yield function f, we makes use of Eqs. (3.64), (3.67), (3.68), (4.4), and (4.11) in $\dot{p}$ and $\overline{\nabla^2 p}$ equations in Table 3 along with the chain rule, such that the following expressions can be written :

(4.13)
$$\dot{p} = \frac{\dot{\bar{p}}}{1 - r} \left(1 - k_3 \bar{R}\right),$$

(4.14)
$$\overline{\dot{\nabla^2 p}} = \frac{\nabla^2 \dot{\bar{p}}}{1 - r} \left(1 - k_4 \bar{R}^g\right).$$

By operating on R and R^g relations in Table 2, and $\bar{R}$ and $\bar{R}^g$ relations (Eqs. (3.67) and (3.68)) with the Lie derivative for a given temperature and keeping the damage history constant (i.e. the damage internal state tensor $\boldsymbol{\phi}$ and temperature T are kept constant), one can write the following evolution equations for $\bar{R}$ and $\bar{R}^g$, respectively, as follows:

(4.15)
$$\dot{\bar{R}} = \frac{a_1 \dot{\bar{p}}}{(1 - r)^2} \left(1 - k_3 \bar{R}\right) \vartheta$$

(4.16)
$$\dot{\bar{R}}^g = \frac{a_2 \nabla^2 \dot{\bar{p}}}{(1 - r)^2} \left(1 - k_4 \bar{R}^g\right) \vartheta$$

Now in order to derive the kinematic hardening evolution equations associated with viscoplasticity, one makes use of $L_v \alpha$ equation in Table 3 along with the chain rule and Eq. (114)$_1$, such that one can write the following:

(4.17)
$$L_v(\alpha_{ij}) = -\dot{\Lambda}^{vp} M_{minj} \frac{\partial F}{\partial \bar{X}_{mn}},$$

Substitution of Eq. (3.64) into the above equation yields:

(4.18)
$$L_v(\alpha_{ij}) = -\dot{\Lambda}^{vp} M_{minj} \left(\frac{\partial f}{\partial \bar{X}_{mn}} + k_1 \bar{X}_{mn}\right),$$

Since $\partial f / \partial \bar{X} = -\partial f / \partial \bar{\tau}$ as it is clear from Eq. (4.1), it can be easily shown by using Eqs. (4.10) and (4.11) that Eq. (4.18) can be rewritten as follows:

$$(4.19) \qquad L_v(\alpha_{ij}) = M_{minj} \left(\bar{d}_{mn}^{vp} - k_1 \dot{\bar{p}} \bar{X}_{mn} \right).$$

However, operating on the X relation in Table 2, and $\bar{X}$ relation Eq. $(114)_1$ with the Lie derivative for a given temperature and keeping the damage history constant (i.e. the viscodamage effective tensor $\mathbf{M}$ and temperature T are kept constant), one can write the following evolution equation for $\bar{X}$ as follows:

$$(4.20) \qquad L_v(\bar{X}_{ij}) = \mathbf{M}_{ikjl}\mathbf{M}_{mknl} \left(a_3 \bar{d}_{mn}^{vp} - k_1 a_3 \dot{\bar{p}} \bar{X}_{mn} \right) \vartheta.$$

When the infinitesimal deformations and rate and temperature-independent response of a material are assumed and the micro-damage effects are neglected then the kinematic hardening law (Eq. (4.20)) reduces to that proposed by Armstrong and Frederick [6].

Similarly, by utilizing $L_v \nabla^2 \alpha$ equation in Table 3 and the Lie derivative of X^g relation in Table 2, one can write the gradient-dependent evolution equation of viscoplasticity kinematic hardening ($L_v X^g$) as:

$$(4.21) \qquad L_v(\bar{X}_{ij}^g) = M_{ikjl}M_{mknl} \left(a_4 \frac{\partial f}{\partial \bar{\tau}_{mn}} - k_2 a_4 \bar{X}_{mn}^g \right) \nabla^2 \dot{\bar{p}} \vartheta$$

It is noteworthy to point out here that the derived evolution equations of viscoplasticity hardening ($\dot{\bar{R}}$, $\dot{\bar{R}}^g$, $L_v X$, and $L_v X^g$) containing hardening terms that represent the strengthening mechanism as well as recovery terms that represent the softening mechanism. Both the hardening and recovery terms are affected by the static (thermal) recovery term ϑ in such a way that the functional dependence of the hardening of materials could be multiplicatively decomposed into two functions: thermal (static) and stress (dynamic). Therefore, those evolution equations characterize the time and thermal effects due to the rate and temperature dependence. Freed et. al. [38] pointed out that in viscoplasticity at elevated temperatures, thermal recovery of materials usually plays an important rule in the deformation process. In impact-related problems, the thermomechanical response of the assumed internal state variables is strongly dependent on the temperature history. This requires existence of thermal recovery (softening) terms in the evolution equations of the internal state variables.

4.4. Damage evolution criterion and its corresponding flow rules

4.4.1. Damage evolution criterion.
By combining viscoplasticity with viscodamage, it seems natural that damage can only affect the undamaged material skeleton. Thus the viscodamage function g is also defined in terms of the effective

stresses. The anisotropic viscodamage calculation is formulated in spirit of viscoplasticity; therefore, analogous to the dynamic viscoplastic yield surface presented in the previous section, the dynamic viscodamage surface g which evolves from a static damage surface g_s can be postulated as follows:

$$(4.22) \qquad g = g_s - L(T, r, \dot{r}, \overline{\nabla^2 r}) = 0$$

$$(4.23) \quad g = \sqrt{J_2(Y_{ij} - H_{ij} - H_{ij}^g)} - l(T) - K(r,T) - K^g\left(\nabla^2 r, T\right)$$

$$- L(T, r, \nabla^2 r, \dot{r}, \overline{\dot{\nabla^2 r}}) = 0$$

where $J_2(Y - H - H^g)$ denotes the second invariant of the damage force $(Y - H - H^g)$, $l(T)$ is the initial damage threshold as a function of temperature, and L is the threshold damage force increment for microdamge nucleation, growth, and coalescence, which is dependent on the temperature, damage accumulation, and rate of damage accumulation.

Similar expressions for $l(T)$ and $\dot{\Lambda}^d$ can be postulated as presented by Eqs. (4.6) and (4.8), respectively, such that:

$$(4.24) \qquad l(T) = l_o \vartheta,$$

$$(4.25) \qquad \dot{\Lambda}^d = \frac{1}{\eta_{vd}} \left[\frac{\langle L \rangle}{l + K + K^g} \right]^{m_2},$$

where l_o is the initial damage threshold at zero absolute temperature, zero damage strain, and static damage strain rate, and $\vartheta = 1 - (T/T_m)^n$. m_2 is the damage rate sensitivity parameter and η_{vd} is the viscosity parameter or the damage relaxation time, which can be different than η_{vp}.

By making use of Eq. (4.25), one can write an expression for the overforce damage function L as follows:

$$(4.26) \qquad L = (\eta_{vd}\dot{r})^{1/m_2} (l + K + K^g).$$

The postulated dynamic viscodamage function $g = 0$ (Eq. (4.23)) is a generalization of the static damage surface as proposed by Voyiadjis and Deliktas (2000b) for rate-dependent materials. Similar to the yield surface, the static damage surface can be simply recovered by imposing $\eta_{vd} = 0$ (rate-independent), so that one has the instantaneous damage case $g \leq 0$. In the undamaged domain, both the static and dynamic damage growth conditions are equivalent since, in that case, $\dot{r} = 0$. Therefore, the admissible damage forces are constrained to

remain on or within the undamaged domain ($g \leq 0$). The model response in the viscodamage domain is then characterized as follows:

(4.27) $$g = g_s < 0 \Leftrightarrow \text{undamaged state} \Rightarrow \dot{\nabla}^d = 0,$$

(4.28) $$g \leq 0 \quad \text{and} \quad \dot{g} \begin{cases} < 0 \;\Rightarrow \dot{\Lambda}^d = 0 \\ = 0 \;\Rightarrow \dot{\Lambda}^d = 0 \\ > 0 \;\Rightarrow \dot{\Lambda}^d > 0 \end{cases} \Leftrightarrow \begin{cases} \text{undamaged state} \\ \text{damage initiation} \\ \text{damage growth} \end{cases}$$

4.4.2. Damage hardening rules. Now, in order to derive the hardening evolution equations associated with viscodamage process, we follow the same procedure presented in the previous section for viscoplasticity. The evolution equations for the viscodamage isotropic hardening functions K and K^g are obtained by making use of Eqs. (3.65) and (4.23) in $\dot{r}$ and $\overline{\nabla^2 r}$ relations in Table 3 along with the Lie derivative of K and K^g equations in Table 2 for a given temperature, such that the following expressions are obtained:

(4.29) $$\dot{K} = b_1 \left(1 - k_7 K \right) \dot{r} \vartheta$$

(4.30) $$\dot{K}^g = b_2 \left(1 - k_8 K^g \right) \nabla^2 \dot{r} \vartheta$$

Moreover, the viscodamage kinematic hardening evolutions equations can be obtained by using $L_v \Gamma$ and $L_v \nabla^2 \Gamma$ relations in Table 3 and Eq. (3.65), such that one obtains the following:

(4.31) $$L_v(\Gamma_{ij}) = -\dot{\Lambda}^d \left(\frac{\partial g}{\partial H_{ij}} + k_5 H_{ij} \right),$$

(4.32) $$L_v(\nabla^2 \Gamma_{ij}) = -\nabla^2 \dot{\Lambda}^d \left(\frac{\partial g}{\partial H_{ij}^g} + k_6 H_{ij}^g \right),$$

Since $\partial g / \partial H = \partial g / \partial H^g = -\partial g / \partial Y$ as it is clear from Eq. (4.23) it can be easily shown by taking the Lie derivative of H and H^g equations in Table 2 for a given temperature that Eqs. (4.31) and (4.32) can be rewritten as follows:

(4.33) $$L_v(H_{ij}) = \left(b_3 \frac{\partial g}{\partial Y_{ij}} - k_5 b_3 H_{ij} \right) \dot{r} \vartheta$$

(4.34) $$L_v(H_{ij}^g) = \left(b_4 \frac{\partial g}{\partial Y_{ij}} - k_6 b_4 H_{ij}^g \right) \nabla^2 \dot{r} \vartheta.$$

Similar to the viscoplastic hardening evolution equations, the derived visco-damage evolution equations consider the dynamic recovery as well as the static recovery (thermal recovery). Besides, the static recovery occurs in both the hardening and the dynamic recovery terms. The hardening term of each assumed internal state variable accounts for strengthening mechanisms, while the recovery terms account for softening mechanisms.

For the sake of completeness, we outline in the Appendix the derivatives that are necessary to calculate the above derived evolution equations.

5. Rate-type constitutive relation

As previously mentioned, the elasticity tensor $\Im$ can be considered constant either in the material or current configuration. This leads to the definition of different materials. We are going to consider here that the elasticity tensor $\Im$ is constant in the reference configuration C^o. Therefore, operating on the stress relation Eq. (3.30) with the Lie derivative and keeping the damage history constant (i.e. the internal state variables ϕ constant), one can obtain for a general thermo-elasto-viscoplastic and viscodamage flow processes the following relation:

$$(5.1) \qquad L_v(\tau_{ij}) = \Im_{ijkl} \left(d_{kl} - d_{kl}^{vp} - d_{kl}^{id} \right) - \beta_{ij}\dot{T}$$

where $\Im$ is the spatial elasticity-damage tensor related to the fourth-order tensor of elastic-damaged constants C (Eq. (2.31)) by the (push-forward) transformation as follows:

$$(5.2) \qquad \Im_{ijkl} = F_{ir}F_{js}F_{km}F_{ln}C_{rsmn}.$$

Similarly, Eqs. (5.1) and (5.2) can be defined, respectively, using the Jaumann-Zaremba stress rate tensor as follows:

$$(5.3) \qquad \tau_{ij}^o = a_{ijkl} \left(d_{kl} - d_{kl}^{vp} - d_{kl}^{id} \right) - \beta_{ij}\dot{T}$$

where

$$(5.4) \qquad a_{ijkl} = \Im_{ijkl} + \delta_{il}\tau_{jk} + \delta_{jl}\tau_{ik}.$$

Substitution of Eqs. (4. 48), (4.8) and (4.25) into Eq. (5.3), yields:

$$(5.5) \quad \tau_{ij}^o = a_{ijkl}$$

$$\cdot \left[d_{kl} - \frac{1}{\eta_v} \left\langle \frac{\bar{\tau}_v}{\bar{\tau}_{yp} + \bar{R} + \bar{R}^g} \right\rangle^{m_1} \frac{\partial f}{\partial \tau_{kl}} - \frac{1}{\eta_{vd}} \left\langle \frac{L}{l + K + K^g} \right\rangle^{m_2} \frac{\partial g}{\partial \tau_{kl}} \right] - \beta_{ij}\dot{T}.$$

6. Thermomechanical couplings

Substituting Ψ from Eq. (3.14) into Eq. (3.11) yields the following energy balance equation:

$$(6.1) \qquad \rho\left(\dot{\Psi} + \eta\dot{T} + \dot{\eta}T\right) - \frac{\rho}{\rho^o}\tau_{ij}d_{ij} - \rho h + q_{i,i} = 0$$

Substituting $\dot{\Psi}$ (Eq. (3.17)) into the above equality and taking into account the results presented by Eqs. (3.23) and (3.40) we obtain the following:

$$(6.2) \qquad \rho^o\dot{\eta}T = \tau_{ij}\left(d_{ij}^{vp} + d_{ij}^{id}\right) - \Pi_{int} + \rho^o h + Jq_{i,i} = 0$$

Operating on the entropy relation Eq. (3.23)$_2$ with the Lie derivative and substituting the result into Eq. (6.2), we obtain:

$$(6.3) \quad \rho c_p\dot{T} = \Upsilon\frac{\rho}{\rho^o}\tau'_{ij}\left(d_{ij}^{vp} + d_{ij}^{id}\right) + \Upsilon\frac{\rho}{\rho^o}P\left(d_{ij}^{vp} + d_{ij}^{id}\right) - \rho\Pi_{int} + \rho\frac{\partial\Sigma_k}{\partial T}\dot{\aleph}_k T$$

$$- \frac{\rho}{\rho^o}\beta_{ij}d_{ij}^E T + \rho^o r_{ext} + k\nabla^2 T$$

where $c_p = T\partial\eta/\partial T$ is the specific heat at constant pressure.

The rate type-equations (Eqs. (5.5) and (6.3)) take into account effects of the viscoplastic and viscodamage strain-induced anisotropy (i.e. kinematic hardening), flow stress temperature and strain-rate sensitivity (i.e. isotropic hardening), anisotropic damage (i.e. softening generated by damage defects nucleation and growth mechanisms), thermomechanical couplings (i.e. thermal viscoplastic and viscodamage softening), strong viscoplasticity and viscodamage coupling, and heterogeneity in the material behavior (i.e. the non-local influence).

7. Conclusions

A thermodynamically consistent nonlocal gradient-enhanced framework is presented here with strong viscoplasticity and anisotropic viscodamage coupling for impact-related problems. Thermodynamically consistent constitutive equations are derived in order to introduce issues such as the statistical inhomogeneity in the evolution-related viscoplasticity and viscodamage variables associated with the RVE, localization and size effects of deformation defects on the macroscopic response of heterogeneous materials, and temperature and strain-rate sensitivity.

This model is general enough to describe the evolution of visco-inelasticity in a material body accounting for physical discontinuities through the use of

a nonlocal approach based on the gradient-dependent theory of viscoplasticity and viscodamage. The interaction of the length-scales is a crucial factor in understanding and controlling the material defects such as mobile and immobile dislocation densities, voids, and cracks influence on the macroscopic response. The behavior of these defects is captured not only individually, but also the enhanced strong coupling between the two dissipative processes takes into account the interaction between these defects and their ability to create spatio-temporal patterns under different loading conditions. An equation of state is presented in this work for high impact loading that accounts for compressibility effects (change in density) in terms of the nominal volumetric damage strain.

Length-scale parameters are implicitly and explicitly introduced into the present dynamical formulation. Implicit length-scale measure is introduced through the use of the rate-dependent theory, while explicit length-scale measures are introduced through the use of the gradient-dependent theory.

The computational issue of this theoretical formulation with proper explanation of the proper boundary conditions associated with the gradients and evaluation of respective material parameters will be presented in a forthcoming work. Calibration for the different material properties in the proposed approach may be difficult, or impossible for certain cases. However, the proposed framework is generalized to that of viscoplasticity coupled with viscodamage, and one needs more studies to be performed in order to effectively assess the potential applications of this framework.

Appendix

The following relations are necessary for model implementation:

$$(A.1) \qquad Y_{ij} = 2\bar{\Psi}^E M_{kplq} J_{kplqij}$$

where $\bar{\Psi}^E$ is given by:

$$(A.2) \qquad \bar{\Psi}^E = \frac{1}{2}\bar{\tau}_{ij}\bar{\mathfrak{S}}^{-1}_{ijkl}\bar{\tau}_{kl} = \frac{1}{2}\tau_{ij}\mathfrak{S}^{-1}_{ijkl}\tau_{kl},$$

and J is a sixth-order tensor and is given by:

$$(A.3) \qquad J_{kplqab} = -\frac{\partial M^{-1}_{kplq}}{\partial \phi_{ab}} = \frac{1}{2}\left(\delta_{lq}\delta_{ka}\delta_{pb} + \delta_{kp}\delta_{la}\delta_{qb}\right),$$

$$(A.4) \qquad \frac{\partial f}{\partial \tau_{ij}} = \frac{\partial f}{\partial \bar{\tau}_{mn}}\frac{\partial \bar{\tau}_{mn}}{\partial \tau_{ij}} = M_{minj}\frac{\partial f}{\partial \bar{\tau}_{mn}},$$

(A.5)
$$\frac{\partial g}{\partial \tau_{ij}} \equiv \frac{\partial g}{\partial Y_{mn}} \frac{\partial Y_{mn}}{\partial \tau_{ij}} = e_{ij}^{E} \frac{\partial g}{\partial Y_{mn}} M_{kplq} J_{kplqmn},$$

(A.6)
$$\frac{\partial f}{\partial Y_{ij}} \equiv \frac{\partial f}{\partial \tau_{mn}} \frac{\partial \tau_{mn}}{\partial Y_{ij}} = \frac{\partial f}{\partial \tau_{mn}} \left(\frac{\partial Y_{ij}}{\partial \tau_{mn}} \right)^{-1},$$

where

(A.7)
$$\frac{\partial Y_{ij}}{\partial \tau_{mn}} = M_{kplq} J_{kplqij} e_{mn}^{E},$$

(A.8)
$$\frac{\partial g}{\partial Y_{ij}} \equiv \frac{Y_{ij} - H_{ij} - H_{ij}^{g}}{\sqrt{J_2 \left(Y_{kl} - H_{kl} - H_{kl}^{g} \right)}},$$

(A.9)
$$\frac{\partial f}{\partial \bar{\tau}_{ij}} \equiv \frac{3}{2} \frac{\bar{\tau}_{ij}' - \bar{X}_{ij} - \bar{X}_{ij}^{g}}{\sqrt{3 J_2 \left(\bar{\tau}_{ij}' - \bar{X}_{kl} - \bar{X}_{ij}^{g} \right)}}.$$

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