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Gradient field theory of material instabilities

Dedicated to Professor Zenon Mróz
on the occasion of his 70^{th} birthday

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PREVIOUSLY, we developed a gradient thermodynamic theory of internal fields (migratory motions). The theory predicts the observed periodic deformation structures, in material domains under uniform tractions. More recently we showed, in a uniform stress field, that the theory has the proper mathematical framework for the prediction of Portevin-Le Chatelier (PLC for short) instabilities.
Here we review our previous work and address the more difficult problem of a non-uniform stress field. Specifically, we predict the points of instability of a solid cylinder under torsion, with the experiments of Dillon as backdrop. Again, we find close agreement between theory and experiment.

1. Introduction

IN A RECENT SERIES of paper, VALANIS [1 – 4], we presented an isothermal, gradient thermodynamic theory of internal fields. The theory provided a theoretical basis for the appearance of *non-uniform strain fields*, in homogeneous material domains, under *uniform surface tractions*, in situations where local theories would predict otherwise. In the most recent paper, VALANIS [4], we demonstrated that imbedded in the theory is a *mathematical framework*, for the theoretical treatment of 'unstable solids'.

Specific attention was given to the Portevin-Le Chatelier effect, whereby a macroscopically uniform domain under uniform, monotonically increasing tractions, suffers *spontaneous* changes in deformation, at specific *discrete* values of the tractions while the resulting strain becomes non-uniform.

It was further shown that this *metastable* behavior is caused by the presence of *particular* internal field ξ_i. These are continuous and twice differentiable and bounded in the material domain D in the sense that $\|\xi_i\| < \infty$ in D, double bars denoting the Euclidean norm. The fields satisfy the partial differential equation:

$$C\nabla^2\boldsymbol{\xi} + \boldsymbol{\Sigma}.\boldsymbol{\xi} = 0 \tag{1.1}$$

in D, where C is a material constant, ∇^2 is the Laplacian and $\boldsymbol{\Sigma}(\boldsymbol{\sigma})$ is a tensor function of the stress $\boldsymbol{\sigma}$, to be discussed in the text. We shall refer to this equation as the *Instability Equation*. In the specific case treated here, ξ_i signify *end-points of diffusion and/or rearrangement processes.* They represent, therefore, *equilibrium states* in the sense that their dual internal forces Ξ_i are zero.

The boundary conditions are:

$$\xi_i = 0 \quad \text{on} \quad S_i,$$

(1.2)

$$\xi_{i',j} n_j = 0 \quad \text{on} \quad S_p,$$

where S_i and S_p denote a permeable and impermeable surface, respectively. Quite clearly the solution of Eq. (1.1) gives rise to an eigenvalue problem. If $\boldsymbol{\Sigma}$ is constant in D, then non-null solutions to Eq. (1.1) will exist only for specific *characteristic* values of $\boldsymbol{\sigma}$.

In our most recent work, cited previously, VALANIS [4], we applied the theory to the experiments by LUBAHN [5], who tested flat steel specimens in tension, DILLON [6] who did experiments on hollow aluminium cylinders under torsion and SHARPE [13] who tested solid aluminium cylinders under tension. All these experiments were done under load control and at a *slow rate* of loading.

All three authors observed metastable behavior of the Portevin-Le Chatelier type in the sense that the deformation was a monotonic continuous function of the tractions, except for specific discrete values of the latter, at which the material body suffered a sudden and spontaneous change in the deformation.

The application of the theory to the above experiments resulted in the demonstration that the Instability Equation (1.1) contains the appropriate physics of metastable behaviour and that its solution, in all three cases, given *precise* predictive values of the tractions at which the instabilities were observed.

Moreover it was shown that the collapse tractions are, within a geometric factor, the *eigenvalues* of the solutions of the Instability Equation (1.1).
We refer the reader to VALANIS [4] for details but treat the flat bar in tension in some detail in Sec. 3.

In this paper we illustrate the ability of the theory to deal with the more complex problem where the tractions are *not uniform*. The work was motivated by the researches of DILLON [6], who carried out experiments in load control, on solid aluminium cylinders in torsion. As in previous cases, material instabilities of the Portevin-Le Chatelier type were observed, in the sense that the cylinders suffered spontaneous changes in twist at discrete values of the applied torque.

As is well known, local theories will predict that the twist is a continuous function of the applied torque. Such theories, therefore, are incapable of describing, let alone predicting, this type of phenomenon.

2. Gradient thermodynamics

The theory is expressed in terms of gradients of internal variables, more appropriately, internal fields. It was given previously, in a Helmoltz and a Gibbs formulation in earlier work by VALANIS [1 – 3]. It is basically the following. The Helmotz (Gibbs) free energy density denoted by ψ (denoted by ϕ) is a function of the strain tensor ϵ_{ij} (stress tensor σ_{ij}) and *three* different types of internal variables:

(i) Second order tensors p_{ij}, which are *dissipative* and obey evolution equations of the local type.

(ii) Vectors q_i and their gradients $q_{i,j}$ which are also *dissipative* and give rise to inhomogeneous strain fields in the presence of uniform surface tractions.

(iii) Vectors ξ_i and their gradients $\xi_{i,j}$. These are a subset of q_i in the sense that their dual internal forces Ξ_i are zero! Thus, either ξ_i are *inviscid* or they represent *terminal equilibrium points* of an irreversible process.

Both q_i and ξ_i are mathematical representations of non-affine migration of subsets of particles, into material subdomains that are exterior to the initial neighborhood of the particles (see VALANIS [3]). Such motions are brought about by diffusion of dislocations, voids, interstitials and/or other processes such as particle diffusion or microslip.

2.1. Helmotz formulation

The formulation is based on the global variational inequality (2.1), that pertains to a material domain D of volume V and surface S under prevailing isothermal conditions:

$$\delta\Psi \leq \int_S T_i \delta u_i dS + \int_V f_i \delta u_i dV, \tag{2.1}$$

where Ψ is the total free energy of the domain, T_i – the surface tractions, f_i – the body forces in D, u_i – the displacement field and δ is the variation operator.

The free energy density ψ such that

$$\int_V \psi dV = \Psi \tag{2.2}$$

is then introduced where

$$\psi = \psi(\epsilon_{ij};\ p_{ij};\ q_i;\ q_{i,j}). \tag{2.3}$$

Since ξ_i are a sub-class of q_i they do not appear explicitly in Eq. (2.3).

In variance with local theories, ψ is a function of the strain tensor, m local internal variables p_{ij} and n vectorial internal field variables q_i and their gradients

$q_{i,j}$, subject to the constitutive constraint that for all rigid body displacement variations δu_i^*:

$$\delta\Psi = 0; \quad \delta q_i = 0. \tag{2.4}$$

2.2. Field equations

The following relations then hold in D, VALANIS [1 – 3], for all p_{ij} and q_i:

$$\sigma_{ij} = \partial\psi/\partial\epsilon_{ij}, \tag{2.5}$$

$$(2.6) \qquad \begin{aligned} Q_{ij} + \partial\psi/\partial p_{ij} = 0. \\ Q_i = (\partial\psi/q_{i,j})_{,j} - \partial\psi/\partial q_i, \end{aligned}$$

$$\sigma_{ij,j} + f_i = 0, \tag{2.7}$$

where Q_{ij} is the internal force dual to p_{ij} and Q_i the internal force, dual to q_i. Note that Q_{ij} and q_i must satisfy the dissipation inequalities:

$$Q_{ij}\dot{q}_{ij} > 0 \tag{2.8}$$

$$Q_i\dot{q}_i > 0 \tag{2.9}$$

whenever $\|Q_{ij}\| \neq 0$, $\|Q_i\| \neq 0$, double bars denoting a Euclidean norm. As mentioned previously, the sub-class of q_i such that $Q_i = 0$, is denoted by ξ_i.

The theory is made complete by the addition of "internal constitutive equations". These are relations between Q_{ij} and $\dot{q}_{ij}$ on the one hand and Q_i and $\dot{q}_i$ on the other, whose existence is necessitated by the inequalities (2.8) and (2.9). More will be said about this point in the text to follow.

2.3. Boundary conditions

We shall limit the analysis to the case where the configuration deformation is diffusive in the sense that q_i and ξ_i are migratory motions. The following are then the boundary conditions.

On S_T, the part of the surface where tractions are given:

$$\sigma_{ij}n_j = T_i. \tag{2.10}$$

On S_u, the complement of S_T, where displacements U_i are given:

$$u_i = U_i. \tag{2.11}$$

On S_p, the permeable part of the surface:

$$\partial\psi/\partial q_{i,j}n_j = 0, \tag{2.12}$$

while on S_i, the impermeable complement of S_p:

(2.13) $$q_i = 0.$$

No boundary condition is necessary for p_{ij}.

2.4. Gibbs formulation

The Gibbs free energy density ϕ is given by Eq. (2.16) where:

(2.14) $$\phi(\sigma_{ij};\ p_{ij}:\ q_i;\ q_{i,j}) = \psi - \sigma_{ij}\epsilon_{ij}.$$

Equation (2.16) in conjuction with inequality (2.1) and the equilibrium equation, leads to the following variational inequality that pertrains to the Gibbs formulation:

(2.15) $$-\int_S u_i \delta T_i - \int_V u_i \delta f_i \geq \int_V \delta\phi dV.$$

The pertinent equations that follow, VALANIS [3], are given below. In D:

(2.16) $$\epsilon_{ij} = -\partial\phi/\partial\sigma_{ij},$$

(2.17) $$Q_{ij} = -\partial\phi/P_{ij},$$

(2.18) $$Q_i = (\partial\phi/\partial q_{i,j})_{,j} - \partial\phi/\partial q_i,$$

while the constitutive equations for Q_{ij} and Q_i remain the same. The form of the boundary conditions remains unchanged. Thus on S_T:

(2.19) $$\sigma_{ij}n_j = T_i.$$

On S_p:

(2.20) $$\partial\phi/\partial q_{i,j}n_j = 0.$$

On S_u:

(2.21) $$u_i = U_i.$$

On S_i:

(2.22) $$q_i = 0.$$

2.5. A partitioned form of ϕ

In this paper we shall posit that the energies associated with the pertaining deformation mechanisms are additive in the sense that:

(2.23) $$\phi = \phi_e + \phi_p + \phi_q$$

where

$$\phi_e = \phi_e(\sigma_{ij}), \qquad \phi_p = \phi_p(\sigma_{ij}, p_{ij}), \tag{2.24}$$

$$\phi_q = \phi_q(\sigma_{ij};\ q_{i,j};\ q_i). \tag{2.25}$$

Thus, in view of Eq. (2.15):

$$\epsilon_{ij} = \epsilon^e_{ij} + \epsilon^p_{ij} + \epsilon^q_{ij} \tag{2.26}$$

i.e., the strains are also additive.

We shall focus our attention on the variables ξ_i, the dual thermolynamic forces Ξ_i of which are zero. Thus, it suffices to write ϕ in the form:

$$\phi = \phi^*(\sigma_{ij}; p_{ij}; q_i; q_{i,j}) + \phi_\xi(\sigma_{ij}; \xi_i; \xi_{j};_j). \tag{2.27}$$

We shall, however, put mathematical constraints on ϕ^* and ϕ_ξ. Ve shall require that ϕ^* be continuous in its variables but that ϕ_ξ be continuous n ξ_i and $\xi_{i \cdot j}$. The physical ramifications of these constraints will become evident in what follows.

2.6. The form of ϕ_ξ

The theory proposed here is linear and thus ϕ is quadratic in its arguments. In the full expansion of ϕ, the part attributed to ξ_i and its gradient is given in Eq. (2.30):

$$\phi_\xi = -(1/2)\Sigma_{ij}\xi_i\xi_j + (1/2)C\xi_{i,j}\xi_{i,j} \tag{2.28}$$

where Σ_{ij} is a material function of stress and C is a positive material constant. This form gives rise to material instabilities of the Portevin Le-Chatelier type as shown previously by VALANIS [4].

2.7. Constitutive questions

Inequality (2.1) becomes an equality when $\delta\Psi$ is expressed in terns of the variation δD^* of the dissipation D^*:

$$\delta\Psi = \int\limits_S T_i \delta u_i dS + \int\limits_V f_i \delta u_i dV - \int\limits_V \delta D^* dV \tag{2.29}$$

where

$$\delta D^* = Q_i \delta q_1 > 0 \qquad \text{for all} \quad \|Q_i\| \neq 0. \tag{2.30}$$

Inequality (2.30) is a *Variational Inequality* that ensures that no free energy may be extrated from a material by means of an 'external agency' to which the variations δq_i are due. Its mathematical significance is that of all the variations δq_i only some, a subset, are admissible, i.e., those that satisfy the inequality $Q_i \delta q_i > 0$, for all $\|Q_i\| \neq 0$.

In the event that $\delta q_i = \dot{q}_i \delta t$, where $\dot{q}_i$ is the actual rate of q_i due to the actual physical process, then the variational inequality becomes the Dissipation Inequality, i.e.,

(2.31) $$\dot{D}^* = Q_i \dot{q}_i > 0.$$

As argued previously, also for local thermodynamics, VALANIS [7], Q_i and $\dot{q}_i$ are thus related. The most obvious such relation is the one where Q_i is linear and homogeneous in $\dot{q}_i$, i.e.,

(2.32) $$Q_i = b_{ij}\dot{q}_j; \qquad b_{ij}\dot{q}_i\dot{q}_j > 0$$

which may or may not be a good description of the process at hand.

This relation is, however, a member of a broader class of relations for which:

(2.33) $$Q_i = 0 \quad \text{for all} \quad \dot{q}_i = 0.$$

This is the class that we use in this paper. Most probably, the number of relations in this class is infinite and we shall not attempt to enumerate all these in this paper, even if we could.

Since, however, ξ_i are subsets of q_i that represent terminal points in the metastable process, they may be defined by the operational relation:

(2.34) $$q_i = \xi_i, \quad Q_i = 0.$$

3. The Portevin-Le Chatelier effect

Previous theoretical work on this effect appears in the papers by AIFANTIS [8, 9, 10]. There it was demonstrated that the field equations that govern the spatial and temporal evolution of the dislocation densities, have analytical features that give them the capability for describing this effect.

Here, the deformation mechanism of interest results in a series of *contained* collapse events, not associated with a continuous plastic process brought about by coordinated intergranular slip. A collapse process is a spontaneous structural rearranngement that *takes place at constant stress.* We conclude that the internal variables of the collapse process are not those of continuous plastic deformation, but others that obey different internal constitutive laws.

REMARK

It is of historical interest that DILLON [6], observed a small amount of plastic strain between metastable states but LUBAHN [5] did not. Thus in the "Lubahn solid" the deformation between metastable states is elastic. In this "ideal" solid: $\phi = \phi_e + \phi_\xi$, i.e., ϕ is independent of p_{ij} and q_i. More generally, by virtue of the constraints on ϕ^* and ϕ_ξ, the collapse to a new configuration is brought about through the internal fields ξ_i, which are not present in the continuous plastic process.

In effect: (i) $\phi_\xi = 0$ when $\xi_i = 0$; (ii) the variables ξ_i are constant during (continuous) plastic deformation; (iii) during collapse, $\boldsymbol{\sigma}$, hence $\boldsymbol{\epsilon}^e$, is constant; moreover, p_{ij} and q_i are also constant, since $\boldsymbol{\sigma}$ is *continuous* in these variables, the agents of the plastic process (local and non-local). Hence $\boldsymbol{\epsilon}^p$ and $\boldsymbol{\epsilon}^q$ are also constant. Thus, in Eq. (2.27), ϕ^* is constant during the collapse process.

To proceed with the analysis, we begin with Eq. $(2.6)_2$, which in view of Eq. (2.34), gives rise to the following defining equation for ξ_i:

$$(\partial\phi_\xi/\partial\xi_{i,j})_{,j} - \partial\phi_\xi/\partial\xi_i = 0 \tag{3.1}$$

in D. We further set Σ_{ij} to be linear and homogeneous in σ_{ij}. But since all non-elastic mechanisms are assumed to lead to isochoric deformation, then Σ_{ij} must be linear and homogeneous in the deviatoric stress tensor s_{ij}. Thus, Eq. (2.28) becomes:

$$\phi_\xi = -(B/2)s_{ij}\xi_i\xi_j + (C/2)\xi_{i,j}\xi_{i,j}. \tag{3.1$'$}$$

Without loss of generality, we set B equal to unity. Then in view of Eq. (3.1), the following second order partial differential equation for ξ_1 is obtained in D:

$$C\xi_{i,jj} + s_{ij}\xi_j = 0. \tag{3.2}$$

3.1. The boundary condition

We shall take the position that the boundary is permeable. This must be true if the material is uniform. In essence, since there is internal diffusion, the boundaries of all internal sub-domains are permeable. Hence the external boundary is also permeable. This conclusion is confirmed by the demonstrated agreement between the theory and the experimental results.

Hence, $S = S_p$ and thus on S:

$$\xi_{i,j}n_j = 0. \tag{3.3}$$

The strain ϵ^ξ_{ij}, which the ξ-field contributes to the total strain is given, in the light of Eq. (2.18), by Eqs. (3.4):

$$\epsilon^\xi_{ij} = \xi_i\xi_j - (1/3)\xi_k\xi_k\delta_{ij}. \tag{3.4}$$

Note that $\epsilon^\xi_{ii} = 0$.

4. Metastable behavior of a flat bar under tension

The problem of the flat bar under tension was solved in a previous work, VALANIS [4]. Because of its relative simplicity and the fact that it illustrates the essential points of metastable behavior, it merits a (brief) review which we give below.

We consider the case of a rectangular domain of length a in direction $x(x_1)$, width b in direction $y(x_2)$ and of small thickness c in direction $z(x_3)$. Thus, $0 < x < a$, $0 \leq y \leq b$ and $0 \leq z \leq c$. Further, we consider solutions such that ξ_i is constant in the thickness direction z. A uniform *tensile* traction T is applied on the boundaries $x = 0$, a, while the boundaries $y = 0, b$ and $z = 0, c$ are traction-free. A uniform stress state prevails throughout the plate by virtue of the equation of equilibrium. The deviatoric stress components s_{ij} are zero for $i \neq j$, while $s_{11} = s = 2T/3$, $s_{22} = s_{33} = -s/2$ where T is a positive scalar. The boundary condition (3.3) now becomes: $\partial\xi_i/\partial x = 0$, $x = 0, a$; $\partial\xi_i/\partial y = 0$, $y = 0, b$.

We seek solutions that lead to repetitive, patterned material structures and thus are *periodic* in the variables x_i. In light of the boundary condition (3.3), the only admissible solution, VALANIS [4] is the following:

$$\xi_i = A_i \cos(n\pi x/a)\cos(m\pi y/b), \tag{4.1}$$

where m and n are integers and A_i are eigenvectors of the characteristic equation:

$$\lambda \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} s_{11} & & \\ & s_{22} & \\ & & s_{33} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, \tag{4.2}$$

where

$$\lambda = C\{(n\pi/a)^2 + (m\pi/b)^2\}. \tag{4.3}$$

Thus it follows that:

$$\lambda = s = (2/3)T, \tag{4.4}$$

$$A_2 = A_3 = 0. \tag{4.5}$$

In view of Eq. (4.4), *a solution exists only for characteristic values of the traction* T. However, A_1 is indeterminate.

The strain field is shown in Eq. (4.6):

$$\epsilon^{\xi}_{ij} = \begin{pmatrix} 2/3 & & \\ & -1/3 & \\ & & -1/3 \end{pmatrix} A_1^2 \cos^2 m\pi(x/a)\cos^2 n\pi(y/b). \tag{4.6}$$

REMARK

The field equation predicts that spontaneous collapse (and self-organization) occurs when the maximum (tensile) principal stress σ reaches an *eigenstate*, at which it attains a characteristic value T_c

$$T_c = (3/2)C(\{(n/a)^2 + (m/b)^2\}\pi^2, \tag{4.7}$$

where m and n are positive integers. The integers m and n are independent, giving rise to denumerably infinite metastable states. Hovewer, given n, the collapse traction T is a maximum for $m = n$. If the material resists collapse as long as possible then it will until $m = n$. This is the maximal traction hypothesis which we adopted previously, VALANIS [4].

Thus setting $m = n$ in Eq. (4.7) we find the following simple relation for the collapse traction T_c:

$$T_c = T_{co}n^2 \tag{4.8}$$

where

$$T_{co} = (3/2)C\pi^2 \left[(1/a)^2 + 1/b)^2\right], \tag{4.9}$$

or

$$\boxed{\sqrt{T_c} = \sqrt{T_{co}}n.} \tag{4.10}$$

Hence, the square roots of the collapse tractions are proportional to the ordinal number of their occurrence.

Two things remained to be found: (a) the constant $\sqrt{T_{co}}$ and the ordinal number n_c at which the first collapse takes place. To this end, Eq. (4.10) was written in the form of Eq. (4.11):

$$\sqrt{T_c} = \sqrt{T_{co}}(n_c + r). \tag{4.11}$$

Setting the cross-sectional area of the bar equal to unity (with no loss of generality), a plot of $\sqrt{T_c}$ versus r gave a straight line with a slope 1.972, while setting $r = 0$ determined n_c, found to be equal to 14. Thus:

$$\sqrt{T_c} = 1.972(14 + r), \tag{4.12}$$

$r = 0, 1, 2...m$. The match with individual experimental and calculated values of traction loads and ordinal numbers, shown in Table 1 below, is very close despite reading errors as well as experimental errors.

Table 1. Comparison of observed and calculated tractions and ordinal numbers

$r = 0, 1, 2 \ldots m$	0	1	2	3	4	5
Observed P_c	760	860	980	1120	1260	1420
Calculated P_c	764	878	998	1127	1263	1407
Calculated $n_c + r$	14	15	16	17	18	19

5. Metastable states in torsion of a solid cylinder

In this section we apply the theory to predict theoretically the experimental results of DILLON [6], in which solid cylinders made of aluminium 1100 were subjected to pure torsion. Observations were made of the torque at which a cylinder collapsed to a subsequent stable state. A depiction of the experiments is given in Fig. 1.

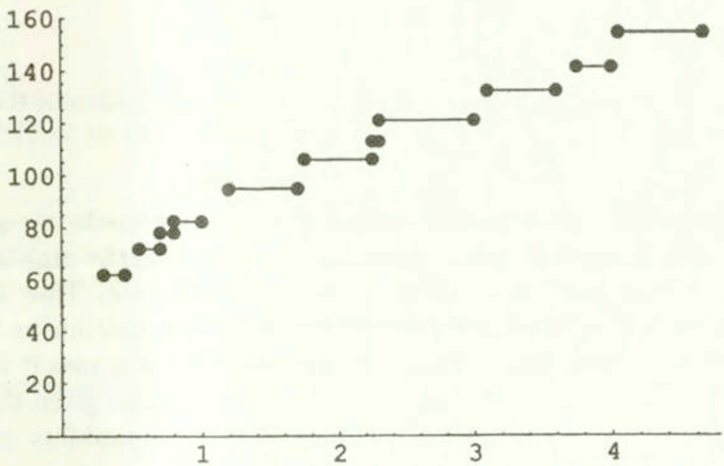

FIG. 1. Experimental values of torque (in 1b) with initial and terminal values of twist (no units) at points of instability.

To this end, we let r, θ and z be the polar coordinates of a cylinder radius a and height h, subjected to pure torsion. We now recall Eq. (3.2), which is the equation of metastable motion:

$$C\nabla^2 \xi_i + s_{ij}\xi_j = 0, \tag{5.1}$$

where ∇^2 is the Laplacian operator, given in cylindrical coordinates by Eq. (5.2):

$$\nabla^2 \xi_i = \partial^2 \xi_i / \partial r^2 + (1/r) \partial \xi_1 / \partial r + (1/r^2) \partial^2 \xi_i / \partial \theta^2 + \partial \xi_i / \partial z^2. \tag{5.2}$$

We exclude solutions that depend on θ and are thus periodic in θ, because then either ξ_i or $\partial \xi_i / \partial \theta$ will vanish at specific points on the circumference. This is not warranted since all points on the circumference of a continuum have equal status. Equation (5.2) thus becomes:

$$\nabla^2 \xi_i = \partial^2 \xi_i / \partial r^2 + (1/r) \partial \xi_i / \partial r + \partial^2 \xi_i \partial z^2. \tag{5.3}$$

The boundary condition is given by Eq. (3.3), i.e., $\xi_{i,j} n_j = 0$ on S.

It remains to determine the stress tensor s_{ij} in the Cartesian coordinate system x_i. The components of s in the polar coordinate system are zero with the exception of $s_{\theta z} = s_{z\theta} = \tau$. A tensor transformation gives then the following values of s_{ij} in the x_i-system:

$$s_{ij} = \begin{pmatrix} 0 & 0 & -\tau \sin\theta \\ 0 & 0 & \tau \cos\theta \\ -\tau \sin\theta & \tau \cos\theta & 0 \end{pmatrix}. \tag{5.4}$$

DIGRESSION

When the stress field σ_{ij} is not explicity known, the solution of the equation,

$$C \nabla^2 \xi_i + s_{ij} \xi_j = 0 \tag{5.5}$$

faces impossible difficulties unless certain hypotheses are made in regard to the mechanism of collapse. The difficulty lies in the fact that the amplitude of the inelastic strain field in all metastable states is indeterminate. Thus the concept of a constitutive equation whereby the stress tensor is a continuous function of the history of the strain tensor, must be abandoned. In this case it is no longer possible to determine, in general, the stress field in a domain given the tractions on the surface. The reason is that the condition of compatibility is no longer applicable since the deformation is not known.

These considerations do not apply in our particular case, because the stress field in a solid cylinder under torsion is statically determinate – what will be discussed shortly.

5.1. Stress field cylinder under torsion

In a cylindrical coordinate system (r, θ, z) the components of the stress tensor are zero with the exception of $\sigma_{\theta z} \equiv \tau$. As a result of the stress field, the equilibrium conditions are:

$$\partial\sigma_{\theta z}/\partial z = \partial\sigma_{\theta z}/\partial\theta = 0 \tag{5.6}$$

and therefore $\sigma_{\theta z} = \tau(r)$. It follows that if on the free ends ($z = 0, h$), the tractions $\tau(0, r) = \tau(h, r)$ are prescribed as functions of r, then $\sigma_{\theta z}$ is (statically) determinate.

In the experiments by Dillon, cited previously, a torque T_E (experimental) was applied to one cylinder end, while the other was held fixed. It is not known whether the experimental shear stress distribution $\tau(r)$ was *exactly* linear in r, resulting in a torque $T_L(= T_E)$. If it was not, then the experiment consisted in applying a torque T_L together with a self-equilibrating stress system, very localized at the cylinder end. We will thus proceed on the basis that a torque T_L was applied in the form that $\tau = \tau_0 r/a$, where a is the radius of the cylinder. Our position will be justified by the findings.

Substitution of Eq. (5.1) in (5.4) gives rise to the following set of equations:

$$\begin{aligned} C\nabla^2\xi_1 &= +\tau\sin\theta\,\xi_3, \\ C\nabla^2\xi_2 &= -\tau\cos\theta\,\xi_3, \\ c\nabla^2\xi_3 &= +\tau\sin\theta\,\xi_1 - \tau\cos\theta\,\xi_2. \end{aligned} \tag{5.7}$$

It is shown in the Appendix that this set of simultaneous partial differential equations reduces to the following uncoupled set of equations in ξ_i:

$$C\nabla^2\xi_i + \tau\xi_i = 0 \tag{5.8}$$

which in the light of the linear dependence of τ on r becomes:

$$C\nabla^2\xi_i + \tau_0(r/a)\xi_i = 0. \tag{5.9}$$

The boundary conditions in the cylindrical coordinate system are:

$$\begin{aligned} &\partial\xi_i/\partial r = 0 \quad (r = a); \\ &\partial\xi_i/\partial z = 0 \quad (z = 0, h). \end{aligned} \tag{5.10}$$

5.2. Solution of equation (5.9)

Because the shear stress τ is a function of r only, it is useful to begin with the case where ξ_i are functions of r only, i.e., $\xi_i = \xi_i(r)$. It is pointed out that this is not a maximal field solution. However, it will be shown that the collapse stresses of the maximal field solution are very close to the ones obtained here.

In view of Eq. (5.3), the differential equation for ξ_i is given below:

$$C[d^2\xi_i/dr^2 + (1/r)d\xi_i/dr] + \tau_0(r/a)\xi_i = 0 \tag{5.11}$$

with the boundary conditions $d\xi_i/dr = 0$, $r = a$, and the boundedness condition $\|\xi_i\| < \infty$, $0 \le r \le a$.

We introduce the transformation $\rho = r/a$, whereupon Eq. (5.11) becomes:

$$d^2\xi_i/d\rho^2 + (1/\rho)d\xi_i/d\rho + \tau^*\rho\xi_i = 0 \tag{5.12}$$

where $\tau^* = \tau_0 a^2/C$.

To solve this equation we appeal to the "generic" equation:

$$d^2\xi_i/d\rho^2 + (1/\rho)d\xi_i/dr + (bc\rho^{c-1})^2\, xi_i = 0 \tag{5.13}$$

which has the solution, BOAS [11]:

$$\xi_i = A_i J_0(br^c) + B_i Y_0(br^c), \tag{5.14}$$

where J_0 and Y_0 are the Bessel functions of order zero, and A_i and B_i are undetermined constants of the solution. To obtain Eq. (5.11) from Eq. (5.14) we set $c = 3/2$ and thus:

$$2\sqrt{\tau^*/3} = b = (2a/3)\sqrt{(\tau_0/C)}. \tag{5.15}$$

Because ξ_i are bounded in the domain $0 \le r \le a$ and in light of the fact that $Y(0) = \infty$, we set $B_i = 0$ and thus:

$$\xi_i = A_i J_0(b\rho^{3/2}). \tag{5.16}$$

The remaining boundary condition, at $\rho = 1$, is $d\xi_i/d\rho = 0$. Hence:

$$J_1(b) = 0 \tag{5.17}$$

where J_1 is the Bessel function of order one.

It follows that a non-null solution for ξ_i exists only for characteristic values of b, these being the zeros of J_1. The amplitudes A_i remain *indeterminate.*

In physical terms, the deformation of the cylinder is continuous in the applied torque, except for specific values of b when the deformation is augmented by the spontaneous appearance of the gradient fields ξ_i. The magnitude of the augmentation, however, is indeterminate. These characteristic values of b are points of instability.

5.3. Values of the toeque at points of instability

The relation between τ_0 and the applied torque is given below:

$$\tau_0 = (2/\pi a^4)T. \tag{5.18}$$

In view of Eq. (5.15),

$$kj_1^{(n)} = \sqrt{T_n}, \tag{5.19}$$

where the constant k is given by Eq. (5.20),

$$k = (3a/2)\sqrt{(C\pi/2)}\,. \tag{5.20}$$

Moreover, $j_1^{(n)}$ is the n-th zero of J_1 and T_n is the value of the applied torque at which the n-th instability occurs. In other words: *the square root of the n-th collapse torque is proportional to the n-th zero of the Bessel function* J_1.

Two issues arise at this juncture: (a) the first measurable collapse torque and (b) the determination of the constant k. In our previous work, in connection with instabilities in a uniform stresses field, as in the case of in tension of a uniform bar and torsion of a hollow cylinder, we found that the first collapse traction was found to occur at a high value of n, generally greater than ten. For such values of n and over substantial intervals Δn, $j_1^{(n)}$ may be approximated by the formula:

$$j_1^{(n)} = j_1^* + \pi n, \qquad n > n_0;\ n_0 > 10 \tag{5.21}$$

where j_1^* is a constant. For these values of n and in view of Eq. (5.21):

$$\sqrt{T_n} = k(j_1^* + \pi n). \tag{5.22}$$

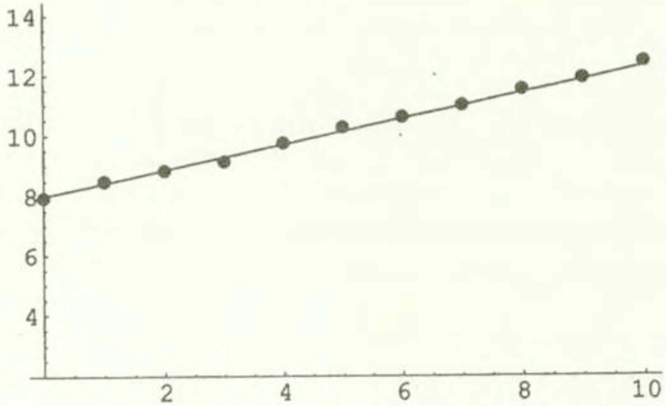

FIG. 2. Square root of experimental values of torque vs. their sequence.

In light of the above considerations, a plot of the experimental values of $\sqrt{T_n}$ versus n, should give a straight line with a slope $k\pi$. Such a plot is shown in Fig. 2, where the experimental points follow closely a straight line with a slope $k\pi$ equal to 0.421 and hence, $k = 0.134$. With this experimentally determined value of k, we have the following theoretical expression for the collapse torque T_n:

$$\sqrt{T_n} = 0.134 j_1^{(n)}. \tag{5.23}$$

This is a remarkable result.

The first observable value of T_n was found by inspection to be T_{19}. The theoretical values of T_n were then calculated and are given together with the experimental values in Table 2. The agreement is quite close.

Table 2. Experimental and theoretical values of collapse torque ($ft - lb$).

n	19	20	21	22	23	24	25	26	27	28	29
$(\sqrt{T_n})_{ex}$	7.94	8.50	8.83	9.24	9.75	10.3	10.6	11.0	11.5	11.9	12.4
$(\sqrt{T_n})_{th}$	8.10	8.52	8.94	9.36	9.79	10.2	10.6	11.0	11.5	11.9	12.3
$j_1^{(n)}$	60.47	63.61	66.75	69.90	73.04	76.18	79.32	82.46	85.60	88.71	91.89

5.4. Maximal internal fields

We recall that in the case of a flat bar (Sec. 1), the solution for the internal vector field was given by Eq. (4.1), shown here as Eq. (5.24):

$$\xi_i = A_i \cos(n\pi x/a)\cos(m\pi y/b). \tag{5.24}$$

Given n, denumerably infinite set of m may be chosen in the solution. However, it was argued in that section that the *maximal traction* hypothesis requires that $m = n$. The fields that satisfy this conditions were called *maximal fields*.

The solution obtained above is *not* a maximal field solution, yet the agreement between the theoretical collapse torque values and their experimental counterparts was excellent. Is there any inconsistency? We proceed to examine this question.

To this end, we write Eq. (5.11), in cylindrical coordinate system:

$$C[\partial^2\xi_i/\partial r^2 + (1/r)\partial\xi_i/\partial r + \partial^2\xi_i/\partial z^2] + \tau_0(r/a)\xi_i = 0. \tag{5.25}$$

We introduce the transformation, $\rho = (r/a)$, $\zeta = (z/h)$, $0 \le \rho \le 1$, $0 \le \zeta \le 1$, whereupon Eq. (5.25) becomes:

$$\partial^2\xi_i/\partial\rho^2 + (1/\rho)\partial\xi_i/\partial\rho + (a/h)^2\partial^2\xi_i/\partial\zeta^2 + \tau^*\rho\xi_i = 0 \tag{5.26}$$

where $\tau^* = (a^2\tau_0/C)$. The following set of maximal field solutions satisfy the boundary condition $(5.10)_2$:

$$\xi_i^{(n)} = R_n(\rho)\cos(n\pi\zeta), \tag{5.27}$$

$n = 1, 2, ...\infty$, where $R^{(n)}$ satisfies the differential Eq. (5.28):

$$d^2R^{(n)}/d\rho^2 + (1/\rho)dR^{(n)}/d\rho - (a\pi/h)^2 n^2 R^{(n)} + \tau_n^* \rho R^{(n)} = 0. \tag{5.28}$$

Evidently R_n is not a periodic function and thus it must be interpreted as having n (albeit decreasing) half-oscillations in the domain $0 \le \rho \le 1$, just like $\cos n\pi\zeta$ in the domain $0 \le \zeta \le 1$.

REMARK

Note that as $h \to \infty$ at constant radius a, Eq. (5.28) tends to (5.12). We can thus say that the solution $\xi_i = \xi_i(r)$ pertains to a cylinder of infinite length. We call the n-th eigenvalue of this solution $(\tau_n^*)_\infty$.

5.5. Solution of equation (5.28)

DILLON [6] tested specimens of diameter of 0.5 inch. ($a = 0.5$) and length of either 9 or 19 inches. He did not specify the specimen length in the test reported. We shall take the conservative position that $h = 9$ – this will become clear shortly – in which event $(h/\pi a) = h^* = 11.46$. The object then is to determine τ_n^* such that the solution $R^{(n)}(\rho)$ has (a) n half-oscillations and (b) a derivative $dR^{(n)}/d\rho$ that vanishes at $\rho = 1$.

REMARK

Because the solution $\xi_i(r)$, independent of z, depicts collapse torque values that were close to their experimental counterparts, we reason that τ_n^* must be close to $(1.5j_1^n)^2$, which are the values of $(\tau_n^*)_\infty$. Furthermore, since:

$$\sqrt{T_n} = 0.134(2/3)\sqrt{\tau_n^*} \tag{5.29}$$

and because the first observable T_n was T_{19}, we also reason that the first τ_n^* is in fact τ_{19}^*.

The solution then consisted in finding the eigenvalue τ_{19}^* such that $R^{(n)}$ had 19 half-oscillations and its derivative $dR^{(n)}/d\rho$ vanished at $\rho = 1$. The solution was found numerically by means of the software NDSolve of MATHEMATICA and the function $R^{(19)}$ is shown in Fig. 3. It has 19 half-oscillations. Its derivative function $dR^{(n)}/d\rho$ shown in Fig. 4, passes through zero at $\rho = 1$ The eigenvalue τ_{19}^* was found to be equal to 8235. This value compares with 8227 which was the value of $(\tau_{19}^*)_\infty$. The difference between the two is of the order of 0.1% which is well within the experimental error. The same was found to be true of all other τ_n^*. For instance τ_{29}^* was found to be 19017. The corresponding value of $(\tau_{29}^*)_\infty$ was 18998. Again the difference between them is about 0.1%.

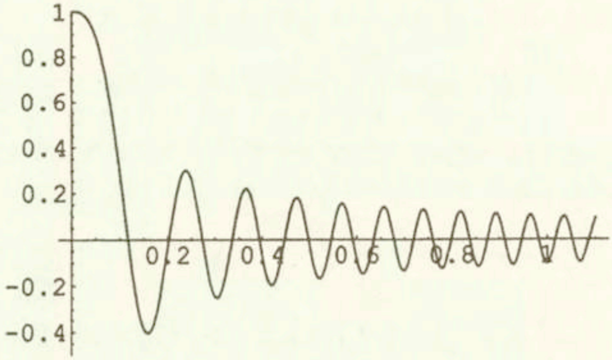

FIG. 3. Eigenfunction $Rn(r); n = 19$.

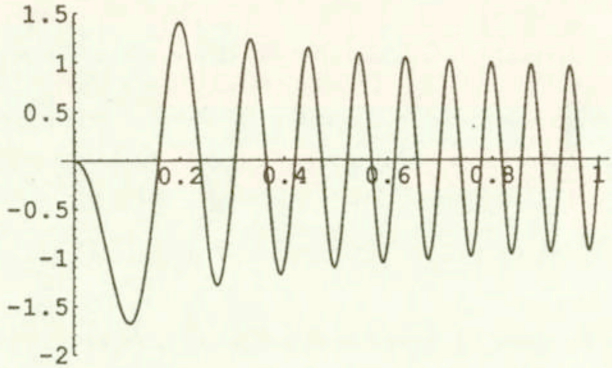

FIG. 4. First derivative of $Rn(r) : n = 19$.

CONCLUSION 1

A parametric study showed that the two values tend to come closer together as $(a\pi/h)^2$ tends to zero, not surprisingly. Thus below a certain value of this parameter, the length of the specimen does not affect the values of the collapse torques. This theoretical finding suggests an interesting experimental program to determine the effect of the cylinder geometry on the values of the collapse torques. The theory predicts that they are proportional to the square of the cylinder radius and become independent of the lenght as the latter becomes asymptotically infinite.

CONCLUSION 2

In the case of the maximal field solutions and for a cylinder geometry such that $(h/\pi a) > 10$, as in the case of the specimens tested, the values of the collapse torques are given within experimental error, by the relation:

$$\sqrt{T_n} = kj_1^{(n)}, \tag{5.30}$$

where k is a material constant, in this case 0.134. This is an unexpected and remarkable result.

The agreement between the predictions of Eq. (5.30) and the experimental data of DILLON [6], is illustrated in Fig. 5. There, the dots are the experimental points and the lines are the eigenstates, i.e., the "torque states" at points of instability. Use was made of Eq. (5.22).

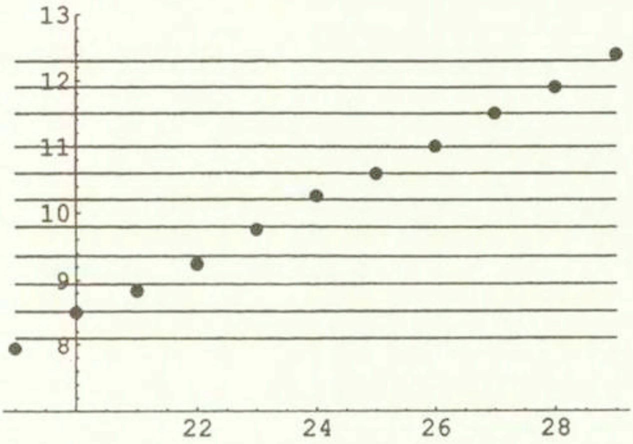

FIG. 5. Experimental points of instability vs. theoretical eigenstates (lines) ordinate: square root of torque; abscissa: ordinal numbers.

Apendix

We begin with Eqs. (5.7) in D, i.e.,

$$\begin{aligned} C\nabla^2\xi_1 &= +\tau\sin\theta\,\xi_3, \\ C\nabla^2\xi_2 &= -\tau\cos\theta\,\xi_3, \\ C\nabla^2\xi_3 &= +\tau\sin\theta\,\xi_1 - \tau\cos\theta\,\xi_2. \end{aligned} \tag{A.1}$$

A vector transformation gives the following relation between (ξ_1, ξ_2) and (ξ_r, ξ_θ):

$$\begin{aligned} \xi_r &= \xi_1\cos\theta + \xi_2\sin\theta, \\ \xi_\theta &= -\xi_1\sin\theta + \xi_2\cos\theta. \end{aligned} \tag{A.2}$$

Also:

$$\begin{aligned} \xi_1 &= \xi_r\cos\theta - \xi_\theta\sin\theta, \\ \xi_2 &= \xi_r\sin\theta + \xi_\theta\sin\theta. \end{aligned} \tag{A.3}$$

Multiply Eqs. $(A.1)_1$ and $(A.1)_2$ by $\cos\theta$ and $\sin\theta$ respectively, add and use Eqs. $(A.2)_1$ to obtain:

$$\nabla^2 \xi_r = 0. \tag{A.4}$$

A. 1.1. Boundary condition

The boundary condition on S is:

$$\xi_{i,j} n_j = 0. \tag{A.5}$$

In the cylindrical coordinate system Eq. (A.5) on the cylindrical surface $r = a$ becomes:

$$(\partial \xi_i / \partial r) r_{,j} n_j + (\partial \xi_i / \partial \theta) \theta_{,j} n_j = 0. \tag{A.6}$$

However, ξ_i is independent of θ and $r_{,i} = n_i$. Thus Eq. (A.6) reduces to:

$$\partial \xi_i / \partial r = 0. \tag{A.7}$$

Thus and in ligth of Eqs. $(A.2)_1$ and $(A.2)_2$:

$$\partial \xi_r / \partial r = 0; \quad \partial \xi_\theta / \partial r = 0 \tag{A.8}$$

while on the flat surface $z = 0, h$:

$$\partial \xi_i / \partial z = 0 \tag{A.9}$$

in view of which Eqs. (A.4):

$$\partial \xi_r / \partial z = 0, \quad \partial \xi_\theta / \partial z = 0. \tag{A.10}$$

We now use the classical result that, if the Laplacian of a function f vanishes in D while on S: $\partial f / \partial n = 0$, then the function is at most a constant in D. We set this constant equal to zero. Thus in view of Eqs. (A.8) and (A.10) it follows that:

$$\xi_r = 0. \tag{A.11}$$

Hence, in light of Eqs. (A.3) we find that:

$$\xi_1 = -\xi_\theta \sin\theta; \quad \xi_2 = \xi_\theta \cos\theta. \tag{A.12}$$

Hence, from Eq. $(A.1)_3$ on the one hand and Eq. $(A.1)_1$ and $(A.1)_2$ on the other:

$$C\nabla^2 \xi_3 + \tau \xi_\theta = 0; \quad C\nabla^2 \xi_\theta + \tau \xi_3 = 0. \tag{A.13}$$

A solution to the Eq. (A.13) is:

$$\xi_3 = \xi_\theta. \tag{A.14}$$

It is now a short step to show from Eqs. (A.1) and (A.12) that:

$$C\nabla^2\xi_i + \tau\xi_i = 0 \tag{A.15}$$

which is the result that was to be shown.

A. 1.2. Uniqueness of the solution

If another solution exists then $\xi_3 \neq \xi_\theta$. Let $\xi = \xi_3 - \xi_\theta$. Then in view of Eqs. (A.13) and setting $\tau^* = \tau_0/Ca$:

$$\nabla^2\xi = \tau^* r\xi \tag{A.16}$$

or,

$$\partial[r(\partial\xi/\partial r)]/\partial r + r\partial^2\xi/\partial z^2 = \tau^* r^2\xi. \tag{A.17}$$

Thus, if a solution $\xi \neq 0$ exists, it must satisfy Eq. (A.17) with $\tau^* > 0$ and the boundary conditions $\partial\xi/\partial r =$ on $r = a$, and $\partial\xi/\partial z = 0$ on $z = 0, h$ by virtue of Eqs. (A.9) and (A.10).

We multiply both sides of Eq. (A.17) by ξ and integrate over the domain D using integration by parts and the indicated boundary conditions to find the following result:

$$-\int_0^h\int_0^a r(\partial\xi/\partial r)^2 dr dz - \int_0^h\int_0^a r(\partial\xi/\partial z)^2 dr dz = \tau^*\int_0^h\int_0^a r^2\xi^2 dr dz. \tag{A.18}$$

Since the left-hand side of Eq. (A.18) is non-positive (allowing the set of solutions to include $\xi =$ constant) and the integral on the right-hand side is positive, a solution can exist only for non-positive values of τ^*. But τ^* is positive and thus a non-null solution for ξ does not exist.

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