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"Bottom crystal" and possibility of water wave attenuation

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THE INFLUENCE of periodic bottom structure ("bottom crystal") on surface water waves is considered. The problem reduces to a two-dimensional Helmholtz operator with periodic potential. Zero-range potential method based on the theory of self-adjoint extensions of symmetric operators is used. It is shown that there is a gap in the spectrum. An application of this spectral property to the problem of wave attenuation is discussed.

1. Introduction

THE PROBLEM of surface water waves near a coastline, in harbours and channels, is very interesting both from the theoretical point of view and from the point of view of applications in ocean engineering. There is a number of works concerning edge waves and trapped modes (i.e. modes of oscillation which occur at discrete frequences below a certain cut-off frequency and consist of motion which is confined to some localized region of water near an obstacle or variable bottom topography). The oldest example of such a mode was provided by STOKES [1] who constructed an explicit expression for a wave travelling along a beach of constant slope (edge wave). Ursell extended the results of Stokes to show that there is a set of trapped modes for a beach of constant slope with the number of possible modes increasing as the beach angle becomes small. It has been shown that trapped modes can exist due to a submerged obstacle [2-7] or geometric properties of the system-form of the coastline, coupling apertures, crest at the bottom, etc. [8-13].

In the present paper we shall deal with periodic bottom structures (a periodic system of hills or crests at the bottom). It is convenient to use the term "bottom crystal" for such structures, because the corresponding system has properties which are analogous to that of a two-dimensional crystal. From a mathematical point of view the linearized problem of water waves reduces to the investigation of the two-dimensional Helmholtz equation. Cartesian coordinates are chosen with x, y in the undisturbed free surface and z directed vertically upwards. First suppose that the fluid is of uniform depth h and the usual assumption of classical water-wave theory is made. Thus, we seek a time-harmonic velocity potential $\Phi(x,y,z,t)$ and we write

$$\Phi(x,y,z,t) = \Re\{\phi(x,y)\cosh(k(z+h))\exp(-i\omega t)\}$$

to ensure that the velocity of the fluid normal to the bottom vanishes on $z=-h$. Here, in order to satisfy the convential linearized free-surface condition on $z=0$, k is a positive root of the equation

$$\omega^2 = gk\tanh(kh) \tag{1.1}$$

and ω is the radiation frequency, g is the gravitational acceleration. As a result, we get the two-dimensional Helmholtz equation for function ϕ:

$$\Delta\phi(x,y) + k^2\phi(x,y) = 0. \tag{1.2}$$

Now, suppose that the depth is not uniform. For example, suppose there is system of small circles Ω_s on the plane (x,y) with the centres at the nodes of a doubly-periodic lattice on the plane. Suppose the depth to be equal to h for the points (x,y) outside the circles and to $h_1, h_1 < h$, for $(x,y)\in\Omega_s$, for some s (see Fig. 1).

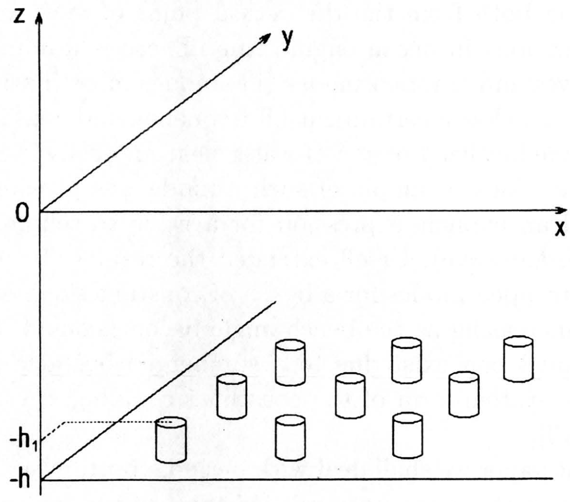

FIG. 1. Periodic bottom structure.

In this situation we have the Helmholtz Equation (1.2) with periodically varying coefficient k (see (1.1)). If the diameters of the circles are small, we can use a model in which the perturbations of k^2 are replaced by point-like ones – the zero-range potential approach. This method is widely used in quantum mechanics

[14, 15], diffraction theory [16], fluid mechanics [17]. The approach is based on the theory of self-adjoint extensions of symmetric operators. We deal with the Laplace operator perturbed by periodic array of zero-range potentials, k^2 is the spectral parameter. We analyse spectral properties of our periodic system in the framework of the method and show that there is a gap in the spectrum for some parameters of the "bottom crystal". It means that some wave frequencies are prohibited. This effect can be used for wave attenuation. Namely, for a concrete harbour some wave frequencies are the most dangerous and powerful. Suppose we make a "bottom crystal" in this harbour with such parameters that these frequencies lie in the gap. Hence, these frequencies are prohibited, and we get essential wave attenuation.

The dispersion equation for a "bottom crystal" is obtained. A "bottom crystal waveguide" (a system in which one or several lines of nodes of the lattice are empty) is considered. It can be used for wave concentration in some regions. The application of the model to the description of a system of thin submerged cylinders is discussed.

2. Spectral properties of a "bottom crystal"

Let Λ be the two-dimensional lattice

$$\Lambda = \{n_1 a_1 + n_2 a_2 \in \mathbf{R}^2; (n_1, n_2) \in \mathbf{Z}^2\},$$

where

$$a_j = (a_j^1, a_j^2), \quad j = 1, 2$$

are two linearly independent vectors in $\mathbf{R}^2$,

$$\Gamma = \{n_1 b_1 + n_2 b_2 \in \mathbf{R}^2; (n_1, n_2) \in \mathbf{Z}^2\}$$

is the dual lattice ($a_i b_{j'} = 2\pi\delta_{jj'}, j, j' = 1, 2$), $\widehat{\Lambda}$ is the Brillouin zone,

$$\widehat{\Lambda} = \{s_1 b_1 + s_2 b_2 \in \mathbf{R}^2; s_j \in [-1/2, 1/2), j = 1, 2\}.$$

To construct an operator with periodic point-like interactions we start from the closure of the Laplace operator in $L_2(\mathbf{R}^2)$ restricted to the set of smooth functions which vanish at the nodes of the lattice Λ. It is a symmetric non-self-adjoint operator. To switch on the point-like interaction means to construct its self-adjoint extension. Taking into account the periodicity condition, one obtains a model operator $-\Delta_\Lambda$, more precisely, one-parameter (α) family of model operators (self-adjoint extensions). It is known [14] that the spectrum of $-\Delta_\Lambda$ is absolutely continuous and has a band structure. The dispersion equation has the form

$$g_k(0, \theta) = \alpha,$$

where

(2.1) $$g_k(0,\theta) = -\frac{1}{2\pi}(\ln(-ik/2) - C_E) + \sum_{\lambda\in\Lambda,\lambda\neq 0} \frac{i}{4} H_0^{(1)}(k \mid \lambda \mid) \exp(-i\theta\lambda),$$

α is a model parameter which is related to "the strength" of the potentials (in our case to $h - h_1$ and diameters of the circles). The sum can be computed using the Poisson summation formula [14]. Note that one can consider the sum using arguments analogous to that for the three-dimensional lattice sum in [18]. As a result, we can describe the spectrum of $-\Delta_\Lambda$. Namely, if the basic cell contains only one centre, it is the following:

(2.2) $$\sigma(-\Delta_\Lambda) = \sigma_{ac}(-\Delta_\Lambda) = [E_0^{\alpha,\Lambda}(0), E_0^{\alpha,\Lambda}(\theta_0)] \cup [E_1^{\alpha,\Lambda}, \infty),$$

where

$$\theta_0 = -\frac{1}{2}(b_1 + b_2),$$

$$E_1^{\alpha,\Lambda} = \min\left\{E_{b_-}^{\alpha,\Lambda}(0), \frac{1}{4} \mid b_- \mid^2\right\}.$$

Here b_- is the member of the pair $\{b_1, b_2\}$ of least magnitude. $E_0^{\alpha,\Lambda}(\theta)$ is the first root ($E_{b_-}^{\alpha,\Lambda}(\theta)$ is the second root) of the equation

(2.3) $$\alpha = (C_E + \ln 2)/(2\pi) + (2\pi)^{-2} \lim_{\omega\to\infty}\Big[\sum_{\gamma\in\Gamma,|\gamma+\theta|\leq\omega} \frac{\mid\widehat{\Lambda}\mid}{\mid \gamma+\theta\mid^2 - k^2} - 2\pi\ln\omega\Big], \qquad \theta\in\widehat{\Lambda},$$

θ is a quasimomentum, α is a model parameter (related to the "strength" of the potential).

The following inequalities are valid:

$$E_1^{\alpha,\Lambda} > 0, \alpha\in\mathbf{R},$$

$$E_0^{\alpha,\Lambda}(\theta_0) < 0 \Longleftrightarrow \alpha \leq g_0(0,\theta_0).$$

Moreover,

$$E_0^{\alpha,\Lambda}(\theta_0) \to \begin{cases} \mid\theta_0\mid^2, & \alpha\to\infty, \\ -\infty, & \alpha\to-\infty, \end{cases}$$

$$E_0^{\alpha,\Lambda}(0) \to \begin{cases} 0, & \alpha\to\infty, \\ -\infty, & \alpha\to-\infty, \end{cases}$$

$$E_1^{\alpha,\Lambda} \to \begin{cases} |\, b_- \,|^2 /4, & \alpha \to \infty, \\ 0, & \alpha \to -\infty. \end{cases}$$

It means that in the generic case we have a gap in the spectrum $(E_0^{\alpha,\Lambda}(\theta_0), E_1^{\alpha,\Lambda})$, a part of which is on positive half-axis. But there exists a model parameter $\alpha_{1,\Lambda} \in \mathbf{R}$, such that there is no gap:

$$\sigma(-\Delta_\Lambda) = \sigma_{ac}(-\Delta_\Lambda) = [E_0^{\alpha,\Lambda}(0), \infty), \alpha \geq \alpha_{1,\Lambda}.$$

3. "Bottom crystal waveguide"

Consider a "bottom crystal" with one empty line of nodes. To study the spectral properties of the system it is convenient to investigate, firstly, a periodic chain of zero-range potentials in $\mathbf{R}^2$. Let Λ_1 be

$$\Lambda_1 = \{(0, na) \in \mathbf{R}^2; n \in \mathbf{Z}\},$$

where $a > 0$, $\widehat{\Lambda_1} = [-\pi/a, \pi/a), \Gamma_1 = \{(0, 2\pi n/a) \in \mathbf{R}^2; n \in \mathbf{Z}\}$. Suppose, as earlier, that the basic cell contains only one centre. Using the "restriction- extension" procedure described above one obtains the spectrum of the corresponding operator $-\Delta_{\Lambda_1}$. The dispersion equation has a form analogous to Eq. (2.1):

$$(3.1)\quad \alpha = g_k(0,\theta) = -\frac{1}{2\pi}(\ln(-ik/2) - C_E) + \sum_{\lambda \in \Lambda_1, \lambda \neq 0} \frac{i}{4} H_0^{(1)}(k \mid \lambda \mid) \exp(-i\theta\lambda).$$

Using the Poisson summation formula [14], one can compute the lattice sum. The result is that the spectrum is absolutely continuous and has the following structure:

$$(3.2)\quad \sigma(-\Delta_{\Lambda_1}) = \sigma_{ac}(-\Delta_{\Lambda_1}) = \begin{cases} [E^{\alpha,\Lambda_1}(0), \infty), & \alpha \geq \alpha_{\Lambda_1}, \\ [E^{\alpha,\Lambda_1}(0), E^{\alpha,\Lambda_1}(-\pi/a)] \cup [0,\infty), & \alpha < \alpha_{\Lambda_1}, \end{cases}$$

where $E^{\alpha,\Lambda_1}(\theta)$, $E^{\alpha,\Lambda_1}(\theta) = k^2$, is the root of the equation

$$(3.3)\quad \alpha = (C_E + \ln 2)/(2\pi) + (2\pi)^{-2} \lim_{\omega \to \infty} \Big[\sum_{\gamma \in \Gamma_1, |\gamma+\theta| \leq \omega} \frac{a}{\sqrt{|\, \gamma + \theta \,|^2 - k^2}} - 2\pi \ln \omega \Big],$$

(3.3) [cont.]
$$\theta \in \widehat{\Lambda_1}, \Im k \geq 0, \ \Re k \geq 0.$$

Moreover, $E^{\alpha,\Lambda_1}(0) < 0, \alpha \in \mathbf{R}, E^{\alpha,\Lambda_1}(0) < E^{\alpha,\Lambda_1}(-\pi/a) < 0$ for $\alpha < \alpha_{\Lambda_1}$, where

$$\alpha_{\Lambda_1} = (C_E + \ln 2)/(2\pi) + (2\pi)^{-2} \lim_{\omega\to\infty} \left[\sum_{\gamma\in\Gamma_1, |\gamma-\pi/a|\leq\omega} \frac{a}{|\gamma - \pi/a|} - 2\pi \ln \omega \right].$$

Hence, for some values of the parameter we have two bands and a gap.

To construct the model of a waveguide in a bottom crystal we deal with a lattice of zero-range potentials with one empty line of nodes $\Lambda \setminus \Lambda_1$. Following the described procedure, one obtains the dispersion equation in the form

$$-\frac{1}{2\pi}(\ln(k/2) + C_E) + i/4 + \sum_{\gamma\in\Lambda,\gamma\neq0} \frac{i}{4} H_0^{(1)}(ik \mid \gamma \mid) \exp(-i\theta\gamma)$$
$$- \sum_{\gamma\in\Lambda_1,\gamma\neq0} \frac{i}{4} H_0^{(1)}(ik \mid \gamma \mid) \exp(-i\theta\gamma) = \alpha,$$

where C_E is the Euler constant, $C_E = 0.5772....$ One can see that each term of the right-hand part has been considered earlier (3.1), (2.1), and we obtain the following form of the dispersion equation:

$$\alpha = (C_E + \ln 2)/(2\pi) + (2\pi)^{-2} \lim_{\omega\to\infty} \left[\sum_{\gamma\in\Gamma, |\gamma+\theta|\leq\omega} \frac{|\widehat{\Lambda}|}{|\gamma+\theta|^2 - k^2} - 2\pi \ln \omega \right]$$
$$- (2\pi)^{-2} \lim_{\omega\to\infty} \left[\sum_{\gamma\in\Gamma_1, |\gamma+\theta|\leq\omega} \frac{a}{\sqrt{|\gamma+\theta|^2 - k^2}} - 2\pi \ln \omega \right].$$

Taking into account known information about each term of the right-hand part (see above), we come to the conclusion that, generally speaking, there are two bands ("crystal" band and "waveguide" band), which may intersect. The position of bands depends on the correlation between $E^{\alpha,\Lambda_1}(0)$, $E^{\alpha,\Lambda_1}(-\pi/a)$, $E_0^{\alpha,\Lambda}(0)$, $E_0^{\alpha,\Lambda}(\theta_0)$, $E_1^{\alpha,\Lambda}$(see (2.2), (3.2)). States corresponding to the "waveguide" band describe waves spreading along the empty line of nodes, i.e. waveguide effect.

Analogous consideration takes place in a case when there are several (for example, three) empty lines. Evidently, in this situation we have additional bands. The number of "waveguide bands" coincides with the number, n, of empty lines of centres (of course, the bands can intersect), because in this situation we have a periodic chain with basic cell consisting of n centres.

One can consider more complicated structure – two coupled bottom crystal waveguides. Namely, one deals with a lattice with two empty layers of nodes and one additional empty node between them. Consider one centre Λ_0 in $\mathbf{R}^2$ as a simple preliminary stage. To introduce zero-range potential means to state a relation between coefficients of asymptotics of functions near the point Λ_0 [14]. Taking into account that the Green function for free two-dimensional space $\frac{i}{4}H_0^{(1)}(kr)$ has the following asymptotics near zero

$$\frac{i}{4}H_0^{(1)}(kr) = -\frac{1}{2\pi}(\ln r + \ln(-ik/2) - C_E/2) + o(r), r \to 0,$$

one obtains the following dispersion equation:

$$\alpha - C_E/2 = \ln\frac{k}{2i}. \tag{3.4}$$

Here α is model parameter ("strength" of the potential). One can see that (3.4) has one imaginary root k. Hence, there is one bound state $k^2, k^2 < 0$.

Using the above arguments, one comes to the conclusion that there are two "waveguide" bands for the system of coupled waveguides, because the basic cell for 1D lattice consists of two centres. There is also an eigenvalue (bound state) which corresponds to "coupling aperture" (empty node). Note that if there are several (n) empty nodes ("coupling windows") then there are, generally speaking, n bound states.

4. Discussion

Let us discuss the problem of the choice of the model parameters. For this purpose we consider the problem for single "cylinder" of radius a. The solution of the corresponding two-dimensional problem of scattering of the plane wave u_0 should be continuous together with its derivative on the circle $r = a$:

$$(u^+ - u^-)\,|_{r=a} = 0, \tag{4.1}$$

$$\left(\frac{\partial}{\partial r}u^+ - \frac{\partial}{\partial r}u^-\right)|_{r=a} = 0. \tag{4.2}$$

The function u satisfies the following conditions:

$$\Delta u + k^2 u = 0, \quad r > a,$$

$$\Delta u + k_1^2 u = 0, \quad r < a.$$

We seek the solution in the form of series:

$$u = \begin{cases} \sum_{m=0}^{\infty} B_m J_m(k_1 r)\cos(m\varphi), & r \le a, \\ u_0 + \sum_{m=0}^{\infty} A_m H_m^{(1)}(kr)\cos(m\varphi), & r \ge a. \end{cases}$$

Due to the conditions on the circle one obtains the system:

$$(4.3) \qquad A_m H_m^{(1)}(ka) - B_m J_m(k_1 a) = -(\pi(1+\delta_{mo}))^{-1} \int_0^{2\pi} u_0 \,|_{r=a} \cos(m\varphi)d\varphi,$$

$$(4.4) \qquad kA_m H_m^{(1)\prime}(ka) - k_1 B_m J_m'(k_1 a)$$
$$= -(\pi(1+\delta_{mo}))^{-1} \int_0^{2\pi} \frac{\partial}{\partial r} u_0 \,|_{r=a} \cos(m\varphi)d\varphi.$$

Solving the system, one gets

$$(4.5) \quad A_m = (\pi(1+\delta_{mo})D_m)^{-1}\Bigg(\int_0^{2\pi} u_0 \,|_{r=a} \cos(m\varphi)d\varphi k_1 J_m'(k_1 a)$$
$$- \int_0^{2\pi} \frac{\partial}{\partial r} u_0 \,|_{r=a} \cos(m\varphi)d\varphi J_m(k_1 a)\Bigg),$$

$$(4.6) \qquad D_m = J_m(k_1 a)kH_m^{(1)\prime}(ka) - J_m'(k_1 a)k_1 H_m^{(1)}(ka).$$

The model described above deals with the first term of the series only ($m = 0$). For $a \to 0$ integrals in (4.5) transform:

$$\int_0^{2\pi} u_0 \,|_{r=a} d\varphi = 2\pi u_0(0) + o(1), \quad a \to 0, \quad u_0(0) = u_0 \,|_{r=0} .$$

Using the Green formula, one obtains for small a

$$\int_0^{2\pi} \frac{\partial}{\partial r} u_0 \mid_{r=a} d\varphi = \frac{1}{a} \int_{r=a} \frac{\partial}{\partial r} u_0 \mid_{r=a} ds$$

$$= \frac{1}{a} \int\int_{r<a} (\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}) dxdy = -\frac{k^2}{a} \int\int_{r<a} u_0 dxdy$$

$$= -\frac{k^2}{a}(u_0(0) + o(1)) \int\int_{r<a} dxdy = -\pi k^2 u_0(0) a + o(a).$$

Thus, using the asymptotics of the cylindrical functions, one gets from (4.5), (4.6) under the condition $a \to 0$:

(4.7) $$A_0 = \frac{u_0(0)}{((k_1^2 - k^2)a^2)^{-1} - \ln a - \ln k - \ln(-i/2) - C_E/2}.$$

Compare the result with the corresponding result in the model. The solution of the scattering problem in the model has the following form:

$$u = u_0 + \tilde{A}_0 H_0^{(1)}(kr).$$

The solution has the following asymptotics in the neighbourhood of zero:

$$u = c_+ \ln r + c_- + o(r^0).$$

To construct the model one should assume a relation between the asymptotics coefficients:

$$c_- = \alpha c_+.$$

Taking into account the asymptotics of the Hankel function, one finds:

(4.8) $$\tilde{A}_0 = \frac{u_0(0)}{\alpha - \ln k - \ln(-i/2) - C_E/2}.$$

The comparison of (4.7) and (4.8) gives the following condition for choosing the model parameter:

$$\alpha = ((k_1^2 - k^2)a^2)^{-1} - \ln a.$$

Note that $k_1^2 - k^2$ is related with the vertical size of the cylinder $h - h_1$ (1.1). Using two parameters (k_1^2, a), one can choose α in such a way that it gives us the appropriate correlation between the model and the realistic solutions in the fixed range of frequencies. For example, for $ka = \text{const} = M$:

(4.9) $$\alpha = (k_1^2 a^2 - M^2)^{-1} - \ln a.$$

Taking into account the locality, one can believe that this choice of the parameters is appropriate in a more complicated problem of periodic system of cylinders.

Numerical analysis of the dispersion equation (3.1) is made. The results is in Fig. 2, where the dependence of the first roots of the equation of one of the quasi-momentum components is shown (for fixed second component). There are three curves for three fixed values of the second component on the figure for better seeing. Marked strip in the picture does not contain roots for any values of any components of the quasi-momentum – it is really a gap in the spectrum.

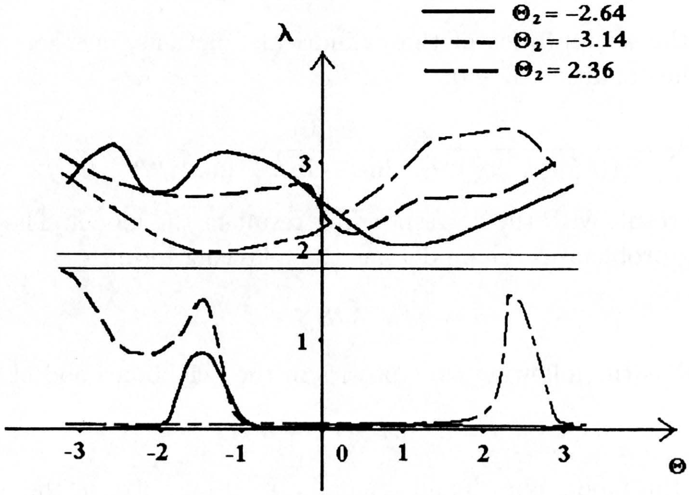

FIG. 2. The dependence of the first roots of the equation on one of the quasi-momentum components for fixed second component. The gap is marked.

The described effects can be applied to wave attenuation in harbours and channels and near some submerged or semi-submerged constructions. Namely, if dangerous frequencies for a concrete body (pier of bridge or derrick, etc.), that coincide with characteristic resonant frequencies of the object, are in the gap of the "bottom crystal". Then, these frequencies are prohibited, and, consequently, there will be attenuation of the surface wave. The same effect occurs for a channel. Moreover, one can use "bottom crystal waveguide" and coupled "bottom crystal waveguides" to concentrate waves in some regions and to create a waveguide effect in a part of the channel.

We use the periodicity condition in the model. It is an idealization. There is no periodic structure in reality. Every structure is finite, we have only a part of a lattice. But calculations show that effects, analogous to those for periodic system appear when there are not very many of centres (30-40). Hence, it seems to be realistic to use the effect for engineering applications.

One can use the model for investigation of trapped modes for a system of thin submerged cylinders. In this case the problem reduces to the two-dimensional Helmholtz equation in a cross-section of the system [4, 6, 7]. It is easy to show that single zero-range potential in a strip gives us a mode (see above the description of zero-range potential in a free space). It corresponds to a trapped mode for single thin submerged cylinder [2]. Now, suppose there is a periodic chain of thin cylinders. Hence, in the model one has the two-dimensional Helmholtz problem in a strip with a chain of zero-range potentials. The dispersion equation is analogous. The only difference is that we should replace the Green function G_k , $G_k = \frac{i}{4} H_0^{(1)}(k \mid \lambda \mid)$, for free space by the Green function for the strip. The corresponding model operator has a band analogous to that in previous section ("bottom crystal waveguide").

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