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Stationary thermoelasticity and stochastic homogenization

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THE AIM OF THE PAPER is twofold: first, the stochastic homogenization theorem formulated by DAL MASO and MODICA [9, 10] is extended to the case applicable to a class of nonlinear problems of mechanics. Second, this new theorem is applied to determine the effective thermoelastic response of the material body with stochastically periodic microstructure. As a result, one obtains the closed form of effective (homogenized) stored energy function. As a specific case, one-dimensional problem is solved analytically.

1. Introduction

EFFECTIVE MATERIAL moduli of nonhomogeneous, linear, thermoelastic solids were derived by various authors. FRANCFORT [11] solved the problem of homogenization of a thermoelastic solid with microperiodic structure. The method of two-scale asymptotic expansions was used to obtain effective thermoelastic constants, cf. [22]. The idea was further developed by GAŁKA *at al.* [12] in the case of diffusion in a thermoelastic body. Thermopiezoelectric composites were investigated in [13].

In the case of random microstructure various, rather engineering-type, stochastic approaches were used, cf. [23]. The method of conditional moments due to KHOROSHUN [16] was applied to predict the effective properties of stochastic composites. Particularly, the effective thermoelastic moduli of anisotropic composites with ellipsoidal inclusions were determined in [17]. Thermoelastic properties of porous anisotropic materials were investigated in [20]. The micromechanical approach based on the Green function technique, as well as the interfacial Hill operators, was applied in [6] to the analysis of thermoelastostatic behaviour of composites with coated randomly distributed inclusions. The local effective thermoelastic properties of graded random structure matrix composites were considered in [5] under the hypothesis of effective field homogeneity near the inclusions.

Papers dealing with the application of rigorous homogenization methods to randomly inhomogeneous materials are not numerous [9, 10], cf. also [8, 14]. The aim our paper is to perform homogenization of randomly inhomogeneous thermoelastic media in the case of stochastically periodic microstructure. To this end we apply the method of stochastic Γ-convergence. As a specific case, one-dimensional problem is solved analytically.

We observe that the Γ-convergence method is applicable only to stationary thermoelasticity, the case investigated in this paper.

2. Stochastic homogenization theorem

The aim of the present paper is to determine the global thermoelastic response of the material body with stochastically periodic microstructure. To this end the method of stochastic Γ-convergence is used. As a result, one obtains the closed form of effective (homogenized) stored energy function. To find this function explicitly, provided that a stochastic microstructure is prescribed, one has to solve a counterpart of a so-called cell problem. Unfortunately, this can be done only in specific cases, Sec. 5. The microstructure is understood here as a real heterogeneous thermoelastic body whose properties vary rapidly and are stochastically periodic in space, see below. The real dimension of a single cell of periodicity is large enough to permit the application of the concept of continuum, but the number of cells is too large to apply any numerical procedure for solving the proper system of partial differential equations. To cope with such a difficulty, a passage to the limit with suitably defined small parameter is performed. The limit procedure is nothing else but smearing out the microheterogeneities, i.e. the number of cells goes to infinity and at the same time their characteristic dimension becomes infinitely small.

Our considerations are based on employing the notion of stochastic Γ - convergence. An alternative approach would consist in applying stochastic G - convergence [15, 18] or stochastic two-scale convergence in the mean [2, 3]. Suitable comments will be provided at the end of Sec. 3. In the present section we are going to formulate a general stochastic homogenization theorem. In Sec. 3 we provide the proof and comments.

Let us pass to the formulation of general stochastic homogenization theorem applicable to performinig homogenization of equations of stationary thermoelesticity. We denote by $\mathcal{A}_0$ the family of all bounded open subsets of $\mathbb{R}^N$. Obviously, from the physical point of view $N = 1, 2$, or 3. Nevertheless no such restriction on the space dimension is needed in Secs. 2 or 3. For every $A \in \mathcal{A}_0$ we denote by $W^{1,\alpha}(A)$ the Sobolev space of functions of $L^\alpha(A)$ whose first-order weak derivatives belong to $L^\alpha(A)$.

Let us fix $\alpha > 1$, $\beta > 1$, $c_1 \geq c_0 > 0$. We denote by $\mathcal{F} = \mathcal{F}(c_0, c_1, \alpha, \beta)$ the class of all functionals

$$F : (L^{\alpha}_{loc}(\mathbb{R}^N))^N \times L^{\beta}_{loc}(\mathbb{R}^N) \times \mathcal{A}_0 \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$$

such that

$$F(\mathbf{u}, T, A) \tag{2.1}$$

$$= \begin{cases} \int\limits_A f[\mathbf{x}, \mathbf{e}(\mathbf{u}(\mathbf{x})), T(\mathbf{x}), \nabla T(\mathbf{x})] d\mathbf{x} & \text{if } \begin{cases} \mathbf{u}_{|A} \in W^{1,\alpha}(A)^N, \\ T_{|A} \in W^{1,\beta}(A), \end{cases} \\ +\infty, & \text{otherwise} \end{cases}$$

Here $f : \mathbb{R}^N \times \mathbb{E}^N_s \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is any function satisfying the following conditions:

(i) $f(\mathbf{x}, \boldsymbol{\epsilon}, \xi, \mathbf{q})$ is Lebesgue measurable in $\mathbf{x}$ and convex in $\boldsymbol{\epsilon}$, ξ and $\mathbf{q}$;

(ii) $c_0(|\boldsymbol{\epsilon}|^{\alpha} + |\xi|^{\beta} + |\mathbf{q}|^{\beta} \leq f(\mathbf{x}, \boldsymbol{\epsilon}, \xi, \mathbf{q}) \leq c_1(|\boldsymbol{\epsilon}|^{\alpha} + |\xi|^{\beta} + |\mathbf{q}|^{\beta+1})$
for each $(\mathbf{x}, \boldsymbol{\epsilon}, \xi, \mathbf{q}) \in \mathbb{R}^N \times \mathbb{E}^N_s \times \mathbb{R} \times \mathbb{R}^N$.

We denote by $\mathbb{E}^N_s$ the space of symmetric $N \times N$ matrices; in the case of linear thermoelasticity $\alpha = \beta = 2$. Moreover $\mathbf{e}(\mathbf{u})$ denotes the small strain tensor

$$e_{ij}(\mathbf{u}) = u_{(i,j)} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.2}$$

where, as usual, $u_{(i,j)} = \dfrac{\partial u_i}{\partial x_j}$.

We observe that DAL MASO and MODICA [9, 10] studied only the integrands of the form $f(\mathbf{x}, \nabla u(\mathbf{x}))$, cf. also SAB [21].

In order to perform stochastic homogenization of equation of stationary thermoelasticity, a more general approach is obviously needed.

In Sec. 3 we shall consider a possibility of weakening the assumptions (i), (ii). After DAL MASO and MODICA [9, 10], we equip $\mathcal{F}$ with the metric d so that the $\mathcal{F}$ is a compact metric space. To define the metric d, we first introduce the ϵ-Yosida ($\epsilon > 0$) transform of $F \in \mathcal{F}$:

$$T_{\varepsilon} F(\mathbf{u}, T, A) = \inf\Big\{F(\mathbf{v}, R, A) + \frac{1}{\varepsilon} \int\limits_A |\mathbf{v} - \mathbf{u}|^{\alpha} d\mathbf{x} \tag{2.3}$$

$$+ \frac{1}{\varepsilon} \int\limits_A |R - T|^{\beta} d\mathbf{x} \,|\, \mathbf{v} \in L^{\alpha}_{loc}(\mathbb{R}^N)^N, \;\; R \in L^{\beta}_{loc}(\mathbb{R}^N)\Big\}.$$

Now we are in a position to define a distance on $\mathcal{F}$. Let us choose a countable dense subset $\mathcal{W} = \{\mathbf{w}_j |\ j \in \mathbb{N}\} \times \{g_j |\ j \in \mathbb{N}\}$ of $W^{1,\alpha}(\mathbb{R}^N)^N \times W^{1,\beta}(\mathbb{R}^N)$ and a countable subfamily $\mathcal{B} = \{B_k | k \in \mathbb{N}\}$ of $\mathcal{A}_0$. Here $\mathbb{N}$ denotes the set of natural numbers. For instance, $\mathcal{B}$ could be chosen as the family of all bounded open subsets of $\mathbb{R}^N$ which are finite unions of rectangles with rational vertices. Let us define for $F, G \in \mathcal{F}$

$$d(F,G) = \sum_{i,j,k=1}^{+\infty} \frac{1}{2^{i+j+k}} |\phi(T_{1/i}F(\mathbf{w}_j, g_j, B_k)) - \phi(T_{1/i}G(\mathbf{w}_j, g_j, B_k))|. \tag{2.4}$$

Here $\phi : \overline{\mathbb{R}} \to \mathbb{R}$ is any increasing, continuous bounded function. For instance, we may take $\phi = \arctan$ [9].

To prove that d is a distance on $\mathcal{F}$ it suffices to show that if $d(F,G) = 0$ then $F = G$. Indeed, Proposition 1.11 and Corollary 1.6 due to DAL MASO and MODICA [9], now extended to our more general case, are formulated as follows.

PROPOSITION 1.

(a) Let $F \in \mathcal{F}$, $\mathbf{u} \in L^\alpha_{loc}(\mathbb{R}^N)^N$, $T \in L^\beta_{loc}(\mathbb{R}^N)$, $A \in \mathcal{A}_0$.
Then

$$\lim_{\varepsilon \to 0^+} T_\varepsilon F(\mathbf{u}, T, A) = \sup_{\varepsilon > 0} T_\varepsilon F(\mathbf{u}, T, A) = F(\mathbf{u}, T, A).$$

(b) Let $\mathcal{W}$ be a dense subset of $W^{1,\alpha}(\mathbb{R}^N)^N \times W^{1,\beta}(\mathbb{R}^N)$ and $\mathcal{B}$ a dense subfamily of $\mathcal{A}_0$. If $F, G \in \mathcal{F}$ and $F(\mathbf{w}, g, B) = g(\mathbf{w}, g, B)$ $\forall(\mathbf{w}, g) \in \mathcal{W}$, $\forall B \in \mathcal{B}$ then $F = G$. □

Now we have to show that the metric space $(\mathcal{F}, d)$ is compact, hence complete and separable. To this end we have to introduce the notion of Γ-convergence. For more details the reader is referred to [7,10] and the relevant references cited therein. We observe that this type of variational convergence was introduced by E. De Giorgi and profoundly developed by the Italian School of the Calculus of Variations.

Let X be a metric space and let $\{F_\delta\}_{\delta > 0}$ be a sequence of functions defined on X with values in $\overline{\mathbb{R}}$. For instance, in our case $X = L^\alpha(A)^N \times L^\beta(A)$. We say that $\{F_\delta\}$ $\Gamma(X)$-converges at a point $z_\infty \in X$ to $\lambda \in \overline{\mathbb{R}}$ if the following two conditions are satisfied:

$(\mathrm{A})_1$ $\lambda \le \liminf\limits_{\delta \to 0^+} F_\delta(z_\delta)$ for any sequence $\{z_\delta\}_{\delta>0}$ converging in X to z_∞;

$(\mathrm{A})_2$ there exists a sequence $\{z_\delta\}_{\delta>0}$ converging in X to z_∞ such that $\limsup_{\delta \to 0^+} F_\delta(z_\delta) \le \lambda$.

In such a case we write $\lambda = \Gamma(X) \lim\limits_{\delta \to 0^+} F_\delta(z_\infty)$. More precisely, we should write $\Gamma(X^-)$ instead of $\Gamma(X)$, cf. [7,9]. Since only the above notion of Γ-convergence is used in this paper, therefore we prefer to use our simpler notation.

If there exists $F_\infty : X \to \overline{\mathbb{R}}$ such that

$$F_\infty(z) = \Gamma(X) \lim_{\delta\to 0^+} f_\delta(z), \quad \forall z \in X$$

we say that $\{F_\delta\}$ $\Gamma(X)$-converges to F_∞. Then from (A_1) and (A_2) we conclude that

$$F_\infty(z_\infty) = \min\{\liminf_{\delta\to 0^+} F_\delta(z_\delta) | z_\delta \text{ converges in } X \text{to } z_\infty\} \tag{2.5}$$

for every $z_\infty \in X$. Consequently, the $\Gamma(X)$-limit F_∞ is determined uniquelly.

Let now $\{F_\delta\}$ be a sequence in $\mathcal{F}$. Then we write

$$\Gamma(L^\alpha(A)^N \times L^\beta(A)) \lim_{\delta\to 0^+} F_\delta(\mathbf{u}, T) = F_\infty(\mathbf{u}, T) \tag{2.6}$$

$\forall(\mathbf{u}, T) \in L^\alpha(A)^N \times L^\beta(A)$, whenever $A \in \mathcal{A}_0$. More precisely, in (2.6) we should write $(F_\delta)_A$ and $(F_\infty)_A$ instead of F_δ and F_∞, cf. [9]. Indeed, each $F \in \mathcal{F}$ defines, for every $A \in \mathcal{A}_0$, a functional $F_A: L^\alpha(A)^N \times L^\beta(A) \to \overline{\mathbb{R}}$, cf. [9]. It suffices to extend $(\mathbf{u}, T) \in L^\alpha(A)^N \times L^\beta(A)$ to an element $(\widetilde{\mathbf{u}}, \widetilde{T})$ of $L^\alpha_{loc}(\mathbb{R}^N)^N \times L^\beta_{loc}(\mathbb{R}^N)$. We observe that the value of $F(\widetilde{\mathbf{u}}, \widetilde{T}, A)$ does not depend on the extension $(\widetilde{\mathbf{u}}, \widetilde{T})$ of $(\mathbf{u}, T)$.

As we shall see, the distance d on $\mathcal{F}$ has been chosen to be defined by (2.4) since then there is a link between d and Γ-convergence. Primarily, however, we formulate a compactness result.

PROPOSITION 2. The class $\mathcal{F}$ is compact for the $\Gamma(L^\alpha \times L^\beta)$-convergence, i.e. every sequence $\{F_\delta\}_{\delta>0}$ in $\mathcal{F}$ contains a subsequence that $\Gamma(L^\alpha \times L^\beta)$-converges to a functional $F_\infty \in \mathcal{F}$.

Proof. Let $\{F_\delta\}_{\delta>0}$ be a sequence in $\mathcal{F}$. By Theorems 2.4 and 4.3 of [7] there exists a subsequence $\{F_{\delta'}\}$ and a function $F_\infty : \mathbb{R}^N \times \mathbb{E}^N_s \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, non-negative, Lebesgue measurable in $\mathbf{x}$ and convex in the remaining variables, such that

$$\Gamma(L^\alpha(A)^N \times L^\beta(A)) \lim_{\delta'\to 0} (F_{\delta'})_A(\mathbf{u}, T) = \int_A f_\infty(\mathbf{x}, \mathbf{e}(\mathbf{u}(\mathbf{x})), T(\mathbf{x}), \nabla T(\mathbf{x}))d\mathbf{x} \tag{2.7}$$

for every $A \in \mathcal{A}_0$ and $(\mathbf{u}, T) \in W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$. If $(\mathbf{u}, T) \in (L^\alpha(A)^N \setminus W^{1,\alpha}(A)^N) \times (L^\beta(A) \setminus W^{1,\beta}(A))$ and $\{\mathbf{u}_\delta, T_\delta\}_{\delta>0}$ is a sequence converging in $L^\alpha(A)^N \times L^\beta(A)$ to $(\mathbf{u}, T)$, then $\{\mathbf{u}_\delta, T_\delta\}$ cannot have bounded subsequences in $W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$. Indeed, extending slightly Corollary 1.4 of DAL MASO and MODICA [9] we conclude that if $A \in \mathcal{A}_0$ then any bounded sequence in

$W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$ contains a subsequence that converges in $L^{\alpha}_{loc}(A)^N \times L^{\beta}_{loc}(A)$, weakly in $W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$ and pointwise almost everywhere in A.

Consequently, either $\{\mathbf{u}_\delta, T_\delta\}_{\delta>0} \notin W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$, or

$$\lim_{\delta''\to 0} \int_A (|\mathbf{e}(\mathbf{u}_{\delta''})|^\alpha + |T_{\delta''}|^\beta + |\nabla T_{\delta''}|^\beta) d\mathbf{x} = +\infty$$

for each subsequence $\{\mathbf{u}_{\delta''}, T_{\delta''}\}_{\delta''>0}$ contained in $W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$.

In both cases we get

$$\liminf_{\delta\to 0^+} (F_{\delta'})_A(u_\delta) = +\infty.$$

We recall that $\{\delta'\}$ is a subsequence of $\{\delta\}$. For instance, $\{\delta\} = \left\{\frac{1}{n}\right\}$, $n \in \mathbb{N}$, $\{\delta'\} = \left\{\frac{1}{n_k}\right\}$.

Thus we arrive at

$$\Gamma(L^\alpha(A)^N \times L^\beta(A)) \lim_{\delta'\to 0^+} (F_{\delta'})_A(\mathbf{u}, T)$$
$$= \begin{cases} \int_A f_\infty(\mathbf{x}, \mathbf{e}(\mathbf{u}(\mathbf{x})), T(\mathbf{x}), \nabla T(\mathbf{x})) dx & \text{if } (\mathbf{u}, T) \in W^{1,\alpha}(A)^N \times W^{1,\beta}(A) \\ +\infty & \text{if } (\mathbf{u}, T) \in (L^\alpha(A)^N \setminus W^{1,\alpha}(A)^N) \times (L^\beta(A) \setminus W^{1,\beta}(A)) \end{cases}$$

for every $A \in \mathcal{A}_0$.

The r.h.s. of the last equality defines a functional $F_\infty:\ L^\alpha_{loc}(\mathbb{R}^N)^N \times L^\beta_{loc}(A) \times \mathcal{A}_0 \to \overline{\mathbb{R}}$ which is the $\Gamma(L^\alpha \times L^\beta)$-limit of $\{F_{\delta'}\}_{\delta'>0}$. It now remains to prove that $F_\infty \in \mathcal{F}$, i.e. that condition (ii) following formula (2.1) is satisfied. Indeed, we have

$$c_0 \int_A (|\mathbf{e}(\mathbf{u})|^\alpha + |T|^\beta + |\nabla T|^\beta) d\mathbf{x} \le F_{\delta'}(\mathbf{u}, T, A)$$
$$\le c_1 \int_A (1 + |\mathbf{e}(\mathbf{u})|^\alpha + |T|^\beta + |\nabla T|^\beta) d\mathbf{x}$$

for every $A \in \mathcal{A}_0$, $(\mathbf{u}, T) \in W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$. By taking the $\Gamma(L^\alpha(A)^N \times L^\beta(A))$-limit of these three terms we obtain

$$c_0 \int_A (|\mathbf{e}(\mathbf{u})|^\alpha + |T|^\beta + |\nabla T|^\beta) d\mathbf{x} \le F_\infty(\mathbf{u}, T, A)$$
$$\le c_1 \int_A (1 + |\mathbf{e}(\mathbf{u})|^\alpha + |T|^\beta + |\nabla T|^\beta) d\mathbf{x}$$

for every $A \in \mathcal{A}_0$ and $(\mathbf{u}, T) \in W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$.

Let now $B_\rho(\mathbf{u})$ be the ball in $\mathbb{R}^N$ with center at $\mathbf{x}$ and radius ρ, $|B_\rho(\mathbf{u})|$ denotes its Lebesgue measure. Furthermore, let us denote by $l_\mathbf{q}$ and l_ϵ the linear functions such that $l_\mathbf{q}: \mathbb{R}^N \to \mathbb{R}$, $l_\mathbf{q} = \mathbf{q} \cdot \mathbf{x}$, $l_\epsilon: \mathbb{E}^N_s \to \mathbb{R}$, $l_\epsilon = \boldsymbol{\epsilon}\, \mathbf{x}$. Then we get, cf. Remark 1.1 of [9]

$$(A) \qquad \lim_{\rho\to 0^+} \frac{1}{|B_\rho(\mathbf{u})|} \int\limits_{B_\rho(\mathbf{x})} f_\infty(\mathbf{x}, \boldsymbol{\epsilon}, \xi, \mathbf{q}) d\mathbf{x} = f_\infty(\mathbf{x}, \boldsymbol{\epsilon}, \xi, \mathbf{q}),$$

a. e. $\mathbf{x} \in \mathbb{R}^N$, where $\xi \in \mathbb{R}$.

In virtue of the last relation we finally obtain

$$c_0(|\boldsymbol{\epsilon}|^\alpha + |\xi|^\beta + |\mathbf{q}|^\beta) \le f_\infty(\mathbf{x}, \boldsymbol{\epsilon}, \xi, \mathbf{q}) \le c_1(1 + |\boldsymbol{\epsilon}|^\alpha + |\xi|^\beta + |\mathbf{q}|^\beta).$$

It means that $F_\infty \in \mathcal{F}$ and the proof is complete. □

Now we are in a position to formulate a theorem which links d, Γ-convergence and ε-Yosida transform.

THEOREM 1. *Let $\{F_\delta\}_{\delta>0}$ be a sequence in $\mathcal{F}$ and $F_\infty \in \mathcal{F}$. Then the following conditions are equivalent:*

(1) $\lim_{\delta\to 0^+} d(F_\delta, F_\infty) = 0$;
(2) $\Gamma(L^\alpha \times L^\beta) \lim_{\delta\to 0^+} F_\delta = F_\infty$;
(3) $\lim_{\delta\to 0^+} (T_\varepsilon F_\delta)(\mathbf{u}, T, A) = (T_\varepsilon F_\infty)(\mathbf{u}, T, A)$
for each $\varepsilon > 0, (\mathbf{u}, T) \in L^\alpha_{loc}(A)^N \times L^\beta_{loc}(A), A \in \mathcal{A}_0$.

P r o o f. The proof parallels that of Proposition 1 of DAL MASO and MODICA [9], with obvious extensions. Therefore it is omitted here. □

Random integral functionals

Let (Ω, Σ, P) be a fixed probability space, that is Ω is a set of elementary events, Σ is a σ-field of subsets of Ω and P is a probability measure on Σ.

A random integral functional is any measurable function $F: \Omega \to \mathcal{F}$ when Ω is equipped with the σ-field Σ and $\mathcal{F}$ with the Borel σ-field Σ_F generator by the distance d defined by Eq. (2.4), cf. [9].

If F is a random integral functional, the image measure $F_\# P$ on $\mathcal{F}$ defined by $(F_\# P)(S) = P(F^{-1}(S))$ for every $S \in \Sigma_F$, is called the *distribution law* of F. We shall write $F \sim G$ if F and G are random integral functionals having the same distribution law.

The additive group $\mathbb{Z}^N$ and the multiplicative group $\mathbb{R}^+$ act on $\mathcal{F}$ by the translation operator $\tau_\mathbf{z}$ $(\mathbf{z} \in \mathbb{Z}^N)$ defined by

$$(2.8) \qquad (\tau_\mathbf{z} F)(\mathbf{u}, T, A) = \int\limits_A f(\mathbf{x} + \mathbf{z}, \mathbf{e}(\mathbf{u}), T, \nabla T) d\mathbf{x}$$

and by the homothety operator ρ_ε ($\varepsilon > 0$) defined by

$$(\rho_\varepsilon F)(\mathbf{u}, T, A) = \int_A f\Big(\frac{\mathbf{x}}{\varepsilon}, \mathbf{e}(\mathbf{u}), T, \nabla T\Big) d\mathbf{x} \tag{2.9}$$

for every $(\mathbf{u}, T) \in W^{\alpha}_{loc}(\mathbb{R}^N)^N \times W^{1,\beta}_{loc}(\mathbb{R}^N)$, and $A \in \mathcal{A}_0$. We recall that $\mathbb{Z}$ stands for the set of integers. We observe that if the integrand f does not depend on T, but still depends on ∇T, then

$$(\tau_\mathbf{z} F)(\mathbf{u}, T, A) = F(\tau_\mathbf{z}\mathbf{u}, \tau_\mathbf{z} T, \tau_\mathbf{z} A),$$

where $(\tau_\mathbf{z}\mathbf{u})(\mathbf{x}) = \mathbf{u}(\mathbf{x}-\mathbf{z})$, $\tau_\mathbf{z} T(\mathbf{x}) = T(\mathbf{x}-\mathbf{z})$, $\tau_\mathbf{z} A = \{\mathbf{x} \in \mathbb{R}^N | \mathbf{x}-\mathbf{z} \in A\}$, and

$$(\rho_\varepsilon F)(\mathbf{u}, A) = \varepsilon^N(\rho_\varepsilon\mathbf{u}, \rho_\varepsilon T, \rho_\varepsilon A),$$

where $(\rho_\varepsilon\mathbf{u})(\mathbf{x}) = \dfrac{1}{\varepsilon}\mathbf{u}(\varepsilon\mathbf{x})$, $(\rho_\varepsilon T)(\mathbf{x}) = \dfrac{1}{\varepsilon}T(\varepsilon\mathbf{x})$, $\rho_\varepsilon A = \{\mathbf{x} \in \mathbb{R}^N | \varepsilon\mathbf{x} \in A\}$. In other words, during translation and homothety, T in (2.8) and (2.9) is treated as a parameter similarly to the case of periodic homogenization, cf. [4].

By virtue of Corollary 2.4 due to DAL MASO AND MODICA [9], we conclude that if F is a random integral functional and $\mathbf{z} \in \mathbb{R}^N$, $\varepsilon > 0$, then the functions $\tau_\mathbf{z} F$, $\rho_\varepsilon F : \Omega \to \mathcal{F}$ defined by

$$(\tau_\mathbf{z} F)(\omega) = \tau_\mathbf{z}(F(\omega)), \quad (\rho_\varepsilon F)(\omega) = \rho_\varepsilon(F(\omega)), \ \ \forall\omega \in \Omega, \tag{2.10}$$

are random integral functionals. Furthemore, if G is another random integral functional such that $F \sim G$, then we have $\tau_\mathbf{z} F \sim \tau_\mathbf{z} G$ and $\rho_\varepsilon F \sim \rho_\varepsilon G$.

We say that $\{F_\varepsilon\}_{\varepsilon>0}$ is a *stochastic homogenization process* modelled on a fixed random integral functional F on Ω if $F_\varepsilon \sim \rho_\varepsilon F$ for every $\varepsilon > 0$, that is F_ε and $\rho_\varepsilon F$ have the same distribution law.

Let F be a random integral functional. We say that F is stochastically periodic if F and $\tau_\mathbf{z} F$ have the same law for every $\mathbf{z} \in \mathbb{Z}^N$.

Ergodicity is a well-established notion when applied to integrands. Here we need ergodicity in $\mathcal{F}$ with respect to $\mathbb{Z}^N$. After DAL MASO and MODICA [10] we say that a random integral functional $F \in \mathcal{F}$ is ergodic if $P[F \in S] = 0$ or 1 for every Σ_F-measurable subset S of $\mathcal{F}$ such that $\tau_\mathbf{z}(S) = S$ for every $\mathbf{z} \in \mathbb{Z}^N$.

For $F \in \mathcal{F}$, $A \in \mathcal{A}_0$, $\xi \in \mathbb{R}$ and $(\mathbf{u}_0, T_0) \in W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$ we may consider the following Dirichlet problem:

$$m_\xi(F, \mathbf{u}_0, T_0, A) = \min\Big\{\int_A f(\mathbf{x}, \mathbf{e}(\mathbf{u}(\mathbf{x})), \xi, \nabla T(\mathbf{x}))d\mathbf{x} \ \Big|\, (\mathbf{u}-\mathbf{u}_0, T-T_0) \in W^{1,\alpha}_0(A)^N \times W^{1,\beta}_0(A)\Big\} \tag{2.11}$$

We conclude that $m_\xi(F, \mathbf{u}_0, T_0)$ is continuous in F with respect to the metric d. We stress that in (2.11) $\xi \in \mathbb{R}$ plays the role of a parameter.

Let $Q_{1/\varepsilon}$ be the cube

$$Q_{1/\varepsilon} = \{\mathbf{x} \in \mathbb{R}^N : |x_i| < 1/\varepsilon,\ i = 1, ..., N\}$$

and $|Q_{1/\varepsilon}| = (2/\varepsilon)^N$ its Lebesgue measure. We recall that $l_\mathbf{q} = \mathbf{q}\cdot\mathbf{x}$ and $l_\epsilon = \boldsymbol{\epsilon}\,\mathbf{x}$, where $\mathbf{q} \in \mathbb{R}^N$, $\boldsymbol{\epsilon} \in \mathbb{E}^N_s$.

After these lengthy, yet necessary preparations, we are in a position to state our main homogenization theorem.

THEOREM 2. *Let F be a random integral functional and define $F_\varepsilon = \rho_\varepsilon F$. If f is periodic in law, then F_ε converges P-almost everywhere as $\varepsilon \to 0^+$ to a random integral F_0. Moreover, there exist $\Omega' \subset \Omega$ of full measure such that the limit*

$$\lim_{\varepsilon\to 0^+} \frac{m_\xi(F(\omega), l_\epsilon, l_\mathbf{q}, Q_{1/\varepsilon})}{|Q_{1/\varepsilon}|} = f_0(\omega, \boldsymbol{\epsilon}, \xi, \mathbf{q}) \tag{2.12}$$

exists for every $\omega \in \Omega'$, $\xi \in \mathbb{R}$, $\mathbf{q} \in \mathbb{R}^N$, $\boldsymbol{\epsilon} \in \mathbb{E}^N_s$ and

$$F_0(\omega)(\mathbf{u}, T, A) = \int_A f_0[\omega, \mathbf{e}(\mathbf{u}(\mathbf{x})), T(\mathbf{x}), \nabla T(\mathbf{x})]d\mathbf{x} \tag{2.13}$$

for every $\omega \in \Omega'$, $A \in \mathcal{A}_0$, $(\mathbf{u}, T) \in L^\alpha_{loc}(A)^N \times L^\beta_{loc}(A)$ with $\mathbf{u}|_A \in W^{1,\alpha}(A)^N$, $T_{|A} \in W^{1,\beta}(A)$. Additionally, if F is ergodic, then F_0 is or equivalently $f_0(\omega, \boldsymbol{\epsilon}, \xi, \mathbf{q})$ does not depend on ω and

$$f_0(\boldsymbol{\epsilon}, \xi, \mathbf{q}) = \lim_{\varepsilon\to 0} \int_\Omega \frac{m_\xi(F(\omega), l_\epsilon, l_\mathbf{q}, Q_{1/\varepsilon})}{|Q_{1/\varepsilon}|} \tag{2.14}$$

for every $\xi \in \mathbb{R}$, $\mathbf{q} \in \mathbb{R}^N$, $\boldsymbol{\epsilon} \in \mathbb{E}^N_s$.

3. Proof the stochastic homogenization theorem and comments

Prior to passing to the proof of Theorem 2 we are going to provide useful comments and additional indispensable tools.

First we observe that similar theorem was formulated by DAL MASO and MODICA [10] for a much simpler case where

$$(\rho_\varepsilon F)(\omega)(T, A) = \int_A f\Big(\omega, \frac{\mathbf{x}}{\varepsilon}, \nabla T(\mathbf{x})\Big)d\mathbf{x}. \tag{3.1}$$

The same authors stated a stronger result as Theorem 3 in their another paper [9]. More precisely, for $\{F_\varepsilon\}_{\varepsilon>0}$ a stochastic process modelled on a stochastically periodic random integral functional F, in [9], it was assumed that there exists $M > 0$ such that the two families of random functions

$$(F(\cdot)(T,A))_{T\in L^\beta_{loc}(\mathbb{R}^N)}, \qquad (F(\cdot)(T,B))_{T\in L^\beta_{loc}(\mathbb{R}^N)}$$

are independent wherever $A, B \in \mathcal{A}_0$ with $\mathrm{dist}(A,B) > M$. Then a counter-part of formula (2.14) holds. In fact, $\{F_\varepsilon\}$ converges in probability as $\varepsilon \to 0^+$ to the single functional $F_0 \in \mathcal{F}$ independent of ω. Now, the functional F_0 is easily deduced from (2.14) by deleting $\boldsymbol{\epsilon}$ and ξ.

Let us recall the motions of convergence in probability and convergence in law cf. [9] and the relevant references cited therein.

We say that a sequence of random integral functional $\{F_\varepsilon\}_{\varepsilon>0}$ converges in probability to a random integral functional F_∞ if

$$\lim_{\varepsilon\to 0^+} P\Big\{\omega \in \Omega | d(F_\varepsilon(\omega), F_\infty(\omega)) > \eta\Big\} = 0, \forall \eta > 0 \tag{3.2}$$

where d is the distance on $\mathcal{F}$. It is well-known that any sequence converging in probability contains a subsequence which converges pointwise almost everywhere.

We say that $\{F_\varepsilon\}_{\varepsilon>0}$ converges in law to F_∞ if the corresponding laws $\mu_\varepsilon = F_{\varepsilon\#}P$ converge weakly $-*$ as $\varepsilon \to 0^+$ to $\mu_\infty = F_{\infty\#}P$, i.e.,

$$\lim_{\varepsilon\to 0^+} \int\limits_{\mathcal{F}} \varphi(F) d\mu_\varepsilon(F) = \int\limits_{\mathcal{F}} \varphi(F) d\mu_\infty(F) \tag{3.3}$$

for every continuous function $\varphi: \ F \to \mathbb{R}$.

Equivalently we may write

$$\langle \mu_\varepsilon, \varphi\rangle \to \langle \mu_\infty, \varphi\rangle \quad \text{as} \quad \varepsilon \to 0$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing defined on $C^*(\mathcal{F}) \times C(\mathcal{F})$; $C(\mathcal{F})$ denotes the space of continuous functions on the compact space $\mathcal{F}$ with the supremum norm and $C^*(\mathcal{F})$ is its dual.

The interrelationship between these two types of covergence is well-known, cf. [9, Prop. 2.9].

PROPOSITION 3. Let F_∞ be a constant random integral functional, that is there exists $F_0 \in \mathcal{F}$ such that $F_\infty(\omega) = F_0$ for P-almost all $\omega \in \Omega$. Then convergence in law and convergence in probability toward F_∞ are equivalent.

Let us comment on the stronger version of the stochastic homogenization theorem due to DAL MASO and MODICA [9]. This theorem is unsatisfactory for two important reasons. First, the indenpedence at large distances is not always verified. Such is the case of chessboard structure with cells of random size sketched in Fig. 3 of [10]. Second, whilst convergence in probability is the best possible if we give as in [9] the hypotheses in terms of laws, the problem arises whether there is almost everywhere convergence in the case $F_\varepsilon = \rho_\varepsilon F$, a result well-known in the case of linear stochastic homogenization, cf. [9, 10] and the relevant references therein.

Both these difficulties are overcome by Theorem 1 of DAL MASO and MODICA [10] and our more general Theorem 2.

Nonlinear stochastic homogenization of random media was also performed by SAB [21]. In essence, this author considers integrands of the type $f(\omega, \mathbf{x}, \boldsymbol{\epsilon})$, $\boldsymbol{\epsilon} \in \mathbb{E}^N_s$, convex with respect to $\boldsymbol{\epsilon}$. Essential novelty lies in admitting the linear growth in $\boldsymbol{\epsilon}$, thus allowing for the study of homogenization of perfectly plastic media with random distribution of microheterogeneities. Such an approach is confined to deformational theory of plasticity, sometimes called Hencky plasticity. SAB [21] observed a correspondence between periodic media and statistically homogeneous ergodic (S.H.E.) media. We observe that this class is larger than the class of media described by stochastically periodic random integral functionals. SAB'S [21] approach involves an N-dimensional dynamical system on Ω, sometimes called the measure preserving flow. This dynamical system is assumed to be ergodic. Having introduced the dynamical system, not necessarily ergodic, one can exploit the stochastic differential calculus.

In order to prove our Theorem 2 we need a few additional results, cf. [1, 10].

A set function $\mu: \mathcal{A}_0 \to \mathbb{R}$ is said to be *subadditive* if

$$\mu(A) \leq \sum_{k \in K} \mu(A_k) \tag{3.4}$$

for every $A \in \mathcal{A}_0$ and for every finite family $\{A_k\}_{k\in K}$ in $\mathcal{A}_0$ such that

$$A_k \subset A \ \ \forall k \in K, \qquad A_j \cap A_k = \emptyset \ \ \forall j, k \in K, \ j \neq k, \qquad |A - \bigcup_{k\in K} A_k| = 0.$$

Let $\mathcal{M} = \mathcal{M}(c)$ be the family of subadditive functions $\mu: \mathcal{A}_0 \to \mathbb{R}$ such that

$$0 \leq \mu(A) \leq c|A| \qquad \forall A \in \mathcal{A}_0$$

where $c > 0$ is a fixed constant. We denote by Σ_M the trace on $\mathcal{M}$ of the product Σ-algebra of $\mathbb{R}^{\mathcal{A}_0}$.

Let (Ω, Σ, P) be a given probability space. $A\,(\Sigma, \Sigma_M)$-measurable map $\mu : \Omega \to \mathcal{M}$ is called a *subadditive process*.

The group $\mathbb{Z}^N$ acts on $\mathcal{M}$ by the formula

$$(\tau_{\mathbf{z}}\mu)(A) = \mu(\tau_{\mathbf{z}}A). \tag{3.5}$$

If $(-\mu)$ is subadditive then μ is called superadditive.

We say that a subadditive process is ergodic if $P[\mu \in S] = 0$ or 1 for Σ_M-measurable subset S of $\mathcal{M}$ such that $\tau_{\mathbf{z}}S = S$ for every $\mathbf{z} \in \mathbb{Z}^N$.

Essential role in the proof of Theorem 2. will play the following proposition, which is substantially the subadditive ergodic theorem due to AKCOGLU and KRENGEL [1].

PROPOSITION 4. Let $\mu : \Omega \to \mathcal{M}$ be a subadditive process. If μ is periodic in law, that is μ and $\tau_{\mathbf{z}}\mu$ have the same law for every $\mathbf{z} \in \mathbb{Z}^N$, then there exists a Σ-measurable function $\Phi : \Omega \to \mathbb{R}$ and a subset $\Omega' \subset \Omega$ of full measure such that

$$\lim_{\varepsilon\to 0^+} \frac{\mu(\omega)\left(\frac{1}{\varepsilon}Q\right)}{\left|\frac{1}{\varepsilon}Q\right|} = \lim_{t\to+\infty} \frac{\mu(\omega)(tQ)}{|tQ|} = \Phi(\omega) \tag{3.6}$$

for every $\omega \in \Omega'$ and for every cube $Q \in \mathbb{R}^N$. Moreover, if μ is ergodic then Φ is constant. □

For the proof the reader is refered to DAL MASO and MODICA [10].

Proof of Theorem 2. We divide it into two steps.

STEP 1. The random integrand $f(\omega, \mathbf{x}, \boldsymbol{\epsilon}, \xi, \mathbf{q})$ does not depend on ξ. Then $m_\xi(F(\omega), l_\epsilon, l_{\mathbf{q}}, Q_{1/\varepsilon})$ appearing in Eq. (2.12) does not involve ξ and simply write $m(F(\omega), l_\epsilon, l_{\mathbf{q}}, Q_{1/\varepsilon})$. Now our proof is an extension of the proof of Theorem I due to DAL MASO and MODICA [10]. We recall that $l_{\mathbf{q}} = \mathbf{q}\cdot\mathbf{x}$, $l_\epsilon = \boldsymbol{\epsilon}\,\mathbf{x}$.

Let us fix $\mathbf{q} \in \mathbb{R}^N$, $\boldsymbol{\epsilon} \in \mathbb{E}^N_s$ and define

$$\mu_{\mathbf{p}}(\omega)(A) = m(F(\omega), l_\epsilon, l_{\mathbf{q}}, A), \quad \forall\omega \in \Omega,\ \forall A \in \mathcal{A}_0$$

where $\mathbf{p} = (\boldsymbol{\epsilon}, \mathbf{q})$. Then $\mu_{\mathbf{p}}(\omega) \in \mathcal{M}(c)$ with $c = c_1(1 + |\mathbf{q}|^\alpha + |\boldsymbol{\epsilon}|^\beta)$ for every $\omega \in \Omega$, and $\mu_{\mathbf{p}} : \Omega \to \mathcal{M}$ is (Σ, Σ_M)-measurable since $m(\cdot, l_\epsilon, l_{\mathbf{q}}, A)$ is continuous on $\mathcal{F}$ equipped with the distance d. For every $\mathbf{z} \in \mathbb{Z}^N$, $\omega \in \Omega$, $A \in \mathcal{A}_0$ we have

$$(\tau_{\mathbf{z}}\mu_{\mathbf{p}})(\omega)(A) = \mu_{\mathbf{p}}(\omega)(\tau_{\mathbf{z}}A) = \min_{(\mathbf{u},T)}\Big\{(\tau_{\mathbf{z}}F)(\omega)(\tau_{-\mathbf{z}}\mathbf{u}, \tau_{-\mathbf{z}}T, A)$$

$$| \ \tau_{-\mathbf{z}}\mathbf{u} - \tau_{-\mathbf{z}}l_{\epsilon} \in W_0^{1,\alpha}(A)^N,\ \tau_{-\mathbf{z}}T - \tau_{-\mathbf{z}}l_{\mathbf{q}} \in W_0^{1,\beta}(A)\Big\}$$

$$= \min_{(\mathbf{v},R)}\Big\{(\tau_{\mathbf{z}}F)(\omega)(\mathbf{v} + l_{\epsilon}(\mathbf{z}), R + l_{\mathbf{q}}(\mathbf{z}), A)$$

$$| \ \mathbf{v} - l_{\epsilon} \in W_0^{1,\alpha}(A)^N,\ R - l_{\mathbf{q}} \in W_0^{1,\beta}(A)\Big\}.$$

Since the integrand of F depends only on $\mathbf{x}$, $\mathbf{e}(\mathbf{u})$ and ∇T, therefore

$$(\tau_{\mathbf{z}}F)(\mathbf{v} + l_{\epsilon}(\mathbf{z}), R + l_{\mathbf{q}}(\mathbf{z}), A) = (\tau_{\mathbf{z}}F)(\omega)(\mathbf{v}, R, A).$$

Hence

$$(\tau_{\mathbf{z}}\mu_{\mathbf{p}})(\omega)(A) = m((\tau_{\mathbf{z}}F)(\omega), l_{\epsilon}, l_{\mathbf{q}}, A),$$

for every $\mathbf{z} \in \mathbb{Z}^N$, $\omega \in \Omega$, $A \in \mathcal{A}_0$. Thus $\mu_{\mathbf{p}}$ is periodic in law because $\tau_{\mathbf{z}}F$ and F have the same law and $m(\cdot, l_{\epsilon}, l_{\mathbf{q}}, A)$ is continuous on $\mathcal{F}$.

In virtue of Proposition 4 we conclude that there exist a subset $\Omega'_{\mathbf{p}} \subset \Omega$ of full measure and a Σ-measurable function $\Phi_{\mathbf{p}} : \Omega \to \mathbb{R}$ such that

$$\lim_{t\to+\infty} \frac{\mu_{\mathbf{p}}(\omega)(tQ)}{|tQ|} = \Phi_{\mathbf{p}}(\omega)$$

for every $\omega \in \Omega'_{\mathbf{p}}$ and for every cube $Q \in \mathbb{R}^N$. Let now $Q_{1/\varepsilon}$ be the cube defined in Sec. 2 and let $f_0 : \Omega \times \mathbb{E}^N_s \times \mathbb{R}^N \to \mathbb{R}$ be the function defined by

$$f_0(\omega, \boldsymbol{\epsilon}, \mathbf{q}) = \limsup_{\varepsilon\to 0^+} \frac{\mu_{\mathbf{p}}(\omega)(Q_{1/\varepsilon})}{|Q_{1/\varepsilon}|} \quad \forall(\omega, \boldsymbol{\epsilon}, \mathbf{q}) \in \Omega \times \mathbb{E}^N_s \times \mathbb{R}^N.$$

We observe that the functions

$$\mathbf{p} = (\boldsymbol{\epsilon}, \mathbf{q}) \to \frac{\mu_{\mathbf{p}}(\omega)(A)}{|A|} \quad (\omega \in \Omega,\ A \in \mathcal{A}_0)$$

are convex and equibounded between 0 and $c_1(1 + |\mathbf{q}|^{\beta} + |\boldsymbol{\epsilon}|^{\alpha})$, hence locally equicontinuous. The convexity follows from the convexity in $(\mathbf{u}, T)$ of $F(\omega)(\mathbf{u}, T, A)$. Consequently $f_0(\omega, \boldsymbol{\epsilon}, \mathbf{q})$ is convex in $(\boldsymbol{\epsilon}, \mathbf{q})$. Let us set

$$\Omega' = \bigcap_{\mathbf{p}\in\mathbb{Q}^N} \Omega'_{\mathbf{p}}$$

where $\mathbb{Q}$ is the set of rational numbers. We have $P(\Omega') = 1$ and

$$\lim_{\varepsilon\to 0^+} \frac{\mu_{\mathbf{p}}(\omega)\left(\frac{1}{\varepsilon}Q\right)}{\left|\frac{1}{\varepsilon}Q\right|} = f_0(\omega, \boldsymbol{\epsilon}, \mathbf{q})$$

for every $\omega \in \Omega'$, $\mathbf{p} = (\boldsymbol{\epsilon}, \mathbf{q}) \in \mathbb{E}^N_s \times \mathbb{R}^N$ and for every cube Q in $\mathbb{R}^N$. Furthermore, we get

$$\mu_{\mathbf{p}}(\omega)\left(\frac{1}{\varepsilon}Q\right) = \left(\frac{1}{\varepsilon}\right)^N m((\rho_\varepsilon F)(\omega), l_\epsilon, l_{\mathbf{q}}, Q).$$

Hence, since $\rho_\varepsilon F = F_\varepsilon$, we obtain

$$\lim_{\varepsilon\to 0^+} \frac{m(F_\varepsilon(\omega), l_\epsilon, l_{\mathbf{q}}, Q)}{|Q|} = f_0(\omega, \boldsymbol{\epsilon}, \mathbf{q})$$

for every $\omega \in \Omega'$, $\boldsymbol{\epsilon} \in \mathbb{E}^N_s$, $\mathbf{q} \in \mathbb{R}^N$ and for every cube in $\mathbb{R}^N$. In virtue of Proposition 2, for every $\omega \in \Omega'$ there exists an integral functional $F_0(\omega) \in \mathcal{F}$ such that $F_\varepsilon(\omega)$ $\Gamma(L^\alpha \times L^\beta)$ converges to $F_0(\omega)$ as $\varepsilon \to 0^+$. More precisely, there exists such a subsequence still denoted by F_ε.

Let us calculate the integrand $g_0(\omega, \mathbf{x}, \boldsymbol{\epsilon}, \mathbf{q})$ of $F_0(\omega)$. Fix $\omega \in \Omega'$ and set

$$Q_\rho(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^N : |y_i - x_i| < \rho,\ i = 1, ..., N\}.$$

Taking into account formula (A) of Sec. 2 and the continuity of $m(\cdot, l_\epsilon, l_{\mathbf{q}}, A)$, we conclude that there exist a subset $\mathcal{N}$ of $\mathbb{R}^N$ with $|\mathcal{N}| = 0$ such that

$$\begin{aligned} g_0(\omega, \mathbf{x}, \boldsymbol{\epsilon}, \mathbf{q}) &= \lim_{\rho\to 0^+} \frac{m(F_0(\omega), l_\epsilon, l_{\mathbf{q}}, Q_\rho(\mathbf{x}))}{|Q_\rho(\mathbf{x})|} \\ &= \lim_{\rho\to 0^+}\lim_{\varepsilon\to 0^+} \frac{m(F_\varepsilon(\omega), l_\epsilon, l_{\mathbf{q}}, Q_\rho(\mathbf{x}))}{|Q_\rho(\mathbf{x})|} \\ &= \lim_{\rho\to 0^+}\lim_{\varepsilon\to 0^+} \frac{\mu_{\mathbf{p}}(\omega)\left(\frac{1}{\varepsilon}Q_\rho(\mathbf{x})\right)}{\left|\frac{1}{\varepsilon}Q_\rho(\mathbf{x})\right|} = f_0(\omega, \boldsymbol{\epsilon}, \mathbf{q}) \end{aligned}$$

for every $\mathbf{x} \in \mathbb{R}^N \setminus \mathcal{N}$, $\boldsymbol{\epsilon} \in \mathbb{E}^N_s$, $\mathbf{q} \in \mathbb{R}^N$. Thus we get

$$F_0(\omega)(\mathbf{u}, T, A) = \int_A f_0[\omega, \mathbf{e}(\mathbf{u}(\mathbf{x})), \nabla T(\mathbf{x})]d\mathbf{x}$$

for every $\omega \in \Omega'$, $A \in \mathcal{A}_0$, $(\mathbf{u}, T) \in L^\alpha_{loc}(\mathbb{R}^N)^N \times L^\beta_{loc}(\mathbb{R}^N)$ such that $(\mathbf{u}, T)|_A \in W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$.

If F is ergodic, then $\mu_{\mathbf{p}}$ is ergodic and on account of Proposition 4, $\Phi_{\mathbf{p}}$, and thus also f_0 do not depend on ω.

STEP 2. Let now $f = f(\omega, \mathbf{x}, \boldsymbol{\epsilon}, T, \mathbf{q})$. On account of Proposition 2, for a fixed $\omega \in \Omega$, there exists a subsequence of $\{F_\varepsilon\}_{\varepsilon>0}$, still denoted by $\{F_\varepsilon\}$ such that $F_\varepsilon(\omega)\ \Gamma(L^\alpha \times L^\beta)$ converges to

$$F_\infty(\omega)(\mathbf{u}, T, A) = \int_\Omega f_0[\omega, \mathbf{x}, \mathbf{e}(\mathbf{u}(\mathbf{x})), T(\mathbf{x}), \nabla T(\mathbf{x})]d\mathbf{x}$$

for each $A \in \mathcal{A}_0$, $(\mathbf{u}, T) \in L^\alpha_{loc}(\mathbb{R}^N)^N \times L^\beta_{loc}(\mathbb{R}^N)$ with $(\mathbf{u}, T)|_A \in W^{1,\alpha}(A)^N \times W^{1,\beta}(A)$.

Let now $T \in C^1(\mathbb{R}^N)$ and consider the function, cf. BRAIDES [4] in the case of periodic homogenization

$$f_T(\omega, \mathbf{x}, \boldsymbol{\epsilon}, \mathbf{q}) = f(\omega, \mathbf{x}, \boldsymbol{\epsilon}, T(\mathbf{x}), \mathbf{q}).$$

To the function f_T we may apply Step 1 and write

$$\lim_{\varepsilon\to 0^+} \frac{m(F_T(\omega), l_\epsilon, l_\mathbf{q}, Q_{1/\varepsilon})}{|Q_{1/\varepsilon}|} = f_{0T}(\omega, \boldsymbol{\epsilon}, \mathbf{q}) \tag{3.7}$$

for every $\omega \in \Omega'$, $(\boldsymbol{\epsilon}, \mathbf{q}) \in \mathbb{E}^N_s \times \mathbb{R}^N$, and

$$F_{0T}(\mathbf{u}, R, A) = \int_A f_0[\omega, \mathbf{e}(\mathbf{u}(\mathbf{x})), T(\mathbf{x}), \nabla R(\mathbf{x})]d\mathbf{x}$$

Still by Step 1 and (3.7) we get

$$f_\infty(\omega, \mathbf{x}, \boldsymbol{\epsilon}, \xi, \mathbf{q}) = f_{0\xi}(\omega, \boldsymbol{\epsilon}, \mathbf{q}) = f_0(\omega, \boldsymbol{\epsilon}, \xi, \mathbf{q}).$$

Ergodicity implies that f_0 does not depend on ω. Thus the proof of Theorem 2 is complete. □

REMARK 1.

(i) The easiest way of proving ergodicity of F is to verify a mixing condition (or independence at large distances), cf. [9].

(ii) A random integrand f is ergodic if it satisfies the following mixing condition [9]:

$$\lim_{\substack{|\mathbf{z}|\to+\infty \\ \mathbf{z}\in\mathbb{Z}^N}} P\Big(\{\omega \in \Omega | f(\omega, \mathbf{x}_i, \boldsymbol{\epsilon}_i, \xi, \mathbf{q}_i) > s_i\ \forall i \in I,$$

$$f(\omega, \mathbf{y}_j + \mathbf{z}, \boldsymbol{\Delta}_j, \xi, \mathbf{r}_j) > t_j\ \forall j \in J\}\Big)$$

$$= P\Big(\{\omega \in \Omega | f(\omega, \mathbf{x}_i, \boldsymbol{\epsilon}_i, \xi, \mathbf{q}_i) > s_i\ \forall i \in I\}\Big)$$

$$\times P\Big(\{\omega \in \Omega | f(\omega, \mathbf{y}_j, \boldsymbol{\Delta}_j, \xi, \mathbf{r}_j) > t_j\ \forall j \in J\}\Big)$$

for every pair of finite families $\{(\mathbf{x}_i, \epsilon_i, \mathbf{q}_i\}_{i\in I}$ and $\{(\mathbf{y}_j, \mathbf{\Delta}_j, \mathbf{r}_j\}_{j\in J}$ in $\mathbb{R}^N \times \mathbb{E}^N_s \times \mathbb{R}^N$. Here $\xi \in \mathbb{R}$ is treated as a parameter. Then one can extend Theorem III due to DAL MASO and MODICA [10] and prove that, for instance, if f is ergodic, then F is ergodic.
Ergodicity of F also follows if a measure preserving ergodic flow on Ω is introduced. Then the integrand is ergodic and the ergodicity of F is satisfied.

(iii) It seems that Theorem 2 can be weakened by assuming the convexity of integrands only with respect to $\boldsymbol{\epsilon} \in \mathbb{E}^N_s$ and $\mathbf{q} \in \mathbb{R}^N$. Then appropriate conditions on f are specified by BRAIDES [4]. This author performed the so-called periodic nonuniform homogenization. It means that after homogenization the integrand f_0 depends additionally on the macroscopic variable $\mathbf{x} \in V$, where V denotes a domain in $\mathbb{R}^N$ occupied by the considered body. The authors of the present paper are not aware whether any results of this type are available for random media; we mean here non-uniform stochastic homogenization of functionals. The only available approach is due to BOURGEAT *et al.* [3]. These authors introduced the motion of stochastic two-scale convergence in the mean which allows for treating the media remaining macroscopically inhomogeneous.

4. Thermoelastic stochastically periodic composite

Theorem 1 is general and covers a broad class of stochastic microstructures. In the remaining part of the paper we will focus on specific microstructures. More precisely, we consider a two-phase thermoelastic composite, occupying a domain $A \subset \mathbb{R}^N$. The phases are located randomly in periodic cubic cells with a given distribution. The dimensionless parameter ε is equal to the ratio of the length of the cell l and the characteristic dimension of the body L, $\varepsilon = l/L$.

The classical Duhamel-Neumann relations are satisfied at an arbitrary point $\mathbf{x} \in A$:

$$\boldsymbol{\sigma} = \boldsymbol{\lambda}^\varepsilon(\omega, \mathbf{x})\mathbf{e} - \beta^\varepsilon(\omega, \mathbf{x})T, \quad \boldsymbol{\beta}^\varepsilon(\omega, \mathbf{x}) \equiv \boldsymbol{\lambda}^\varepsilon(\omega, \mathbf{x})\boldsymbol{\alpha}^\varepsilon(\omega, \mathbf{x}), \tag{4.1}$$

where the strain-displacement is linear

$$e_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{4.2}$$

The Fourier relation takes now the form

$$\mathbf{q}^\varepsilon(\omega, \mathbf{x}) = -\varkappa^\varepsilon(\omega, \mathbf{x})\nabla T, \tag{4.3}$$

where $\omega \in \Omega$. Here, $\boldsymbol{\sigma}$ is the stress tensor; $\boldsymbol{\lambda}$ and $\boldsymbol{\alpha}$ are the tensors of the elastic moduli and thermal expansion; T is the temperature increment; $\mathbf{q}$ is the heat flux whiles $\varkappa$ denotes the conductivity tensor.

We recall that

$$\boldsymbol{\lambda}^{\varepsilon}(\omega,\mathbf{x}) = \boldsymbol{\lambda}\Big(\omega,\frac{\mathbf{x}}{\varepsilon}\Big), \quad \varkappa^{\varepsilon}(\omega,\mathbf{x}) = \varkappa\Big(\omega,\frac{\mathbf{x}}{\varepsilon}\Big), \quad etc.$$

The random functional F^{ε}, given by:

$$(4.4)\quad F^{\varepsilon}(\omega)(\mathbf{u},T,A) \equiv \begin{cases} \int_A f^{\varepsilon}(\omega,\mathbf{x},\mathbf{e}(\mathbf{u});T,\nabla T)d\mathbf{x} & \text{if } (\mathbf{u},T)\in H^1(A)^N\times H^1(A) \\ +\infty & \text{otherwise} \end{cases}$$

where

$$(4.5)\quad f^{\varepsilon}(\omega,\mathbf{x},\mathbf{e}(\mathbf{u});T,\nabla T) = \frac{1}{2}\{[\mathbf{e}(\mathbf{u}) - T\boldsymbol{\alpha}^{\varepsilon}(\omega,\mathbf{x})]^{\top}\boldsymbol{\lambda}^{\varepsilon}(\omega,\mathbf{x}) \cdot [\mathbf{e}(\mathbf{u}) - T\boldsymbol{\alpha}^{\varepsilon}(\omega,\mathbf{x})] + (\nabla T)^{\top}\varkappa^{\varepsilon}(\omega,\mathbf{x})\nabla T\}.$$

Under usual symmetry and coercivity assumptions pertaining to matrices λ and κ the integrand $f(\omega,\mathbf{x},\boldsymbol{\epsilon},\xi,\mathbf{q})$ is convex in $(\boldsymbol{\epsilon},\xi,\mathbf{q})\in\mathbb{E}^N_s\times\mathbb{R}\times\mathbb{R}^N$ and $f\in\mathcal{F}$ where $\alpha=\beta=2$. The convexivity in ξ results from linearity of the transformation $(\boldsymbol{\epsilon},\xi)\longrightarrow(\boldsymbol{\epsilon}-\xi\boldsymbol{\alpha})$. Here the superscript $\top$ stands for transposition.

The moduli $\boldsymbol{\lambda}$, $\varkappa$, $\boldsymbol{\alpha}$ possess the usual properties, cf. [11-13].

We have

$$(4.6)\qquad \boldsymbol{\sigma} = \frac{\partial f^{\varepsilon}}{\partial \mathbf{e}}, \qquad -\mathbf{q}^{\varepsilon} = \frac{\partial f^{\varepsilon}}{\partial \nabla T}.$$

Now we introduce the stochastically periodic structure as follows: let $(X^{\varepsilon}_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^N}$ be a family of independent random variables defined on a probability space (Ω,Σ,P)

$$(4.7)\qquad \begin{aligned} &P\{\omega\in\Omega:\ \ X^{\varepsilon}_{\mathbf{k}}(\omega)=1\} = c_1,\\ &P\{\omega\in\Omega:\ \ X^{\varepsilon}_{\mathbf{k}}(\omega)=0\} = 1-c_1=c_2 \end{aligned}$$

for every $\varepsilon>0$, $\mathbf{k}\in\mathbb{Z}^N$ and for $c_1\in]0,1[$ fixed.

For every $\varepsilon>0$ and $\mathbf{k}\in\mathbb{Z}^N$, let $Q^{\varepsilon}_{\mathbf{k}}$ be the cube in $\mathbb{R}^N$ defined by

$$(4.8)\qquad Q^{\varepsilon}_{\mathbf{k}} = \{\mathbf{x}\in\mathbb{R}^N:\quad \varepsilon k_i\le x_i<\varepsilon(k_i+1),\quad i=1,...,N\}$$

and denote by $I^{\varepsilon}_{\mathbf{k}}$ its characteristic function.

Furthermore, let us define the stochastically periodic characteristic function

$$\chi^{\varepsilon}(\omega, \mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^N} X^{\varepsilon}_{\mathbf{k}}(\omega) I^{\varepsilon}_{\mathbf{k}}(\mathbf{x}), \qquad \omega \in \Omega,\ \mathbf{x} \in \mathbb{R}^N \tag{4.9}$$

where $I^{\varepsilon}_{\mathbf{k}}(\mathbf{x})$ is periodic in $\mathbf{x}$. More precisely, $I^{\varepsilon}_{\mathbf{k}}$ is the characteristic function of the cube $Q^{\varepsilon}_{\mathbf{k}}$:

$$I^{\varepsilon}_{\mathbf{k}}(\mathbf{u}) = \begin{cases} 1, & \text{if } \mathbf{x} \in Q^{\varepsilon}_{\mathbf{k}}, \\ 0, & \text{otherwise.} \end{cases}$$

We also set

$$\chi^{\varepsilon}(\omega, \mathbf{x}) = \chi\Big(\omega, \frac{\mathbf{x}}{\varepsilon}\Big), \tag{4.10}$$

where $\chi(\omega, \cdot)$ is a 1-periodic function, since

$$I^{\varepsilon}_{\mathbf{k}}(\mathbf{x}) = I^{1}_{\mathbf{k}}\Big(\frac{\mathbf{x}}{\varepsilon}\Big).$$

REMARK 2. If $\mathbf{y} = \mathbf{x}/\varepsilon$, one recovers the microvariable well-known in periodic homogenization [22]. □

For $\varepsilon > 0$ the values of the coefficients $\boldsymbol{\lambda}^{\varepsilon}$, $\boldsymbol{\alpha}^{\varepsilon}$ and $\varkappa^{\varepsilon}$ may be determined by the function $\chi^{\varepsilon}(\omega, \mathbf{x})$ and positive definite moduli $\boldsymbol{\lambda}^{(i)}$, $\boldsymbol{\beta}^{(i)}$, $\boldsymbol{\alpha}^{(i)}$, $\varkappa^{(i)}$, $i = 1, 2$, characterizing each of the two phases we write

$$\begin{aligned}
\boldsymbol{\lambda}^{\varepsilon}(\omega, \mathbf{x}) &= \boldsymbol{\lambda}^{(1)}\chi^{\varepsilon}(\omega, \mathbf{x}) + \boldsymbol{\lambda}^{(2)}(1 - \chi^{\varepsilon}(\omega, \mathbf{x})),\\
\boldsymbol{\beta}^{\varepsilon}(\omega, \mathbf{x}) &= \boldsymbol{\beta}^{(1)}\chi^{\varepsilon}(\omega, \mathbf{x}) + \boldsymbol{\beta}^{(2)}(1 - \chi^{\varepsilon}(\omega, \mathbf{x})),\\
\boldsymbol{\alpha}^{\varepsilon}(\omega, \mathbf{x}) &= \boldsymbol{\alpha}^{(1)}\chi^{\varepsilon}(\omega, \mathbf{x}) + \boldsymbol{\alpha}^{(2)}(1 - \chi^{\varepsilon}(\omega, \mathbf{x})),\\
\varkappa^{\varepsilon}(\omega, \mathbf{x}) &= \varkappa^{(1)}\chi^{\varepsilon}(\omega, \mathbf{x}) + \varkappa^{(2)}(1 - \chi^{\varepsilon}(\omega, \mathbf{x})).
\end{aligned}$$

The moduli $\lambda^{(i)}$, $\beta^{(i)}$, $\alpha^{(i)}$, $\varkappa^{(i)}$ are constant.

Now we are in a position to apply Theorem 1. Particularly, we have

$$(\rho_{\varepsilon}F)(\omega)(\mathbf{u}, T, A) = \int_A f\Big(\omega, \frac{\mathbf{x}}{\varepsilon}, \mathbf{e}(\mathbf{u}); T, \nabla T) d\mathbf{x}, \tag{4.11}$$

where

$$f^{\varepsilon}(\omega, \mathbf{x}, \mathbf{e}(\mathbf{u}); T, \nabla T) = f\Big(\omega, \frac{\mathbf{x}}{\varepsilon}, \mathbf{e}(\mathbf{u}); T, \nabla T\Big),$$

$$\begin{aligned}
f\Big(\omega, \frac{\mathbf{x}}{\varepsilon}, \mathbf{e}(\mathbf{u}); T, \nabla T\Big) &= \frac{1}{2}\Big[\mathbf{e}(\mathbf{u}) - T\boldsymbol{\alpha}\Big(\omega, \frac{\mathbf{x}}{\varepsilon}\Big)\Big]^{\top} \cdot \boldsymbol{\lambda}\Big(\omega, \frac{\mathbf{x}}{\varepsilon}\Big)\\
&\quad \cdot \Big[\mathbf{e}(\mathbf{u}) - T\boldsymbol{\alpha}\Big(\omega, \frac{\mathbf{x}}{\varepsilon}\Big)\Big] + + \frac{1}{2}(\nabla T)^{\top}\varkappa\Big(\omega, \frac{\mathbf{x}}{\varepsilon}\Big)\nabla T.
\end{aligned} \tag{4.12}$$

The Γ-limit of $F^{\varepsilon}_{\varepsilon>0}$ is now given by Theorem 1, where now f^{ε} is defined by (3.12).

The limit functional F_0 is given by a non-random integral functional:

$$(4.13)\quad F_0(\omega)(\mathbf{u},T,A) \equiv \begin{cases} \int\limits_A f_0(\mathbf{e}(\mathbf{u});T,\nabla T)d\mathbf{x} & \text{if } (\mathbf{u},T)\in H^1(A)^3\times H^1(A), \\ +\infty & \text{otherwise.} \end{cases}$$

The integrand $f_0(\cdot;\cdot,\cdot)$ is given by

$$(4.14)\quad f_0(\mathbf{E};\theta,\mathbf{S}) = \lim_{\varepsilon\to 0}\int\limits_{\Omega}\min_{\mathbf{u},T}\Big\{\frac{1}{|Q_{1/\varepsilon}|}\int\limits_{Q_{1/\varepsilon}} f(\omega,\mathbf{x},\mathbf{e}(\mathbf{u});\theta,,\nabla T)d\mathbf{x}dP(\omega)$$
$$|\mathbf{u}=\mathbf{E}\cdot\mathbf{x},\quad T=\mathbf{S}\cdot\mathbf{x}\quad\text{on}\quad \partial Q_{1/\varepsilon},(\mathbf{u},T)\in\mathcal{U}\Big\}$$

where $\mathbf{E}\in\mathbb{E}^N_s$, $\theta\in\mathbb{R}$, $\mathbf{S}\in\mathbb{R}^N$.

REMARK 3. The necessary conditions for the existence of minimum in (4.14) are the following Euler equations

$$(4.15)\quad \left.\begin{aligned} &\operatorname{div}[\boldsymbol{\lambda}(\mathbf{x},\omega)\cdot(\mathbf{e}(\widetilde{\mathbf{u}})+\mathbf{E}-\boldsymbol{\alpha}(\mathbf{x},\omega)\theta)]=0\\ &\operatorname{div}[\varkappa(\mathbf{x},\omega)\cdot(\nabla\widetilde{T}+\mathbf{S})]=0\end{aligned}\right\}\quad\text{in } Q_{1/\varepsilon}$$

with

$$\widetilde{\mathbf{u}}=\mathbf{0}\qquad \widetilde{T}=0\quad\text{on}\quad\partial Q_{1/\varepsilon}.$$

Since the problem is linear, we look for the fields $\widetilde{\mathbf{u}}$ and $\widetilde{T}$ in the following abberative form

$$(4.16)\quad \widetilde{u}_i=-\varphi^{(mn)}_i E_{mn}+\phi_i\theta,\qquad \widetilde{T}=\psi^m S_m.$$

Then

$$(4.17)\quad \begin{aligned} &e_{ij}(\widetilde{\mathbf{u}})=-e_{ij}(\varphi^{(mn)})E_{mn}+e_{ij}(\phi)\theta,\\ &\nabla_i\widetilde{T}=-\partial_i\psi^m Q_m.\end{aligned}$$

Substituting (4.17) into (4.15) we get:

$$(4.18)\quad \begin{cases} -[\partial_i\lambda_{ijkl}e_{kl}(\varphi^{(mn)})]E_{mn}+(\partial_i\lambda_{ijmn})E_{mn}=0 & \forall\mathbf{E}\in\mathbb{E}^N_s\\ -(\partial_i\lambda_{ijkl}\alpha_{kl})\theta+[\partial_i\lambda_{ijmn}e_{mn}(\phi)]\theta=0 & \forall\theta\in\mathbb{R}\\ \partial_i[\varkappa_{ij}(-\partial_j\psi^m)]S_m+\partial_i(\varkappa_{ij})S_i=0 & \forall\mathbf{S}\in\mathbb{R}^N.\end{cases}$$

Hence we get the system of equations posed on $Q_{1/\varepsilon}$

$$\begin{aligned} &\partial_i \lambda_{ijkl} \partial_{(k} \varphi_{l)}^{(mn)} = \partial_i \lambda_{ijmn} \\ (4.19) \qquad &\partial_i \lambda_{ijmn} \partial_{(k} \phi_{l)} = \partial_i \lambda_{ijkl} \alpha_{kl} \\ &\partial_i \varkappa_{ij} \partial_j \psi^m = \partial_i \varkappa_{im} \end{aligned}$$

with the homogeneous boundary conditions on $\partial Q_{1/ve}$ for the unknown fields $\varphi_i^{(mn)}$, ϕ_i, ψ. □

Knowing the solution of the "cell problems" i.e. the functions $\varphi_i^{(mn)}$, ϕ_i, ψ on $\partial Q_{1/\varepsilon}$, which obviously are the functions of $\mathbf{u}$, ω and ε, we obtain the unique fields

$$\begin{aligned} \mathbf{e}(\mathbf{u}) &= \mathbf{e}(\widetilde{\mathbf{u}}) + \mathbf{E}, \\ \nabla T &= \nabla \widetilde{T}_{\mathbf{S}} \end{aligned}$$

and

$$\begin{aligned} (4.20) \quad f(\omega, \mathbf{x}, \mathbf{e}(\mathbf{u}); \theta, \nabla T) &= \frac{1}{2}[(e_{ij} - \alpha_{ij}\theta)\lambda_{ijkl}(e_{kl} - \alpha_{kl}\theta) + \partial_k T \varkappa_{kl} \partial_l T] \\ &= \frac{1}{2}[\lambda_{ijkl}(-\partial_{(k}\varphi_{l)}^{(mn)} E_{mn} + E_{kl} + \partial_{(k}\phi_{l)}\theta - \alpha_{kl}\theta) \\ &\cdot (-\partial_{(i}\varphi_{j)}^{(mn)} E_{mn} + E_{ij} + \partial_{(i}\phi_{j)}\theta - \alpha_{ij}\theta) \\ &+ (-\partial_i \psi^j S_j + S_i)\varkappa_{ik}(-\partial_k \psi^l S_l + S_k)] \\ &= \frac{1}{2}\Big\{\lambda_{ijkl}[(I_{klmn} - \partial_{(k}\varphi_{l)}^{(mn)})E_{mn} + (\partial_{(k}\phi_{l)} - \alpha_{kl})\theta] \\ &\cdot [(I_{ijmn} - \partial_{(i}\varphi_{j)}^{(mn)})E_{mn} + (\partial_{(i}\phi_{j)} - \alpha_{ij})\theta] \\ &\qquad + \varkappa_{ik}[(I_{kl} - \partial_k \psi^l)S_l][(I_{mi} - \partial_m \psi^i)S_m]\Big\}. \end{aligned}$$

The tensors I_{kl} and I_{klmn} are unit tensors in the proper spaces, namely:

$$I_{ij} = \delta_{ij} \qquad I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

Substituting (4.20) into (4.14) we arrive at:

$$(4.21) \qquad f_0(\mathbf{E}; \theta, \mathbf{S}) = \frac{1}{2}[\boldsymbol{\lambda}^* \cdot (\mathbf{E} - \boldsymbol{\alpha}^*\theta) \cdot (\mathbf{E} - \boldsymbol{\alpha}^*\theta) + \boldsymbol{\varkappa}^* \cdot \mathbf{S} \cdot \mathbf{S}],$$

where the *macroscopic moduli* are determined by

$$\boldsymbol{\lambda}^* = \lim_{\varepsilon\to 0}\int_\Omega \frac{1}{|Q_{1/\varepsilon}|}\int_{Q_{1/\varepsilon}} \boldsymbol{\lambda}\cdot(\mathbf{I}-\nabla\varphi)d\mathbf{x}dP(\omega),$$

$$\boldsymbol{\alpha}^* = \lim_{\varepsilon\to 0}\int_\Omega \frac{1}{|Q_{1/\varepsilon}|}\int_{Q_{1/\varepsilon}} \boldsymbol{\lambda}\cdot(\boldsymbol{\alpha}-\nabla\phi)d\mathbf{x}dP(\omega),$$

(4.22)

$$\boldsymbol{\beta}^* = \lim_{\varepsilon\to 0}\int_\Omega \frac{1}{|Q_{1/\varepsilon}|}\int_{Q_{1/\varepsilon}} \boldsymbol{\lambda}\cdot(\mathbf{I}-\nabla\varphi)\cdot\boldsymbol{\alpha}d\mathbf{x}dP(\omega),$$

$$\varkappa^* = \lim_{\varepsilon\to 0}\int_\Omega \frac{1}{|Q_{1/\varepsilon}|}\int_{Q_{1/\varepsilon}} \varkappa\cdot(\mathbf{I}-\nabla\psi)d\mathbf{x}dP(\omega).$$

The above abbreviated notation should be understood as, e.g.,

$$\lambda^*_{ijkl} = \lim_{\varepsilon\to 0}\int_\Omega \frac{1}{|Q_{1/\varepsilon}|}\int_{Q_{1/\varepsilon}} \lambda_{ijmn}(\omega,\mathbf{x})[I_{mnkl} - \partial_{(m}\varphi^{(kl)}_{n)}(\varepsilon,\omega,\mathbf{x})]d\mathbf{x}dP(\omega).$$

Here we have exploited the following properties of the solutions of the cell problems

$$\int_{Q_{1/\varepsilon}} \boldsymbol{\lambda}\cdot(\mathbf{I}-\nabla\varphi)\cdot\nabla\varphi d\mathbf{x} = 0, \qquad \int_{Q_{1/\varepsilon}} \boldsymbol{\lambda}\cdot(\mathbf{I}-\nabla\varphi)\cdot\nabla\phi d\mathbf{x} = 0,$$

$$\int_{Q_{1/\varepsilon}} \boldsymbol{\lambda}\cdot(\mathbf{I}-\nabla\phi)\cdot\nabla\phi d\mathbf{x} = 0, \qquad \int_{Q_{1/\varepsilon}} \boldsymbol{\lambda}\cdot(\mathbf{I}-\nabla\phi)\cdot\nabla\varphi d\mathbf{x} = 0,$$

$$\int_{Q_{1/\varepsilon}} \boldsymbol{\lambda}\cdot(\mathbf{I}-\nabla\psi)\cdot\nabla\psi d\mathbf{x} = 0.$$

One can set: $\mathbf{E} = \mathbf{e}(\mathbf{u})$, $\theta = T$, $\mathbf{S} = \nabla T$ and then the macroscopic potential given by (4.21) yields the macroscopic stresses and heat flux fields in the form:

(4.23)
$$\boldsymbol{\sigma}^0 = \frac{\partial f_0}{\partial \mathbf{e}}, \qquad \mathbf{q}^0 = -\frac{\partial f_0}{\partial \nabla T},$$

where

(4.24)
$$\sigma^0 = \boldsymbol{\lambda}^*\cdot[\mathbf{e}(\mathbf{u}) - \boldsymbol{\alpha}^* T], \qquad q^0 = \varkappa^*\nabla T.$$

5. One-dimensional case

Let assume that all material properties as well as displacement and temperature fields are scalar functions of one variable, denoted further by x. Then for every $\varepsilon > 0$, let $(X_k^\varepsilon)_{k\in Z}$ be a family of independent random variables defined on the probabilistic space (Ω, Σ, P)

(5.1)
$$P\{\omega \in \Omega: \ X_k^\varepsilon(\omega) = 1\} = c_1, \qquad P\{\omega \in \Omega: \ X_k^\varepsilon(\omega) = 0\} = 1 - c_1 = c_2$$

for every $k \in Z$ and for $c_1 \in]0,1[$ fixed.

For every $\varepsilon > 0$ and $k \in Z$, let Q_k^ε be the interval in $\mathbb{R}$ defined by

$$Q_{1/\varepsilon} = \{-j \le x < j\}, \qquad |Q_{1/\varepsilon}| = \frac{1}{2j}. \tag{5.2}$$

Denote by I_k^ε its characteristic function. The stochastically periodic characteristic function is given by

$$\chi^\varepsilon(\omega, x) = \sum_{k\in Z} X_k^\varepsilon(\omega) I_k^\varepsilon(x) \qquad \omega \in \Omega, \ x \in \mathbb{R}, \tag{5.3}$$

where $I_k^\varepsilon(x)$ is periodic in x. Moreover, we set

$$\chi^\varepsilon(\omega, x) = \chi\Big(\omega, \frac{x}{\varepsilon}\Big), \tag{5.4}$$

where $\chi(\omega, \cdot)$ is 1-periodic function, because

$$I_k^\varepsilon(x) = I_k^1\Big(\frac{x}{\varepsilon}\Big).$$

Then the coefficients λ^ε, α^ε and $\varkappa^\varepsilon$ are determined by the function $\chi^\varepsilon(\omega, x)$ and positive constants $\lambda^{(i)}$, $\beta^{(i)}$, $\alpha^{(i)}$, $\varkappa^{(i)}$, $i = 1, 2$, as follows

$$\begin{aligned}
\lambda^\varepsilon(\omega, x) &= \lambda^{(1)}\chi^\varepsilon(\omega, x) + \lambda^{(2)}(1 - \chi^\varepsilon(\omega, x)),\\
\beta^\varepsilon(\omega, x) &= \beta^{(1)}\chi^\varepsilon(\omega, x) + \beta^{(2)}(1 - \chi^\varepsilon(\omega, x)),\\
\alpha^\varepsilon(\omega, x) &= \alpha^{(1)}\chi^\varepsilon(\omega, x) + \alpha^{(2)}(1 - \chi^\varepsilon(\omega, x)),\\
\varkappa^\varepsilon(\omega, x) &= \varkappa^{(1)}\chi^\varepsilon(\omega, x) + \varkappa^{(2)}(1 - \chi^\varepsilon(\omega, x)).
\end{aligned} \tag{5.5}$$

In this case the macroscopic potential is expressed by

$$(5.6)\quad f_0(E;\theta,Q) = \lim_{j\to\infty}\int_\Omega \min_{u,T}\frac{1}{2}\frac{1}{2j}\int_{-j}^{j}[(u'-\alpha(\omega,x)\theta)\lambda(\omega,x) \cdot (u'-\alpha(\omega,x)\theta) + T'\varkappa(\omega,x)T']dx\,dP(\omega),$$

subject to

$$(5.7)\qquad u(-j) = -jE,\qquad u(j) = jE,\qquad T(-j) = -jS,\qquad T(j) = jS.$$

Here $u' = du/dx$, etc. To find the minimum in (5.6) we solve the following Euler equations

$$(5.8)\qquad [\lambda(\omega,x)u'(x) - \beta(\omega,x)\theta]' = 0 \ \text{ and } \ [\varkappa(\omega,x)T'(x)]' = 0,\ \forall x \in]-j;j[$$

with the following boundary conditions

$$(5.9)\qquad \begin{aligned} u(-j) &= -jE, & u(j) &= jE,\\ T(-j) &= -jS, & T(j) &= jS. \end{aligned}$$

After straightforward calculations we get

$$(5.10)\qquad u' = \frac{d_1}{\lambda(\omega,x)} + \alpha(\omega,x)\theta,$$

where

$$(5.11)\qquad d_1 = \Big[2Ej - \theta\int_{-j}^{j}\alpha(\omega,x)dx\Big]\Big(\int_{-j}^{j}\frac{dx}{\lambda(\omega,x)}\Big)^{-1},$$

and

$$(5.12)\qquad T' = \frac{d_2}{\varkappa(\omega,x)},$$

$$(5.13)\qquad d_2 = 2Qj\Big(\int_{-j}^{j}\frac{dx}{\varkappa(\omega,x)}\Big)^{-1}.$$

Hence

$$\int_\Omega \min_{u,T}\Big\{\frac{1}{2}\frac{1}{2j}\int_{-j}^{j}[(u'-\alpha(\omega,x)\theta)\lambda(\omega,x)(u'-\alpha(\omega,x)\theta)+T'\varkappa(\omega,x)T']dxdP(\omega)$$

$$|u(-j)=-jE,\ \ u(j)=jE,\ \ T(-j)=-jS,\ \ T(j)=jS\Big\}$$

$$=\frac{1}{2}\frac{1}{2j}\Big[\Big(\int_{-j}^{j}[\lambda(\omega,x)]^{-1}dx\Big)^{-1}\Big(2Ej-\theta\int_{-j}^{j}\alpha(\omega,x)dx\Big)^2$$

$$+\Big(\int_{-j}^{j}[\varkappa(\omega,x)]^{-1}dx\Big)^{-1}S^2j\Big]$$

$$=\frac{1}{2}\Big\{\Big[\frac{1}{2j}\sum_{k=-j}^{j}\Big(\lambda^{(1)}X_k^\varepsilon(\omega)I_k^1(x)+\lambda^{(2)}(1-X_k^\varepsilon(\omega)I_k^1(x))\Big)^{-1}\Big]^{-1}E^2$$

$$+2\Big[\frac{1}{2j}\sum_{k=-j}^{j}\Big(\lambda^{(1)}X_k^\varepsilon(\omega)I_k^1(x)+\lambda^{(2)}(1-X_k^\varepsilon(\omega)I_k^1(x))\Big)^{-1}\Big]^{-1}E$$

$$\cdot\Big[\frac{1}{2j}\sum_{k=-j}^{j}\Big(\alpha^{(1)}X_k^\varepsilon(\omega)I_k^1(x)+\alpha^{(2)}(1-X_k^\varepsilon(\omega)I_k^1(x))\Big)\Big]\theta$$

$$+\Big[\frac{1}{2j}\sum_{k=-j}^{j}\Big(\lambda^{(1)}X_k^\varepsilon(\omega)+\lambda^{(2)}(1-X_k^\varepsilon(\omega))\Big)^{-1}\Big]^{-1}$$

$$\cdot\Big[\frac{1}{2j}\sum_{k=-j}^{j}\Big(\alpha^{(1)}X_k^\varepsilon(\omega)I_k^1(x)+\alpha^{(2)}(1-X_k^\varepsilon(\omega)I_k^1(x))\Big)\Big]^2\theta^2$$

$$+\Big[\frac{1}{2j}\sum_{k=-j}^{j}\Big(\varkappa^{(1)}X_k^\varepsilon(\omega)I_k^1(x)+\varkappa^{(2)}(1-X_k^\varepsilon(\omega)I_k^1(x))\Big)^{-1}\Big]^{-1}S^2\Big\}$$

for any $\varepsilon\in\Omega$.

Taking into account that $(X_k^\varepsilon)_{k\in Z}$ is the family of independent random variables for every ε, $I_k^1(x)$ is one-periodic characteristic function and applying the strong law of large numbers [9] to the limit $j\to\infty$, we obtain:

(5.14) $f_0(E;\theta,S)$

$$=(c_1\lambda_1^{-1}+c_2\lambda_2^{-1})^{-1}E^2-2(c_1\lambda_1^{-1}+c_2\lambda_2^{-1})^{-1}(c_1\alpha_1+c_2\alpha_2)E\theta$$

$$+(c_1\lambda_1^{-1}+c_2\lambda_2^{-1})^{-1}(c_1\alpha_1+c_2\alpha_2)^2\theta^2+(c_1\varkappa_1^{-1}+c_2\varkappa_2^{-1})^{-1}S^2.$$

The final form of the integrand f_0 is expressed by

$$f_0(E;\theta,S) = \lambda^* E^2 - 2\lambda^*\alpha^* E\theta + \lambda^*\alpha^{*2}\theta^2 + \varkappa^* S^2, \tag{5.15}$$

where

$$\lambda^* = (c_1\lambda_1^{-1} + c_2\lambda_2^{-1})^{-1}, \qquad \beta^* = \lambda^*\alpha^*$$
$$\alpha^* = c_1\alpha_1 + c_2\alpha_2, \qquad \varkappa^* = (c_1\varkappa_1^{-1} + c_2\varkappa_2^{-1})^{-1}.$$

6. Final remarks

Except cases such as one-dimensional, one has to resort to approximation of effective moduli. Elaboration of upper and lower bounds for effective thermoelastic moduli remains open. Though media with random microstructure were studied by many authors, see for instance TORQUATO [23], yet application of mathematically rigorous homogenization methods seem to be at its very beginning. The lack of lucid bounding methods is thus not surprising. From Theorem 1 and its specific form applied to two-phase composites with random microstructure we conclude that the method applied is not applicable to nonstationary thermoelasticity.

To solve the problem of stochastic homogenization of equations of coupled, nonstationary thermoelasticity, one has to use either the method of G-convergence or the stochastic two-scale convergence in the mean, cf. [2,3,15,18].

The general stochastic homogenization Theorem 1 was applied to classical, linear, stationary thermoelasticity. This theorem can also be used to physically nonlinear thermoelestic materials with random microstructures provided the the deformations are small.

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