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On interactions of frictional cracks

M. BASISTA

Polish Academy of Sciences
Institute of Fundamental Technological Research
ul. Świętokrzyska 21, 00-049 Warszawa, Poland

THIS PAPER DEALS with the estimation of the stress intensity factors for interacting Mode-II cracks undergoing frictional sliding under overall compressive stresses. Crack interaction effects are examined via the Kachanov method that is extended here to account for frictional and cohesive resistance on crack faces. The accuracy of the obtained results is verified through comparison with the "exact" numerical solutions obtained using a boundary element method.

1. Introduction

THE PROBLEM OF A LINEAR ELASTIC solid with a multitude of interacting, arbitrarily located, open cracks under uniform remote loading σ^{∞} has been considered by many authors over the past 20 years. A general algorithm of solving such a problem is as follows. Using the superposition principle, a multiple crack problem is reduced to a subproblem of a single crack but loaded by unknown tractions that are induced by the other cracks. The unknown tractions are then interrelated through singular integral equations which, except for a handful of special cases, do not lend themselves to rigorous analytical methods of solution. Consequently, one has to solve the problem numerically or resort to approximate techniques. As for the numerical approach, a boundary element method (BEM) proved quite effective in solving 2D multiple crack problems when formulated in terms of the complex variable method and the dislocation distribution along crack contours (cf. [1, 2]). A rather obvious reason behind the popularity of the BEM vs. FEM in crack problems is that in the BEM approach only the boundary of the problem geometry requires discretization. For multiple cracks, where a large number of model runs are necessary, this feature is of primary importance. On the other hand, the tractions on the interacting cracks were accurately approximated using Legendre or Chebyshev orthogonal polynomials ([3, 4, 5, 6, 7]). Recently, JU and TSENG [8] presented a comprehensive appraisal of these asymptotic polynomial techniques for crack interaction problems in plane elasticity.

The present note is focused on strong interactions of straight cracks randomly distributed in an infinite, homogeneous, linear elastic, two-dimensional matrix under the action of compressive loading. It is assumed that the cracks are endowed with frictional-cohesive resistance and are constrained against normal opening. In compressive stress fields these cracks may slide in Mode-II provided they are inclined at nonzero angles to the direction of maximum compression. This problem is of practical importance for geotechnical applications where materials are typically deformed under compressive overall stresses. The basic mechanisms underlying inelastic deformation of brittle solids under compression are those of tension cracking and frictional slip. Usually, these two mechanisms are coupled but there are instances where frictional sliding takes the upper hand or precedes the crack growth. For example, in low-porosity rocks some of the preexisting flaws will close under external compression and slide with friction before they sprout curvilinear wings growing subparallel to the direction of maximum compression (the so-called sliding crack mechanism, e.g. [1, 9]).

The objective of this note is to compute the stress intensity factors (SIFs) at the tips of interacting closed cracks once sliding along their faces has been initiated. For this purpose, a relatively simple yet remarkably accurate method developed by KACHANOV [6] will be extended to account for the frictional contact of cracks faces. In order to illustrate the capabilities of the proposed framework, a number of test examples will be solved and the results confronted with the numerical solutions obtained using the boundary element method. Some aspects of this note are also addressed in the conference paper [10].

2. Method

In essence, the method devised in KACHANOV [6] is based on the following central assumption: *The unknown tractions induced on a considered crack by the presence of other cracks can be approximated by the tractions that would have been acted on the considered crack if the other cracks were loaded by uniform average (normal and shear) tractions.*

To assess the validity of this statement and to fully understand its impact on the calculation of SIFs, the reader is referred to the original paper [6] or related papers [11, 12]. On the other hand, since it is intended to keep this note self-contained, the key points of the Kachanov method will be made clear in the course of the analysis to follow. Note that this method yields very good predictions for freely opening cracks even if their tips are very close one to another. There are, however, some limitations to the method which will be discussed later on.

Consider two closed cracks in an infinite, linear elastic plate (Fig. 1) with the local (crack-attached) coordinate systems (x^L, y^L) and the global coordinate system $(\mathrm{x}_1, \mathrm{x}_2)$. To facilitate drawing, only two cracks are shown but the ensuing equations are formulated for an arbitrary 2D crack array $(L = 1, 2, ..., N)$. The actual (contact) shear and normal stresses existing on the faces of L-th crack are denoted by τ_{xy}^L, σ_y^L. The boundary-value problem A can trivially be decomposed into two subproblems B, C. On the other hand, the problem C can be represented as a superposition of N subproblems, each involving only a single crack but subject to *unknown* shear and normal stresses τ_{xy}^{*L}, σ_y^{*L}. Unlike for open cracks considered in [6] where the sign convention for stresses was inconsequential thus omitted, the signs of the superimposed stresses in the compression case (Fig. 1 A-E) are strictly observed. Here, the adopted sign convention is that of continuum mechanics, i.e. compression is viewed negative. Consequently, for closed frictional cracks it follows from Fig. 1 that

$$\tau_{xy}^{*L} = \tau_{xy}^L - \left(\tau_{xy}^{\infty L} + \Delta\tau_{xy}^L\right), \tag{2.1}$$

$$\sigma_y^{*L} = \sigma_y^L - \left(\sigma_y^{\infty L} + \Delta\sigma_y^L\right), \tag{2.2}$$

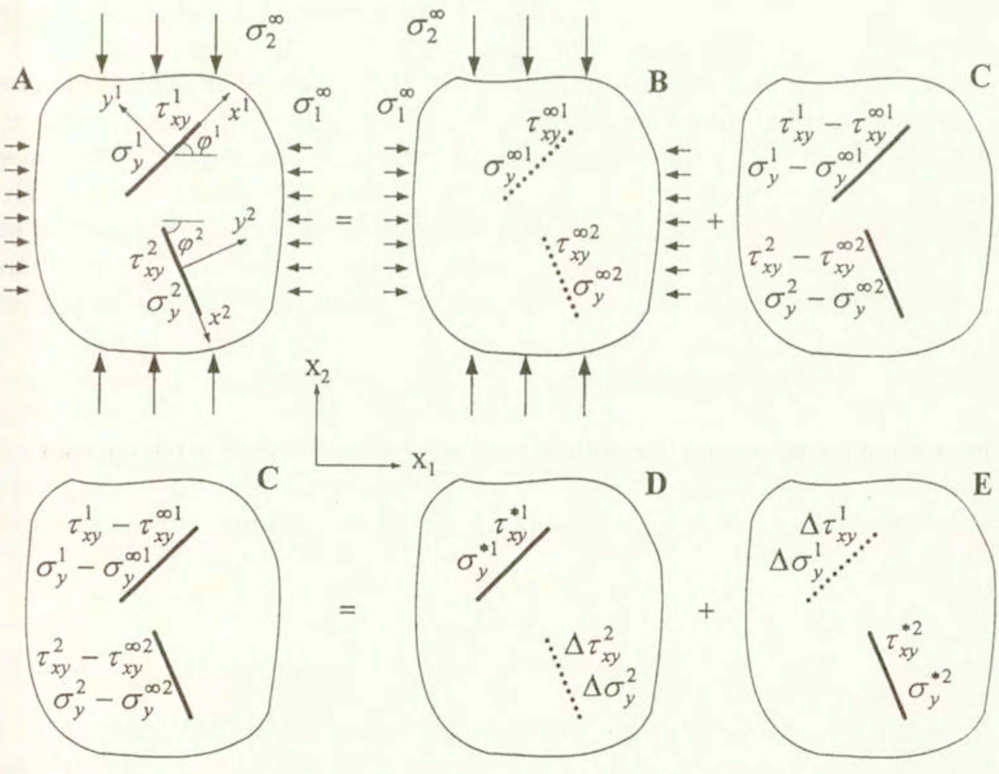

FIG. 1. Superposition of stress for interacting frictional cracks in infinite, linear elastic plate under compression.

where $\tau_{xy}^{\infty L}$, $\sigma_y^{\infty L}$ are the resolved (due to the remote loading $\boldsymbol{\sigma}^{\infty}$ shear and normal stresses in the continuous material along the line of L-th crack; $\Delta\tau_{xy}^L$, $\Delta\sigma_y^L$ are the interaction terms, i.e. shear and normal stresses generated by all other cracks along the line of L-th crack in the continuous material. Note that as long as the crack remains closed, it holds $\sigma_y^{*L} = 0$. Otherwise, $\sigma_y^{*L} = -\left(\sigma_y^{\infty L} + \Delta\sigma_y^L\right)$.

A necessary prerequisite for the computation of the SIFs is the determination of the loading of crack faces $(\tau_{xy}^{*L}, \sigma_y^{*L})$. Once τ_{xy}^{*L}, σ_y^{*L} are known, the SIFs can be computed according to the well-known formulas, [13]:

$$K_I^L(\pm c^L) = \frac{1}{\sqrt{\pi c^L}} \int\limits_{-c^L}^{c^L} \sqrt{\frac{c^L \pm \zeta}{c^L \mp \zeta}}\, p^{*L}(\zeta)d\zeta, \tag{2.3}$$

$$K_{II}^L(\pm c^L) = \frac{1}{\sqrt{\pi c^L}} \int\limits_{-c^L}^{c^L} \sqrt{\frac{c^L \pm \zeta}{c^L \mp \zeta}}\, \tau^{*L}(\zeta)d\zeta, \tag{2.4}$$

where $p^{*L} = -\sigma_y^{*L}$, $\tau^{*L} = -\tau_{xy}^{*L}$, due to the sign convention and c^L denotes the half-length of a straight crack. For closed cracks only (2.4) is relevant.

It is claimed here that irrespective of whether the interacting cracks are open or frictional, the central assumption of the Kachanov method remains valid: the unknown crack interaction stresses $\Delta\sigma_y^L$, $\Delta\tau_{xy}^L$ are induced only by uniform average tractions (as yet unknown) acting on the other cracks' faces. A far-reaching consequence of this assumption is that it yields a functional form for the unknown stresses $\Delta\sigma_y^L$, $\Delta\tau_{xy}^L$ since the problem of a single crack loaded by uniform tractions has analytical solution. Following KACHANOV [6], denote by P_{ij}^K and T_{ij}^K the standard stress fields that are generated in the continuous material by the K-th crack loaded by uniform normal and shear tractions of *unit intensity*, respectively. These standard stress fields can be computed using a suitable Westergaard function (as it is done in this paper) or can be found in textbooks on linear fracture mechanics. Hence, the crack interaction terms $\Delta\sigma_y^L$, $\Delta\tau_{xy}^L$, (generated in the continuous material) can be expressed in the following general form

$$\Delta\sigma_y^L = -n_i^{(L)} \left[P_{ij}^K \langle\sigma_y^{*K}\rangle + T_{ij}^K \langle\tau_{xy}^{*K}\rangle\right] n_j^{(L)}, \; K, L = 1, ..., N; (K \neq L), \tag{2.5}$$

$$\Delta\tau_{xy}^L = -n_i^{(L)} \left[P_{ij}^K \langle\sigma_y^{*K}\rangle + T_{ij}^K \langle\tau_{xy}^{*K}\rangle\right] m_j^{(L)}, \; K, L = 1, ..., N; (K \neq L). \tag{2.6}$$

In (2.5) and (2.6), the summation convention applies to the repeated indices K while it does not apply to the indices placed in parentheses i.e. (L); $\mathbf{n}^L$, $\mathbf{m}^L$ are crack-attached normal and tangential unit vectors; the Macauley bracket $\langle\ \rangle$ denotes the average value of the bracketed quantity. The standard stress fields

P_{ij}^K and T_{ij}^K are usually written assuming positive unit intensities of normal and shear tractions. However, in the considered case, the average normal stress (internal pressure) $\langle \sigma_y^{*K} \rangle$ and the average shear stress $\langle \tau_{xy}^{*K} \rangle$ acting on the crack faces are subject to the adopted sign convention. Hence, the minus signs in (2.5) and (2.6).

If the Eqs. (2.5) and (2.6) are to hold for the averages $\langle \sigma_y^{*K} \rangle$, $\langle \tau_{xy}^{*K} \rangle$ of an arbitrary crack, they also have to hold for $\langle \sigma_y^{*L} \rangle$, $\langle \tau_{xy}^{*L} \rangle$ of the L-th crack itself. This is nothing else but a rule of self-consistency. Applying this rule to the Eqs. (2.5) and (2.6), i.e. averaging them, leads to

$$\langle \Delta\sigma_y^L \rangle = -\Lambda_{11}^{KL} \langle \sigma_y^{*K} \rangle - \Lambda_{12}^{KL} \langle \tau_{xy}^{*K} \rangle, \tag{2.7}$$

$$\langle \Delta\tau_{xy}^L \rangle = -\Lambda_{21}^{KL} \langle \sigma_y^{*K} \rangle - \Lambda_{22}^{KL} \langle \tau_{xy}^{*K} \rangle, \tag{2.8}$$

where Λ_{ij}^{KL} are the Kachanov transmission factors (interaction matrices) defined as follows:

$$\left.\begin{aligned} \Lambda_{11}^{KL} &= n_i^{(L)} \langle P_{ij}^K \rangle^{(L)} n_j^{(L)} \\ \Lambda_{12}^{KL} &= n_i^{(L)} \langle T_{ij}^K \rangle^{(L)} n_j^{(L)} \\ \Lambda_{21}^{KL} &= n_i^{(L)} \langle P_{ij}^K \rangle^{(L)} m_j^{(L)} \\ \Lambda_{22}^{KL} &= n_i^{(L)} \langle T_{ij}^K \rangle^{(L)} m_j^{(L)} \end{aligned}\right\} (K \neq L) \tag{2.9}$$

$$\Lambda_{ij}^{KL} = 0; \quad (K = L). \tag{2.10}$$

For convenience, the notation of the transmission factors in (2.9) has been slightly changed as compared to that in the original paper [6]. For example, Λ_{21}^{KL} denotes the average shear stress (lower index 2) on crack L due to unit normal stress (lower index 1) on crack K. Incidentally, it seems that there is a misprint with regard to the formula (13b) in the original paper. Namely, in the limit case of $K = L$ (crack interaction with itself) it should be $\Lambda_{ij}^{KK} = 0$, (2.10), and not $\Lambda_{ij}^{KK} = \delta_{ij}$, as in [6]. Note that to compute the transmission factors, the standard stress fields generated by the uniformly loaded K-th crack have to be integrated along the line of the L-th crack. For a given configuration of N cracks this is usually done by numerical integration.

The actual stresses τ_{xy}, σ_y induced by the frictional-cohesive contact on the crack faces are interrelated through a law of dry friction. A simple Coulomb-Mohr law is adopted for this purpose:

$$\tau_{xy}^L = \mp\tau_c \pm \mu\sigma_y^L, \tag{2.11}$$

where τ_c is the cohesion and μ is the coefficient of dry friction, both being positive constants. In Eq. (2.11) and the equations to follow, the upper signs hold for cracks oriented at $0 < \varphi^L < \pi/2$ while the lower ones for $-\pi/2 < \varphi^L < 0$, Fig. 1.

Making use of (2.1), (2.2) and (2.11), it follows that

$$\tau_{xy}^{*L} = \mp\tau_c \pm \mu\sigma_y^{\infty L} - \tau_{xy}^{\infty L} \pm \mu\Delta\sigma_y^L - \Delta\tau_{xy}^L. \tag{2.12}$$

Averaging (2.12) and using (2.7), (2.8), the following system of N linear equations is obtained

$$\left(\delta^{KL} \pm \mu\Lambda_{12}^{KL} - \Lambda_{22}^{KL}\right)\langle\tau_{xy}^{*K}\rangle = \mp\tau_c \pm \mu\sigma_y^{\infty L} - \tau_{xy}^{\infty L}, \qquad (K, L = 1, 2, ..., N). \tag{2.13}$$

The right-hand side of (2.13) specifies the remote loading conditions and the friction-cohesion resistance on each crack's faces. In other words, it represents the effective (net) shear stress that drives the crack sliding. The crack array geometry and the influence of friction on the transmission of shear stresses are reflected by the left-hand side. The system of equations (2.13) with the transmission factors (2.9), (2.10) and a given load $(\sigma_y^{\infty L}, \tau_{xy}^{\infty L})$ of the crack faces constitute the governing system of linear algebraic equations from which the average shear stresses $\langle\tau_{xy}^{*K}\rangle$ can be computed.

If the average values of shear stresses $\langle\tau_{xy}^{*K}\rangle$ are known, it is straightforward to compute the whole distribution of τ_{xy}^{*K}. From (2.12), when combined with (2.5) and (2.6), it follows that

$$\tau_{xy}^{*L} = \mp\tau_c \pm \mu\sigma_y^{\infty L} - \tau_{xy}^{\infty L} + n_i^{(L)}T_{ij}^K\langle\tau_{xy}^{*K}\rangle\left(\mp\mu n_j^{(L)} + m_j^{(L)}\right). \tag{2.14}$$

Finally, combining (2.14) and (2.4), the K_{II} factors for interacting frictional cracks read

$$\begin{aligned} K_{II}^L(+c) &= \sqrt{\pi c^{(L)}}\left(\tau_{xy}^{\infty(L)} \pm \tau_c \mp \mu\sigma_y^{\infty(L)}\right) \\ &\quad + \frac{\langle\tau_{xy}^{*K}\rangle}{\sqrt{\pi c^{(L)}}}\left[n_i^{(L)}\left(\int_{-c^L}^{c^L}\sqrt{\frac{c^L+\zeta}{c^L-\zeta}}T_{ij}^K\,d\zeta\right)\left(\pm\mu n_j^{(L)} - m_j^{(L)}\right)\right] \\ K_{II}^L(-c) &= \sqrt{\pi c^{(L)}}\left(\tau_{xy}^{\infty(L)} \pm \tau_c \mp \mu\sigma_y^{\infty(L)}\right) \\ &\quad + \frac{\langle\tau_{xy}^{*K}\rangle}{\sqrt{\pi c^{(L)}}}\left[n_i^{(L)}\left(\int_{-c^L}^{c^L}\sqrt{\frac{c^L-\zeta}{c^L+\zeta}}T_{ij}^K\,d\zeta\right)\left(\pm\mu n_j^{(L)} - m_j^{(L)}\right)\right]. \end{aligned} \tag{2.15}$$

3. Test examples

To evaluate the predictive capability of the present model, the basic Eqs. (2.13), (2.14) (2.15) have been implemented numerically. For this purpose, a source FORTRAN code has been assembled based on a similar code for open cracks [14]. The numerical algorithm is relatively simple except for the weighted integrals in (2.15) which required some special treatment. The following three test examples of an infinite, linear elastic plate under uniaxial compression with:

- a pair of collinear inclined cracks (Fig. 2, insert),
- a pair of symmetrically inclined cracks (Fig. 3, insert),
- a pair of stacked cracks (Fig. 5, insert),

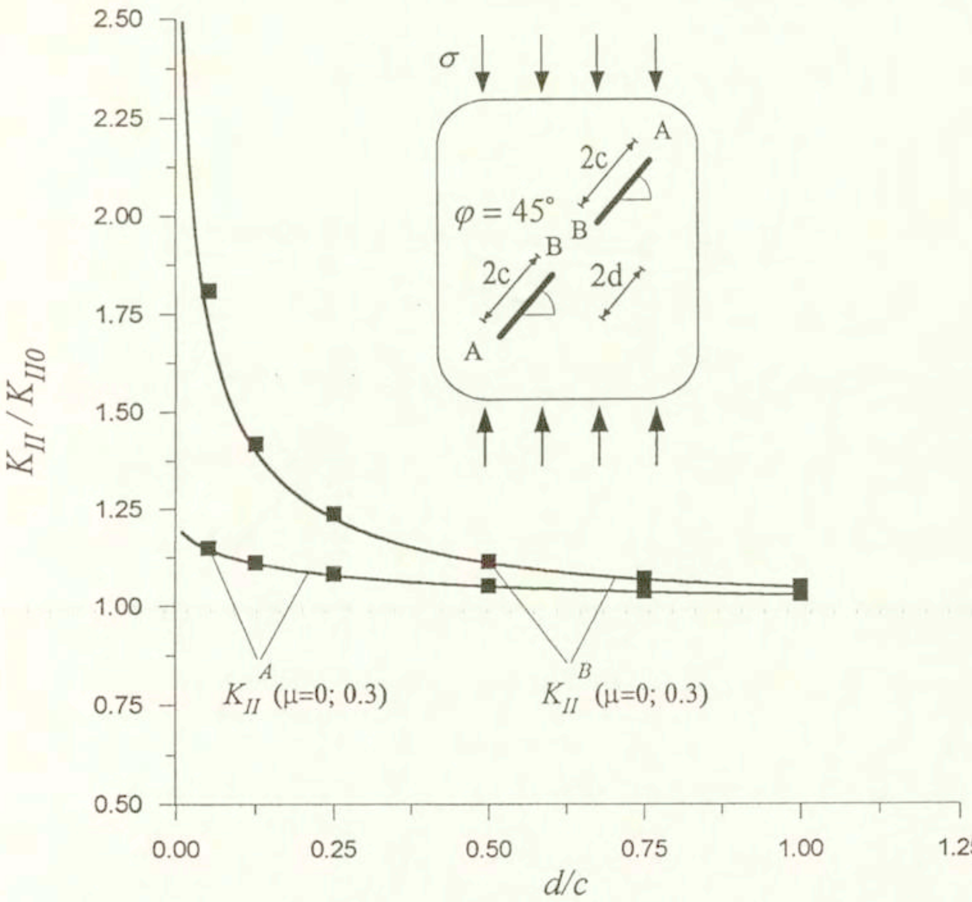

FIG. 2. Two collinear cracks under uniaxial compression. Normalized K_{II} factors vs. relative distance of crack tips for $\mu = 0.3$ and $\tau_c = 0$. Solid lines depict present solution, squares – BEM solution.

have been solved using the developed code. In parallel, the same three test problems have been solved using the BEM source code [15, 16].

In Figures 2, 3, 4, 5 the present solutions are compared with the 'exact' numerical ones obtained by means of the BEM. For the selected test examples the normalized K_{II} factors vs. the relative distance of crack tips d/c are plotted for frictionless ($\mu = 0$) and frictional contact ($\mu = 0.3$) on crack faces. The normalization factor K_{II0} is the stress intensity factor for the respective single crack under a given load with all other cracks absent. On the example of the inclined cracks (insert in Fig. 3), an intermediate step of the method, namely the computed average shear stress $\langle \tau^{*1}_{xy} \rangle$ vs. d/c, is also illustrated and compared with the respective BEM data in Fig. 4.

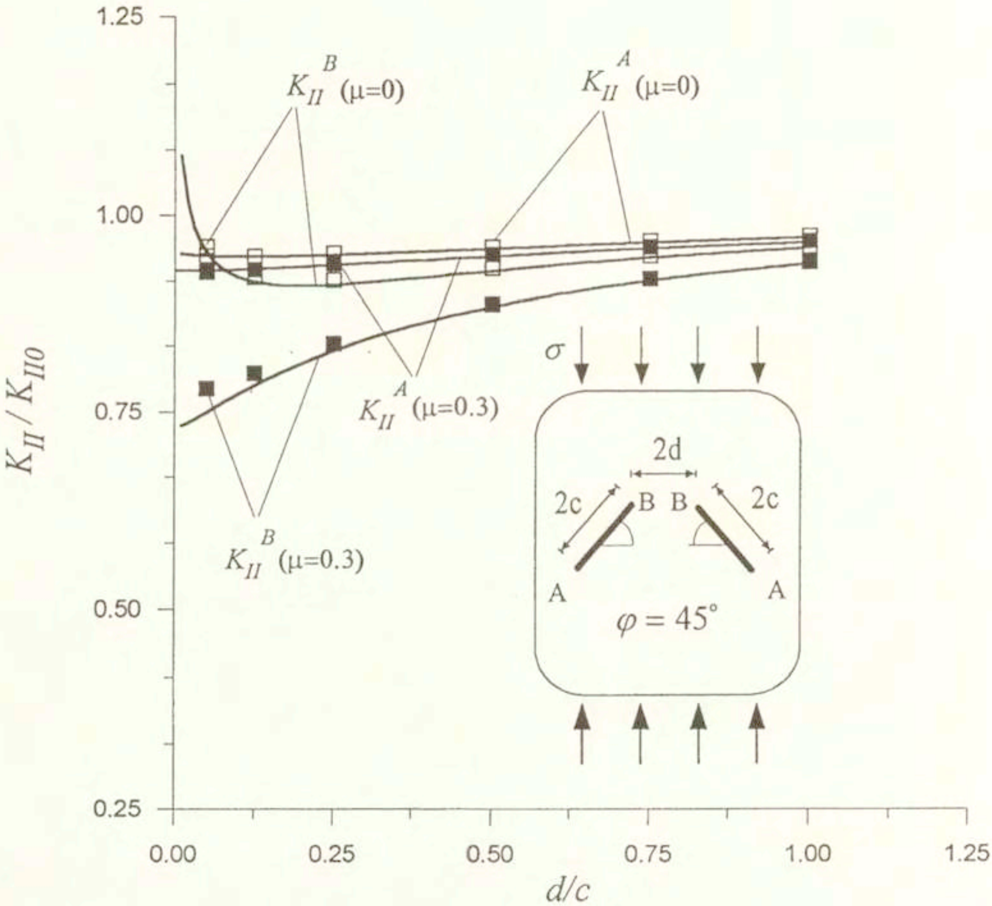

FIG. 3. Two symmetrically inclined crack under uniaxial compression. Normalized K_{II} factors vs. relative distance of crack tips. Solid curves – present solution, squares – BEM data for $\mu = 0$, $\mu = 0.3$ at $\tau_c = 0$.

Typically of open cracks, collinear configurations induce amplification of stresses and stress intensity factors as cracks approach one another. In the case of frictional cracks this effect is maintained, as can be seen in Fig. 2. On the other hand, the stacked crack configurations (Figs. 3, 4, 5) promote shielding (for the most part) as d/c becomes smaller.

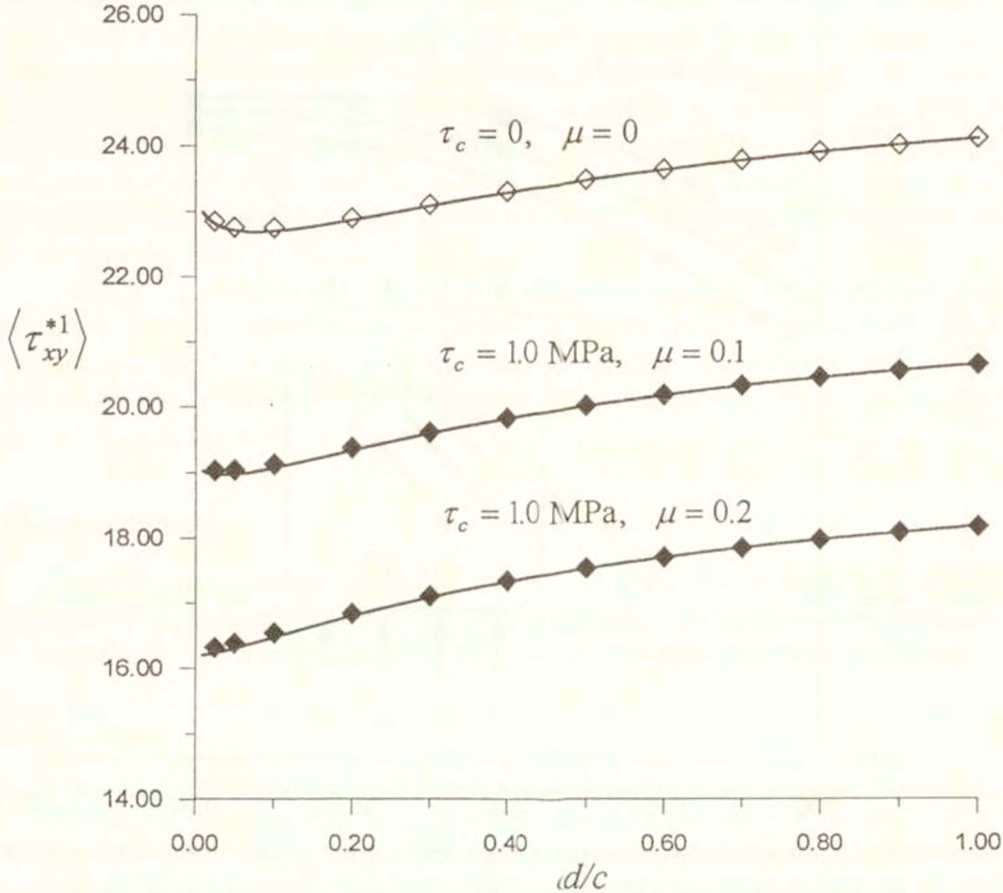

FIG. 4. Average shear stress $\langle \tau_{xy}^{*1} \rangle$ as predicted by Eq. (2.13) vs. BEM data (diamonds) for inclined cracks of Fig. 3.

All figures clearly show that the accuracy of the extended Kachanov method is excellent even at very small distances between the crack tips. This could have been expected for the aligned cracks since Kachanov's method has always been working fine for collinear crack arrays. However, stacked crack configurations are an acid test for the method. Surprisingly enough, for two parallel stacked cracks (Fig. 5) the present results and the BEM data are practically indistinguishable with the exception of the outer tip at $\mu = 0.3$ and $d/c = 0.05$. But even there the error is merely 0.8%. The least accurate results are obtained in the case of inclined cracks for inner tips (labeled B in Fig. 3) at $\mu = 0.3$ and $d/c = 0.05$. Nevertheless, the error in this case is less than 3% which is still reasonable for practical applications.

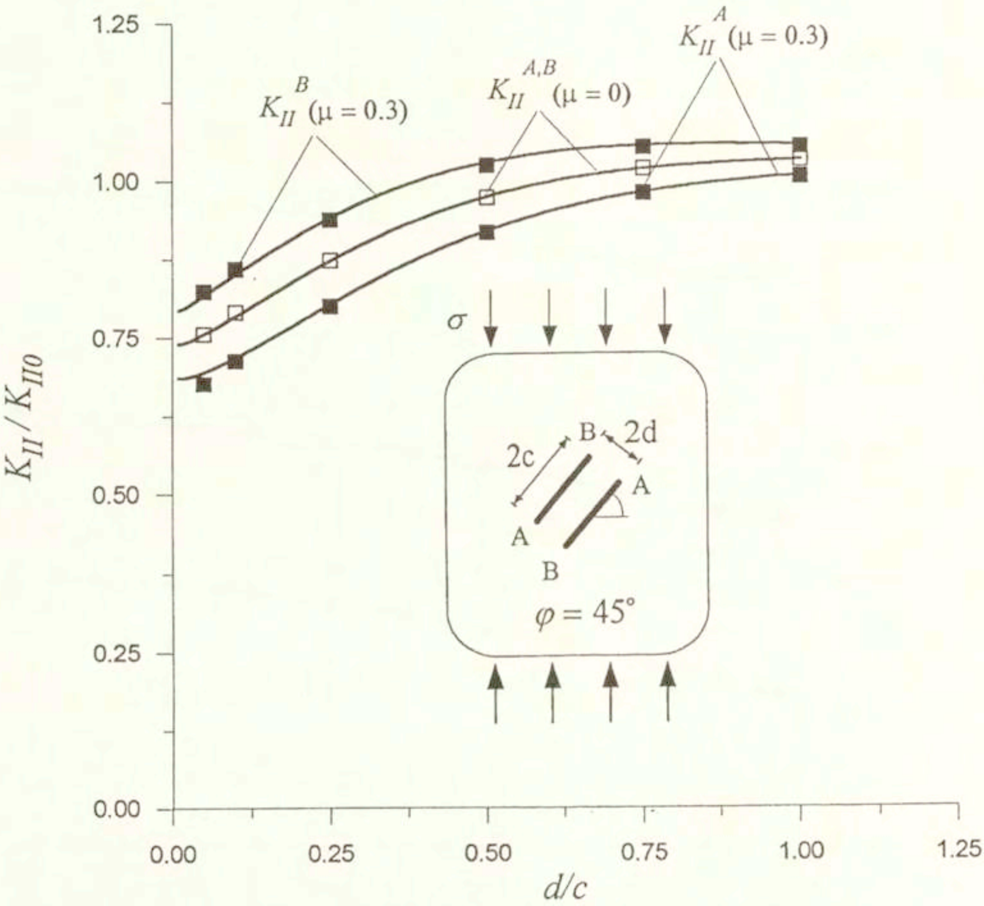

FIG. 5. Two stacked cracks under uniaxial compression. Normalized K_{II} factors vs. relative distance of crack tips. Solid curves – present solution, squares – BEM data, for $\mu = 0$, $\mu = 0.3$ at $\tau_c = 0$.

4. Conclusions

It has been shown in this paper that the Kachanov method of crack interaction analysis can successfully be extended to the case of frictionally sliding cracks. Introducing the Coulomb-Mohr law for the frictional contact on crack faces, the basic equations of the original model were modified accordingly and implemented numerically. The test examples were solved in a twofold manner: using the present code and the BEM program. The agreement between the obtained results is remarkably good even at crack tip distances as small as 0.05 of the crack length.

There are at least two reasons for the observed accuracy of the present model. First of all we have been dealing with somewhat biased, though important, situations of straight cracks and distributed loading on crack faces. For such situations the Kachanov scheme is best suited. The applicability of the method in case of point-force loads on crack faces has yet to be checked. To this end, some preliminary results are presented in [17]. Secondly, the formulas for the SIFs (2.3), (2.4) involve the unknown crack-face tractions only in an integral sense. This is in tune with the basic assumption of the method stating that it is the average tractions that contribute most to the SIFs. As for 3D crack configurations, it is known from the analysis of open cracks (cf. [11]) that the method performs in 3D even better than in 2D cases.

It should be pointed out though that this effective and elegant method has its limitations. It is contingent on the knowledge of standard stress fields, thus confined actually to straight (or penny-shaped) cracks. If the cracks grow out of their original planes, which is often the case in real situations, the simplicity and applicability of the method decrease.

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