

Some more inverse solutions for steady flows of a second-grade fluid

A. M. SIDDIQUI⁽¹⁾, M. R. MOHYUDDIN⁽²⁾, T. HAYAT⁽²⁾,
S. ASGHAR⁽³⁾

⁽¹⁾*Pennsylvania State University, York Campus,
York, PA 17403, USA*

⁽²⁾*Department of Mathematics, Quaid-i-Azam University,
Islamabad 45320, Pakistan*

⁽³⁾*COMSATS, Institute of Information Technology,
Abbottabad, Pakistan*

INVERSE SOLUTIONS of the equations of motion of an incompressible second-grade fluid are obtained by assuming certain forms of the stream function. Expressions for streamlines, velocity components and pressure distributions are given in each case and are compared with the known results.

Key words: second-grade fluid; exact solutions; two moving parallel disks.

Notations

\mathbf{T}	Cauchy stress tensor,
$-p\mathbf{I}$	indeterminate spherical stress,
μ	viscosity,
α_1	elasticity,
α_2	cross-viscosity,
$\mathbf{A}_1, \mathbf{A}_2$	Rivlin–Ericksen tensors,
\mathbf{V}	velocity,
grad	the gradient operator,
d/dt	material time derivative,
\mathbf{T}	the transpose,
ρ	density,
χ	the body force,
∇^2	the Laplacian operator,
\mathbf{V}_t	$\partial\mathbf{V}/\partial t$,
$ \mathbf{A}_1 $	the usual norm of matrix \mathbf{A}_1 ,
u, v, w	the velocity components,
x, y, z	the coordinate axis,
ϕ	relative velocity of the disk,
\hat{p}	modified pressure,
$\{\cdot, \cdot\}$	Poisson bracket,
ω	vorticity vector,

- ν kinematic viscosity,
- Λ second-grade parameter,
- ψ the stream function.

1. Introduction

RHEOLOGICAL PROPERTIES of materials are specified in general by their so called constitutive equations. The simplest constitutive equation for a fluid is a Newtonian one and the classical Navier–Stokes theory is based on this equation. The mechanical behaviour of many real fluids, especially those of low molecular weight, is well enough described by this theory. However, in many fields, such as food industry, drilling operations and bio-engineering, the fluids, either synthetic or natural, are mixtures of different stuffs such as water, particle, oils, red cells and other long chain molecules; this combination imparts strong non-Newtonian characteristics to the resulting liquids; the viscosity function varies non-linearly with the shear rate; elasticity is felt through elongational effects and time-dependent effects. In these cases, the fluids have been treated as viscoelastic fluids. Because of the difficulty to suggest a single model which exhibits all properties of viscoelastic fluids, they cannot be described simply as Newtonian fluids. For this reason, many models or constitutive equations have been proposed and most of them are empirical or semi-empirical. One of the simplest types of models to account for the rheological effects of viscoelastic fluid is the second-grade model. Further, the equations governing the flow of a second-grade fluids are one order higher than the Navier–Stokes equations. A marked difference between the case of the Navier–Stokes theory and that for fluids of second-grade is that, ignoring the non-linearity in the Navier–Stokes equation does not lower the order of the equation; however, ignoring the higher order non-linearities in the case of second-grade fluid reduces the order of the equation. The no-slip boundary condition is insufficient for a second-grade fluid and therefore, one needs an additional boundary condition. A critical review on the boundary condition, the existence and uniqueness of the solution has been given by RAJAGOPAL [1].

The governing equations that describe the flow of a Newtonian fluid is the Navier–Stokes equations. These equations are nonlinear partial differential equations and known exact solutions are few in number. Exact solutions are very important not only because they are solutions of some fundamental flows but also because they serve as accuracy checks for experimental, numerical and asymptotic methods. Since the equations of motion of non-Newtonian fluids are more complicated and nonlinear than the Navier–Stokes equations, so the inverse methods described by NEMENYI [2] have become attractive. In these methods, solutions are found by assuming certain physical or geometrical properties of the flow field. KALONI and HUSCHILT [3], SIDDIQUI and KALONI [4], SIDDIQUI [5],

BENHARBIT and SIDDIQUI [6] and LABROPULU [7] used this method to study the flow problems of a second-grade fluid.

In this paper we discuss the second-grade fluid motion between two parallel disks/plates, moving towards each other or in opposite directions with a constant disk velocity. For such a fluid equations are modeled for a grade of fluid two and are solved by assuming certain form of the stream function. The graphs are plotted explicitly in the functional form to see the behaviour of the flow field.

The paper is organized as follows. In Sec. 2, basic equations and formulation of the problem is given. Section 3 consists of some special flows called the Riabouchinsky type flows and finally, in Sec. 4, concluding remarks are given. Stream function, velocity components and the pressure fields are derived in each case. Moreover, the streamlines are plotted in each case to see the flow behaviour.

2. Governing equations

The constitutive equation of an incompressible fluid of second-grade is of the form [8]

$$(2.1) \quad \mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2,$$

where \mathbf{T} is the Cauchy stress tensor, $-p\mathbf{I}$ denotes the indeterminate spherical stress and μ , α_1 and α_2 are measurable material constants. They denote, respectively, the viscosity, elasticity and cross-viscosity. These material constants can be determined from viscometric flows for any real fluid. \mathbf{A}_1 and \mathbf{A}_2 are Rivlin-Ericksen tensors [8] and they denote, respectively, the rate of strain and acceleration. \mathbf{A}_1 and \mathbf{A}_2 are defined by

$$(2.2) \quad \mathbf{A}_1 = (\text{grad}\mathbf{V}) + (\text{grad}\mathbf{V})^\top,$$

$$(2.3) \quad \mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1(\text{grad}\mathbf{V}) + (\text{grad}\mathbf{V})^\top \mathbf{A}_1.$$

Here \mathbf{V} is the velocity, grad the gradient operator, \top the transpose, and d/dt the material time derivative.

The basic equations governing the motion of an incompressible fluid are

$$(2.4) \quad \text{div}\mathbf{V} = 0,$$

$$(2.5) \quad \rho \frac{d\mathbf{V}}{dt} = \rho\mathbf{x} + \text{div}\mathbf{T},$$

where ρ is the density and \mathbf{x} the body force.

Inserting (2.1) in (2.5) and making use of (2.2) and (2.3) we obtain the following vector equation

$$(2.6) \quad \text{grad} \left[\frac{1}{2} \rho |\mathbf{V}|^2 + p - \alpha_1 \left(\mathbf{V} \cdot \nabla^2 \mathbf{V} + \frac{1}{4} |\mathbf{A}_1|^2 \right) \right] + \rho [\mathbf{V}_t - \mathbf{V} \times (\nabla \times \mathbf{V})] \\ = \mu \nabla^2 \mathbf{V} + \alpha_1 [\nabla^2 \mathbf{V}_t + \nabla^2 (\nabla \times \mathbf{V}) \times \mathbf{V}] + (\alpha_1 + \alpha_2) \text{div} \mathbf{A}_1^2 + \rho \mathbf{X},$$

in which ∇^2 is the Laplacian operator, $\mathbf{V}_t = \partial \mathbf{V} / \partial t$, and $|\mathbf{A}_1|$ is the usual norm of matrix \mathbf{A}_1 . If this model is required to be compatible with thermodynamics, then the material constants must meet the restrictions [9, 10]

$$(2.7) \quad \mu \geq 0, \quad \alpha_1 \geq 0, \quad \alpha_1 + \alpha_2 = 0.$$

On the other hand, experimental results of the tested fluids of second-grade showed that $\alpha_1 < 0$ and $\alpha_1 + \alpha_2 \neq 0$ which contradicts the above conditions and implies that such fluids are unstable. This controversy is discussed in detail in [1]. However, in this paper we will discuss both cases, $\alpha_1 \geq 0$ and $\alpha_1 < 0$.

We consider two parallel disks/plates in water and start moving them towards each other or in opposite directions (considering the size of the disks much larger than the distance between them). One can observe that when the disks are approaching each other, the effort required is smaller than that when the disks are moving apart. It can be discussed and explained by considering the different nature of the fluid motion. When the disks are approaching each other it is of potential type and when they are moving away then that is of rotational nature.

For such consideration, various authors [11, 13, 14] assumed that the horizontal components of the velocity u , v , do not depend on the vertical coordinate, z , whereas the vertical velocity w depends linearly on the distance between the disks. Thus the velocity field is of the following form [15]:

$$(2.8) \quad \mathbf{V}(x, y, z, t) = [u(x, y, t), v(x, y, t), -2\phi z],$$

where ϕ is the relative velocity of the disk, considered here to be constant.

Inserting (2.8) in (2.4) and (2.6) and making use of the assumption (2.7) we obtain, in the absence of body forces, the following equations:

$$(2.9) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2\phi,$$

$$(2.10) \quad \frac{\partial \hat{p}}{\partial x} + \rho \left[\frac{\partial u}{\partial t} - v\omega \right] = \left(\mu + \alpha_1 \frac{\partial}{\partial t} \right) \nabla^2 u - \alpha_1 v \nabla^2 \omega,$$

$$(2.11) \quad \frac{\partial \hat{p}}{\partial y} + \rho \left[\frac{\partial v}{\partial t} + u\omega \right] = \left(\mu + \alpha_1 \frac{\partial}{\partial t} \right) \nabla^2 v + \alpha_1 u \nabla^2 \omega,$$

where

$$(2.12a) \quad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

$$(2.12b) \quad \hat{p} = p + \frac{1}{2}\rho(u^2 + v^2 + 4\phi^2 z^2) - \alpha_1 \left[u \nabla^2 u + v \nabla^2 v + \frac{1}{4} |\mathbf{A}_1^2| \right],$$

$$(2.12c) \quad |\mathbf{A}_1^2| = 4 \left(\frac{\partial u}{\partial x} \right)^2 + 4 \left(\frac{\partial v}{\partial y} \right)^2 + 16\phi^2 + 2 \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2.$$

REMARK 1. On setting $\alpha_1 = 0$ in (2.10) and (2.11) we recover the equations for Newtonian fluid [11].

Equations (2.9)–(2.11) are three partial differential equations for three unknown functions u , v and \hat{p} of the variables (x, y) . Once the velocity field is determined, the pressure field (2.12b) can be calculated by integrating (2.10) and (2.11). Note that the equation for the vertical component w is identically zero.

Eliminating pressure in (2.10) and (2.11), by applying the integrability condition $\partial^2 \hat{p} / \partial x \partial y = \partial^2 \hat{p} / \partial y \partial x$, we get the compatibility equation

$$(2.13) \quad \rho \left[\frac{\partial \omega}{\partial t} + 2\phi\omega + \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \omega \right] \\ = \left(\mu + \alpha_1 \frac{\partial}{\partial t} \right) \nabla^2 \omega + \alpha_1 \left[\left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \nabla^2 \omega + 2\phi \nabla^2 \omega \right].$$

Let us consider the potential component from the horizontal components of the velocity and introduce the flow function of the following form:

$$(2.14) \quad u = \phi x + \frac{\partial \psi}{\partial y}, \quad v = \phi y - \frac{\partial \psi}{\partial x},$$

where $\psi(x, y)$ is the stream function. We see that the continuity equation (2.9) is satisfied identically and (2.14) in (2.13) yields the following equation:

$$(2.15) \quad \rho \left[\left(2\phi + \frac{\partial}{\partial t} \right) \nabla^2 \psi + \phi \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \nabla^2 \psi - \{ \psi, \nabla^2 \psi \} \right] \\ = \left(\mu + \alpha_1 \frac{\partial}{\partial t} \right) \nabla^4 \psi + \alpha_1 \left[2\phi \nabla^4 \psi + \phi \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \nabla^4 \psi - \{ \psi, \nabla^4 \psi \} \right],$$

in which

$$\nabla^4 = \nabla^2 \cdot \nabla^2, \quad \omega = -\nabla^2 \psi$$

and

$$\{\psi, \nabla^2 \psi\} = \frac{\partial \psi}{\partial x} \frac{\partial (\nabla^2 \psi)}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial (\nabla^2 \psi)}{\partial x}$$

is the Poisson bracket.

REMARK 2. The solution $\psi = 0$ of (2.15), corresponds to liquid potential motion, known as the motion near the stagnation point.

3. Solutions of some special types

We consider *Riabouchinsky type flows* in order to solve (2.15).

3.1. Solution of the type $\psi(x, y) = y\xi(x)$

In order to obtain a class of solution of (2.15) we substitute

$$(3.1) \quad \psi(x, y) = y\xi(x)$$

into (2.15) and get the following equation

$$(3.2) \quad \rho [\phi (3\xi'' + x\xi''') - (\xi'\xi'' - \xi\xi''')] = \mu\xi^{IV} + \alpha_1 \begin{bmatrix} \phi (3\xi^{IV} + x\xi^V) \\ -(\xi'\xi^{IV} - \xi\xi^V) \end{bmatrix},$$

where $\xi(x)$ is an arbitrary function of x and primes denote the derivative with respect to x .

Integrating (3.2) once and equating the constant of integration equal to zero we obtain

$$(3.3) \quad \mu\xi''' + \rho [(\xi'^2 - \xi\xi'') - \phi (2\xi' + x\xi'')] - \alpha_1 \begin{bmatrix} (-\xi\xi^{IV} + 2\xi'\xi''' - \xi''^2) \\ -\phi (2\xi''' + x\xi^{IV}) \end{bmatrix} = 0.$$

For the solution of the equation (3.2) we write

$$(3.4) \quad \xi(x) = \delta (1 + \lambda e^{\sigma x}) - \phi x$$

in which δ , σ and λ are arbitrary real constants. Making use of (3.4) into (3.2) we have

$$(3.5) \quad \delta = \frac{\mu\sigma}{\rho - \alpha_1\sigma^2} - \frac{4\phi}{\sigma}$$

and thus from (3.1)

$$(3.6) \quad \psi(x, y) = \left[\frac{\mu\sigma}{\rho - \alpha_1\sigma^2} - \frac{4\phi}{\sigma} \right] y(1 + \lambda e^{\sigma x}) - \phi xy.$$

The velocity components (2.14) and the pressure field (2.12b) become

$$(3.7) \quad u = \left[\frac{\mu\sigma}{\rho - \alpha_1\sigma^2} - \frac{4\phi}{\sigma} \right] (1 + \lambda e^{\sigma x}),$$

$$(3.8) \quad v = 2\phi y - \left[\frac{\mu\sigma}{\rho - \alpha_1\sigma^2} - \frac{4\phi}{\sigma} \right] y\lambda\sigma e^{\sigma x}.$$

$$(3.9) \quad p = p_0 + \mu\bar{a}\lambda\sigma \left(1 - \frac{\sigma^2 y^2}{2} \right) e^{\sigma x} - \frac{1}{2}\rho \left[\bar{a}^2 + 4\phi^2(y^2 + z^2) - \bar{a}^2\lambda^2 e^{2\sigma x} \right] \\ + \alpha_1 \left[\bar{a}\lambda\sigma (\bar{a}\sigma - 2\phi\sigma^2 y^2 - 4\phi) e^{\sigma x} + \bar{a}^2\lambda^2\sigma^2 \left(3 + \frac{\sigma^2 y^2}{2} \right) e^{2\sigma x} + 8\phi^2 \right],$$

where p_0 is an arbitrary constant, known as the reference pressure.

The streamline flow for $\psi = \Omega_1$ is given by the functional form

$$(3.10) \quad y = \frac{\Omega_1}{(1 + \lambda e^{\sigma x})\varepsilon - x\phi},$$

where

$$\varepsilon = \frac{\nu\sigma}{1 - \Lambda\sigma^2} - \frac{4\phi}{\sigma}$$

in which ν is the kinematic viscosity and Λ is the second-grade parameter.

Figure 1 shows the streamlines for $\phi = \sigma = \lambda = 1$, $\mu/\rho = 0.5$, $\alpha_1/\rho = 0.1$, $\psi = 15, 20, 25, 30, 40$.

3.2. Solutions of the type $\psi(x, y) = y\xi(x) + \eta(x)$

Inserting

$$(3.11) \quad \psi(x, y) = y\xi(x) + \eta(x)$$

in (2.15) we obtain the following equation

$$(3.12) \quad \rho \left[\begin{array}{l} 2\phi(y\xi'' + \eta'') + \phi\{y(\xi'' + x\xi''') + x\eta'''\} \\ -\{y(\xi'\xi'' - \xi\xi''') + (\eta'\xi'' - \xi\eta''')\} \end{array} \right] = \mu(y\xi^{IV} + \eta^{IV}) \\ + \alpha_1 \left[\begin{array}{l} 2\phi(y\xi^{IV} + \eta^{IV}) + \phi\{y(\xi^{IV} + x\xi^V) + x\eta^V\} \\ -\{y(\xi'\xi^{IV} - \xi\xi^V) + (\eta'\xi^{IV} - \xi\eta^V)\} \end{array} \right].$$

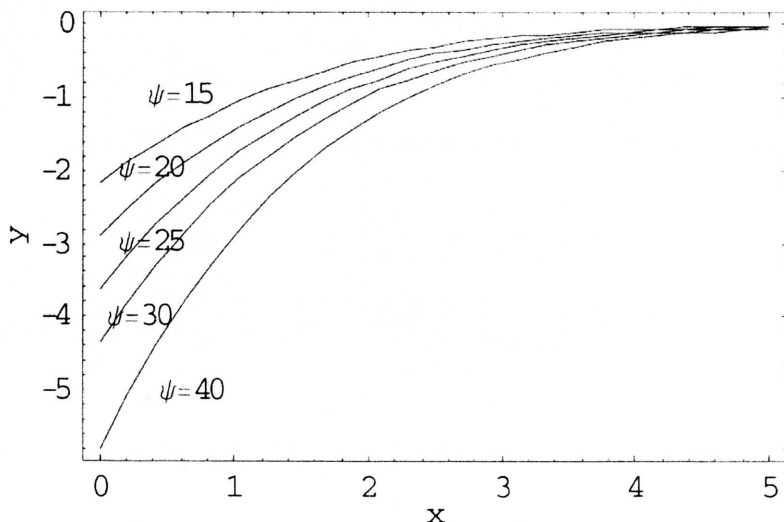


FIG. 1. Streamline flow pattern for $\psi(x, y) = \left[\frac{\mu\sigma}{\rho - \alpha_1\sigma^2} \frac{-4\phi}{\sigma} \right] y (1 + \lambda e^{\sigma x}) - \phi xy$.

From above equation we have

$$(3.13) \quad \rho [(\xi' \xi'' - \xi \xi''') - \phi (3\xi'' + x\xi''')] + \mu \xi^{IV} - \alpha_1 \begin{bmatrix} (\xi' \xi^{IV} - \xi \xi^V) \\ -\phi (3\xi^{IV} + x\xi^V) \end{bmatrix} = 0,$$

and

$$(3.14) \quad \rho [(\eta' \xi'' - \xi \eta''') - \phi (2\eta'' + x\eta''')] + \mu \eta^{IV} - \alpha_1 \begin{bmatrix} (\eta' \xi^{IV} - \xi \eta^V) \\ -\phi (3\eta^{IV} + x\eta^V) \end{bmatrix} = 0,$$

where $\xi(x)$ and $\eta(x)$ are arbitrary functions of its arguments. Integrating (3.13) and (3.14) and then taking the constants of integration equal to zero we have

$$(3.15) \quad \mu \xi''' + \rho [(\xi'^2 - \xi \xi'') - \phi (2\xi' + x\xi'')] - \alpha_1 \begin{bmatrix} (-\xi \xi^{IV} + 2\xi' \xi''' - \xi'^2) \\ -\phi (2\xi''' + x\xi^{IV}) \end{bmatrix} = 0,$$

$$(3.16) \quad \mu \eta''' + \rho [(\eta' \xi' - \xi \eta'') - \phi (2\eta' + x\eta'')] - \alpha_1 \begin{bmatrix} \xi' \eta''' - \xi \eta^{IV} + \eta' \xi''' \\ -\eta'' \xi'' - \phi (2\eta''' + x\eta^{IV}) \end{bmatrix} = 0.$$

We note that (3.13) is similar to (3.2). Its solution is given in (3.4). Substituting (3.4) into (3.13) we have

$$(3.17) \quad \alpha_1 \bar{a} (1 + \lambda e^{\sigma x}) \eta^V + (\mu + 3\alpha_1 \phi) \eta^{IV} - \rho \bar{a} (1 + \lambda e^{\sigma x}) \eta''' - 2\rho \phi \eta'' + \bar{a} \lambda \sigma^2 (\rho - \alpha_1 \sigma^2) e^{\sigma x} \eta' = 0,$$

where

$$\bar{a} = \frac{\mu \sigma}{\rho - \alpha_1 \sigma^2} - 4 \frac{\phi}{\sigma}.$$

We note that it is not easy to obtain the general solution of (3.17). In order to find its solution we consider the following special cases:

CASE 1. When $\alpha_1 \neq 0$, $\phi \neq 0$, $\sigma = 1$, $\lambda = 0$

then (3.17) reduces to

$$(3.18) \quad \alpha_1 \bar{a} \eta^V + (\mu + 3\alpha_1 \phi) \eta^{IV} - \rho \bar{a} \eta''' - 2\rho \phi \eta'' = 0.$$

We see that (3.18) is of fifth order and in order to solve it we reduce its order by putting $\eta'' = \bar{A}(x)$ such that (3.18) becomes

$$(3.19) \quad \alpha_1 \bar{a} \bar{A}''' + (\mu + 3\alpha_1 \phi) \bar{A}'' - \rho \bar{a} \bar{A}' - 2\rho \phi \bar{A} = 0.$$

On substituting $\bar{A}(x) = \hat{P}(x)e^x$, (3.19) takes the form

$$(3.20) \quad \alpha_1 \bar{a} (3\hat{P}' + 3\hat{P}'' + \hat{P}''') e^x + (\mu + 3\alpha_1 \phi) (2\hat{P}' + \hat{P}'') e^x - \rho \bar{a} \hat{P}' e^x = 0.$$

Finally, $\hat{P}'(x) = R(x)$ converts (3.20) into a second order differential equation

$$(3.21) \quad \alpha_1 \bar{a} R'' + (\mu + 3\alpha_1 (\phi + \bar{a})) R' - [(3\alpha_1 - \rho) \bar{a} + 2\mu + 6\alpha_1 \phi] R = 0.$$

The solution of above equation is

$$(3.22) \quad R(x) = A_3 \exp\left(\frac{-c - \sqrt{c^2 - 4d}}{2}\right) x + A_4 \exp\left(\frac{-c + \sqrt{c^2 - 4d}}{2}\right) x,$$

where A_3 and A_4 are arbitrary constants and

$$c = \frac{3\alpha_1 (\bar{a} + \phi) + \mu}{\alpha_1 \bar{a}},$$

$$d = \frac{3\alpha_1 (\bar{a} + 2\phi) + 2\mu - \rho \bar{a}}{\alpha_1 \bar{a}},$$

$$\bar{a} = \frac{\mu}{\rho - \alpha_1} - 4\phi.$$

In order to find $\eta(x)$ we make backward substitutions and finally obtain the form

$$(3.23) \quad \eta(x) = \frac{A_3}{m_1(1+m_1)^2} e^{(1+m_1)x} + \frac{A_4}{m_2(1+m_2)^2} e^{(1+m_2)x} + A_5 e^x + A_6 x + A_7,$$

where A_i ($i = 5, 6, 7$) are constants of integration and

$$m_1 = \frac{-c - \sqrt{c^2 - 4d}}{2}, \quad m_2 = \frac{-c + \sqrt{c^2 - 4d}}{2}.$$

From (3.4), (3.11) and (3.13) we get

$$(3.24) \quad \psi(x, y) = y \left[\frac{\mu}{\rho - \alpha_1} - \phi(4+x) \right] + A_5 e^x + A_6 x + A_7 + \frac{A_3}{m_1(1+m_1)^2} e^{(1+m_1)x} + \frac{A_4}{m_2(1+m_2)^2} e^{(1+m_2)x}.$$

The velocity components and pressure field are

$$(3.25) \quad u = \frac{\mu}{\rho - \alpha_1} - 4\phi,$$

$$(3.26) \quad v = 2\phi y - \left[\frac{A_3}{m_1(1+m_1)} e^{(1+m_1)x} + \frac{A_4}{m_2(1+m_2)} e^{(1+m_2)x} + A_5 e^x + A_6 \right],$$

$$(3.27) \quad p = p_0 - \frac{1}{2}\rho \left[a_1^2 + A_6^2 + 4\phi^2(y^2 + z^2) - 4y\phi A_6 \right] + \alpha_1 \left[\frac{A_3^2}{m_1^2} e^{2(1+m_1)x} + \frac{2A_3 A_4}{m_1 m_2} e^{(2+m_1+m_2)x} + \frac{A_4^2}{m_2^2} e^{2(1+m_2)x} + A_5^2 e^{2x} 4\phi^2 + \frac{2A_3 A_5 (2+3m_1+m_1^2)}{(2+m_1)m_1(1+m_1)} e^{(2+m_1)x} + \frac{2A_4 A_5 (2+3m_2+m_2^2)}{(2+m_2)m_2(1+m_2)} e^{(2+m_2)x} \right].$$

The streamline for $\psi = \Omega_2$ is given by the functional form

$$(3.28) \quad y = -\frac{1}{\varepsilon_1 - x\phi} \times \left[-\Omega_2 + \frac{A_3}{m_1(1+m_1)^2} e^{(1+m_1)x} + \frac{A_4}{m_2(1+m_2)^2} e^{(1+m_2)x} + A_5 e^x + A_6 x + A_7 \right],$$

where

$$\varepsilon_1 = \frac{\nu}{1-\Lambda} - 4\phi, \quad a_1 = \frac{\mu}{\rho - \alpha_1} - 4\phi.$$

Streamline pattern is plotted in Fig. 2 for $\phi = a = \lambda = 1$, $\mu/\rho = 0.5$, $\alpha_1/\rho = 0.1$, $A_3 = A_4 = A_5 = A_6 = A_7 = 1$, $\psi = 15, 20, 25, 30, 40$.

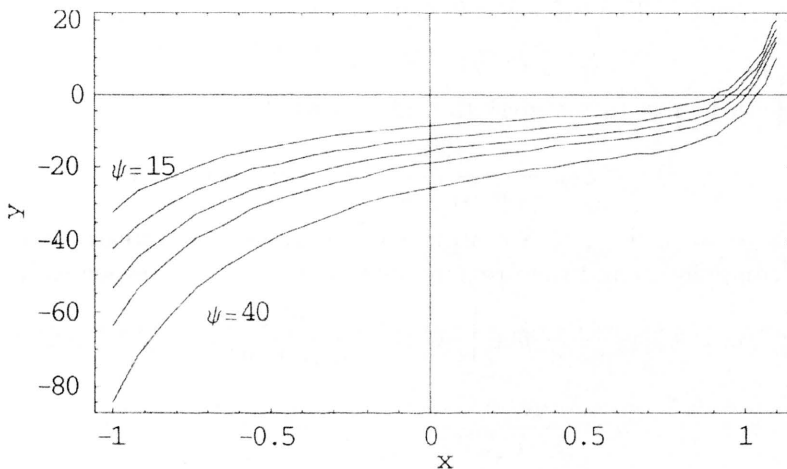


FIG. 2. Streamline flow pattern for $\psi(x, y)$

$$= y \left[\frac{\mu}{\rho - \alpha_1} - \phi(4+x) \right] + A_5 e^x + A_6 x + A_7 + \frac{A_3}{m_1(1+m_1)^2} e^{(1+m_1)x} + \frac{A_4}{m_2(1+m_2)^2} e^{(1+m_2)x}.$$

CASE 2. When $\alpha_1 \neq 0$, $\phi = 0$, $\sigma = 1$, $\lambda \neq 0$

then (3.17) reduces to

$$(3.29) \quad \alpha_1 (1 + \lambda e^x) \eta^V + (\rho - \alpha_1) \eta^{IV} - \rho (1 + \lambda e^x) \eta''' + (\rho - \alpha_1) \lambda e^x \eta' = 0.$$

To obtain the solution of (3.29) we try to reduce its order. For this purpose we put $\eta' = \hat{A}(x)$ which leaves (3.29) into a form which is one order less, that is

$$(3.30) \quad \alpha_1 (1 + \lambda e^x) \hat{A}^{IV} + (\rho - \alpha_1) \hat{A}''' - \rho (1 + \lambda e^x) \hat{A}'' + (\rho - \alpha_1) \lambda e^x \hat{A} = 0.$$

Now substituting $\hat{A}(x) = \bar{P}(x)e^x$ in (3.30) and then $\bar{P}'(x) = R(x)$ into the resulting expression we get

$$(3.31) \quad \alpha_1 \left[\begin{aligned} &(1 + \lambda e^x) R''' + (3 + 4\lambda e^x) R'' \\ &+ (3 + 6\lambda e^x) R' + (1 + 4\lambda e^x) R \end{aligned} \right] \\ = \rho [R'' + (2 - \lambda e^x) R' + (1 - 2\lambda e^x) R].$$

The equation (3.31) is of order three. In order to reduce its order further, we multiply it by e^x and then integrate to obtain

$$(3.32) \quad \alpha_1 (1 + \lambda e^x) R'' + [(2\alpha_1 + \rho) + 2\alpha_1 \lambda e^x] R' \\ + [\alpha_1 + \rho - (2\alpha_1 - \rho) \lambda e^x] R = 0,$$

where we have taken the constant of integration equal to zero.

The solution of (3.32) for $\lambda = 0$ is given by

$$(3.33) \quad R(x) = C_5 e^{-x} + C_6 e^{-(\alpha_1 + \rho)/\alpha_1 x}.$$

The backward substitution gives the value of $\eta(x)$

$$(3.34) \quad \eta(x) = -C_5 x + \frac{\alpha_1^2}{\rho(\alpha_1 + \rho)} C_6 e^{-(\rho/\alpha_1)x} + C_7 e^x + C_8,$$

where C_r ($r = 5, 6, 7, 8$) are arbitrary constants. The stream function, the velocity components and the pressure field in this case are respectively given as

$$(3.35) \quad \psi(x, y) = \frac{\mu}{\rho - \alpha_1} y + \left[-C_5 x + \frac{\alpha_1^2}{\rho(\alpha_1 + \rho)} C_6 e^{-(\rho/\alpha_1)x} + C_7 e^x + C_8 \right],$$

$$(3.36) \quad u = \frac{\mu}{\rho - \alpha_1},$$

$$(3.37) \quad v = C_5 + \frac{\alpha_1}{(\alpha_1 + \rho)} C_6 e^{-(\rho/\alpha_1)x} - C_7 e^x,$$

$$(3.38) \quad p = p_0 - \frac{1}{2}\rho \left[\begin{aligned} &a_2^2 + C_5^2 + 2C_7 \bar{\alpha} e^{(1-\rho/\alpha_1)x} \\ &+ 2 \frac{(1 - \alpha_1)}{(\alpha_1 - \rho)} C_7 \bar{\alpha} e^{(1-\rho/\alpha_1)x} \end{aligned} \right] \\ + \alpha_1 \left[\begin{aligned} &C_7^2 e^{2x} + \frac{\rho^2 \bar{\alpha}^2}{\alpha_1^2} e^{-2(\rho/\alpha_1)x} - C_7 \frac{\rho^2 \bar{\alpha}}{\alpha_1^2} e^{(1-\rho/\alpha_1)x} \\ &+ \left(\frac{\rho}{\alpha_1} - 1 \right) C_7 \bar{\alpha} e^{(1-\rho/\alpha_1)x} \\ &+ C_7 \frac{\bar{\alpha} \alpha_1}{\alpha_1 - \rho} e^{(1-\rho/\alpha_1)x} - C_7 \frac{\bar{\alpha} \rho^3}{\alpha_1^2 (\alpha_1 - \rho)} e^{(1-\rho/\alpha_1)x} \end{aligned} \right],$$

where

$$\bar{\alpha} = \frac{\alpha_1}{\alpha_1 + \rho}, \quad a_2 = \frac{\mu}{\rho - \alpha_1},$$

and the streamline for $\psi = \Omega_3$ is given by the functional form

$$(3.39) \quad y = -\frac{1}{\varepsilon_2} \left[-\Omega_3 - C_5 x + \frac{\Lambda^2}{1 + \Lambda} C_6 e^{-(1/\Lambda)x} + C_7 e^x + C_8 \right],$$

where

$$\varepsilon_2 = \frac{\nu}{1 - \Lambda}.$$

Streamlines are sketched in Fig. 3 for $\phi = \lambda = 0$, $\sigma = 1$, $\mu/\rho = 0.5$, $\alpha_1/\rho = 0.1$, $C_5 = C_6 = C_7 = C_8 = 1$, $\psi = 15, 20, 25, 30, 40$.

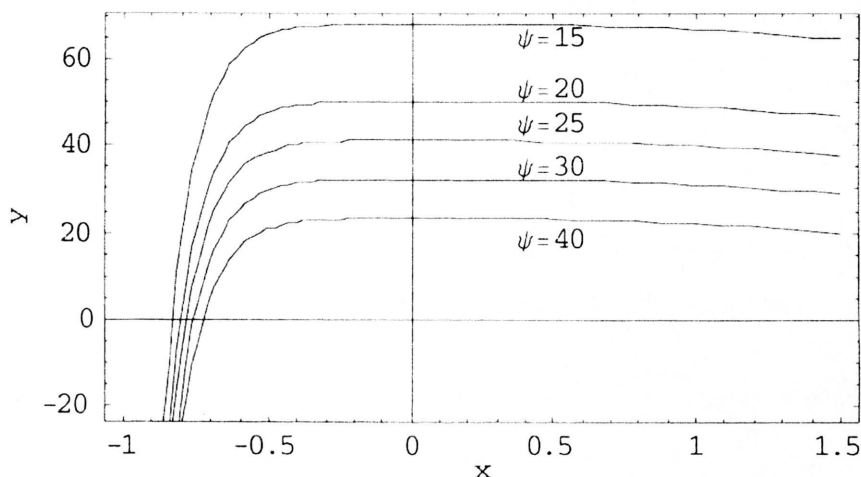


FIG. 3. Streamline flow pattern for

$$\psi(x, y) = \frac{\mu}{\rho - \alpha_1} y + \left[-C_5 x + \frac{\alpha_1^2}{\rho(\alpha_1 + \rho)} C_6 e^{-(\rho/\alpha_1)x} + C_7 e^x + C_8 \right].$$

4. Concluding remarks

In this paper, the analytical solutions of nonlinear equations governing the flow for a second-grade fluid are obtained by assuming different forms of the stream function (already used by various authors in different situations). The expressions for velocity profile, streamline and pressure distribution are constructed in each case. Our result indicate that velocity, stream function and pressure are strongly dependent upon the material parameter α_1 of the second-grade fluid. It is shown through graphs that increase in second-grade parameter

($\alpha_1 = 0.15$) leads to decrease in velocity and decrease in second-grade parameter ($\alpha_1 = -0.5$) leads to increase in velocity (see Figs. 4 and 5). Also, the present analysis is more general and several results of various authors (as already mentioned in the text) can be recovered in the limiting cases.

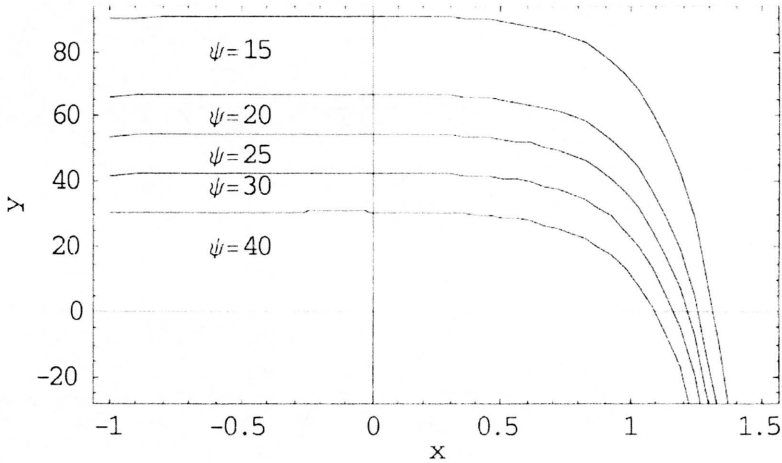


FIG. 4. Streamline flow pattern for negative second-grade parameter for

$$\psi(x, y) = \frac{\mu}{\rho - \alpha_1} y + \left[-C_5 x + \frac{\alpha_1^2}{\rho(\alpha_1 + \rho)} C_6 e^{-(\rho/\alpha_1)x} + C_7 e^x + C_8 \right].$$

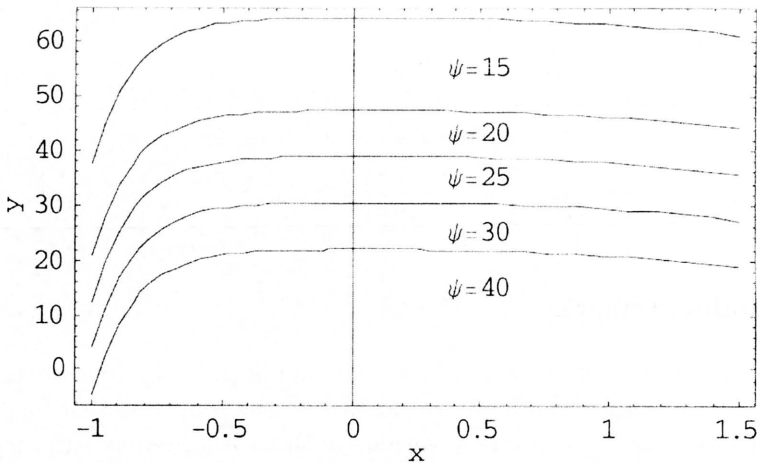


FIG. 5. Streamline flow pattern for positive second-grade parameter for

$$\psi(x, y) = \frac{\mu}{\rho - \alpha_1} y + \left[-C_5 x + \frac{\alpha_1^2}{\rho(\alpha_1 + \rho)} C_6 e^{-(\rho/\alpha_1)x} + C_7 e^x + C_8 \right].$$

References

1. K. R. RAJAGOPAL, *On the boundary conditions for fluids of the differential type*, [in:] A. SEQUEIRA [Ed.], *Navier–Stokes equations and related nonlinear problems*, Plenum Press, New York, 273, 1995.
2. P. N. NEMENYI, *Recent developments in inverse and semi-inverse methods in the mechanics of continua*, *Advances in Applied Mechanics*, 2, New York 1951.
3. P. N. KALONI and K. HUSCHILT, *Semi inverse solutions of a non-Newtonian fluid*, *Int. J. Nonlinear Mech.*, **19**, 373, 1984.
4. A. M. SIDDIQUI and P. N. KALONI, *Certain inverse solutions of a non-Newtonian fluid*, *Int. J. Nonlinear Mech.*, **21**, 459, 1986.
5. A. M. SIDDIQUI, *Some more inverse solutions of a non-Newtonian fluid*, *Mech. Res. Commun.*, **17**, 157, 1990.
6. A. M. BENHARBIT and A. M. SIDDIQUI, *Certain solutions of the equations of the planar motion of a second-grade fluid for steady and unsteady cases*, *Acta Mech.*, **94**, 85, 1992.
7. F. LABROPULU, *Exact solutions of non-Newtonian fluid flows with prescribed vorticity*, *Acta Mech.*, **141**, 11, 2000.
8. R. S. RIVLIN and J. L. ERICKSEN, *Stress deformation relations for isotropic materials*, *J. Rat. Mech. Anal.*, **4**, 323, 1955.
9. J. E. DUNN and R. L. FOSDICK, *Thermodynamics, stability and boundedness of fluids of complexity 2 and fluids of second grade*, *Arch. Rat. Mech. Anal.*, **56**, 191, 1974.
10. R. L. FOSDICK and K. R. RAJAGOPAL, *Anomalous features in the model of second order fluids*, *Arch. Ration. Mech. Anal.*, **70**, 145, 1979.
11. A. CRAIK, *The stability of unbounded two-and three dimensional flows subject to body forces, Some exact solutions*, *J. Fluid Mech.*, **198**, 275, 1989.
12. R. BERKER, *Integration des equations du mouvement d'un fluide visqueux incompressible*, *Handbuk der Physik VII*, Springer, Berlin 1963.
13. A. CRAIK and W. CRIMINALE, *Evolution of wavelike disturbances in shear flows: a class of exact solutions of the Navier–Stokes equations.*, *Proc. R. Soc. Lond., A* **406**, 13–36, 1986.
14. R. LAGNADO, N. PHAN-THIEN and L. LEAL, *The stability of two-dimentional linear flows*, *Phys. Fluids*, **27**, 1094, 1984.
15. S. N. ARISTOV and I. M. GITMAN, *Viscous flow between two moving parallel disks: exact solutions and stability analysis*, *J. Fluid Mech.*, **464**, 209, 2002.

Received January 20, 2003; revised version April 18, 2003.