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Thermoelastic plane problem for material with circular inclusions

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We consider two-dimensional thermoelastic composite materials in the case when the temperature is constant. Using complex potentials and applying a method of functional equations, we construct a simple algorithm to solve the corresponding boundary value problem. The stress tensor is written with the accuracy of up to the term $O\left(R^2\right)$, where $R = \max_{k,m} r_k d_{km}^{-1}$, r_k is the radius of the k-th inclusion, d_{km} is the distance between centers of the k-th and m-th inclusion $(k \neq m)$. The effective elastic constants and the coefficient of thermal expansion are written in analytic form up to $O\left(R^4\right)$.

1. Introduction

We study two-dimensional problems of thermoelastic composite materials. A modern review devoted to homogenization and constructive formulae for such materials is due to Wojnar *et al.* [1] and [2]. We consider the case of the plane strain. Such problems are extensively studied by the method of complex potentials. Muskhelishvili [9], Mikhlin [6] and others reduced the boundary value problem for materials with finite number of inclusions or holes to a system of singular integral equations or to an infinite set of linear algebraic equations. Integral equations and infinite sets of equations can be numerically solved, and the stress and displacement fields can be calculated. The method of complex potentials was extended to periodic problems by Van Fo Fy [10] and Grigilyuk and Fil'shtinskij [3]. In particular, integral equations and infinite systems were constructed and solved numerically for periodic problems.

The closed-form solution is preferable to numerical solution in mechanics of composite materials, since it allows us to obtain analytical formulae for the tensor of effective properties. There are special cases in the books cited above, when the stress and displacement fields were found in analytical form. In the present paper we study a new special case of such problems, the two-dimensional thermoelastic materials with circular inclusions. Each inclusion is modelled by a disk; the position of the center and the radius are arbitrary. Only one essential restriction is imposed, namely, these disks are mutually disjoint. Following [7, 8] we reduce the thermoelastic problem to a system of functional equations, which can be solved by the method of successive approximations. Let us note that the functional equations do not contain integral terms which are hard to calculate analytically. The right-hand sides of the equations involve combinations of functions and their derivatives. This allows us to express the first approximations of the stress and displacement fields in analytic forms.

Using the analytical formulae for the stress field, we can calculate the effective properties of the thermoelastic composite materials. As it is assumed in the homogenization theory (see [1] and [4]), the distribution of the inclusions on the plane is statistically uniform and ergodic. Hence, we can evaluate the effective properties using spatial averaging. Constructive formulae are obtained under the following additional assumptions. Composite material is divided into the same groups of the inclusions. All groups consists of a finite number of inclusions. A group is displayed in Fig. 1. Each group does not interact with others. Moreover, to simplify the calculations we assume that the considered material is isotropic in macroscale as a two-dimensional material. If the group contains only one inclusion, we arrive at the case of the dilute composite materials. In order to obtain a formula for the effective properties in the dilute case, it is sufficient to solve a boundary value problem (conjugate problem) for a single inclusion. This approach is called Maxwell's formalism in the literature. Our case (many inclusions in a group as displayed in Fig. 2) can be called the generalized dilute concentration of inclusions, because we take into account influence of each inclusion on the other ones in any fixed group, and inclusions in different groups do not strike each other. According to Maxwell's formalism, the area fraction of inclusions is a prescribed parameter. In the generalized dilute case we also assume the area concentration v_k of the $k-$th material $(k = 1, 2, ..., n)$ as given parameters. Here n is the number of inclusions in each group. We cannot add all v_k since the inclusions have in general various sizes and thermoelastic properties.

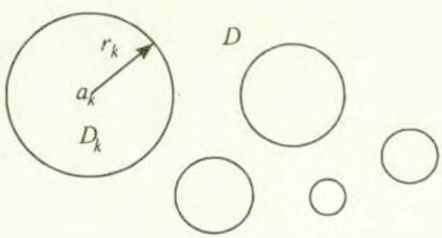

FIG. 1. Representative group of inclusions.

The paper is organized as follows. In Sec. 2 we reduce the problem to a system of functional equations. Section 3 is devoted to solution to this system. The local stress and displacement thermoelastic fields are given in Sec. 4. The

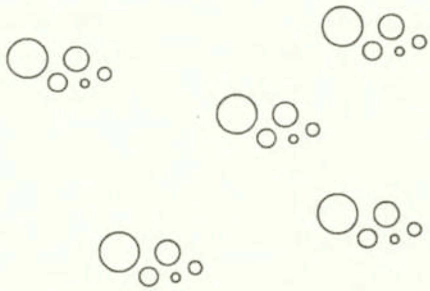

FIG. 2. Composite material with diluted groups of inclusions.

local fields presented in Sec. 5 concern the case of the pure elastic problem. In Sec. 6 the formulae of Sec. 5 are applied to deduce formulae for the effective elastic constants. Then the effective coefficient of thermal expansion is written in analytic form.

2. Complex potentials and conjugation problem

Let us consider a spatial variable (x, y, z). The (x, y)-plane thermoelasticity stress-strain relations for a linear isotropic material are given by the equations [9]

$$\begin{aligned} \sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right) + E\alpha^T T &= E\frac{\partial u}{\partial x}, \\ \sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right) + E\alpha^T T &= E\frac{\partial v}{\partial y}, \\ \sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right) &= 0, \\ \sigma_{xy} &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \end{aligned} \tag{2.1}$$

where

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix} \tag{2.2}$$

is the stress tensor, $(u, v, 0)$ is the displacement vector, ν is the Poisson ratio, E is the Young modulus, $\mu = \dfrac{E}{2}\left(1+\nu\right)$ is the shear modulus, α^T is the coefficient of thermal expansion, T is the temperature distribution. Here all coefficients depend on the coordinates (x, y). The form of the third Eq. (2.1) is obtained from equation $\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right) + E\alpha^T T = f$, where the external pressure $f = E\alpha^T T$ is applied along the $z-$direction in such a way that the plane strain holds [9].

For the sake of simplicity we assume that T is constant. Then the equations of the steady heat conduction and the conditions of perfect thermal contact between materials are fulfilled. Moreover, we can take $T = 1$ because of the linear character of Eq. (2.1).

Let us consider mutually disjoint disks $D_k := \{\zeta \in \mathbb{C}:\ |\zeta - a_k| < r_k\}$ ($k = 1, 2, ..., n > 1$) in the complex plane $\mathbb{C}$ of the variable $\zeta = x + iy$. Let $D := \mathbb{C} \cup \{\infty\} \setminus (\cup_{k=1}^{n} D_k \cup T_k)$, where $T_k := \{t \in \mathbb{C}:\ |t - a_k| = r_k\}$. We assume that T_k are orientated in clockwise sense. Here and in the sequel we use the letter ζ for a complex variable in a domain, t – on the boundary of a domain. We study the thermoelasticity of the composite material, when the domains D and D_k are occupied by materials with the coefficients μ, α^T, $\kappa = 3 - 4\nu$ and μ_k, α_k^T, $\kappa_k = 3 - 4\nu_k$, respectively. We shall also use the constants E, ν and E_k, ν_k, respectively, to denote the elastic properties of the matrix and inclusions.

The component of the stress tensor can be determined by the Kolosov-Muskhelishvili formulae [9]

$$\sigma_{xx} + \sigma_{yy} = \begin{cases} 4\mathrm{Re}\,\phi_k'(\zeta), & \zeta \in D_k, \\ 4\mathrm{Re}\,\varphi'(\zeta), & \zeta \in D, \end{cases}$$

(2.3)

$$\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = \begin{cases} -2\left[\zeta\overline{\phi_k''(\zeta)} + \overline{\psi_k'(\zeta)}\right], & \zeta \in D_k, \\ -2\left[\zeta\overline{\varphi''(\zeta)} + \overline{\psi'(\zeta)}\right], & \zeta \in D, \end{cases}$$

where Re denotes the real part. The functions $\phi_k(\zeta)$ and $\psi_k(\zeta)$, $\varphi(\zeta)$ and $\psi(\zeta)$ are analytical in D_k and D, respectively, and twice differentiable in the closures of the considered domains. The normal forces on T_k are given by the expression

$$\mathbf{T}_n^- = \phi_k(t) + t\overline{\phi_k'(t)} + \overline{\psi_k(t)},\ \mathbf{T}_n^+ = \varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)}. \tag{2.4}$$

The plane displacements $U = (u, v)$ are expressed by the complex potentials

$$U = \begin{cases} \dfrac{1}{2\mu_k}\left[\kappa_k\phi_k(\zeta) - \zeta\overline{\phi_k'(\zeta)} - \overline{\psi_k(\zeta)} + 2\alpha_k^T\mu_k\zeta\right], & \zeta \in D_k, \\ \dfrac{1}{2\mu}\left[\kappa\varphi(\zeta) - \zeta\overline{\varphi'(\zeta)} - \overline{\psi(\zeta)} + 2\alpha^T\mu\zeta\right], & \zeta \in D. \end{cases} \tag{2.5}$$

We assume that the contact between different materials is perfect, i.e.,

$$\mathbf{T}_n^+ = \mathbf{T}_n^-,\ U^+ = U^- \ \text{ on } \ \partial D, \tag{2.6}$$

where ∂D is the boundary of D, $U^+(t) := \lim_{\zeta\to t,\,\zeta\in D} U(\zeta)$, $U^-(t) := \lim_{\zeta\to t,\,\zeta\in D_k} U(\zeta)$. Using the relation (2.4) – (2.5) we write the boundary condition (2.6) in the form

$$\begin{gathered}
\phi_k(t) + t\overline{\phi_k'(t)} + \overline{\psi_k(t)} = \varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)},\\
\frac{1}{\mu_k}\left[\kappa_k\phi_k(t) - t\overline{\phi_k'(t)} - \overline{\psi_k(t)}\right] = \frac{1}{\mu}\left[\kappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)}\right] + 2\eta_k t,\\
|t-a_k| = r_k, \qquad k = 1,2,...,n,
\end{gathered} \tag{2.7}$$

where $\eta_k = \alpha^T - \alpha_k^T$. Introduce the new unknown functions

$$\Phi_k(\zeta) = \left(\frac{r_k^2}{\zeta - a_k} + \overline{a_k}\right)\phi_k'(\zeta) + \psi_k(\zeta),\ |\zeta - a_k| \le r_k.$$

Then condition (2.7) becomes

$$\phi_k(t) + \overline{\Phi_k(t)} = \varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)},$$

$$\frac{\mu}{\mu_k}\left[\kappa_k\phi_k(t) - \overline{\Phi_k(t)}\right] = \kappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} + 2\eta_k\mu t, \quad |t-a_k| = r_k.$$

These relations can be written as follows:

$$\left(1 + \frac{\mu}{\mu_k}\kappa_k\right)\phi_k(t) + \left(1 - \frac{\mu}{\mu_k}\right)\overline{\Phi_k(t)} = (1+\kappa)\,\varphi(t) + 2\mu\eta_k t, \tag{2.8}$$

$$\left(\kappa - \frac{\mu}{\mu_k}\kappa_k\right)\overline{\phi_k(t)} + \left(\kappa + \frac{\mu}{\mu_k}\right)\Phi_k(t) = (1+\kappa)\left(\bar{t}\varphi'(t) + \psi(t)\right) - 2\mu\eta_k\bar{t}. \tag{2.9}$$

The form of Eqs. (2.8) and (2.9) is similar to the $\mathbb{R}$-linear condition [7], since φ and ψ are analytic in D; ϕ_k is analytic in D_k, Φ_k is analytic in D_k except $\zeta = a_k$, where its principal part has the form $r_k^2\,(\zeta - a_k)^{-1}\,\phi_k'(a_k)$. Following [7], we reduce the problem (2.8), (2.9) to a system of functional equations.

Let $\zeta_{(k)}^* = r_k^2\,(\zeta - a_k)^{-1} + \overline{a_k}$ denote the inversion of ζ with respect to the circle T_k. If a function $f(\zeta)$ is analytic in $|\zeta - a_k| < r_k$, then $\overline{f(\zeta_{(k)}^*)}$ is analytic in $|\zeta - a_k| > r_k$. Basing on (2.8), we introduce the function analytic in D and all D_k:

$$\Omega(\zeta) := \begin{cases} \left(1 + \frac{\mu}{\mu_k}\kappa_k\right)\phi_k(\zeta) - \sum_{m\neq k}\left(1 - \frac{\mu}{\mu_m}\right)\left[\overline{\Phi_m(\zeta^*_{(m)})}\right. \\ \left. - (\zeta - a_m)\,\overline{\phi'_m(a_m)}\right] - 2\mu\eta_k\zeta + \left(1 - \frac{\mu}{\mu_k}\right)(\zeta - a_k)\,\overline{\phi'_k(a_k)}, \\ \qquad |\zeta - a_k| \le r_k, \quad k = 1, 2, ..., n, \\ (1 + \kappa)\,\varphi(\zeta) - \sum_{m=1}^{n}\left(1 - \frac{\mu}{\mu_m}\right)\left[\overline{\Phi_m(\zeta^*_{(m)})}\right. \\ \left. - (\zeta - a_m)\,\overline{\phi'_m(a_m)}\right], \quad \zeta \in D. \end{cases}$$

Let us calculate the jump of $\Omega(\zeta)$ across T_k

$$\Omega^+(t) - \Omega^-(t) = (1 + \kappa)\,\varphi(t) - \left(1 - \frac{\mu}{\mu_k}\right)\left[\overline{\Phi_k(t)} - (t - a_k)\,\overline{\phi'_k(a_k)}\right]$$
$$- \left(1 + \frac{\mu}{\mu_k}\kappa_k\right)\phi_k(t) + 2\mu\left(\alpha^T - \alpha^T_k\right)t - \left(1 - \frac{\mu}{\mu_k}\right)(t - a_k)\,\overline{\phi'_k(a_k)}.$$

It follows from Eq. (2.8) that this jump is zero. Hence, $\Omega(\zeta)$ is analytic in $\mathbb{C} \cup \{\infty\}$ by the principle of analytic continuation. Then the Liouville theorem implies that the function $\Omega(\zeta) = p_0$, where p_0 is a constant. It follows from the definition of $\Omega(\zeta)$ in D_k that

$$\text{(2.10)} \qquad \left(1 + \frac{\mu}{\mu_k}\kappa_k\right)\phi_k(\zeta) = \sum_{m\neq k}\left(1 - \frac{\mu}{\mu_m}\right)\left[\overline{\Phi_m(\zeta^*_{(m)})} - (\zeta - a_m)\,\overline{\phi'_m(a_m)}\right]$$
$$+ p_0 + 2\mu\eta_k\zeta - \left(1 - \frac{\mu}{\mu_k}\right)(\zeta - a_k)\,\overline{\phi'_k(a_k)},$$
$$|\zeta - a_k| \le r_k, \quad k = 1, 2, ..., n.$$

This is the first set of functional equations relating the unknown functions $\phi_k(\zeta)$ and $\Phi_k(\zeta)$. The definition of $\Omega(\zeta)$ in D yields

$$\text{(2.11)} \qquad (1 + \kappa)\,\varphi(\zeta) = \sum_{m=1}^{n}\left(1 - \frac{\mu}{\mu_m}\right)\left[\overline{\Phi_m(\zeta^*_{(m)})} - (\zeta - a_m)\,\overline{\phi'_k(a_k)}\right] + p_0,$$
$$\zeta \in D \cup \partial D.$$

Hence, if $\phi_k(\zeta)$ and $\Phi_k(\zeta)$ are determined, φ can be calculated by (2.11).

We now proceed to deduce the second set of functional equations. First we differentiate (2.11) and substitute it in (2.9)

$$\left(\kappa - \frac{\mu}{\mu_k}\kappa_k\right)\overline{\phi_k(t)} + \left(\kappa + \frac{\mu}{\mu_k}\right)\Phi_k(t) = \left(\frac{r_k^2}{t-a_k} + \overline{a_k}\right)\sum_{m=1}^{n}\left(1 - \frac{\mu}{\mu_m}\right)\left[\left(\overline{\Phi_m(t^*_{(m)})}\right)' - \overline{\phi'_k(a_k)}\right] + (1+\kappa)\,\psi(t) - 2\mu\eta_k\left(\frac{r_k^2}{t-a_k} + \overline{a_k}\right), \quad |t-a_k| = r_k. \tag{2.12}$$

Introduce the function analytic in D and meromorphic in D_k

$$\omega(\zeta) := \begin{cases} \left(\kappa + \frac{\mu}{\mu_k}\right)\Phi_k(\zeta) - \sum\limits_{m\neq k}\left(1 - \frac{\mu}{\mu_m}\right)\left(\frac{r_k^2}{\zeta - a_k} + \overline{a_k}\right. \\ \left. - \frac{r_m^2}{\zeta - a_m} - \overline{a_m}\right)\left[\left(\overline{\Phi_m(\zeta^*_{(m)})}\right)' - \overline{\phi'_m(a_m)}\right] \\ - \sum\limits_{m\neq k}\left(\kappa - \frac{\mu}{\mu_m}\kappa_m\right)\overline{\phi_m(\zeta^*_{(m)})} + 2\mu\left(\alpha^T - \alpha_k^T\right) \\ \times\left(\frac{r_k^2}{\zeta - a_k} + \overline{a_k}\right), \quad |\zeta - a_k| \le r_k, \quad k = 1,2,...,n, \\ (1+\kappa)\,\psi(\zeta) + \sum\limits_{m=1}^{n}\left(\frac{r_m^2}{\zeta - a_m} + \overline{a_m}\right)\left(1 - \frac{\mu}{\mu_m}\right) \\ \times\left[\left(\overline{\Phi_m(\zeta^*_{(m)})}\right)' - \overline{\phi'_m(a_m)}\right] - \sum\limits_{m=1}^{n}\left(\kappa - \frac{\mu}{\mu_m}\kappa_m\right)\overline{\phi_m(\zeta^*_{(m)})}, \\ \qquad \zeta \in D. \end{cases}$$

Let us calculate the jump of $\omega(\zeta)$ across T_k

$$\omega^+(t) - \omega^-(t) = (1+\kappa)\,\psi(t) + \left(\frac{r_k^2}{t-a_k} + \overline{a_k}\right)\sum_{m=1}^{n}\left(1 - \frac{\mu}{\mu_m}\right) \times \left[\left(\overline{\Phi_m(t^*_{(m)})}\right)' - \overline{\phi'_k(a_k)}\right] - \left(\kappa + \frac{\mu}{\mu_k}\right)\Phi_k(t) - 2\mu\left(\alpha^T - \alpha_k^T\right) \times\left(\frac{r_k^2}{t-a_k} + \overline{a_k}\right) - \left(\kappa - \frac{\mu}{\mu_k}\kappa_k\right)\overline{\phi_k(t)}.$$

It follows from Eq. (2.12) that $\omega^+(t) - \omega^-(t) = 0$. Hence, by the principle of analytic continuation, $\omega(\zeta)$ is analytic in D and D_k $(k = 1,2,...,n)$ except the points $\zeta = a_k$. The generalized Liouville theorem implies that

$$\omega(\zeta) = \sum_{k=1}^{n} \frac{r_k^2 q_k}{\zeta - a_k} + q_0, \tag{2.13}$$

where q_0 is a constant,

$$q_k = 4\mu\eta_k + \phi_k'(a_k)\left(\kappa + \frac{\mu}{\mu_k} - 1 - \frac{\mu}{\mu_k}\kappa_k\right) - \overline{\phi_k'(a_k)}\left(1 - \frac{\mu}{\mu_k}\right), \quad k = 1, 2, ..., n. \tag{2.14}$$

It follows from the definition of $\omega(\zeta)$ that

$$\begin{aligned}\left(\kappa + \frac{\mu}{\mu_k}\right)\Phi_k(\zeta) = \sum_{m \neq k} \Bigg\{ \left(\kappa - \frac{\mu}{\mu_m}\kappa_m\right)\overline{\phi_m(\zeta_{(m)}^*)} \\ + \left(\frac{r_k^2}{\zeta - a_k} + \overline{a_k} - \frac{r_m^2}{\zeta - a_m} + \overline{a_m}\right)\left(1 - \frac{\mu}{\mu_m}\right)\left[\left(\overline{\Phi_m(\zeta_{(m)}^*)}\right)' - \overline{\phi_m'(a_m)}\right]\Bigg\} \\ - 2\mu\eta_k\left(\frac{r_k^2}{\zeta - a_k} + \overline{a_k}\right) + \omega(\zeta), \quad |\zeta - a_k| \le r_k, \quad k = 1, 2, ..., n.\end{aligned} \tag{2.15}$$

This is the second set of functional equations. $2n$ relations (2.10), (2.15) constitute a system of functional equations with respect to ϕ_k and Φ_k $(k = 1, 2, ..., n)$, where $\phi_k(\zeta), \Phi_k(\zeta) - r_k^2 (\zeta - a_k)^{-1} \phi_k'(a_k)$ are analytic in D_k and continuously differentiable in $D_k \cup T_k$. The definition of $\omega(\zeta)$ in D yields the relation

$$\begin{aligned}(1 + \kappa)\psi(\zeta) = \omega(\zeta) - \sum_{m=1}^{n}\left(\frac{r_m^2}{\zeta - a_m} + \overline{a_m}\right)\left(1 - \frac{\mu}{\mu_m}\right) \\ \times \left[\left(\overline{\Phi_m(\zeta_{(m)}^*)}\right)' - \overline{\phi_m'(a_m)}\right] + \sum_{m=1}^{n}\left(\kappa - \frac{\mu}{\mu_m}\kappa_m\right)\overline{\phi_m(\zeta_{(m)}^*)}, \quad \zeta \in D.\end{aligned} \tag{2.16}$$

3. Solution to the functional equations

It is possible to solve (2.10), (2.15) by the method of successive approximations. Here we only note that this method is applied at least when $1 - \dfrac{\mu}{\mu_m}$ and $\kappa - \dfrac{\mu}{\mu_m}\kappa_m$ are sufficiently small. This case corresponds to weakly inhomogeneous materials, when $\mu \approx \mu_m$ and $\kappa \approx \kappa_m$.

Now we would like to present another method, the method of undetermined coefficients, based on the addition theorems [7]. Put

$$\phi_k(\zeta) = \sum_{l=0}^{\infty} \alpha_{lk} (\zeta - a_k)^l, \ \Phi_k(\zeta) = \frac{r_k^2 \alpha_{1k}}{\zeta - a_k} + \sum_{l=0}^{\infty} \beta_{lk} (\zeta - a_k)^l, \quad k = 1, 2, ..., n, \tag{3.1}$$

where α_{lk}, β_{lk} are unknown constants. Then

(3.2)
$$\overline{\phi_m(\zeta^*_{(m)})} = \sum_{l=0}^{\infty} \overline{\alpha_{lm}} r_m^{2l} (\zeta - a_m)^{-l}, \ \overline{\Phi_m(\zeta^*_{(m)})} - (\zeta - a_m)\, \overline{\phi'_m(a_m)} = \sum_{l=0}^{\infty} \overline{\beta_{lm}} r_m^{2l} (\zeta - a_m)^{-l},$$

$$\left[\overline{\Phi_m(\zeta^*_{(m)})}\right]' - \overline{\phi'_m(a_m)} = -\sum_{l=1}^{\infty} l \overline{\beta_{lm}} r_m^{2l} (\zeta - a_m)^{-(l+1)}, \quad m = 1, 2, ..., n.$$

Re-expand the functions (3.2) in the powers of $(\zeta - a_k)$ with $m \neq k$, using the relation

$$\frac{1}{\zeta - a_m} = -\frac{1}{a_m - a_k} \sum_{j=0}^{\infty} \left(\frac{\zeta - a_k}{a_m - a_k} \right)^j,$$

which can be considered as an addition theorem. Then

$$\overline{\phi_m(\zeta^*_{(m)})} = \sum_{l=0}^{\infty} \overline{\alpha_{lm}} \frac{(-1)^l r_m^{2l}}{(a_m - a_k)^l} \left[\sum_{j=0}^{\infty} \left(\frac{\zeta - a_k}{a_m - a_k} \right)^j \right]^l,$$

(3.3)
$$\overline{\Phi_m(\zeta^*_{(m)})} - (\zeta - a_m)\, \overline{\phi'_m(a_m)} = \sum_{l=0}^{\infty} \overline{\beta_{lm}} \frac{(-1)^l r_m^{2l}}{(a_l - a_k)^l} \left[\sum_{j=0}^{\infty} \left(\frac{\zeta - a_k}{a_m - a_k} \right)^j \right]^l,$$

$$\left[\overline{\Phi_m(\zeta^*_{(m)})}\right]' - \overline{\phi'_m(a_m)} = \sum_{m=0}^{\infty} \overline{\beta_{mk}} \frac{(-1)^l l r_m^{2l}}{(a_m - a_k)^{l+1}} \left[\sum_{j=0}^{\infty} \left(\frac{\zeta - a_k}{a_m - a_k} \right)^j \right]^{l+1}.$$

Substitute these expansions in (2.10) and (2.15), select and equate the coefficients of the same powers of $\zeta - a_k$. As a result, we obtain a system of linear algebraic equations with respect to α_{lm} and β_{lm}.

This system can be solved by the method of reduction [5] which consists in replacing the infinite sum $\sum_{m=0}^{\infty}$ in Eqs. (3.1) – (3.3) by the finite one $\sum_{m=0}^{N}$. The number N is determined according to the desired accuracy. Here we consider the simple case $N = 1$. Then Eq. (3.1) becomes

(3.4)
$$\phi_k(\zeta) \approx \alpha_{0k} + \alpha_k (\zeta - a_k), \ \Phi_k(\zeta) \approx \frac{r_k^2 \alpha_k}{\zeta - a_k} + \beta_{0k} + \beta_k (\zeta - a_k).$$

Substituting (3.4) in (2.10) and (2.15) and selecting the coefficients on $(\zeta - a_k)^s, (s = 0, 1)$, we obtain a system of $\mathbb{R}$–linear algebraic equations with respect to p_0, q_0, α_{0k}, $\alpha_k = \alpha_{1k}$, β_{0k}, $\beta_k = \beta_{1k}$. It is easy to check that the coefficient with $(\zeta - a_k)^{-1}$ in (2.15) gives an identity. Further we consider the case when the inclusions are sufficiently far away from each other, i.e., the value $R := \max_{k,m} r_m |a_k - a_m|^{-1}$ for $m \neq k$ is sufficiently small. The remaining equations up to $O(R^2)$ are

$$\left(1 + \frac{\mu}{\mu_k}\kappa_k\right)\alpha_k = -\sum_{m\neq k}\left(1 - \frac{\mu}{\mu_m}\right)\frac{r_m^2\overline{\beta_m}}{(a_k - a_m)^2} + 2\mu\eta_k - \left(1 - \frac{\mu}{\mu_k}\right)\overline{\alpha_k}, \tag{3.5}$$

$$\left(\kappa + \frac{\mu}{\mu_k}\right)\beta_k = -\sum_{m\neq k}\left(\kappa - \frac{\mu}{\mu_m}\kappa_m\right)\frac{r_m^2\overline{\alpha_m}}{(a_k - a_m)^2} - \sum_{m\neq k}\left(1 - \frac{\mu}{\mu_m}\right)\frac{2r_m^2}{(a_k - a_m)^3}\overline{\beta_m}\,\overline{a_k}. \tag{3.6}$$

The coefficients corresponding to the constant terms give equations for the values α_{0k}, β_{0k}, p_0, q_0. The values α_{0k}, β_{0k}, p_0, q_0 do no affect the stress distribution. In accordance with the general theory [9], some of them remain undetermined. Thus we concentrate our attention on the system (3.5), (3.6) and do not determine α_{0k}, β_{0k}, p_0, q_0.

The zero order approximation of R^2 for Eqs. (3.5), (3.6) yields

$$\alpha_k^{(0)} = \frac{2\mu\eta_k}{2 + \frac{\mu}{\mu_k}\kappa_k - \frac{\mu}{\mu_k}}, \qquad \beta_k^{(0)} = 0. \tag{3.7}$$

Then Eq. (2.14) implies that

$$q_k^{(0)} = 4\mu\eta_k + \mu X_k\left(\kappa + 2\frac{\mu}{\mu_k} - 2 - \frac{\mu}{\mu_k}\kappa_k\right).$$

Substituting $\alpha_k^{(0)}$, $\beta_k^{(0)}$ and $q_k^{(0)}$ in the right-hand part of Eqs. (3.5), (3.6), we obtain the first order approximation

$$\alpha_k^{(1)} = \alpha_k^{(0)},\ \beta_k^{(1)} = -\frac{4(1+\kappa)\mu}{\kappa + \frac{\mu}{\mu_k}}\sum_{m\neq k}\left(\frac{r_m}{a_k - a_m}\right)^2\eta_m\frac{1}{2 + \frac{\mu}{\mu_m}\kappa_m - \frac{\mu}{\mu_m}}, \tag{3.8}$$

since $\beta_m^{(0)} = 0$ up to $O(R^0)$. This process of the successive approximations can be extended and one can get the arbitrary approximation $O(R^{2N})$ for α_k, β_k. We stop at the formulae (3.8) which are valid up to $O(R^2)$.

It follows from (2.11) that

$$(1+\kappa)\,\varphi'(\zeta) = \sum_{m=1}^{n}\left(1-\frac{\mu}{\mu_m}\right)\left(\frac{r_m}{\zeta - a_m}\right)^2 \overline{\beta_m}.$$

Hence,

$$\varphi'(\zeta) = 0 \quad \text{up to} \quad O(R^2), \tag{3.9}$$

since $\beta_m = 0$ up to $O(R^0)$. The function $\psi'(\zeta)$ is determined from the definition of $\omega(\zeta)$

$$\psi'(\zeta) = -\sum_{m=1}^{n}\gamma_m^{(1)}\left(\frac{r_m}{\zeta - a_m}\right)^2, \tag{3.10}$$

where

$$\gamma_m^{(1)} = \frac{4\eta_m \mu \eta_m}{2 + \dfrac{\mu}{\mu_m}\kappa_m - \dfrac{\mu}{\mu_m}}. \tag{3.11}$$

4. Local thermoelastic fields in the composite material

In the present section we gather the formulae concerning the local fields in the material discussed. Applying (2.3), (2.5) and the results of the previous section, we determine the local fields. The local stresses have the form

$$\sigma_{xx} = \begin{cases} 2\alpha_k^{(1)} - \operatorname{Re}\beta_k^{(1)}, & \zeta \in D_k, \\ \displaystyle\sum_{m=1}^{n}\operatorname{Re}\gamma_m^{(1)}\left(\frac{r_m}{\zeta - a_m}\right)^2, & \zeta \in D, \end{cases} \tag{4.1}$$

$$\sigma_{yy} = \begin{cases} 2\alpha_k^{(1)} + \operatorname{Re}\beta_k^{(1)}, & \zeta \in D_k, \\ \displaystyle -\sum_{m=1}^{n}\operatorname{Re}\gamma_m^{(1)}\left(\frac{r_m}{\zeta - a_m}\right)^2, & \zeta \in D, \end{cases} \tag{4.2}$$

$$\sigma_{xy} = \begin{cases} \operatorname{Im}\beta_k^{(1)}, & \zeta \in D_k, \\ \displaystyle -\sum_{m=1}^{n}\operatorname{Im}\gamma_m^{(1)}\left(\frac{r_m}{\zeta - a_m}\right)^2, & \zeta \in D, \end{cases} \tag{4.3}$$

$$\sigma_{xx} = \begin{cases} 4\alpha_k^{(1)} \dfrac{\mu}{\mu_k}, & \zeta \in D_k, \\ 0, & \zeta \in D, \end{cases} \tag{4.4}$$

where $\alpha_k^{(1)}$ and $\beta_k^{(1)}$ are expressed by Eqs. (3.7), (3.8).

The local displacement $U = u + iv$ is calculated with (2.5) up to an additive constant

$$U = \begin{cases} \dfrac{1}{2\mu_k}\left[\alpha_k^{(1)}(\kappa_k - 1)(\zeta - a_k) - \overline{\beta_k^{(1)}}\,\overline{(\zeta - a_k)}\right] + \alpha_k^T(\zeta - a_k), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \zeta \in D_k, \\ -\dfrac{1}{2\mu}\displaystyle\sum_{k=1}^{n} \overline{\gamma_k^{(1)}} \dfrac{r_k^2}{\overline{\zeta - a_k}} + \alpha^T \zeta, \qquad \zeta \in D. \end{cases}$$

This formula yields the following relations for the deformations:

$$\frac{\partial u}{\partial x} = \begin{cases} \dfrac{1}{2\mu_k}\left[\alpha_k^{(1)}(\kappa_k - 1) - \operatorname{Re}\beta_k^{(1)}\right] + \alpha_k^T, & \zeta \in D_k, \\ \dfrac{1}{2\mu}\displaystyle\sum_{k=1}^{n} \operatorname{Re}\gamma_k^{(1)} \left(\frac{r_k}{\zeta - a_k}\right)^2 + \alpha^T, & \zeta \in D. \end{cases} \tag{4.5}$$

$$\frac{\partial v}{\partial y} = \begin{cases} \dfrac{1}{2\mu_k}\left[\alpha_k^{(1)}(\kappa_k - 1) + \operatorname{Re}\beta_k^{(1)}\right] + \alpha_k^T, & \zeta \in D_k, \\ -\dfrac{1}{2\mu}\displaystyle\sum_{k=1}^{n} \operatorname{Re}\gamma_k^{(1)} \left(\frac{r_k}{\zeta - a_k}\right)^2 + \alpha^T, & \zeta \in D. \end{cases} \tag{4.6}$$

$$\frac{\partial u}{\partial y} = \begin{cases} \dfrac{1}{2\mu_k}\operatorname{Im}\beta_k^{(1)}, & \zeta \in D_k, \\ -\dfrac{1}{2\mu}\displaystyle\sum_{k=1}^{n} \operatorname{Im}\gamma_k^{(1)} \left(\frac{r_k}{\zeta - a_k}\right)^2, & \zeta \in D. \end{cases} \tag{4.7}$$

$$\frac{\partial v}{\partial x} = \begin{cases} \dfrac{1}{2\mu_k}\operatorname{Im}\beta_k^{(1)}, & \zeta \in D_k, \\ -\dfrac{1}{2\mu}\displaystyle\sum_{k=1}^{n} \operatorname{Im}\gamma_k^{(1)} \left(\frac{r_k}{\zeta - a_k}\right)^2, & \zeta \in D. \end{cases} \tag{4.8}$$

5. Local elastic fields in the composite material

It is convenient to determine the effective constants of the composite materials in two steps. First we calculate the pure elastic effective constants. Second we find the effective coefficient of thermal expansion. Let us recall that we discuss the isotropic material in macroscale.

Let us fix the external forces

$$\sigma_{xx}^{(\infty)} = 1,\ \sigma_{yy}^{(\infty)} = \sigma_{xy}^{(\infty)} = 0.$$

Then the complex potentials become [9]

$$\varphi^{(\infty)}(\zeta) = \frac{\zeta}{4}, \qquad \psi^{(\infty)}(\zeta) = -\frac{\zeta}{2},$$

and the disturbed potentials $\widetilde{\varphi}$ and $\widetilde{\psi}$ in the domain D take the form

$$\widetilde{\varphi}(\zeta) = \varphi(\zeta) + \frac{\zeta}{4}, \qquad \widetilde{\psi}(\zeta) = \psi(\zeta) - \frac{\zeta}{2},$$

where $\varphi(\zeta)$ and $\psi(\zeta)$ are bounded at infinity.

The contact condition (2.6) becomes

$$\begin{aligned} \phi_k(t) + t\overline{\phi_k'(t)} + \overline{\psi_k(t)} &= \varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} + \frac{t+\bar{t}}{2}, \\ \frac{\mu}{\mu_k}\left[\kappa_k\phi_k(t) - t\overline{\phi_k'(t)} - \overline{\psi_k(t)}\right] &= \kappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} + \frac{\kappa-1}{4}t + \frac{\bar{t}}{2}, \\ |t-a_k| &= r_k,\ k=1,2,...,n. \end{aligned} \tag{5.1}$$

Equation (5.1) corresponds to Eq. (2.7). The difference between Eqs. (5.1) and (2.7) appears in the known terms. We repeat all arguments of Secs. 2 and 3 to solve Eq. (5.1). If we are looking for ϕ_k and Φ_k in the form (3.4), we obtain, with the accuracy of up to $O(R^2)$,

$$\alpha_k = \frac{1+\kappa}{4(2+\frac{\mu}{\mu_k}\kappa_k - \dfrac{\mu}{\mu_k})}, \tag{5.2}$$

$$\beta_k = \frac{1+\kappa}{\dfrac{\mu}{\mu_k}+\kappa}\sum_{m\neq k}\gamma_m\left(\frac{r_m}{a_m-a_k}\right)^2 - \frac{1+\kappa}{2\left(\dfrac{\mu}{\mu_k}+\kappa\right)}, \tag{5.3}$$

where

$$\gamma_k = \frac{1}{2}\frac{\kappa - 1 - \dfrac{\mu}{\mu_k}\kappa_k + \dfrac{\mu}{\mu_k}}{2+\dfrac{\mu}{\mu_k}\kappa_k - \dfrac{\mu}{\mu_k}}, \qquad k=1,2,...,n. \tag{5.4}$$

Here the coefficients α_k, β_k, γ_k are written for the pure elastic statement. They correspond to the coefficients $\alpha_k^{(1)}$, $\beta_k^{(1)}$, $\gamma_k^{(1)}$ for the thermoelastic statement (see Sec. 4). In this case, the complex potentials (up to additive constants) become

$$\phi_k(\zeta) = \alpha_k(\zeta - a_k),\ \psi_k(\zeta) = \beta_k(\zeta-a_k). \tag{5.5}$$

Moreover, we have

$$\varphi(\zeta) = 0, \ \psi(\zeta) = \sum_{k=1}^{n} \gamma_k \frac{r_k^2}{\zeta - a_k} \tag{5.6}$$

within the accuracy of an additive constant and $O(R^2)$.

Applying the second Kolosov-Muskhelishvili formula (2.3), we obtain

$$\sigma_{xx} - \sigma_{yy} = \begin{cases} -2\mathrm{Re}\,\beta_k, & \zeta \in D_k, \\ 2\mathrm{Re} \sum_{k=1}^{n} \gamma_k \left(\dfrac{r_k}{\zeta - a_k} \right)^2 + 1, & \zeta \in D. \end{cases} \tag{5.7}$$

It follows also from Eq. (2.3) that

$$\sigma_{yy} = \begin{cases} 2\alpha_k + \mathrm{Re}\,\beta_k, & \zeta \in D_k, \\ -\mathrm{Re} \sum_{k=1}^{n} \gamma_k \left(\dfrac{r_k}{\zeta - a_k} \right)^2, & \zeta \in D. \end{cases} \tag{5.8}$$

The displacement vector $U = u + iv$ is calculated (up to an additive constant) by

$$U = \begin{cases} \dfrac{1}{2\mu_k} \left[\kappa_k \phi_k(\zeta) - \zeta \overline{\phi_k'(\zeta)} - \overline{\psi_k(\zeta)} \right], & \zeta \in D_k, \\ \dfrac{1}{2\mu} \left[\kappa\varphi(\zeta) - \zeta\overline{\varphi'(\zeta)} - \overline{\psi(\zeta)} + \dfrac{\kappa - 1}{4}\zeta + \dfrac{\overline{\zeta}}{2} \right], & \zeta \in D. \end{cases} \tag{5.9}$$

Substituting Eqs. (5.4) and (5.6) in (5.9) we calculate

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \begin{cases} -\dfrac{\mathrm{Re}\,\beta_k}{\mu_k}, & \zeta \in D_k, \\ \dfrac{1}{\mu}\mathrm{Re} \sum_{k=1}^{n} \gamma_k \left(\dfrac{r_k}{\zeta - a_k} \right)^2 + \dfrac{1}{2\mu}, & \zeta \in D, \end{cases} \tag{5.10}$$

$$\frac{\partial v}{\partial y} = \begin{cases} \dfrac{1}{2\mu_k} \left[\alpha_k (\kappa_k - 1) + \mathrm{Re}\,\beta_k \right], & \zeta \in D_k, \\ -\dfrac{1}{2\mu}\mathrm{Re} \sum_{k=1}^{n} \gamma_k \left(\dfrac{r_k}{\zeta - a_k} \right)^2 + \dfrac{1}{2\mu}\dfrac{\kappa - 3}{4}, & \zeta \in D, \end{cases} \tag{5.11}$$

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \begin{cases} \dfrac{\kappa_k - 1}{\mu_k}\alpha_k, & \zeta \in D_k, \\ \dfrac{\kappa - 3}{4\mu}, & \zeta \in D. \end{cases} \tag{5.12}$$

6. Effective constants

We apply Maxwell's formalism discussed in Sec. 1 to calculate the effective constants in the generalized dilute case. We assume that the composite material contains groups of circular inclusions and all the groups are the same, one of them is displayed in Fig. 1. The composite material is shown in Fig. 2.

Using the formulae from the previous section concerning the purely elastic field, we first calculate the effective elastic constants. We shall use the following equations of elasticity:

$$\sigma_{xx} = \lambda\theta + 2\mu\frac{\partial u}{\partial x}, \tag{6.1}$$

$$\sigma_{yy} = \lambda\theta + 2\mu\frac{\partial v}{\partial y}. \tag{6.2}$$

Here we use the Lamé coefficients

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)},\ \mu = \frac{E}{2(1+\nu)}$$

in Eqs. (6.1) and (6.2) which are understood as piece-wise constant functions.

Let us subtract (6.2) from (6.1)

$$\sigma_{xx} - \sigma_{yy} = 2\mu\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)$$

and average the expression in the latter equation applying (5.7) and (5.10),

$$\langle\sigma_{xx} - \sigma_{yy}\rangle = -2\sum_{k=1}^{n}\mathrm{Re}\,\beta_k v_k + 2\mathrm{Re}\,\Psi_0 + v, \tag{6.3}$$

$$\left\langle\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right\rangle = -\sum_{k=1}^{n}\frac{1}{\mu_k}\mathrm{Re}\,\beta_k v_k + \frac{1}{\mu}\mathrm{Re}\,\Psi_0 + \frac{1}{2\mu}v, \tag{6.4}$$

where $v = 1 - \sum_{k=1}^{n} v_k$ is the area fraction of the matrix,

$$\Psi_0 = \frac{1}{|U|}\int\int_D \sum_{k=1}^{n}\gamma_k\left(\frac{r_k}{\zeta - a_k}\right)^2 dxdy = -\sum_{k=1}^{n}\sum_{m\neq k}\gamma_k\left(\frac{r_k}{a_m - a_k}\right)^2 v_k. \tag{6.5}$$

We calculate the average $\langle\cdot\rangle = \dfrac{1}{|U|}\displaystyle\int\int_D \cdot dxdy$, where U is a "representative cell" in Maxwell's formalism, i.e., U is a domain which has area $|U| = \dfrac{\pi r_k^2}{v_k}$ for each $k = 1, 2, ..., n$ and which contains only one group of the inclusions. Here v_k is the area fraction of the $k-$th inclusions. In particular, U can be a rectangular cell, doubly periodically continued onto the whole plane.

Use of the averaged equations

$$\langle \sigma_{xx} - \sigma_{yy} \rangle = 2\mu_e \left\langle \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right\rangle$$

yields the formula for the effective shear modulus

$$\mu_e = \mu \frac{\dfrac{v}{2} - \displaystyle\sum_{k=1}^{n} \operatorname{Re} \beta_k v_k + \operatorname{Re} \Psi_0}{\dfrac{v}{2} - \displaystyle\sum_{k=1}^{n} \dfrac{\mu}{\mu_k} \operatorname{Re} \beta_k v_k + \operatorname{Re} \Psi_0}, \tag{6.6}$$

where β_k is calculated by (5.3), Ψ_0 is calculated by (6.5) in which γ_k is calculated according to (5.4).

The effective modulus λ_e is found from the averaged Eq. (6.2)

$$\langle \sigma_{yy} \rangle = \lambda_e \langle \theta \rangle + 2\mu_e \left\langle \frac{\partial v}{\partial y} \right\rangle. \tag{6.7}$$

It follows from Eq. (5.8) that

$$\langle \sigma_{yy} \rangle = \sum_{k=1}^{n} (2\alpha_k + \operatorname{Re} \beta_k) v_k - \operatorname{Re} \Psi_0.$$

Use of Eq. (5.11) yields

$$\left\langle \frac{\partial v}{\partial y} \right\rangle = \sum_{k=1}^{n} (\alpha_k (\kappa_k - 1) + \operatorname{Re} \beta_k) \frac{v_k}{2\mu_k} - \frac{\operatorname{Re} \Psi_0}{2\mu} + \frac{\kappa - 3}{8\mu} v,$$

$$\langle \theta \rangle = \sum_{k=1}^{n} \frac{\kappa_k - 1}{\mu_k} \alpha_k v_k + \frac{\kappa - 1}{4\mu} v.$$

Then Eq. (6.7) implies

$$\lambda_e = \frac{\displaystyle\sum_{k=1}^{n} (2\alpha_k + \operatorname{Re} \beta_k) v_k - \operatorname{Re} \Psi_0}{\dfrac{\kappa - 1}{4\mu} v + \displaystyle\sum_{k=1}^{n} \dfrac{\kappa_k - 1}{\mu_k} \alpha_k v_k} - \mu_e \frac{\displaystyle\sum_{k=1}^{n} (\alpha_k (\kappa_k - 1) + \operatorname{Re} \beta_k) \frac{v_k}{\mu_k} - \frac{\operatorname{Re} \Psi_0}{\mu} + \frac{\kappa - 3}{4\mu} v}{\dfrac{\kappa - 1}{4\mu} v + \displaystyle\sum_{k=1}^{n} \dfrac{\kappa_k - 1}{\mu_k} \alpha_k v_k}. \tag{6.8}$$

In order to calculate the effective coefficients of the thermal expansion, we come back to Eq. (2.1) and local field derived in Sec. 5. First, we calculate

$$\nu_e = \frac{\lambda_e}{2(\lambda_e + \mu_e)}, \qquad E_e = \frac{\mu_e (3\lambda_e + 2\mu_e)}{\lambda_e + \mu_e}. \tag{6.9}$$

The second homogenized Eq. (2.1) yields the formula

$$E_e\alpha_e^T = \langle E\alpha^T \rangle + \sum_{k=1}^{n} \frac{v_k}{1+\nu_k} (\alpha_k(\kappa_k - 1) + \operatorname{Re}\beta_k) - \frac{1}{1+\nu}\operatorname{Re}\Psi, \tag{6.10}$$

where

$$\langle E\alpha^T \rangle = \sum_{k=1}^{n} E_k \alpha_k^T v_k + vE\alpha^T \tag{6.11}$$

is the mean value of the piece-wise constant function $E\alpha^T$,

$$\Psi = -\sum_{k=1}^{n} \sum_{m\neq k} \gamma_k^{(1)} \left(\frac{r_k}{a_m - a_k} \right)^2 v_k. \tag{6.12}$$

Here we also use the relation $\dfrac{E}{2\mu} = \dfrac{1}{1+\nu}$.

Applying (4.1) - (4.4) we calculate

$$\langle \nu(\sigma_{xx} + \sigma_{yy}) - \sigma_{yy} \rangle = \sum_{k=1}^{n} (1+\nu_k)[2\alpha_k(2\nu_k - 1) + \operatorname{Re}\beta_k] v_k + (1+\nu)\operatorname{Re}\Psi.$$

Therefore, Eqs. (6.10) - (6.13) yield the formula

$$E_e\alpha_e^T = \langle E\alpha^T \rangle + \sum_{k=1}^{n} \frac{v_k}{1+\nu_k} \left[\alpha_k^{(1)} (\nu_k^2 - 2\nu_k^3) + (\nu_k^2 + 2\nu_k + 2) \operatorname{Re}\beta_k^{(1)} \right] - \frac{\nu^2}{1+\nu}\operatorname{Re}\Psi, \tag{6.13}$$

where $\alpha_k^{(1)}$ and $\beta_k^{(1)}$ are calculated from the formulae (3.7) and (3.8), Ψ has the form Eq. (6.12), E_e is given by Eq. (6.9).

7. Conclusion

We study two-dimensional thermoelastic composite materials with circular inclusions, when the temperature is constant everywhere in the material. Using the complex potentials of Kolosov-Muskhelishvili, we deduce the problem to a system of functional equations which can be solved by the method of successive approximations. This allows us to construct a simple algorithm to determine the local stress and displacement fields in analytic form with the accuracy of up to the term $O\left(R^2\right)$, where $R = \max_{k,m} r_k d_{km}^{-1}$, r_k is the radius of the k-th inclusion, d_{km} is the distance between centers of the k-th and m-th inclusion ($k \neq m$) (see formulae in Secs. 4 and 5). The effective elastic constants are written also in analytic form (6.6) and (6.8) up to $O(R^4)$. The effective coefficient of thermal expansion is expressed by Eq. (6.13).

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