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A dynamic asymptotic model of linear elastic orthotropic plates: first and second-order terms

A. SŁAWIANOWSKA and J. J. TELEGA

Institute of Fundamental Technological Research,
Polish Academy of Sciences,
Świętokrzyska 21, 00-049 Warsaw, Poland,
e-mail : aslawian@ippt.gov.pl; jtelega@ippt.gov.pl

BY USING the asymptotic method combined with variational approach a new dynamic model of elastic orthotropic plates is derived. This model accounts for rotational inertia and takes into account second order terms $\mathbf{U}^2$ and $\boldsymbol{\sigma}^2$ in the asymptotic expansions of displacements and stresses. To avoid the boundary layer analysis, the boundary conditions for $\mathbf{U}^2$ are satisfied in a suitable averaged sense. Convergence theorem is formulated. Influence of the second-order terms on plate response in the static case is investigated.

1. Introduction

IN OUR RECENT PAPERS [19, 20] we derived a new static model of thin, linear elastic orthotropic plates. In essence, this model takes into account the second-order terms $\mathbf{U}^2$, and $\boldsymbol{\sigma}^2$ of the asymptotic expansions of displacements and stresses. For clamped plates the term $\mathbf{U}^2$ does not satisfy the boundary condition $\mathbf{U}^2 = 0$ on that part of the boundary where the plate is clamped. To overcome this inherent difficulty Destuynder investigated the boundary layer, cf. also DAUGE and GRUAIS [4]. An alternative approach was proposed by RAOULT [15, 16] for isotropic plates. To cope with the second-order term $\mathbf{U}^2$ of the asymptotic expansion, this author appropriately defined the boundary conditions. Their precise, average form appears in the definition of the set X of admissible displacements, cf. formulae (3.2), (3.3). In this manner one avoids the study of boundary layer. In our papers [19, 20] the approach due to RAOULT [16] is extended to orthotropic plates in the static case.

The asymptotic approach to construction of models of rods, beams, plates and shells is usually confined to finding the zero-order terms, which correspond to equations obtained in a more standard manner in engineering literature, cf. [2, 5, 12-20, 23]. However, even in this case no a priori assumptions, like the Kirchhoff – Love hypotheses, are needed. The asymptotic approach to modelling of structures has already a long history, cf. the comments in [2, 10, 13, 17, 22]. The book by LEWIŃSKI and TELEGA [13] summarizes the research on finding effective models of linear and nonlinear structures with microstructures. In this case the asymptotic methods also play a crucial role.

In the present contribution the static model elaborated in [19,20] is extended so as to include the inertia term. A new dynamic model of linear elastic orthotropic plates is derived. An important feature of the model is that the second-order terms $\mathbf{U}^2$ and $\boldsymbol{\sigma}^2$ of asymptotic expansions of displacements and stresses are taken into account. We follow the procedure exploited in the static case thus avoiding the analysis typical for the study of boundary layer. Anyway, our approach is an alternative one. The formal asymptotic procedure is justified by proving a convergence theorem. To exhibit a significant influence of second-order terms on the plate response, the circular isotropic plate is investigated in the static case.

2. Basic equations and scalings

Let $\Omega \subset \mathbb{R}^2$ be the mid-plane of the plate and 2ε its thickness. Here $\varepsilon > 0$ is treated as a small parameter. In the underformed state the plate occupies the region $\overline{\mathcal{B}^\varepsilon} = \overline{\Omega} \times [-\varepsilon, \varepsilon]$. We set: $\Gamma = \partial\Omega$, $\Gamma_0^\varepsilon = \Gamma \times [-\varepsilon, \varepsilon]$, $\Gamma_\pm^\varepsilon = \Omega \times \{\pm\varepsilon\}$. Throughout the paper the Cartesian coordinate system is used, except Sec.8.

It is assumed that the plate is made of a linear elastic material with the density ${}^\varepsilon\varrho$. In the case of orthotropy the elasticity tensor $\mathbf{C} = (C_{ijkl})$ is such that $C_{iijj} \neq 0$, $C_{klkl} \neq 0$ for $k \neq l$; (no summation over repeated indices), the remaining coefficients C_{ijkl} vanish. Roman indices take values in $\{1, 2, 3\}$. The basic equations are given by

$$(2.1) \qquad {}^\varepsilon\varrho\, {}^\varepsilon U_i'' = \partial_j\, {}^\varepsilon\sigma_{ij} + {}^\varepsilon f_i \qquad \text{in } \mathcal{B}^\varepsilon \times [0, T],$$

$$(2.2) \qquad {}^\varepsilon\sigma_{ij} = C_{ijkl}\, \gamma_{kl}({}^\varepsilon\mathbf{U}),$$

$$(2.3) \qquad \gamma_{ij}({}^\varepsilon\mathbf{U}) = \frac{1}{2}(\partial_i\, {}^\varepsilon U_j + \partial_j\, {}^\varepsilon U_i),$$

$$(2.4) \qquad {}^\varepsilon\mathbf{U} = \mathbf{0} \qquad \text{on } \Gamma_0^\varepsilon \times [0, T],$$

$$(2.5) \qquad {}^\varepsilon\mathbf{U}(\mathbf{x}, 0) = {}^\varepsilon\widetilde{\mathbf{U}}(\mathbf{x}), \qquad {}^\varepsilon\mathbf{U}'(\mathbf{x}, 0) = {}^\varepsilon\widetilde{\mathbf{V}}(\mathbf{x}), \qquad \mathbf{x} \in \mathcal{B}^\varepsilon,$$

$$(2.6) \qquad {}^\varepsilon\sigma_{i3} = \pm\, {}^\varepsilon g_i \qquad \text{on } \Gamma_\pm^\varepsilon \times [0, T].$$

The quantities ${}^{\varepsilon}\sigma_{ij}$, ${}^{\varepsilon}\mathbf{U}$ and $\gamma_{ij}({}^{\varepsilon}\mathbf{U})$ denote the stress tensor, the displacement vector and the linearized strain tensor, respectively. Obviously, these quantities depend on the space coordinate $\mathbf{x}$ and time t. Here $(\cdot)' = \partial(\cdot)/\partial t$, etc. The initial data ${}^{\varepsilon}\widetilde{\mathbf{U}}$, ${}^{\varepsilon}\widetilde{\mathbf{V}}$ are prescribed. To use the method of asymptotic expansion, [2,3,12-20], it is convenient to work with the fixed domain, say $\mathcal{B} = \Omega \times (-1,1)$. To this end, for $\varepsilon > 0$ we define the mapping, cf. [3],

$$F^{\varepsilon} : \mathbf{x} = (x^1, x^2, x^3) \in \mathcal{B} \longrightarrow F^{\varepsilon}(\mathbf{x}) = (x^1, x^2, \varepsilon x^3) = \mathbf{x}^{\varepsilon} \in \mathcal{B}^{\varepsilon}. \tag{2.7}$$

Let us introduce a composition $(\varphi \circ F^{\varepsilon}) : \mathcal{B} \longrightarrow \mathbb{R}^3$ defined as follows

$$\forall \mathbf{x}^{\varepsilon} = F^{\varepsilon}(\mathbf{x}), \qquad (\varphi \circ F^{\varepsilon})(\mathbf{x}) = \varphi(\mathbf{x}^{\varepsilon}), \quad \text{where} \quad \varphi : \mathcal{B}^{\varepsilon} \longrightarrow \mathbb{R}^3. \tag{2.8}$$

One can easily verify that the quantities $\boldsymbol{\sigma}^{\varepsilon}$, $\mathbf{U}^{\varepsilon}$, etc., defined in $\mathcal{B}$, are interrelated with ${}^{\varepsilon}\boldsymbol{\sigma}$, ${}^{\varepsilon}\mathbf{U}$, etc., defined in $\mathcal{B}^{\varepsilon}$, as follows:

$$U^{\varepsilon}_{\alpha} = {}^{\varepsilon}U_{\alpha} \circ F^{\varepsilon}, \qquad U^{\varepsilon}_3 = \varepsilon\,({}^{\varepsilon}U_3 \circ F^{\varepsilon}), \qquad \varrho = \varepsilon^{-2}\,({}^{\varepsilon}\varrho \circ F^{\varepsilon}), \tag{2.9}$$

$$\sigma^{\varepsilon}_{\alpha\beta} = {}^{\varepsilon}\sigma_{\alpha\beta} \circ F^{\varepsilon}, \qquad \sigma^{\varepsilon}_{\alpha 3} = \varepsilon^{-1}\,({}^{\varepsilon}\sigma_{\alpha 3} \circ F^{\varepsilon}), \qquad \sigma^{\varepsilon}_{33} = \varepsilon^{-2}\,({}^{\varepsilon}\sigma_{33} \circ F^{\varepsilon}), \tag{2.10}$$

$$\begin{aligned} f^{o}_{\alpha} &= {}^{\varepsilon}f_{\alpha} \circ F^{\varepsilon}, \qquad & f^{o}_3 &= \varepsilon^{-1}\,({}^{\varepsilon}f_3 \circ F^{\varepsilon}),\\ g^{o}_{\alpha} &= \varepsilon^{-1}\,({}^{\varepsilon}g_{\alpha} \circ F^{\varepsilon}), \qquad & g^{o}_3 &= \varepsilon^{-2}\,({}^{\varepsilon}g_3 \circ F^{\varepsilon}). \end{aligned} \tag{2.11}$$

Throughout the paper Greek indices take values in $\{1,2\}$. In the ε-independent domain $\mathcal{B}$ we write $\mathbf{U}^{\varepsilon}(\mathbf{x},t)$ and $\boldsymbol{\sigma}^{\varepsilon}(\mathbf{x},t)$ for displacements and stresses.

3. The variational formulation of the plate dynamics

Prior to passing to the variational formulation, we introduce the spaces of stresses and displacements, cf. [15, 16, 19, 20],

$$\Sigma = L^2(\mathcal{B}, \mathbb{E}^3_s), \tag{3.1}$$

$$X = X_{12} \times X_3, \tag{3.2}$$

where

$$\begin{aligned} X_{12} &= \Big\{\mathbf{V} \in H^1(\mathcal{B})^2 : \int_{-1}^{1} V_{\alpha}\, dx_3 = 0, \quad \int_{-1}^{1} x_3\, V_{\alpha} n_{\alpha}\, dx_3 = 0 \quad \text{on } \Gamma\Big\},\\ X_3 &= \Big\{V_3 \in H^1(\mathcal{B}) : \int_{-1}^{1} (1 - x_3^2)\, V_3\, dx_3 = 0 \quad \text{on } \Gamma\Big\}, \qquad \alpha = 1,2. \end{aligned} \tag{3.3}$$

Here $\mathbb{E}^3_s$ denotes the space of symmetric 3×3 matrices and $H^1(\mathcal{B})^3 = [H^1(\mathcal{B})]^3$.

For a function $f(\mathbf{x}, t)$ $(\mathbf{x} \in \mathcal{B})$ we shall often write $f(t) = \{f(\mathbf{x}, t) | \mathbf{x} \in \mathcal{B}\}$ and similarly if $\mathbf{x} \in \Omega$, cf. [16]. Such a notation is common in mathematical studies of evolution problems.

Throughout the paper it is assumed that $f^0_\alpha(\mathbf{x}, t) = f^0_\alpha(x_1, x_2, t)$.

Let $(\mathbf{x}, t) \in \mathcal{B} \times (0, T)$, where $\mathcal{B} = \Omega \times (-1, 1)$. After rescaling (2.9) – (2.11), problem (2.1) – (2.6) is formulated in the variational form, see [16, 20].

PROBLEM $(\mathcal{P}^\varepsilon)$

Find $(\boldsymbol{\sigma}^\varepsilon, \mathbf{U}^\varepsilon) \in L^\infty(0, T; \Sigma \times X)$, such that $\mathbf{U}^{\varepsilon\prime} \in L^\infty(0, T; L^2(\mathcal{B})^3)$ and

$$\forall \boldsymbol{\tau} \in \Sigma, \quad A^\varepsilon(\boldsymbol{\sigma}^\varepsilon, \boldsymbol{\tau}) + B(\boldsymbol{\tau}, \mathbf{U}^\varepsilon) = 0, \tag{3.4}$$

$$\forall \mathbf{V} \in X, \quad -\varrho\,(U^{\varepsilon\,\prime\prime}_3, V_3) - \varrho\,\varepsilon^2\,(U^{\varepsilon\,\prime\prime}_\alpha, V_\alpha) + B(\boldsymbol{\sigma}^\varepsilon, \mathbf{V}) = F^0(\mathbf{V}), \tag{3.5}$$

$$\mathbf{U}^\varepsilon(\mathbf{x}, 0) = \widetilde{\mathbf{U}}^\varepsilon(\mathbf{x}), \qquad \mathbf{U}^{\varepsilon\prime}(\mathbf{x}, 0) = \widetilde{\mathbf{V}}^\varepsilon(\mathbf{x}), \qquad \mathbf{x} \in \mathcal{B}. \tag{3.6}$$

Here $(\cdot, \cdot)$ denotes the duality $(\mathcal{D}'(0, T; L^2(\mathcal{B})), L^2(\mathcal{B}))$. For the definition and properties of functional spaces used in this paper the reader is referred to the book by ADAMS [1].

If $U^{\varepsilon\,\prime\prime}_\alpha(\cdot, t) \in L^2(\mathcal{B})$ then

$$(U^{\varepsilon\,\prime\prime}_\alpha, V_\alpha) = \int_{\mathcal{B}} U^{\varepsilon\,\prime\prime}_\alpha\, V_\alpha\, d\mathbf{x}. \tag{3.7}$$

The forms A^ε and B are defined as follows, cf. [16, 19, 20],

$$\forall \{\boldsymbol{\sigma}, \boldsymbol{\tau}\} \in \Sigma \times \Sigma, \qquad A^\varepsilon(\boldsymbol{\sigma}, \boldsymbol{\tau}) = A^0(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \varepsilon^2\, A^2(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \varepsilon^4\, A^4(\boldsymbol{\sigma}, \boldsymbol{\tau}), \tag{3.8}$$

$$\forall \boldsymbol{\sigma} \in \varSigma, \quad \forall \mathbf{V} \in H^1(\mathcal{B})^3, \qquad B(\boldsymbol{\sigma}, \mathbf{V}) = -\int_{\mathcal{B}} \gamma_{ij}(\mathbf{V})\, \sigma_{ij}\, d\mathbf{x}, \tag{3.9}$$

where $\mathbf{A} = \mathbf{C}^{-1}$. The functional F^0 of external forces is given by

$$F^0(\mathbf{V}) = -\int_{\mathcal{B}} f^0_i\, V^i\, d\mathbf{x} - \int_{\varGamma_\pm} g^0_i\, V^i\, d\varGamma. \tag{3.10}$$

Obviously, the initial data $\widetilde{\mathbf{U}}^\varepsilon$ and $\widetilde{\mathbf{V}}^\varepsilon$ are prescribed, and $\Gamma_\pm = \varOmega \times \{\pm 1\}$. The bilinear form A^ε is easily derived from a bilinear form defined on the $\mathcal{B}^\varepsilon$ and using Eqs. (2.9), (2.10), cf. [16, 19, 20].

It can easily be shown that

$$(3.11)\qquad \begin{aligned} A^0(\boldsymbol{\sigma},\boldsymbol{\tau}) &= (A_{\alpha\beta\gamma\delta}\,\sigma_{\gamma\delta},\,\tau_{\alpha\beta}), \qquad A^4(\boldsymbol{\sigma},\boldsymbol{\tau}) = (A_{3333}\,\sigma_{33},\,\tau_{33}),\\ A^2(\boldsymbol{\sigma},\boldsymbol{\tau}) &= (A_{\alpha\beta33}\,\sigma_{33},\,\tau_{\alpha\beta}) + 2(A_{\alpha3\delta3}\,\sigma_{3\delta},\,\tau_{3\alpha}) + (A_{33\gamma\delta}\,\sigma_{\gamma\delta},\,\tau_{33}). \end{aligned}$$

Here $(\cdot,\,\cdot)$ denotes the scalar product in $L^2(\mathcal{B})$.

Similarly to the static case we assume the asymptotic expansions of $\{\boldsymbol{\sigma}^\varepsilon,\,\mathbf{U}^\varepsilon\}$ as follows:

$$(3.12)\quad \boldsymbol{\sigma}^\varepsilon = \boldsymbol{\sigma}^0 + \varepsilon^2\boldsymbol{\sigma}^2 + \cdots,$$

$$(3.13)\quad \mathbf{U}^\varepsilon = \mathbf{U}^0 + \varepsilon^2\mathbf{U}^2 + \cdots.$$

Performing now the asymptotic analysis, i.e. substituting (3.12) into (3.4) and (3.5), we arrive at problem $(\mathcal{P}^0)$, linked with $\{\boldsymbol{\sigma}^0,\mathbf{U}^0\}$, and problem $(\mathcal{P}^2)$, linked with $\{\boldsymbol{\sigma}^2,\mathbf{U}^2\}$, cf. [15, 16, 19, 20].

The solution $\{\boldsymbol{\sigma}^2,\mathbf{U}^2\}$ of problem $(\mathcal{P}^2)$ yields the *first corrector* to $\{\boldsymbol{\sigma}^0,\mathbf{U}^0\}$. It is well-known that $\mathbf{U}^2$ does not satisfy, in general, the homogeneous boundary condition on Γ_0^1, [4, 16, 19]. We recall that $\mathbf{U}^0$ vanishes on $\Gamma_0^1 = \Gamma\times[-1,1]$.

In subsequent sections we shall examine both the problem $(\mathcal{P}^0)$ and the problem $(\mathcal{P}^2)$. The choice of the space X for kinematically admissible displacements will prove to be crucial. We observe that the boundary conditions involved in definition (3.2),(3.3) of the space X are satisfied only in an averaged sense.

REMARK 1. Let us assume that $f_i^0\in L^2(0,T;L^2(\mathcal{B}))$ and $g_i^{0\,\pm}\in L^2(0,T;L^2(\Gamma_\pm))$, $\widetilde{\mathbf{U}}^\varepsilon\in X$, $\widetilde{\mathbf{V}}^\varepsilon\in L^2(\mathcal{B})^3$. Then the solution $\{\mathbf{U}^\varepsilon,\boldsymbol{\sigma}^\varepsilon\}$ of problem $(\mathcal{P}^\varepsilon)$ exists and is unique; moreover, $\mathbf{U}^{\varepsilon\,\prime\prime}\in L^2(0,T;X')$, where X' denotes the dual space of X. For the proof the reader is referred to [6].

4. Study of Problem $(\mathcal{P}^0)$

Identifying the terms linked with ε^0 in problem $(\mathcal{P}^\varepsilon)$ we obtain problem $(\mathcal{P}^0)$, which includes a classical dynamic problem for orthotropic plates. The plate problem reads:

Find $\{\boldsymbol{\sigma}^0,\mathbf{U}^0\}\in L^\infty(0,T;\Sigma\times X)$ such that $U_3^{0\,\prime}\in L^\infty(0,T;\,L^2(\mathcal{B}))$ and

$$(4.1)\qquad \forall\,\boldsymbol{\tau}\in\Sigma,\quad A^0(\boldsymbol{\sigma}^0,\boldsymbol{\tau}) + B(\boldsymbol{\tau},\mathbf{U}^0) = 0,$$

$$(4.2)\qquad \forall\,\mathbf{V}\in X,\quad -\varrho\,(U_3^{0\,\prime\prime},\,V_3) + B(\boldsymbol{\sigma}^0,\mathbf{V}) = F^0(\mathbf{V}),$$

$$(4.3)\qquad U_3^0(0) = \widetilde{U}\,{}_3^0,\qquad U_3^{0\,\prime}(0) = \widetilde{V}\,{}_3^0.$$

We recall that $\mathbf{U}^0 = (U^0_\alpha, U^0_3)$. The in-plane displacements U^0_α are specified below. To proceed further, we introduce two spaces

(4.4) $$X_{KL} = \{\, \mathbf{U} \,|\ \ \gamma_{\alpha 3}(\mathbf{U}) = 0, \quad \gamma_{33}(\mathbf{U}) = 0\,\},$$

(4.5) $$S = \{\, \boldsymbol{\tau} \in \Sigma \,|\ \tau_{\alpha 3} = 0, \quad \tau_{33} = 0\,\}.$$

The existence result for problem $(\mathcal{P}^0)$ is classical, cf. [16].

PROPOSITION 1. If $F^0(\mathbf{V})$ is given by (3.10) and if

$$f^0_3 \in H^1(0,T; L^2(\mathcal{B})), \quad f^0_\alpha \in L^\infty(0,T;\, L^2(\Omega)), \quad g^0_i \in H^1(0,\, T; L^2(\Gamma_+ \cup \Gamma_-)),$$

then problem $(\mathcal{P}^0)$ has a unique solution $\{\mathbf{U}^0, \boldsymbol{\sigma}^0\}$. □

Now we consider problem $(\mathcal{P}^0 f)$, that is a particular case of $(\mathcal{P}^0)$ where $\{\boldsymbol{\sigma}^0, \mathbf{U}^0\} \in L^\infty(0,T; S \times X_{KL})$. From Eqs. (4.1) – (4.3), proceeding similarly to the static case, we find

$$U^0_3 \in L^\infty(0,\, T; H^2_0(\Omega)), \quad U^{0\prime}_3 \in L^\infty(0,\, T; L^2(\Omega)),$$

(4.6) $$2\varrho\, U^{0\prime\prime}_3 + \frac{2}{3} D_{\alpha\beta\gamma\delta}\, \partial_{\alpha\beta\gamma\delta}\, U^0_3 = \int\limits_{-1}^{1} f^0_3\, dx_3 \,+\, g^{0+}_3 \,+\, g^{0-}_3 \,+\, \partial_\alpha(g^{0+}_\alpha - g^{0-}_\alpha);$$

$$\alpha, \beta, \gamma, \delta = 1, 2,$$

(4.7) $$U^0_3(0) = \widetilde{U}^0_3, \qquad U^{0\prime}_3(0) = \widetilde{V}^0_3,$$

(4.8) $$U^0_\alpha = u^0_\alpha - x_3\, \partial_\alpha U^0_3,$$

where

(4.9) $$D_{\alpha\beta\gamma\delta} \,=\, C_{\alpha\beta\gamma\delta} - C_{\alpha\beta 33}\, C^{-1}_{3333}\, C_{33\gamma\delta}.$$

Here $\mathbf{u}^0$ is the unique solution of the plane problem:

find $\mathbf{u}^0 \in L^\infty(0,T; H^1_0(\Omega)^2)$ such that

(4.10) $$[\, K\, \mathbf{u}^0,\, \mathbf{v}] = F^0(-\mathbf{v},\, 0), \qquad \forall \mathbf{v} \in L^\infty(0,T; H^1_0(\Omega)^2),$$

where

$$[K\,\mathbf{u},\,\mathbf{v}] := 2\,[D_{\alpha\beta\gamma\delta}\,\gamma_{\gamma\delta}(\mathbf{u}),\,\gamma_{\alpha\beta}(\mathbf{v})], \tag{4.11}$$

and $[\cdot,\cdot]$ denotes the scalar product on $L^2(\Omega)$.

We observe that in the case of transverse homogeneity, the problem $(\mathcal{P}^0)$ (and problem $(\mathcal{P}^2)$ as well) splits into a plane elasticity problem and a plate problem, similarly to the static case considered in [16, 19, 20].

From Eq.(4.1) we conclude that

$$\sigma^0_{\alpha\beta} = D_{\alpha\beta\lambda\delta}\,\gamma_{\lambda\delta}(\mathbf{U}^0). \tag{4.12}$$

Passing to the full problem $(\mathcal{P}^0)$ we have to find the formulas for $\sigma^0_{\alpha 3}$ and σ^0_{33}. To this end we use *Lemma A*, formulated in the Appendix. In the dynamic problem considered, the inertia term $(-\varrho U_3^{0\,\prime\prime})$ is to be added to f_3^0, the third component of the body force vector $\mathbf{f}^0$. Proceeding then similarly to the static case we finally get

$$\sigma^0_{\alpha 3} = -\frac{1}{2}(1-x_3^2)\,D_{\alpha\beta\lambda\delta}\,\partial_{\beta\lambda\delta}\,U_3^0 + \frac{(1+x_3)}{2}g_\alpha^{0+} - \frac{(1-x_3)}{2}g_\alpha^{0-}, \tag{4.13}$$

$$\begin{aligned}\sigma^0_{33} = \varrho U_3^{0\,\prime\prime}\frac{(x_3^3-x_3)}{2} - \int_{-1}^{x_3} f_3^0\,dx_3 + \frac{2+3x_3-x_3^3}{4}\int_{-1}^{1} f_3^0\,dx_3 \\ + \frac{g_3^{0+}-g_3^{0-}}{2} + \frac{(3x_3-x_3^3)(g_3^{0+}+g_3^{0-})}{4} \\ + \frac{(1-x_3^2)}{4}\,\partial_\alpha(g_\alpha^{0+}+g_\alpha^{0-}) + x_3\frac{(1-x_3^2)}{4}\,\partial_\alpha(g_\alpha^{0+}-g_\alpha^{0-}).\end{aligned} \tag{4.14}$$

Formula (4.13) for the stresses $\sigma^0_{\alpha 3}$ is formally similar to the static case; however, now U_3^0 and $\sigma^0_{\alpha 3}$ depend also on time. If $U_3^{0\,\prime\prime}$ vanishes then we recover the formula for σ^0_{33} known from the static analysis, [16, 19, 20].

5. Study of Problem $(\mathcal{P}^2 f)$

Identifying the terms of $A^\varepsilon(\boldsymbol{\sigma}^\varepsilon,\boldsymbol{\tau})$ linked with ε^2 in $(\mathcal{P}^\varepsilon)$ we get problem $(\mathcal{P}^2)$ and its truncated form $(\mathcal{P}^2 f)$, cf. [15, 6, 19, 20]. Here we limit ourselves to the second problem.

Find $(\boldsymbol{\sigma}^2,\mathbf{U}^2) \in L^\infty(0,T;\,S\times X)$ such that $U_3^{2\,\prime} \in L^\infty(0,T;L^2(\mathcal{B}))$ and

$$\forall\,\boldsymbol{\tau}\in\Sigma, \qquad A^0(\boldsymbol{\sigma}^2,\boldsymbol{\tau}) + B(\boldsymbol{\tau},\mathbf{U}^2) = -A^2(\boldsymbol{\sigma}^0,\boldsymbol{\tau}), \tag{5.1}$$

$$(5.2)\qquad \forall \mathbf{V} \in X_{KL}, \qquad -\varrho(U_3^{2\prime\prime}, V_3) + B^2(\boldsymbol{\sigma}^2, \mathbf{V}) = \varrho(U_\alpha^{0\prime\prime}, V_\alpha),$$

$$(5.3)\qquad U_3^2(0) = \widetilde{U}_3^2, \qquad U_3^{2\prime}(0) = \widetilde{V}_3^2.$$

We observe that for $\boldsymbol{\sigma}^2 \in S$ and $\boldsymbol{\sigma}^0 \notin S$ we have

$$A^0(\boldsymbol{\sigma}^2, \boldsymbol{\tau}) = ({D_{\alpha\beta\gamma\delta}}^{-1}\, \sigma^2_{\gamma\delta}, \tau_{\alpha\beta}),$$

$$A^2(\boldsymbol{\sigma}^0, \boldsymbol{\tau}) = (A_{\alpha\beta33}\, \sigma^0_{33}, \tau_{\alpha\beta}),$$

provided that $\boldsymbol{\tau} \in S$.

Let us pass to characterization of problem $(\mathcal{P}^2 f)$. If the external forces $\{\mathbf{f}^0, \mathbf{g}^0\}$ are sufficiently regular, see Sec.6, and if the initial conditions are of the form

$$(5.4)\quad \widetilde{U}_3^2 = \widetilde{w}^2 + A_{33\alpha\beta}\, D_{\alpha\beta\gamma\delta}\left[x_3\, \gamma_{\gamma\delta}(\mathbf{u}^0(0)) - \frac{1}{2}\, x_3^2\, \partial_{\gamma\delta}\, \widetilde{w}^0\right], \qquad \widetilde{w}^2 \equiv \widetilde{u}_3^2,$$

$$(5.5)\qquad \widetilde{V}_3^2 = \widetilde{v}_3^2 + A_{33\alpha\beta}\, D_{\alpha\beta\gamma\delta}\left[x_3\, \gamma_{\gamma\delta}(\mathbf{u}^{0\prime}(0)) - \frac{1}{2}\, x_3^2\, \partial_{\gamma\delta}\, \widetilde{v}_3^0\right],$$

with, see [19, 20],

$$(5.6)\qquad \widetilde{w}^2 \in H^2(\Omega), \quad \widetilde{w}^2 = \frac{1}{10}\, D_{\alpha\beta\lambda\mu}\, A_{33\alpha\beta}\, \partial_{\lambda\mu}\widetilde{w}^0 \qquad \text{on } \Gamma \times (0, T),$$

$$(5.7)\quad \partial_n \widetilde{w}^2 = \frac{1}{10}\, A_{33\alpha\beta}\, D_{\alpha\beta\lambda\mu}\, \partial_n\, \partial_{\lambda\mu}\widetilde{w}^0 - \frac{8}{10} \sum_{\alpha=1,2} A_{\alpha 3\alpha 3}\, D_{\alpha\alpha\gamma\delta}\, \partial_n\, \partial_{\gamma\delta}\, \widetilde{w}^0 \qquad \text{on } \Gamma \times (0, T),$$

then there exists the unique solution $\{\boldsymbol{\sigma}^2, \mathbf{U}^2\} \in L^\infty(0, T; S \times X)$, $U_3^{2\prime} \in L^\infty(0, T; L^2(\mathcal{B}))$ of problem $(\mathcal{P}^2 f)$.

The displacement (U_α^2, U_3^2) is given by

$$(5.8)\qquad U_3^2 = w^2 + A_{33\alpha\beta}\, D_{\alpha\beta\gamma\delta}\left[x_3\, \gamma_{\gamma\delta}(\mathbf{u}^0) - \frac{1}{2}\, x_3^2\, \partial_{\gamma\delta}\, w^0\right], \qquad w^2 \equiv u_3^2,$$

$$(5.9)\quad U_\alpha^2 = u_\alpha^2 - x_3\, \partial_\alpha\, w^2 - A_{\alpha 3\delta 3}\, (x_3 - \frac{1}{3}\, x_3^3)\, D_{\delta\beta\gamma\kappa}\, \partial_{\beta\gamma\kappa}\, w^0 - \frac{1}{2}\, A_{33\xi\zeta}\, D_{\xi\zeta\gamma\delta}\, [x_3^2\, \partial_\alpha\, \gamma_{\gamma\delta}(\mathbf{u}^0) - \frac{1}{3}\, x_3^3\, \partial_{\alpha\gamma\delta}\, w^0],$$

where $w^2 \in L^\infty(0,T;\, H^2(\Omega))$ and $w^{2\prime} \in L^\infty(0,T; L^2(\Omega))$.

Formally, the formulas for the components of displacement vector $\mathbf{U}^2$ are the same as in the static case, [19, 20]. Now, however,all quantities describing the dynamic problem depend also on time.

For the membrane components of the stress tensor we have the expression, cf. [20],

$$\sigma^2_{\gamma\delta} = D_{\alpha\beta\gamma\delta}\, \gamma_{\alpha\beta}\,(\mathbf{U}^2) - A_{\alpha\beta 33}\, D_{\alpha\beta\gamma\delta}\, \sigma^0_{33}. \tag{5.10}$$

Substituting (5.8) into (5.2) and integrating the obtained equation twice by parts, we get the formulation of the plate bending equation in problem $(\mathcal{P}^2 f)$ in the form

$$\begin{aligned}(5.11)\quad & 2\varrho\, w^{2\prime\prime} + \frac{2}{3} D_{\alpha\beta\gamma\delta}\, \partial_{\alpha\beta\gamma\delta}\, w^2 \\ &= -\Big(\frac{3}{10} D_{\alpha\beta\gamma\delta}\, A_{\gamma\delta 33} + \frac{8}{10} \sum_{\gamma=1,2} D_{\alpha\beta\gamma\gamma}\, A_{\gamma 3\gamma 3}\Big)\, \partial_{\alpha\beta}\Big[\int\limits_{-1}^{1} f^0_3 + g^{0+}_3 + g^{0-}_3 \\ &\quad + \partial_\alpha (g^{0+}_\alpha - g^{0-}_\alpha)\Big] + D_{\alpha\beta\gamma\delta}\, A_{\gamma\delta 33}\, \partial_{\alpha\beta} \int\limits_{-1}^{1} x_3\, dx_3 \int\limits_{-1}^{x_3} f^0_3 d\widetilde{x}_3 + \frac{2}{3}\varrho\, \partial_{\alpha\alpha} w^{0\prime\prime} \\ &\quad + \varrho\, \partial_{\alpha\beta} w^{0\prime\prime} \Big[\frac{4}{15} D_{\alpha\beta\gamma\delta}\, A_{\gamma\delta 33} + \frac{8}{5} \sum_{\gamma=1,2} D_{\alpha\beta\gamma\gamma}\, A_{\gamma 3\gamma 3}\Big] \quad \text{in } \Omega \times (0, T),\end{aligned}$$

where the boundary and initial conditions are

$$w^2 = \frac{1}{10}\, D_{\alpha\beta\lambda\mu}\, A_{33\alpha\beta}\, \partial_{\lambda\mu} w^0 \qquad \text{on } \Gamma \times (0, T), \tag{5.12}$$

$$\partial_n w^2 = \frac{1}{10} A_{33\alpha\beta} D_{\alpha\beta\lambda\mu} \partial_n\, \partial_{\lambda\mu} w^0 - \frac{8}{10} \sum_{\alpha=1,2} A_{\alpha 3\alpha 3}\, D_{\alpha\alpha\gamma\delta} \partial_n\, \partial_{\gamma\delta}\, w^0 \qquad \text{on } \Gamma \times (0, T), \tag{5.13}$$

$$w^2(0) = \widetilde{w}^2, \qquad w^{2\prime}(0) = \widetilde{v}^2_3. \tag{5.14}$$

The dynamic membrane problem resulting from $(\mathcal{P}^2 f)$ means finding

$$\mathbf{u} \in L^\infty(0, T;\, H^1(\Omega)^2)$$

such that

$$(5.15)\quad K\,\mathbf{u}^2 = -2\varrho\,\mathbf{u}^{0\,\prime\prime} + \frac{1}{3}\,A_{33\xi\zeta}\,D_{\xi\zeta\gamma\delta}\,\partial_\beta\,\triangle\,\gamma_{\gamma\delta}(\mathbf{u}^0) + A_{\gamma\delta 33}\,D_{\alpha\beta\gamma\delta}\,\partial_\beta \int_{-1}^{1} \sigma^0_{33}\,\delta_{\alpha\beta}\,dx_3,$$

$$(5.16)\quad \mathbf{u}^2 \in L^\infty\,(0,T;\,H^1(\Omega)^2), \qquad u^2_\alpha = \frac{1}{6}\,A_{33\lambda\mu}\,D_{\lambda\mu\gamma\delta}\,\partial_\alpha\,\gamma_{\gamma\delta}(\mathbf{u}^0) \quad \text{on } \Gamma\times(0,T).$$

6. Convergence study

The aim of this section is to prove the second-order convergence of displacements and stresses. Precise meaning of such a convergence follows from Theorem 1. Primarily, however, we formulate the following result, being a slight modification of Corollary 2 proved in Raoult [16].

PROPOSITION 2. Let

$$(6.1)\qquad \psi = \int_{-1}^{1} f^0_3\,dx_3 + g^{0+}_3 + g^{0-}_3 + \partial_\alpha\,(g^{0+}_\alpha - g^{0-}_\alpha),$$

(i)
$$f^0_\alpha \in H^3(0,T;\,H^1(\Omega)),\quad g^{0+}_\alpha \in H^3(0,T;\,H^1(\Gamma^+)),\quad g^{0-}_\alpha \in H^3(0,T;\,H^1(\Gamma^-)),$$

(ii)
$$f^0_3 \in H^3(0,T;\,L^2(\mathcal{B})),\quad g^{0+}_3 \in H^3(0,T;\,L^2(\Gamma^+)),\quad g^{0-}_3 \in H^3(0,T;\,L^2(\Gamma^-)),$$

(iii)
$$\widetilde{U}^0_3 \in H^4(\Omega)\cap H^2_0(\Omega),$$

(iv),
$$\psi(x,0) - \frac{2}{3}\,D_{\alpha\beta\lambda\mu}\,\partial_{\alpha\beta\lambda\mu}\,\widetilde{U}^0_3 \in H^3(\Omega)\cap H^2_0(\Omega),$$

$$(v) \qquad \widetilde{V}_3^0 \in H^4(\Omega) \cap H_0^2(\Omega),$$

$$(vi) \qquad \psi'(x,0) - \frac{2}{3} D_{\alpha\beta\lambda\mu}\, \partial_{\alpha\beta\lambda\mu} \widetilde{V}_3^0 \in H_0^1(\Omega),$$

then

$$\text{(a)} \qquad \begin{gathered} U_3^0 \in C^1(0,T; H^4(\Omega) \cap H_0^2(\Omega)), \quad U_3^{0\,\prime\prime} \in C^0(0,T; H^3(\Omega) \cap H_0^2(\Omega)), \\ U_3^{0\,\prime\prime\prime} \in C^0(0,T; H_0^1(\Omega)), \end{gathered}$$

$$\text{(b)} \qquad u_\alpha^0 \in H^3(0,T;\, H^3(\Omega) \cap H_0^1(\Omega). \qquad \square$$

In the proof of convergence we shall also exploit a priori estimates.

PROPOSITION 3. Under the assumptions of Proposition 2. and if the initial data of problems $(\mathcal{P}^\varepsilon)$ and $(\mathcal{P}^0)$ are such that

$$(6.2) \qquad \|(\sigma_{\alpha\beta}^\varepsilon - \sigma_{\alpha\beta}^0)(0)\| \le C\varepsilon^2, \quad \|(\sigma_{\alpha3}^\varepsilon - \sigma_{\alpha3}^0)(0)\| \le C\varepsilon, \quad \|\sigma_{33}^\varepsilon(0)\| \le C,$$

$$(6.3) \qquad \|U_3^{\varepsilon\,\prime}(0) - w^{0\,\prime}(0)\| \le C\varepsilon^2, \quad \|U_\alpha^{\varepsilon\,\prime}(0) - U_\alpha^{0\,\prime}(0)\| \le C\varepsilon,$$

where $\|\cdot\| = \|\cdot\|_{L^2}$ and C is a generic positive constant, then

$$(6.4) \quad \varepsilon^{-2}(\sigma_{\alpha\beta}^\varepsilon - \sigma_{\alpha\beta}^0),\ \varepsilon^{-1}(\sigma_{\alpha3}^\varepsilon - \sigma_{\alpha3}^0),\ \sigma_{33}^\varepsilon \quad \text{are bounded in } L^\infty(0,T; L^2(\mathcal{B})),$$

$$(6.5) \qquad \varepsilon^{-2}(\mathbf{U}^\varepsilon - \mathbf{U}^0) \quad \text{is bounded in } L^\infty(0,T;X),$$

$$(6.6) \quad \varepsilon^{-2}(U_3^{\varepsilon\,\prime} - w^{0\,\prime}), \quad \varepsilon^{-1}(U_\alpha^{\varepsilon\,\prime} - U_\alpha^{0\,\prime}) \quad \text{are bounded in } L^\infty(0,T; L^2(\mathcal{B})).$$

P r o o f. The proof does not differ from a similar one performed by RAOULT [15, 16] for isotropic plates.

REMARK 2. The assumptions appearing in Proposition 3. can be formulated exclusively in terms of the data of problems $(\mathcal{P}^\varepsilon)$ and $(\mathcal{P}^0)$. More precisely, by using the constitutive equations involved in these problems and if the initial data are such that

$$(6.7) \qquad \|\widetilde{\mathbf{U}}^\varepsilon - \mathbf{U}^0(0)\|_{H^1} \le C\varepsilon^2,$$

(6.8) $\|\varepsilon^{-2}\gamma_{\alpha 3}(\widetilde{\mathbf{U}}^{\varepsilon}) - A_{\alpha 3\alpha 3}\sigma^0_{\alpha 3}(w^0(0))\|_{L^2} \le C\varepsilon$, (no summation over α),

where

$$\sigma^0_{\alpha 3}(w^0) = -\frac{(1-x_3^2)}{2} D_{\alpha\beta\lambda\mu}\,\partial_{\beta\lambda\mu}\, w^0 + \frac{(1+x_3)}{2} g^{0+}_{\alpha} - \frac{(1-x_3)}{2} g^{0-}_{\alpha},$$

(6.9) $$\|\varepsilon^{-2}\gamma_{33}(\widetilde{\mathbf{U}}^{\varepsilon}) - \gamma_{33}(w^0(0))\|_{L^2} \le C\varepsilon^2,$$

where

$$\gamma_{33}(w^0(0)) = -C^{-1}_{3333} \sum_{\mu=1,2} C_{\mu\mu 33}\,\gamma_{\mu\mu}(\mathbf{u}^0(0)),$$

(6.10) $$\|\widetilde{V}^{\varepsilon}_3 - w^{0\,\prime}(0)\|_{L^2} \le C\varepsilon^2, \quad \|\widetilde{V}^{\varepsilon}_{\alpha} - U^{0\,\prime}_{\alpha}(0)\|_{L^2} \le C\varepsilon,$$

then (6.4) – (6.6) are satisfied. □

After these preparations we can formulate the second-order convergence theorem.

THEOREM 1. *Under the assumptions of Proposition 2. and if the data of problems* $(\mathcal{P}^{\varepsilon})$ *and* $(\mathcal{P}^0)$ *are such that*

(i) $$\varepsilon^{-2}(\widetilde{\mathbf{U}} - \mathbf{U}^0(0))$$

converges in $H^1(\mathcal{B})^3$ *to an element* $\mathbf{U}^2$ *of the form*

(6.11) $$\widetilde{U}^2_3 = \widetilde{w}^2 + A_{33\alpha\beta}\, D_{\alpha\beta\gamma\delta}\left[x_3\,\gamma_{\gamma\delta}(\mathbf{u}^0(0)) - \frac{1}{2}\,x_3^2\,\partial_{\gamma\delta}\,\widetilde{w}^0\right],$$

with $\widetilde{w}^2 \in H^2(\Omega)$, *and*

$$\widetilde{w}^2 = \frac{1}{10}\, D_{\alpha\beta\lambda\mu}\, A_{33\alpha\beta}\,\partial_{\lambda\mu}\widetilde{U}^0_3 \qquad \text{on } \Gamma,$$

$$\partial_n\widetilde{w}^2 = \frac{1}{10}\, A_{33\alpha\beta}\, D_{\alpha\beta\lambda\mu}\,\partial_n\,\partial_{\lambda\mu}\widetilde{U}^0_3 - \frac{8}{10}\sum_{\alpha=1,2} A_{\alpha 3\alpha 3}\, D_{\alpha\alpha\gamma\delta}\,\partial_n\,\partial_{\gamma\delta}\,\widetilde{U}^0_3 \qquad \text{on } \Gamma,$$

(6.12) $$\widetilde{U}^2_{\alpha} = \widetilde{u}^2_{\alpha} - x_3\,\partial_{\alpha}\, w^2 - A_{\alpha 3\delta 3}\,(x_3 - \frac{1}{3}x_3^3)\, D_{\delta\beta\gamma\kappa}\,\partial_{\beta\gamma\kappa}\, w^0 - \frac{1}{2}\, A_{33\xi\zeta}\, D_{\xi\zeta\gamma\delta}\left[x_3^2\,\partial_{\alpha}\,\gamma_{\gamma\delta}(\mathbf{u}^0) - \frac{1}{3}\,x_3^3\,\partial_{\alpha\gamma\delta}\, w^0\right],$$

where $\widetilde{u}_\alpha^2 \in H^1(\Omega)$ *is given by*

$$(6.13)\quad K\widetilde{\mathbf{u}}^2 = -2\varrho\,\mathbf{u}^{0\,\prime\prime} + \frac{1}{3}\,A_{33\xi\zeta}\,D_{\xi\zeta\gamma\delta}\,\partial_\beta\,\triangle\,\gamma_{\gamma\delta}(\mathbf{u}^0) + A_{\gamma\delta 33}\,D_{\alpha\beta\gamma\delta}\,\partial_\beta\int_{-1}^{1}\sigma_{33}^0\,\delta_{\alpha\beta}\,dx_3 \quad in\ \Omega,$$

with

$$\widetilde{\mathbf{u}}_\alpha^2 = \frac{1}{6}\,A_{33\lambda\mu}\,D_{\lambda\mu\gamma\delta}\,\partial_\alpha\,\gamma_{\gamma\delta}(\mathbf{u}^0(0) \quad on\ \Gamma,$$

$$\text{(ii)}\quad \begin{aligned} &\varepsilon^{-1}(\sigma_{\alpha3}^\varepsilon(0) - \sigma_{\alpha3}^0(0)) \longrightarrow 0 \quad in\ L^2(\mathcal{B}),\\ &\sigma_{33}^\varepsilon(0) \longrightarrow \sigma_{33}^0(0) \quad in\ L^2(\mathcal{B}),\end{aligned}$$

$$\text{(iii)}\quad \begin{aligned} &\varepsilon^{-1}(\widetilde{V}_\alpha^\varepsilon - U_\alpha^{0\,\prime}(0)) \longrightarrow 0 \quad in\ L^2(\mathcal{B}),\\ &\varepsilon^{-2}(\widetilde{V}_3^\varepsilon - \widetilde{V}_3^0) \longrightarrow \widetilde{V}_3^2 \quad in\ L^2(\mathcal{B}),\end{aligned}$$

with $\widetilde{V}_3^2$ *of the form*

$$(6.14)\qquad \widetilde{V}_3^2 = \widetilde{v}_3^2 + A_{33\alpha\beta}\,D_{\alpha\beta\gamma\delta}\left[x_3\,\gamma_{\gamma\delta}(\mathbf{u}^{0\,\prime}(0)) - \frac{1}{2}\,x_3^2\,\partial_{\gamma\delta}\,\widetilde{v}_3^0\right],$$

then we have

$$\begin{aligned}
&\varepsilon^{-2}(\mathbf{U}^\varepsilon - \mathbf{U}^0) \longrightarrow \mathbf{U}^2 \quad in\ L^2(0,T;\ X),\\
&\varepsilon^{-1}(U_\alpha^\varepsilon - U_\alpha^0)' \longrightarrow 0 \quad in\ L^2(0,T;\ L^2(\mathcal{B})),\\
&\varepsilon^{-2}(U_3^\varepsilon - U_3^0)' \longrightarrow U_3^{2\,\prime} \quad in\ L^2(0,T;\ L^2(\mathcal{B})),\\
&\varepsilon^{-2}(\sigma_{\alpha\beta}^\varepsilon - \sigma_{\alpha\beta}^0) \longrightarrow \sigma_{\alpha\beta}^2 \quad in\ L^2(0,T;\ L^2(\mathcal{B})),\\
&\varepsilon^{-1}(\sigma_{\alpha3}^\varepsilon - \sigma_{\alpha3}^0) \longrightarrow 0 \quad in\ L^2(0,T;\ L^2(\mathcal{B})),\\
&(\sigma_{33}^\varepsilon - \sigma_{33}^0) \longrightarrow 0 \quad in\ L^2(0,T;\ L^2(\mathcal{B})),
\end{aligned}$$

where $((\sigma_{\alpha\beta}^2, 0, 0), \mathbf{U}^2)$ *is the unique solution to problem* $(\mathcal{P}^2 f)$ *with the initial data* $\widetilde{U}_3^2$, $\widetilde{V}_3^2$.

P r o o f. It is divided into two major steps. First, one establishes the weak convergence by using Proposition 3. Next the strong convergence is demonstrated by showing the convergence of norms, i.e.,

$$\int_0^T \Big[\varrho\varepsilon^{-4}\|\overline{U}_3^{\varepsilon\prime}\|_{L^2}^2 + \varrho\varepsilon^{-2}\|\overline{U}_\alpha^{\varepsilon\prime}\|_{L^2}^2 + A(\tilde{\boldsymbol{\sigma}}^\varepsilon, \tilde{\boldsymbol{\sigma}}^\varepsilon)\Big]dt \longrightarrow \int_0^T \Big[\varrho\|U_3^{2\prime}\|_{L^2}^2 + A(\tilde{\boldsymbol{\sigma}}^2, \tilde{\boldsymbol{\sigma}}^2)\Big]dt,$$

where

$$A(\boldsymbol{\sigma}, \boldsymbol{\tau}) = (A^0 + A^2 + A^4)(\boldsymbol{\sigma}, \boldsymbol{\tau}),$$
$$\overline{\mathbf{U}}^\varepsilon = \mathbf{U}^\varepsilon - \mathbf{U}^0, \qquad \overline{\boldsymbol{\sigma}}^\varepsilon = \boldsymbol{\sigma}^\varepsilon - \boldsymbol{\sigma}^0,$$
$$\tilde{\boldsymbol{\sigma}}^\varepsilon = (\varepsilon^{-2}\overline{\sigma}^\varepsilon_{\alpha\beta},\ \varepsilon^{-1}\overline{\sigma}^\varepsilon_{\alpha3},\ \overline{\sigma}^\varepsilon_{33}), \quad \tilde{\boldsymbol{\sigma}}^2 = (\sigma^2_{\alpha\beta}, 0, 0).$$

Otherwise the proof runs similarly to the one devised by RAOULT [16] for isotropic plates. However, formula (148) in [16] is to be replaced by

$$2\,w^{2\prime}(0) - \frac{1}{3}A_{33\gamma\delta}\,D_{\gamma\delta\alpha\beta}\,\partial_{\alpha\beta}\,U_3^{0\prime}(0) = 2\,\widetilde{v}_3^2 - \frac{1}{3}A_{33\gamma\delta}\,D_{\gamma\delta\alpha\beta}\,\partial_{\alpha\beta}\widetilde{V}_3^0,$$

where $U_3^{2\prime}(0) = \widetilde{V}_3^2$.

REMARK 3. Conditions (6.13 (ii)) can be formulated in terms of the data of problems $(\mathcal{P}^\varepsilon)$ and $(\mathcal{P}^0)$ as follows:

$$\varepsilon^{-1}\left[\varepsilon^{-2}\gamma_{\alpha3}(\widetilde{\mathbf{U}}^\varepsilon) - \gamma_{\alpha3}(\widetilde{\mathbf{U}}^0)\right] \longrightarrow 0,$$

where $\gamma_{\alpha3} = A_{\alpha3\alpha3}\,\sigma^0_{\alpha3}(U_3^0)$ (no summation over α),

$$\varepsilon^{-2}\Big[\sum_{\mu=1,2} C_{\mu\mu33}\,\gamma_{\mu\mu}(\widetilde{\mathbf{U}}^\varepsilon) + \varepsilon^{-2}\,C_{3333}\,\gamma_{33}(\widetilde{U}_3^\varepsilon)\Big] \longrightarrow -\frac{x_3^3 - x_3}{6}D_{\alpha\beta\gamma\delta}\,\partial_{\alpha\beta\gamma\delta}\,\widetilde{U}_3^0 + \left\{\frac{(x_3+1)}{2}\int_{-1}^{1} f_3^0 - \int_{-1}^{x_3} f_3^0 + \frac{(1-x_3^2)}{4}\,\partial_\alpha(g_\alpha^{0+} + g_\alpha^{0-}) + \frac{g_3^{0+} - g_3^{0-}}{2} + \frac{x_3(g_3^{0+} + g_3^{0-})}{2}\right\}(0).$$

7. Interpretation of the asymptotic approach for the dynamic problem

Let ${}^{\varepsilon}\varrho = \varepsilon^2\, \varrho, \quad {}^{\varepsilon}g_3^+ = \varepsilon^2\, g_3^+$, see [20]. According to (2.9) we write

$$ {}^{\varepsilon}U_3\ (x_1, x_2, \varepsilon\, x_3, t) = \varepsilon^{-1}\ U_3^{\varepsilon}\ (x_1, x_2, x_3, t). $$

Consequently, for sufficiently small ε we have:

$$ \varepsilon^{-2}({}^{\varepsilon}U_3\ -\ Z_3^0)\ (x_1, x_2, 0, t) \simeq z_3^2\ (x_1, x_2, t), \quad z_3^0(x_1, x_2, t) = Z_3^0(x_1, x_2, 0, t). $$

Here Z_3^0 is the deflection in the real plate with the thickness $2h$. Such a plate is subjected to the gravity forces with the density ${}^{h}\varrho$ and the load ${}^{h}g_3^+$, acting on the upper face. The mid-plane deflection is

$$ (7.1)\quad {}^{h}r_3\ =\ z_3^0\ +\ h^2 z_3^2, \qquad \text{where } z_3^0\ =\ z_3^0\,(x_1, x_2, t), \qquad z_3^2\ =\ z_3^2\,(x_1, x_2, t). $$

The following dynamic plate equations are satisfied by ${}^{h}r$:

$$ (7.2)\quad 2h\ {}^{h}\varrho\ {}^{h}r_3{}'' + \frac{2}{3}\, h^3 D_{\alpha\beta\gamma\delta}\, \partial_{\alpha\beta\gamma\delta}\, {}^{h}r_3\ -\ {}^{h}\varrho\, h^3 \Big[\frac{2}{3}\, \triangle z_3^{0\,\prime\prime} + \partial_{\alpha\beta}\, z_3^{0\,\prime\prime} \Big(\frac{4}{15} D_{\alpha\beta\gamma\delta}\, A_{\gamma\delta 33} + \frac{8}{5} D_{\alpha\beta\gamma\gamma}\, A_{\gamma 3\gamma 3}\Big)\Big] = -2h\ {}^{h}\varrho\ +\ {}^{h}g_3^+, \ \text{ in } \Omega \times (0,\, T). $$

The term in rectangular brackets is the *corrector term*. The boundary conditions implied by the analysis of the second-order term $\{\boldsymbol{\sigma}^2, \mathbf{U}^2\}$ are:

$$ (7.3)\qquad {}^{h}r_3\ = \frac{1}{10}\ h^2\, D_{\alpha\beta\lambda\mu} A_{33\alpha\beta}\ \partial_{\lambda\mu} z_3^0 \qquad \text{on } \Gamma \times (0,\, T), $$

$$ (7.4)\quad \partial_n\ {}^{h}r_3\ =\ \frac{1}{10}\ h^2 A_{33\alpha\beta}\ D_{\alpha\beta\lambda\mu}\ \partial_n\ \partial_{\lambda\mu} z_3^0 - \frac{8}{10}\, h^2 \sum_{\alpha=1,2} A_{\alpha 3\alpha 3}\ D_{\alpha\alpha\gamma\delta}\ \partial_n\ \partial_{\gamma\delta} z_3^0 \quad \text{on } \Gamma \times (0,\, T). $$

Obviously, $z_3^0 \in L^{\infty}(0, T; H_0^2(\Omega))$ is a solution to

$$ (7.5)\quad 2h\ {}^{h}\varrho\, z_3^{0\,\prime\prime} + \frac{2}{3}\ h^3\ D_{\alpha\beta\lambda\mu}\ \partial_{\alpha\beta\lambda\mu}\ z_3^0\ = -\, 2h\, {}^{h}\varrho + {}^{h}g_3^+ \qquad \text{in } \Omega \times (0,\, T), $$

$$ (7.6)\qquad z_3^0(0) = \widetilde{Z}_3^0, \quad z_3^{0\,\prime}(0) = 0 \qquad \text{in } \Omega. $$

We have additionally assumed that the initial velocity of all points of the plate is zero, or ${}^{e}\widetilde{\mathbf{V}} = \mathbf{0}$. The initial conditions for Eq.(7.2) are specified by

$$(7.7) \qquad {}^{h}r_3\,(0) = \widetilde{Z}_3^0 + h^2 \widetilde{z}_3^2, \qquad {}^{h}r_3{}'\,(0) = 0.$$

The initial data $\widetilde{Z}_3^0$ and $\widetilde{Z}_3^2$ are determined by performing the static analysis, see (5.6). Thus we assume that at the beginning of a dynamic process, i.e. for $t = 0$, we have

$$(7.8) \qquad \begin{aligned} &{}^{h}A({}^{h}\boldsymbol{\sigma}(0), \boldsymbol{\tau}) + {}^{h}B(\boldsymbol{\tau}, {}^{h}\mathbf{U}(0)) = 0, \\ &{}^{h}B({}^{h}\boldsymbol{\sigma}(0), \mathbf{V}) = {}^{h}F(0)(\mathbf{V}). \end{aligned}$$

Now, to find $\widetilde{Z}_3^0$ we have to solve the problem

$$(7.9) \qquad \widetilde{Z}_3^0 \in H_0^2(\Omega), \qquad \frac{2}{3} D_{\alpha\beta\gamma\delta}\, \partial_{\alpha\beta\gamma\delta}\, \widetilde{Z}_3^0 = -2\varrho + g_3^{0+}(0).$$

To get the initial quantity $\widetilde{z}_3^2$, we have to consider the boundary-value problem

$$(7.10) \qquad \frac{2}{3} D_{\alpha\beta\gamma\delta}\, \partial_{\alpha\beta\gamma\delta}\, \widetilde{z}_3^2 = 0 \quad \text{in } \varOmega,$$

$$(7.11) \qquad \widetilde{z}_3^2 = \frac{1}{10} D_{\alpha\beta\lambda\mu} A_{33\alpha\beta} \partial_{\lambda\mu} \widetilde{Z}_3^0 \qquad \text{on } \varGamma,$$

$$(7.12) \qquad \partial_n\, \widetilde{z}_3^2 = \frac{1}{10} A_{33\alpha\beta}\, D_{\alpha\beta\lambda\mu}\, \partial_n\, \partial_{\lambda\mu} \widetilde{Z}_3^0 - \frac{8}{10} \sum_{\alpha=1,2} A_{\alpha 3\alpha 3}\, D_{\alpha\alpha\gamma\delta}\, \partial_n\, \partial_{\gamma\delta} \widetilde{Z}_3^0 \quad \text{on } \Gamma.$$

In the case of isotropic plates, Eqs.(7.2)-(7.6) reduce to those derived earlier by RAOULT [16]

$$(7.13) \qquad 2\,{}^{h}\varrho\, h\, {}^{h}r_3'' + \frac{2}{3}\,\frac{E}{1-\nu^2}\, h^3 \triangle^2\, {}^{h}r_3 = -2h\, {}^{h}\varrho + {}^{h}g_3^{+} + {}^{h}\varrho\, h^3\, \frac{34 - 14\nu}{15(1-\nu)} \triangle\, z_3^{0}{}'' \quad \text{in } \varOmega \times (0, T),$$

$$(7.14) \qquad {}^{h}r_3 = -h^2\, \frac{\nu}{10\,(1-\nu)} \triangle z_3^0 \qquad \text{on } \varGamma \times (0, T),$$

$$\partial_n\,{}^h r_3 = -h^2\,\frac{8+\nu}{10(1-\nu)}\,\partial_n \triangle z_3^0 \qquad \text{on } \Gamma\times(0,\,T), \tag{7.15}$$

$$ {}^h r_3\,(0) = \widetilde{Z}_3^0 + h^2 \widetilde{z}_3^2, \qquad {}^h r_3\,'\,(0) = 0. \tag{7.16}$$

Here $z_3^0 \in L^\infty(0,T;\, H_0^2(\Omega))$ is a solution to

$$2h\ {}^h\varrho\, z_3^{0\,\prime\prime} + \frac{2}{3}\,\frac{E}{1-\nu^2}\,h^3\,\triangle^2 z_3^0 = -\,2h\,{}^h\varrho + {}^h g_3^+ \quad \text{in } \Omega\times(0,\,T), \tag{7.17}$$

$$z_3^0(0) = \widetilde{Z}_3^0, \quad z_3^{0\,\prime}(0) = 0 \quad \text{in } \Omega. \tag{7.18}$$

In this simple case of loading, the in-plane components of the displacement in the real plate vanish. Indeed, in such a case Eq. (4.5) yields $u_\alpha^0 = 0$, and since $[K\,\mathbf{u}^2,\,\mathbf{v}] = 0$, we conclude that $u_\alpha^2 = 0,\ \alpha = 1,2$.

8. Illustrative example: evaluation of the influence of the first corrector in the static case

In the present section we shall illustrate quantitatively the influence of the second-order asymptotic term or the *first corrector* on the plate deflection. We shall analyse only the static boundary-value problem for circular plates. Thus all variables do not depend on time and the inertial terms are neglected. Also, initial conditions are obviously absent.

The asymptotic theory is applied to a circular *isotropic* plate of thickness $2h$, of the radius R, and subjected to the uniform vertical load g_3 acting on Γ_+^h. Let $r_3(\rho)$ be the deflection of the plate mid-plane accounting for the *corrector*. In the polar coordinates $(\rho,\,\theta)$ we have

$$r_3(\rho, x_3)_{|x_3=0} = r_3(\rho,0) = r_3 = w^0(\rho,0) + h^2 w^2(\rho,0). \tag{8.1}$$

To find r^3 we have to solve the following boundary-value problem:

$$D\,\frac{1}{\rho}\frac{d}{d\rho}\Big\{\rho\frac{d}{d\rho}\Big[\frac{1}{\rho}\frac{d}{d\rho}\Big(\rho\frac{d\,r_3}{d\rho}\Big)\Big]\Big\} = g_3 \qquad \text{in } \Omega, \tag{8.2}$$

$$r_3 = -\frac{h^2\nu}{10(1-\nu)}\,\Big(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}\Big)w^0 \qquad \text{on } \Gamma, \tag{8.3}$$

$$\partial_n\, r_3 = -\frac{h^2(8+\nu)}{10(1-\nu)}\,\partial_n\,\Big(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}\Big)w^0 \qquad \text{on } \Gamma. \tag{8.4}$$

Here E is the the Young modulus, ν denotes the Poisson ratio, $D = \dfrac{2Eh^3}{3(1-\nu^2)}$, ρ is the radial running coordinate in the polar coordinate system.

First, we solve the classical bending problem

$$D\frac{1}{\rho}\frac{d}{d\rho}\Big\{\rho\frac{d}{d\rho}\Big[\frac{1}{\rho}\frac{d}{d\rho}\Big(\rho\frac{dw^0}{d\rho}\Big)\Big]\Big\} = g_3 \qquad \text{in } \Omega, \tag{8.5}$$

$$w^0 = 0, \qquad \frac{dw^0}{d\rho} = 0 \qquad \text{on } \Gamma. \tag{8.6}$$

It is known that the solution of Eqs.(8.5),(8.6) has the form [22]

$$w^0 = \frac{g_3(R^2-\rho^2)^2}{64D}. \tag{8.7}$$

Taking into account (8.7) in (8.3) and (8.4) we get

$$r_3 = -\frac{h^2\nu}{10(1-\nu)}\cdot\frac{g_3 R^2}{8D} \qquad \text{on } \Gamma, \tag{8.8}$$

$$\partial_n r_3 = -\frac{h^2(8+\nu)}{10(1-\nu)}\cdot\frac{g_3 R}{2D} \qquad \text{on } \Gamma, \tag{8.9}$$

since $\Big(\dfrac{d^2}{d\rho^2} + \dfrac{1}{\rho}\dfrac{d}{d\rho}\Big)w^0 = \dfrac{g_3 R^2}{8D}$ on Γ.

Finally, solving Eq.(8.2) with the boundary conditions (8.8) and (8.9), we obtain

$$r_3 = r_3(\rho) = \frac{g_3(R^2-\rho^2)^2}{64D} + \frac{g_3 h^2}{80D(1-\nu)}\big[R^2(16+\nu) - 2\rho^2(8+\nu)\big]. \tag{8.10}$$

For the central point of the plate, where $\rho = 0$, we get the following formulae:

$$r_3 = r_3(\rho) = \frac{g_3 R^4}{64D} + \frac{g_3 h^2 R^2(16+\nu)}{80D(1-\nu)}, \tag{8.11}$$

$$w^0(0) = \frac{3R^4(1-\nu^2)}{128\, h^3}\cdot g_3/E, \tag{8.12}$$

$$h^2 w^2(0) = \frac{3R^2(1+\nu)(16+\nu)}{160\, h}\cdot g_3/E. \tag{8.13}$$

Table 1. Influence of the *first corrector* on deflection of clamped circular isotropic plate, $\rho = 0$ (centre of the plate)

h/R	$\frac{h^2 w^2}{w^0}$ if $\nu = 0.3$	$\frac{h^2 w^2}{w^0}$ if $\nu = 0.45$
1/20	4.7%	6.0%
1/15	8.3%	10.6%
1/10	18.6%	23.9%

Let us present the above results in the form of Table 1. This table shows a relative participation of the *first corrector* included in $r_3(0)$.

For $\rho = R$ the boundary conditions for the deflections w^0, and w^2, are specified by

$$w^0 = 0, \qquad w^2 = -\frac{g_3 R^2}{80D} \cdot \frac{\nu}{(1-\nu)} \neq 0 \quad \text{on } \Gamma.$$

From Table 1 we conclude that the influence of w^2 may be significant, depending on the Poisson ratio, the thickness and the radius of the plate.

9. Comments on related papers

Refined theories of plates can be derived either with the use of asymptotic expansions or by assuming suitable displacements or stress hypotheses. The aim of this section is to perform a comparison of our results with the approaches used in the papers [7, 8, 9, 11, 21]. In our case the displacement distribution is obtained by using asymptotics. In contrast to the approach used in [9,11,21], the asymptotic method does not require any assumptions on the kinematics.

The papers [9, 21] are devoted to dynamics of thick and moderately thick plates, respectively. In [9] the kinematical assumptions due to Hencky are used:

$$(9.1)\quad u_\alpha(x_i,t) = x_3\,\phi_\alpha(x_\beta,t), \quad u_3(x_i,t) = w(x_\alpha,t), \quad \alpha,\beta = 1,2, \quad i = 1,2,3,$$

where ϕ_α denotes rotation of the plate cross-sections. In the equations describing the free vibrations plate problem appears the term: $-4\,\varrho h^3/3(1-\nu)\cdot A\,\triangle w''$. It coincides with the rotational inertia term obtained in our paper for the isotropic case, see Eq.(7.13), provided that $A = (17-7\nu)/10$. We observe that in [9] four alternate values of A are cited, for which the obtained solution was discussed.

In [21] a special form for shear stresses (not for displacements) was assumed. Moderately thick plate on an elastic foundation yields the equation describing free vibrations of the plate. In this governing equation, a term similar to our second order term(the *corrector*)

$$\frac{34-14\nu}{15(1-\nu)}\,\varrho h^3\,\triangle z_3^{0\,\prime\prime} \tag{9.2}$$

is involved. However, in [21] the numerator is equal to $34-12\nu$.

In the paper [11] Jemielita's kinematical assumptions are admitted. The dynamic problem for thick, isotropic plate is analysed starting from the Hamilton variational principle. First, the energy-consistent model is derived. Then, rational simplifications of the three functionals used yield, in the static case, Reissner equations. In the equation of motion obtained in [11] for the averaged plate deflection the term (9.2) appears. However, in the model studied, also other terms are present in this equation. Such a difference is due to the fact that in [11] the kinematical hypothesis has been a priori assumed.

The developments of the paper [7] are based on using some averaged values in Reissner's sense. Finally, one derives a Kármán-Reissner nonlinear anisotropic plate model and its linear approximation. The elastic anisotropic material is described by means of engineering coefficients, E_i, ν_{ij}.

In [8] the refined 2-D dynamic equations of an isotropic thin plate are derived from 3-D equations of elasticity with the use of asymptotic method. The small parameter λ is defined as follows: $\lambda = \varepsilon^{1/q}$, $\varepsilon = h/l$, $2h$ – the thickness, where l is the characteristic plate dimension, and $r = p/q$, p, q – integers. The asymptotic expansions for displacements are admitted in the form:

$$v_x = \varepsilon^{3r-3}\sum_{s=0}^{\infty}\lambda^s\,v_x^{(s)},\quad v_z = \varepsilon^{4r-4}\sum_{s=0}^{\infty}\lambda^s\,v_z^{(s)},\quad v_x = u_x/h,\quad v_z = u_z/h. \tag{9.3}$$

The formula for v_y is similar to Eq.$(9.3)_1$. The coefficients $v_x^{(s)}$, $v_z^{(s)}$ involve expansions:

$$v_z^{(s)} = \sum_{k=0}^{K}\zeta^k\,v_{zk}^{(s)},\quad v_x^{(s)} = \sum_{k=0}^{K+1}\zeta^k\,v_{xk}^{(s)},\quad \zeta = z/h,$$

where $K = s/q$, if it is an even number, and $K = (s/q-1)$ in the opposite case. For the quantities $v_{zk}^{(s)}, v_{xk}^{(s)}$ the recurrence formulas are derived in [8].

For various approximations, various dynamical models are obtained. For instance, if $s < 6q-6p$ then in the dynamical equation the following term is present:

$$\frac{17-7\nu}{15(1-\nu)}\,\frac{\partial^2}{\partial\tau^2}\,\triangle v_{z0}^{(s-2q-2p)}, \tag{9.4}$$

where $\tau = t/t_0$, $t_0 = l\,\varepsilon^{\omega-1}\sqrt{\varrho/E}$. In this theory an important role is played by the parameter $\omega = 2r$ characterising the time-dependence of stresses. If $\omega < 0$, the quasistatic deformation is described. The remaining two classes of this theory are characterised by: $0 \le \omega < 2$ and $\omega \ge 2$. The second class contains the 3-D dynamic deformation.

At the end of the comments given above, we want to mention that an useful introduction to engineering approaches of formulations of refined plate models may be offered by the monograph [14].

Comparing our results with other contributions we conclude that second order terms (the *correctors*) also appear in refined engineering models. However, these terms are not always exactly the same, though very similar. Also, in our case the *corrector* has been rigorously derived and justified.

Appendix

The following lemma was formulated in [16].
LEMMA A.
Let Y be a space such that $\{v \in H^1(\mathcal{B}) \mid v = 0 \text{ on } \Gamma_0^1\} \subset Y \subset H^1(\mathcal{B})$
and let G be a linear form on Y given by

$$\forall\, v \in Y, \quad G(v) = (p\ ,\ v) - (q_\alpha, \partial_\alpha w) + \int\limits_{\Omega} \Big[rw(1) + sw(-1) \Big]\, dx_1\, dx_2$$

with $p,\ q_\alpha,\ \partial_\alpha q_\alpha \in L^2(\mathcal{B})$ and $r,\ s \in L^2(\Omega)$. Then the problem

$$\sigma \in L^2(\mathcal{B}), \quad \forall\, v \in Y, \quad (\sigma\ , \partial_3 v) = G(v)$$

has a solution if and only if the following compatibility conditions are satisfied:

$$(C_1) \qquad \int\limits_{-1}^{1} (p + \partial_\alpha q_\alpha)\, dx_3\ +\ r\ +\ s = 0,$$

$$(C_2) \qquad \forall\, v \in Y, \qquad \int\limits_{\Gamma_0^1} q_\alpha\, n_\alpha\, v\, d\Gamma = 0.$$

The solution σ is given by

$$\sigma = \int\limits_{-1}^{x_3} (-p - \partial_\alpha q_\alpha)\, dx_3\ +\ \frac{1}{2}\int\limits_{-1}^{1} (p + \partial_\alpha q_\alpha)\, dx_3\ +\ \frac{1}{2}(r - s).$$

Similarly, let $\mathcal{Z}$ be such that $\{(v_\alpha) \in H^1(\mathcal{B})^2 \mid v_\alpha = 0 \text{ on } \Gamma_0^1\} \subset \mathcal{Z} \subset H^1(\mathcal{B})^2$ and let G be a linear form on $\mathcal{Z}$ given by

$$\forall\, (v_\alpha) \in \mathcal{Z}, \quad G_1(v_\alpha) = (p_\alpha, v_\alpha) - (q_{\alpha\beta}, \partial_\alpha v_\beta) + \int\limits_{\Omega} [r_\alpha v_\alpha(1) + s_\alpha v_\alpha(-1)]\, dx_1 dx_2$$

with $p_\alpha,\ q_{\alpha\beta},\ \partial_\beta q_{\alpha\beta} \in L^2(\mathcal{B})$, and $r_\alpha,\ s_\alpha \in L^2(\Omega)$.

Then the problem

$$\sigma_\alpha \in L^2(\mathcal{B}), \quad \forall v_\alpha \in \mathcal{Z}, \quad (\sigma_\alpha\ ,\ \partial_3 v_\alpha) = G_1(v_\alpha)$$

has a solution if and only if the following compatibility conditions are satisfied:

$$(D_1) \qquad \int\limits_{-1}^{1} (p_\alpha + \partial_\beta q_{\alpha\beta})\, dx_3\ +\ r_\alpha\ +\ s_\alpha = 0,$$

$$(D_2) \qquad \forall\ v_\alpha \in \mathcal{Z}, \qquad \int\limits_{\Gamma_0^1} q_{\alpha\beta}\, n_\beta\, v_\alpha\, d\Gamma = 0.$$

The solution σ_α $(\alpha = 1, 2)$ is given by

$$\sigma_\alpha = \int\limits_{-1}^{x_3} (-p_\alpha - \partial_\beta q_{\alpha\beta})\, dx_3\ +\ \frac{1}{2}\int\limits_{-1}^{1} (p_\alpha + \partial_\beta q_{\alpha\beta})\, dx_3\ +\ \frac{1}{2}(r_\alpha - s_\alpha).$$

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