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Homogenization of compressible fluid flow in porous media with interfacial flow barrier

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THIS WORK IS CONCERNED with modelling compressible fluid flow in a composite porous medium with interfacial flow barrier. The macroscopic behaviour and the effective permeability are obtained by homogenization, i.e. by upscaling the description at the heterogeneity scale. Five distinct macroscopic models are derived that relate to five relative orders of magnitude of the interfacial conductance with respect to the permeabilities of the constituents.

1. Introduction

THIS WORK IS AIMED towards deriving the macroscopic governing equations of the flow of a compressible fluid in a porous medium with interfacial flow barrier. Such a medium is locally characterised by a representative elementary volume (REV) whose size is $O(l)$ and that consists of two porous solids (Ω_1 and Ω_2) whose common boundary (Γ) is a thin layer of very low porosity which constitutes a flow barrier (Fig. 1). This situation occurs for example in sedimentary structures that contain shales.

At the local scale, (i.e. at the REV scale), the flow of a compressible fluid in such a medium is governed by the following equations:

$$\nabla \cdot (k_1 p_1 \nabla p_1) = \phi_1 \frac{\partial p_1}{\partial t} \qquad \text{in } \Omega_1, \tag{1.1}$$

$$\nabla \cdot (k_2 p_2 \nabla p_2) = \phi_2 \frac{\partial p_2}{\partial t} \qquad \text{in } \Omega_2, \tag{1.2}$$

$$(k_1 p_1 \nabla p_1) \cdot \mathbf{n} = (k_2 p_2 \nabla p_2) \cdot \mathbf{n} \qquad \text{on } \Gamma, \tag{1.3}$$

$$(k_1 p_1 \nabla p_1) \cdot \mathbf{n}_1 = -\frac{h}{2}(p_1^2 - p_2^2) \qquad \text{on } \Gamma. \tag{1.4}$$

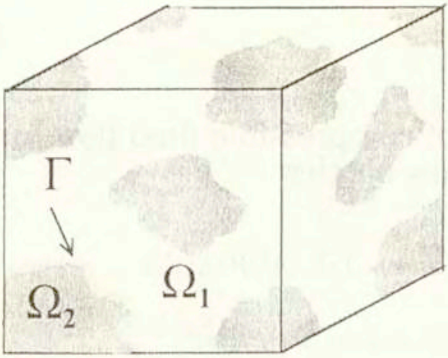

FIG. 1. Periodic cell of the two-constituent porous medium with interfacial flow barrier.

In these equations, p_1 and p_2 are the fluid pressures in Ω_1 and Ω_2, respectively; fluid densities, ρ_1 and ρ_2 are assumed to be such that $\rho_1 = Ap_1$, $\rho_2 = Ap_2$, where A is constant. Equations (1.1) – (1.2) describe the flow of the compressible fluid in Ω_1 and Ω_2, respectively [1]. k_1 and k_2 are the permeabilities of Ω_1 and Ω_2, respectively, and are positive functions of the space variable. For simplicity, they are assumed to be isotropic. ϕ_1 and ϕ_2 are the porosities of Ω_1 and Ω_2, respectively, and are assumed to be constant. Equation (1.3) expresses continuity of fluxes on the interface Γ. The interface flow barrier gives rise to boundary condition (1.4), where h is the interfacial conductance and is a positive function of the space variable. Equation (1.4) is the non-linear counterpart of Eq. (1.8) below.

In the present study, both domains, Ω_1 and Ω_2 are assumed to be connected.

Due to the high compressibility of the fluid, this problem is strongly non-linear. Investigations of similar but linear problems have been carried out via homogenization in the context of heat conduction in composites with heat barrier [2, 3] and of pollutant transfer in a medium with interfacial diffusion barrier [4]. In these studies, the local description is of the following type:

$$\nabla \cdot (\alpha_1 \nabla u_1) = \beta_1 \frac{\partial u_1}{\partial t} \quad \text{in } \Omega_1, \tag{1.5}$$

$$\nabla \cdot (\alpha_2 \nabla u_2) = \beta_2 \frac{\partial u_2}{\partial t} \quad \text{in } \Omega_2, \tag{1.6}$$

$$(\alpha_1 \nabla u_1) \cdot \mathbf{n} = (\alpha_2 \nabla u_2) \cdot \mathbf{n} \quad \text{on } \Gamma, \tag{1.7}$$

$$(\alpha_1 \nabla u_1) \cdot \mathbf{n}_1 = -h(u_1 - u_2) \quad \text{on } \Gamma. \tag{1.8}$$

The non-dimensional number arising from equation (1.8)

$$b = \frac{|h(u_1 - u_2)|}{|\alpha_1 \nabla u_1|} = \frac{hl}{\alpha_1}, \tag{1.9}$$

is the Biot number and characterises the interfacial barrier.

The flow of an incompressible fluid in a medium with interfacial flow barrier is also locally described by the linear set of equations (1.5) – (1.8). Therefore, the originality of the present work relies upon the non-linearities due to the strong compressibility of the fluid.

Whereas equation (1.8) is well admitted and currently used, to our knowledge, equation (1.4) that describes the effect of an interfacial flow barrier on a non-linear process, has never been proposed yet. However we see that equation (1.4) is physically consistent as it is obtained by replacing u_1 and u_2 by u_1^2 and u_2^2, respectively, in (1.8). The validity of both equations (1.4) and (1.8) can be demonstrated via homogenization. The rigorous derivation of (1.4) by homogenization will be the purpose of a further paper.

The essence of homogenization method is to determine an equivalent macroscopic behaviour by upscaling the local description. The purpose of the present study is to homogenize the local description (1.1) – (1.4), in order to determine the influence of the interfacial barrier on the effective permeability and on the structure of the macroscopic seepage equations.

The fundamental assumption behind homogenization theory is that the scales are separated:

$$l << L, \tag{1.10}$$

where l and L are the characteristic lengths at the heterogeneity scale and at the macroscopic scale, respectively. As this definition conjures up a geometrical separation of scales, we shall draw attention to the fact that this fundamental condition must also be checked regarding the phenomenon. For example, in the case of a wave propagation, the microscopic characteristic length, l, must also be small compared to the wavelength.

In this study, we use the method of homogenization for periodic structures – also called method of multiple scales – introduced by [5] and [6]. The key parameter of the method is the small parameter

$$\varepsilon = \frac{l}{L} << 1, \tag{1.11}$$

in which L is the macroscopic characteristic length and, depending on the problem under consideration, is either geometrical (i.e. the sample size) or related to the excitation (e.g. wavelength).

We also assume the medium to be periodic. This assumption is actually not a restriction: it allows determination of the macroscopic behaviour without any prerequisite on the form of the macroscopic equations. In the context of a periodic medium, the REV is simply the period.

In this study, we use the approach suggested in [7], by which the problem is tackled in a physical rather than mathematical manner. Indeed, it offers the additional benefit that the conditions under which homogenization does apply are expressly stated. This formulation of the method is the basis of definition and estimation of the non-dimensional numbers arising from the local description under consideration. This fundamental step is called normalisation and is aimed at specifying all cases that can be homogenized.

In the present work, we show that fluid flow in a porous medium with interfacial flow barrier is not, a priori, described by a single model. The normalisation of the local description (1.1) – (1.4) is presented in Sec. 2. It highlights five distinct physical situations to be homogenized that relate to five distinct relative values of the interfacial conductance with respect to the permeabilities. Section 3 sets out to examine the mathematical formulation of the method as a result of the separation of scales. Finally, in Sec. 4 details are given of the derivation of the five corresponding macroscopic models. Three models are single-pressure field models and the two others are two-pressure field models. We show that besides its link to the relative value of the interfacial conductance with respect to the permeabilities, the choice of the model is also related to the excitation.

2. Normalisation

The purpose of this section is to define the set of non-dimensional numbers that characterise the local description (1.1) – (1.4) and then to estimate them with respect to the small parameter ε.

From equation (1.1) we can define

$$T_1 = \frac{|\nabla \cdot (k_1 p_1 \nabla p_1)|}{\left|\phi_1 \dfrac{\partial p_1}{\partial t}\right|}. \tag{2.1}$$

Similarly, equation (1.2) introduces

$$T_2 = \frac{|\nabla \cdot (k_2 p_2 \nabla p_2)|}{\left|\phi_2 \dfrac{\partial p_2}{\partial t}\right|} = O(T_1) \times O\left(\frac{k_2}{k_1}\right) \times O\left(\frac{p_2}{p_1}\right) \times O\left(\frac{\phi_1}{\phi_2}\right). \tag{2.2}$$

Now, from Eq. (1.3) arises

$$A = \frac{|(k_1 p_1 \nabla p_1) \cdot \mathbf{n}|}{|(k_2 p_2 \nabla p_2) \cdot \mathbf{n}|} = O\left(\frac{k_1}{k_2}\right) \times O\left(\frac{(p_1)^2}{(p_2)^2}\right). \tag{2.3}$$

Finally, from Eq. (1.4) we get the following non-dimensional number:

$$B = \frac{\left|\dfrac{h}{2}\left(p_1^2 - p_2^2\right)\right|}{|(k_1 p_1 \nabla p_1) \cdot \mathbf{n}_1|}, \tag{2.4}$$

which is similar to the Biot number defined in (1.9) and which characterises the relative value of the interfacial conductance with respect to the permeabilities of the constituents.

For estimating these non-dimensional numbers, let us consider l as the reference characteristic length. This arbitrary choice does not affect the final result. When using l as the reference length, the estimations of T_1, T_2, A and B are denoted by T_{1_l}, T_{2_l}, A_l and B_l . By assuming for simplicity that:

$$\left|\frac{k_1}{k_2}\right|_l = O(1); \qquad \left|\frac{p_1}{p_2}\right|_l = O(1); \qquad \left|\frac{\phi_1}{\phi_2}\right|_l = O(1), \tag{2.5}$$

it turns out that

$$T_{1_l} = O(T_{2_l}), \tag{2.6}$$

$$A_l = O(1), \tag{2.7}$$

and

$$B_l = O\left(\frac{hl}{2k_1}\right). \tag{2.8}$$

Since we want to describe a transient flow at the macroscopic scale, we may consider that:

$$|\nabla \cdot (k_1 p_1 \nabla p_1)|_L = O\left(\left|\phi_1 \frac{\partial p_1}{\partial t}\right|_L\right), \tag{2.9}$$

$$|\nabla \cdot (k_2 p_2 \nabla p_2)|_L = O\left(\left|\phi_2 \frac{\partial p_2}{\partial t}\right|_L\right), \tag{2.10}$$

which means

$$T_{1_L} = O(T_{2_L}) = O(1). \tag{2.11}$$

Thus, when using l for estimating T_1 and T_2, we get

$$T_{1_l} = O(T_{2_l}) = O(\varepsilon^{-2}). \tag{2.12}$$

The orders of magnitude (2.12) are actually related to the previously mentioned condition of separation of scales regarding the excitation. In effect, they express the fact that the characteristic time of the flow must be sufficiently large to ensure a good separation of scales. Estimations (2.12) are required for applying homogenization as there would not exist any equivalent macroscopic continuous description for $T_{1_l} = O(T_{2_l}) < O(\varepsilon^{-2})$, whereas orders of magnitude (2.5) are pure assumptions.

Once the orders of magnitude of T_{1_l}, T_{2_l} and A_l have been fixed, the local description remains only conditioned by the order of magnitude of B_l, which is actually a measure of the influence of the interfacial conductance. The non-dimensional description is the following, in which all quantities are now non-dimensional quantities:

$$\nabla \cdot (k_1 p_1 \nabla p_1) = \varepsilon^2 \phi_1 \frac{\partial p_1}{\partial t} \qquad \text{in } \Omega_1, \tag{2.13}$$

$$\nabla \cdot (k_2 p_2 \nabla p_2) = \varepsilon^2 \phi_2 \frac{\partial p_2}{\partial t} \qquad \text{in } \Omega_2, \tag{2.14}$$

$$(k_1 p_1 \nabla p_1) \cdot \mathbf{n} = (k_2 p_2 \nabla p_2) \cdot \mathbf{n} \qquad \text{on } \Gamma, \tag{2.15}$$

$$(k_1 p_1 \nabla p_1) \cdot \mathbf{n}_1 = -O(B_l) \frac{h}{2} (p_1^2 - p_2^2) \qquad \text{on } \Gamma. \tag{2.16}$$

We shall now apply the homogenization procedure to this local description.

3. Mathematical formulation of the method

As a result of the separation of scales, two non-dimensional space variables may be defined:

$$\mathbf{y} = \frac{\mathbf{X}}{l}, \qquad \mathbf{x} = \frac{\mathbf{X}}{L},$$

where $\mathbf{X}$ is the physical space variable.

If the condition of separation of scales is verified, then $\mathbf{y}$ and $\mathbf{x}$ appear as two independent space variables: $\mathbf{y}$ is the microscopic variable and describes the heterogeneity scale whereas $\mathbf{x}$ is the macroscopic variable.
As a consequence, the physical variables of the problem, p_1 and p_2, are *a priori* functions of $\mathbf{y}$ and $\mathbf{x}$:

$$p_1 = p_1(\mathbf{y}, \mathbf{x}, t),$$

$$p_2 = p_2(\mathbf{y}, \mathbf{x}, t).$$

Moreover, the partial derivative with respect to the physical space variable $\mathbf{X}$ can be written as:

$$\frac{\partial}{\partial X_i} = \frac{1}{l}\frac{\partial}{\partial y_i} + \frac{1}{L}\frac{\partial}{\partial x_i}. \tag{3.1}$$

Since l is the reference characteristic length, the non-dimensional gradient operator is therefore given by

$$\nabla_y + \varepsilon \nabla_x, \tag{3.2}$$

where ∇_y and ∇_x are the gradient operators with respect to variables $\mathbf{y}$ and $\mathbf{x}$, respectively. The homogenization method of multiple scales is based on the fundamental statement that if the scales are well separated, then all physical variables can be looked for in the form of asymptotic expansions in powers of ε:

$$p_1 = p_1^0(\mathbf{y},\mathbf{x},t) + \varepsilon p_1^1(\mathbf{y},\mathbf{x},t) + ..., \tag{3.3}$$

$$p_2 = p_2^0(\mathbf{y},\mathbf{x},t) + \varepsilon p_2^1(\mathbf{y},\mathbf{x},t) + ..., \tag{3.4}$$

in which the functions p_1^i and p_2^i are $\mathbf{y}$-periodic.

The method consists in incorporating expansions (3.3) and (3.4) in the non-dimensional local description (2.13) – (2.16). Solving the boundary-value problems arising at the successive orders of ε leads to the macroscopic description. It turns out that five distinct macroscopic descriptions can be derived from the local description (2.13) – (2.16), that correspond to five distinct orders of magnitude for B_l:

$$B_l = O(\varepsilon^p), \qquad p = -1, 0, 1, 2, 3. \tag{3.5}$$

The derivation of these macroscopic models via homogenization is the purpose of the next section.

4. Derivation of the macroscopic models

4.1. Model I: $B_l = O(\varepsilon^{-1})$

First-order problem (Eqs. (2.13), (2.14) and (2.16) at the order of ε^0 and Eq. (2.16) at the order of ε^{-1}):

$$\nabla_y \cdot \left(k_1 p_1^0 \nabla_y p_1^0\right) = 0 \qquad \text{in } \Omega_1, \tag{4.1}$$

$$\nabla_y \cdot \left(k_2 p_2^0 \nabla_y p_2^0\right) = 0 \qquad \text{in } \Omega_2, \tag{4.2}$$

$$\left(k_1 p_1^0 \nabla_y p_1^0\right) \cdot \mathbf{n} = \left(k_2 p_2^0 \nabla_y p_2^0\right) \cdot \mathbf{n} \qquad \text{on } \Gamma, \tag{4.3}$$

$$p_1^0 = p_2^0 \qquad \text{on } \Gamma, \tag{4.4}$$

(p_1^0 and p_2^0 are $\mathbf{y}$-periodic).

Let $\mathcal{V}(\Omega)$ be the Hilbert space of functions θ defined and continuous over Ω, that are $\mathbf{y}$-periodic and that satisfy the condition

$$\int\limits_{\Omega} \theta \, d\Omega = 0. \tag{4.5}$$

$\mathcal{V}(\Omega)$ is equipped with the following inner product:

$$(\alpha,\beta)_{\mathcal{V}(\Omega)} = \int_{\Omega} \nabla_y \alpha \; k \; \nabla_y \beta \; d\Omega. \tag{4.6}$$

The equivalent variational formulation of (4.1) – (4.4) is given by

$$\forall \theta \in \mathcal{V}(\Omega) : (\theta, (p^{\circ})^2)_{\mathcal{V}(\Omega)} = \text{o}. \tag{4.7}$$

Existence and uniqueness of the solution to (4.7) are proved by Lax-Milgram Lemma. Equation (4.7) gives

$$(p^0)^2 = (p^0)^2(\mathbf{x}, t). \tag{4.8}$$

Therefore, we get

$$p_1^0 = p_2^0 = p^0(\mathbf{x}, t). \tag{4.9}$$

Second-order problem (Eqs. (2.13), (2.14) and (2.15) at the order of ε^1 and Eq. (2.16) at the order of ε^0).

Since $p^0 = p^0(\mathbf{x}, t)$, the system reduces to the following linear boundary value problem:

$$\nabla_y \cdot \left[k_1 \left(\nabla_y p_1^1 + \nabla_x p^0\right)\right] = 0 \qquad \text{in } \Omega_1, \tag{4.10}$$

$$\nabla_y \cdot \left[k_2 \left(\nabla_y p_2^1 + \nabla_x p^0\right)\right] = 0 \qquad \text{in } \Omega_2, \tag{4.11}$$

$$\left[k_1 \left(\nabla_y p_1^1 + \nabla_x p^0\right)\right] \cdot \mathbf{n} = \left[k_2 \left(\nabla_y p_2^1 + \nabla_x p^0\right)\right] \cdot \mathbf{n} \qquad \text{on } \Gamma, \tag{4.12}$$

$$p_1^1 = p_2^1 \qquad \text{on } \Gamma. \tag{4.13}$$

(p_1^1 and p_2^1 are $\mathbf{y}$-periodic).

The equivalent variational formulation of this system is:

$$\forall \theta \in \mathcal{V}(\Omega) : \left(\theta, p^1\right)_{\mathcal{V}(\Omega)} = -\int_{\Omega} \nabla_y \theta \; k \; \nabla_x p^0 \; d\Omega. \tag{4.14}$$

Hence, by the Lax-Milgram Lemma, there is a unique solution to (4.14). As a result, there is a solution modulo a constant to the system (4.10) – (4.14). Let τ_i^{I} be the particular solution of (4.14) for $\dfrac{\partial p^0}{\partial x_j} = \delta_{ij}$. Thus, p^1 is written as:

$$p^1 = \boldsymbol{\tau}^{\mathrm{I}} \cdot \nabla_x p^0 + \bar{p}^1(\mathbf{x}, t), \tag{4.15}$$

where p^1 stands for p_1^1 in Ω_1 and for p_2^1 in Ω_2 and where $p^1(\mathbf{x}, t)$ is an arbitrary function. $\boldsymbol{\tau}^{\mathrm{I}}$ is $\mathbf{y}$-periodic, its mean value is zero,

$$< \boldsymbol{\tau}^{\mathrm{I}} >_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega} \boldsymbol{\tau}^{\mathrm{I}} d\Omega = \mathbf{0}, \tag{4.16}$$

and $\boldsymbol{\tau}^I$ is the solution of the following boundary-value problem:

$$\frac{\partial}{\partial y_i}\left(\frac{\partial \tau_k^I}{\partial y_i}+\delta_{ik}\right)=0 \qquad \text{in } \Omega, \tag{4.17}$$

$$k_1\left(\frac{\partial \tau_k^I}{\partial y_i}+\delta_{ik}\right)n_i=k_2\left(\frac{\partial \tau_k^I}{\partial y_i}+\delta_{ik}\right)n_i \qquad \text{on } \Gamma, \tag{4.18}$$

in which k stands for k_1 in Ω_1 and for k_2 in Ω_2.

Third-order problem (Eqs. (2.13), (2.14) and (2.15) at the order of ε^2):

$$\nabla_y\cdot\left[k_1\left(p^0\left(\nabla_y p_1^2+\nabla_x p_1^1\right)+p_1^1\left(\nabla_y p_1^1+\nabla_x p^0\right)\right)\right] \tag{4.19}$$
$$+\nabla_x\cdot\left[k_1\left(p^0\left(\nabla_y p_1^1+\nabla_x p^0\right)\right)\right]=\phi_1\frac{\partial p^0}{\partial t} \qquad \text{in } \Omega_1,$$

$$\nabla_y\cdot\left[k_2\left(p^0\left(\nabla_y p_2^2+\nabla_x p_2^1\right)+p_2^1\left(\nabla_y p_2^1+\nabla_x p^0\right)\right)\right] \tag{4.20}$$
$$+\nabla_x\cdot\left[k_2\left(p^0\left(\nabla_y p_2^1+\nabla_x p^0\right)\right)\right]=\phi_2\frac{\partial p^0}{\partial t} \qquad \text{in } \Omega_2,$$

$$\left[k_1\left(p^0\left(\nabla_y p_1^2+\nabla_x p_1^1\right)+p_1^1\left(\nabla_y p_1^1+\nabla_x p^0\right)\right)\right]\cdot\mathbf{n} \tag{4.21}$$
$$=\left[k_2\left(p^0\left(\nabla_y p_2^2+\nabla_x p_2^1\right)+p_2^1\left(\nabla_y p_2^1+\nabla_x p^0\right)\right)\right]\cdot\mathbf{n} \qquad \text{on } \Gamma.$$

(p_1^2 and p_2^2 are $\mathbf{y}$-periodic).

Integrating (4.19) over Ω_1 and (4.20) over Ω_2 and then using the divergence theorem, the condition of periodicity and boundary condition (4.21) leads to

$$\nabla_x\cdot\left(p^0 < k(\nabla_y p^1+\nabla_x p^0>_\Omega\right)=<\phi>\frac{\partial p^0}{\partial t}, \tag{4.22}$$

where

$$<\phi>=n_1\phi_1+n_2\phi_2; \qquad n_i=\frac{|\Omega_i|}{\Omega}. \tag{4.23}$$

Equation (4.22) can also be written as:

$$\nabla_x\cdot\left(\tilde{K}^I p^0\nabla_x p^0\right)=<\phi>\frac{\partial p^0}{\partial t}, \tag{4.24}$$

in which $\tilde{K}^I$ is componentwise defined by:

$$K_{ij}^I=\frac{1}{|\Omega|}\int\limits_{\Omega} k\left(\frac{\partial \tau_i^I}{\partial y_j}+\delta_{ij}\right)d\Omega. \tag{4.25}$$

It can be shown that $\tilde{K}$ is positive definite. When $B_l = O(\varepsilon^p), p < -1$, the macroscopic behaviour is also described by Model I.

4.2. Model II: $B_l = O(\varepsilon^0)$

First-order problem (Eqs. (2.13), (2.14), (2.15) and (2.16) at the order of ε^0):

$$\nabla_y \cdot \left(k_1 p_1^0 \nabla_y p_1^0\right) = 0 \qquad \text{in } \Omega_1, \tag{4.26}$$

$$\nabla_y \cdot \left(k_2 p_2^0 \nabla_y p_2^0\right) = 0 \qquad \text{in } \Omega_2, \tag{4.27}$$

$$\left(k_1 p_1^0 \nabla_y p_1^0\right) \cdot \mathbf{n} = \left(k_2 p_2^0 \nabla_y p_2^0\right) \cdot \mathbf{n} \qquad \text{on } \Gamma, \tag{4.28}$$

$$\left(k_1 p_1^0 \nabla_y p_1^0\right) \cdot \mathbf{n}_1 = -\frac{h}{2}\left[(p_1^0)^2 - (p_2^0)^2\right] \qquad \text{on } \Gamma. \tag{4.29}$$

(p_1^0 and p_2^0 are $\mathbf{y}$-periodic).
Let $\mathcal{W}(\Omega)$ be the Hilbert space of functions θ that are $\mathbf{y}$-periodic, continuous over Ω_1 and Ω_2 and possibly discontinuous over Γ and such that

$$\int_\Omega \theta\, \Omega = 0. \tag{4.30}$$

$\mathcal{W}(\Omega)$ is equipped with the following inner product:

$$(\alpha, \beta)_{\mathcal{W}(\Omega)} = \int_\Omega \nabla_y \alpha\; k\; \nabla_y \beta\; d\Omega + \int_\Gamma h(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)\; dS. \tag{4.31}$$

The equivalent formulation of (4.26) – (4.29) is

$$\forall \theta \in \mathcal{W}(\Omega) : \left(\theta, (p^0)^2\right)_{\mathcal{W}(\Omega)} = 0, \tag{4.32}$$

from which we get

$$(p^0)^2 = (p^0)^2(\mathbf{x}, t). \tag{4.33}$$

Thus, the solution is

$$p_1^0 = p_2^0 = p^0(\mathbf{x}, t). \tag{4.34}$$

Second-order problem (Eqs. (2.13), (2.14), (2.15) and (2.16) at the order of ε^1).
This system reduces to the following linear boundary value problem:

$$\nabla_y \cdot \left[k_1 \left(\nabla_y p_1^1 + \nabla_x p^0\right)\right] = 0 \qquad \text{in } \Omega_1, \tag{4.35}$$

$$\nabla_y \cdot \left[k_2\left(\nabla_y p_2^1 + \nabla_x p^0\right)\right] = 0 \qquad \text{in } \Omega_2, \tag{4.36}$$

$$\left[k_1\left(\nabla_y p_1^1 + \nabla_x p^0\right)\right] \cdot \mathbf{n} = \left[k_2\left(\nabla_y p_2^1 + \nabla_x p^0\right)\right] \cdot \mathbf{n} \qquad \text{on } \Gamma, \tag{4.37}$$

$$\left[k_1\left(\nabla_y p_1^1 + \nabla_x p^0\right)\right] \cdot \mathbf{n}_1 = -h(p_1^1 - p_2^1) \qquad \text{on } \Gamma. \tag{4.38}$$

(p_1^1 and p_2^1 are $\mathbf{y}$-periodic).

The equivalent variational formulation of (4.35) – (4.38) is the following:

$$\forall \theta \in \mathcal{W}(\Omega) : \left(\theta, p^1\right)_{\mathcal{W}(\Omega)} = -\int_{\Omega} \nabla_y \theta \; k \; \nabla_x p^0 \; d\Omega. \tag{4.39}$$

Let $\boldsymbol{\tau}_{1i}^{\mathrm{II}}$ and $\boldsymbol{\tau}_{2i}^{\mathrm{II}}$ be the particular solutions in Ω_1 and Ω_2 for $\dfrac{\partial p^0}{\partial x_j} = \delta_{ij}$. Then, p_1^1 and p_2^1 are given by:

$$p_1^1 = \boldsymbol{\tau}_1^{\mathrm{II}} . \nabla_x p^0 + \bar{p}_1^1(\mathbf{x}, t), \tag{4.40}$$

$$p_2^1 = \boldsymbol{\tau}_2^{\mathrm{II}} . \nabla_x p^0 + \bar{p}_2^1(\mathbf{x}, t). \tag{4.41}$$

$\boldsymbol{\tau}_1^{\mathrm{II}}$ and $\boldsymbol{\tau}_2^{\mathrm{II}}$ are $\mathbf{y}$-periodic and are the solutions of the following system:

$$\frac{\partial}{\partial y_i}\left[k_1\left(\frac{\partial \tau_{1k}^{\mathrm{II}}}{\partial y_i} + \delta_{ik}\right)\right] = 0 \qquad \text{in } \Omega_1, \tag{4.42}$$

$$\frac{\partial}{\partial y_i}\left[k_2\left(\frac{\partial \tau_{2k}^{\mathrm{II}}}{\partial y_i} + \delta_{ik}\right)\right] = 0 \qquad \text{in } \Omega_2, \tag{4.43}$$

$$k_1\left(\frac{\partial \tau_{1k}^{\mathrm{II}}}{\partial y_i} + \delta_{ik}\right) n_i = k_2\left(\frac{\partial \tau_{2k}^{\mathrm{II}}}{\partial y_i} + \delta_{ik}\right) n_i \qquad \text{on } \Gamma, \tag{4.44}$$

$$k_1\left(\frac{\partial \tau_{1k}^{\mathrm{II}}}{\partial y_i} + \delta_{ik}\right) n_{1i} = -h\left(\tau_{1k}^{\mathrm{II}} - \tau_{2k}^{\mathrm{II}}\right) \qquad \text{on } \Gamma. \tag{4.45}$$

$\boldsymbol{\tau}_1^{\mathrm{II}}$ and $\boldsymbol{\tau}_2^{\mathrm{II}}$ also verify

$$\int_{\Omega_1} \boldsymbol{\tau}_1^{\mathrm{II}} \, d\Omega = 0 + \int_{\Omega_2} \boldsymbol{\tau}_2^{\mathrm{II}} \, d\Omega = 0. \tag{4.46}$$

It turns out that $\boldsymbol{\tau}_1^{\mathrm{II}}$ and $\boldsymbol{\tau}_2^{\mathrm{II}}$, and therefore p_1^1 and p_2^1 are h-dependent.

Third-order problem (Eqs. (2.13), (2.14) and (2.15) at the order of ε^2): This problem is similar to the third-order problem in Sec. 4.1. Using identical reasoning to that outlined in Sec. 4.1, we arrive at the following macroscopic description:

$$\nabla_x \cdot \left(\tilde{K}^{\mathrm{II}} p^0 \nabla_x p^0\right) = <\phi> \frac{\partial p^0}{\partial t}, \tag{4.47}$$

in which

$$K_{ij}^{\mathrm{II}} = \frac{1}{|\Omega|} \int_{\Omega} k \left(\frac{\partial \tau_i^{\mathrm{II}}}{\partial y_j} + \delta_{ij}\right) d\Omega. \tag{4.48}$$

$\tilde{K}^{\mathrm{II}}$ is h-dependent.

4.3. Model III: $B_l = O(\varepsilon^1)$

First-order problem (Eqs. (2.13), (2.14), (2.15) and (2.16) at the order of ε^0). It consists of two non-coupled boundary value problems:

$$\nabla_y \cdot \left(k_1 p_1^0 \nabla_y p_1^0\right) = 0 \qquad \text{in } \Omega_1, \tag{4.49}$$

$$\left(k_1 p_1^0 \nabla_y p_1^0\right) \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma, \tag{4.50}$$

$$\nabla_y \cdot \left(k_2 p_2^0 \nabla_y p_2^0\right) = 0 \qquad \text{in } \Omega_2, \tag{4.51}$$

$$\left(k_2 p_2^0 \nabla_y p_2^0\right) \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma. \tag{4.52}$$

(p_1^0 and p_2^0 are $\mathbf{y}$-periodic).
The equivalent variational formulations are

$$\forall \theta_1 \in \mathcal{V}(\Omega_1) : \left(\theta_1, (p_1^0)^2)\right)_{\mathcal{V}} (\Omega_1) = 0, \tag{4.53}$$

$$\forall \theta_2 \in \mathcal{V}(\Omega_2) : \left(\theta_2, (p_2^0)^2)\right)_{\mathcal{V}} (\Omega_2) = 0. \tag{4.54}$$

($\mathcal{V}(\Omega)$ and $(.,.)_{\mathcal{V}(\Omega)}$ are defined in Subsec. 4.1). Therefore, the solutions are

$$p_1^0 = p_1^0(\mathbf{x}, t), \tag{4.55}$$

$$p_2^0 = p_2^0(\mathbf{x}, t). \tag{4.56}$$

Second-order problem (Eqs. (2.13), (2.14), (2.15) and (2.16) at the order of ε^1):

$$\nabla_y \cdot \left[k_1 p_1^0 \left(\nabla_y p_1^1 + \nabla_x p_1^0\right)\right] = 0 \qquad \text{in } \Omega_1, \tag{4.57}$$

$$\nabla_y \cdot \left[k_2 p_2^0 \left(\nabla_y p_2^1 + \nabla_x p_2^0\right)\right] = 0 \qquad \text{in } \Omega_2, \tag{4.58}$$

$$\left[k_1 p_1^0 \left(\nabla_y p_1^1 + \nabla_x p_1^0\right)\right] \cdot \mathbf{n} = \left[k_2 p_2^0 \left(\nabla_y p_2^1 + \nabla_x p_2^0\right)\right] \cdot \mathbf{n} \qquad \text{on } \Gamma, \tag{4.59}$$

$$\left[k_1 p_1^0 \left(\nabla_y p_1^1 + \nabla_x p_1^0\right)\right] \cdot \mathbf{n}_1 = -\frac{h}{2} \left[(p_1^0)^2 - (p_2^0)^2\right] \qquad \text{on } \Gamma. \tag{4.60}$$

(p_1^1 and p_2^1 are $\mathbf{y}$-periodic).

Integrating (4.57) over Ω_1 and using the divergence theorem, the periodicity and (4.60) leads to

$$\frac{1}{|\Omega|}\int_{\Gamma} hdS \; \left[\left(p_1^0 - p_2^0\right)\right] = 0, \tag{4.61}$$

from which we deduce

$$p_1^0 = p_2^0 = p^0(\mathbf{x}, t). \tag{4.62}$$

As a consequence, the set (4.57) – (4.60) reduces to the two following linear non-coupled boundary value problems:

$$\nabla_y \cdot \left[k_1\left(\nabla_y p_1^1 + \nabla_x p^0\right)\right] = 0 \qquad \text{in } \Omega_1, \tag{4.63}$$

$$\left[k_1\left(\nabla_y p_1^1 + \nabla_x p^0\right)\right] \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma, \tag{4.64}$$

$$\nabla_y \cdot \left[k_2\left(\nabla_y p_2^1 + \nabla_x p^0\right)\right] = 0 \qquad \text{in } \Omega_2, \tag{4.65}$$

$$\left[k_2\left(\nabla_y p_2^1 + \nabla_x p^0\right)\right] \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma. \tag{4.66}$$

The equivalent variational formulations are

$$\forall\theta_1 \in \mathcal{V}(\Omega_1) : (\theta_1, p_1^1)_{\mathcal{V}(\Omega_1)} = -\int_{\Omega_1} \nabla_y\theta_1 \; k_1 \; \nabla_x p^0 \; d\Omega, \tag{4.67}$$

$$\forall\theta_2 \in \mathcal{V}(\Omega_2) : (\theta_2, p_2^1)_{\mathcal{V}(\Omega_2)} = -\int_{\Omega_2} \nabla_y\theta_2 \; k_2 \; \nabla_x p^0 \; d\Omega. \tag{4.68}$$

Let $\boldsymbol{\tau}_{1i}^{\mathrm{III}}$ and $\boldsymbol{\tau}_{2i}^{\mathrm{III}}$ be the particular solutions for $\dfrac{\partial p^0}{\partial x_j} = \delta_{ij}$. The solutions are:

$$p_1^1 = \boldsymbol{\tau}_1^{\mathrm{III}} \cdot \nabla_x p^0 + \bar{p}_1^1(\mathbf{x}, t), \tag{4.69}$$

$$p_2^1 = \boldsymbol{\tau}_2^{\mathrm{III}} \cdot \nabla_x p^0 + \bar{p}_2^1(\mathbf{x}, t), \tag{4.70}$$

where $\boldsymbol{\tau}_1^{\mathrm{III}}$ and $\boldsymbol{\tau}_2^{\mathrm{III}}$ are $\mathbf{y}$-periodic, and they satisfy

$$\int_{\Omega_1} \boldsymbol{\tau}_1^{\mathrm{III}} \; d\Omega = \mathbf{0}, \qquad \int_{\Omega_2} \boldsymbol{\tau}_2^{\mathrm{III}} \; d\Omega = \mathbf{0}, \tag{4.71}$$

and are the solutions of:

$$\frac{\partial}{\partial y_i}\left(\frac{\partial \tau_{1k}^{\mathrm{III}}}{\partial y_i} + \delta_{ik}\right) = 0 \qquad \text{in } \Omega_1, \tag{4.72}$$

$$\left(\frac{\partial \tau_{1k}^{\mathrm{III}}}{\partial y_i} + \delta_{ik}\right) n_i = 0 \qquad \text{on } \Gamma, \tag{4.73}$$

$$\frac{\partial}{\partial y_i}\left(\frac{\partial \tau_{2k}^{\mathrm{III}}}{\partial y_i} + \delta_{ik}\right) = 0, \qquad \text{in } \Omega_2, \tag{4.74}$$

$$\left(\frac{\partial \tau_{2k}^{\mathrm{III}}}{\partial y_i} + \delta_{ik}\right) n_i = 0 \qquad \text{on } \Gamma. \tag{4.75}$$

The macroscopic behaviour is derived from the third-order problem as in Subsecs. 4.1 and 4.2 and is given by:

$$\nabla_x \cdot \left(\tilde{K}^{\mathrm{III}} p^0 \nabla_x p^0\right) = <\phi> \frac{\partial p^0}{\partial t}, \tag{4.76}$$

in which

$$\tilde{K}^{\mathrm{III}} = \tilde{K}_1^{\mathrm{III}} + \tilde{K}_2^{\mathrm{III}}, \tag{4.77}$$

where $\tilde{K}_1^{\mathrm{III}}$ and $\tilde{K}_2^{\mathrm{III}}$ are the effective permeabilities of both constituents:

$$K_{1ij}^{\mathrm{III}} = \frac{1}{|\Omega|}\int_{\Omega_1} k_1 \left(\frac{\partial \tau_{1i}^{\mathrm{III}}}{\partial y_j} + \delta_{ij}\right) d\Omega, \tag{4.78}$$

$$K_{2ij}^{\mathrm{III}} = \frac{1}{|\Omega|}\int_{\Omega_2} k_2 \left(\frac{\partial \tau_{2i}^{\mathrm{III}}}{\partial y_j} + \delta_{ij}\right) d\Omega. \tag{4.79}$$

$\tilde{K}^{\mathrm{III}}$ is independent of h.

4.4. Model IV: $B_l = O(\varepsilon^2)$

First-order problem (Eqs. (2.13), (2.14), (2.15) and (2.16) at the order of ε^0). This system is identical to the first-order problem in 4.3 and therefore leads to

$$p_1^0 = p_1^0(\mathbf{x}, t), \tag{4.80}$$

$$p_2^0 = p_2^0(\mathbf{x}, t). \tag{4.81}$$

Second-order problem (Eqs. (2.13), (2.14), (2.15) and (2.16) at the order of ε^1). This problem reduces to two non-coupled linear boundary value problems that are identical to (4.63) – (4.66). The solutions are

$$p_1^1 = \boldsymbol{\tau}_1^{\mathrm{IV}} \cdot \nabla_x p_1^0 + \bar{p}_1^1(\mathbf{x}, t), \tag{4.82}$$

$$p_2^1 = \boldsymbol{\tau}_2^{\mathrm{IV}} \cdot \nabla_x p_2^0 + \bar{p}_2^1(\mathbf{x}, t), \tag{4.83}$$

where $\boldsymbol{\tau}_1^{\mathrm{IV}} = \boldsymbol{\tau}_1^{\mathrm{III}}$ and $\boldsymbol{\tau}_2^{\mathrm{IV}} = \boldsymbol{\tau}_2^{\mathrm{III}}$.

Third-order problem (Eqs. (2.13), (2.14), (2.15) and (2.16) at the order of ε^2). This problem can be expressed as follows:

$$\nabla_y \cdot \left[k_1 \left(p_1^0 \left(\nabla_y p_1^2 + \nabla_x p_1^1\right) + p_1^1 \left(\nabla_y p_1^1 + \nabla_x p^0\right)\right)\right] + \nabla_x \cdot \left[k_1 \left(p_1^0 \left(\nabla_y p_1^1 + \nabla_x p_1^0\right)\right)\right] = \phi_1 \frac{\partial p_1^0}{\partial t} \qquad \text{in } \Omega_1, \tag{4.84}$$

$$\left[k_1 \left(p_1^0 \left(\nabla_y p_1^2 + \nabla_x p_1^1\right) + p_1^1 \left(\nabla_y p_1^1 + \nabla_x p_1^0\right)\right)\right] \cdot \mathbf{n} = -\frac{h}{2}\left[\left(p_1^0\right)^2 - \left(p_2^0\right)^2\right] \qquad \text{on } \Gamma, \tag{4.85}$$

$$\nabla_y \cdot \left[k_2 \left(p_2^0 \left(\nabla_y p_2^2 + \nabla_x p_2^1\right) + p_2^1 \left(\nabla_y p_2^1 + \nabla_x p_2^0\right)\right)\right] + \nabla_x \cdot \left[k_2 \left(p_2^0 \left(\nabla_y p_2^1 + \nabla_x p_2^0\right)\right)\right] = \phi_2 \frac{\partial p_2^0}{\partial t} \qquad \text{in } \Omega_2, \tag{4.86}$$

$$\left[k_2 \left(p_2^0 \left(\nabla_y p_2^2 + \nabla_x p_2^1\right) + p_2^1 \left(\nabla_y p_2^1 + \nabla_x p_2^0\right)\right)\right] \cdot \mathbf{n} = +\frac{h}{2}\left[\left(p_1^0\right)^2 - \left(p_2^0\right)^2\right] \qquad \text{on } \Gamma. \tag{4.87}$$

(p_1^2 and p_2^2 are $\mathbf{y}$-periodic).

Integrating (4.84) and (4.86) over Ω_1 and Ω_2, respectively, applying the divergence theorem and using the condition of periodicity and boundary conditions (4.85) and (4.87) yields:

$$\nabla_x \cdot \left(\tilde{K}_1^{\mathrm{IV}} p_1^0 \nabla_x p_1^0\right) - \frac{H}{2}\left[\left(p_1^0\right)^2 - \left(p_2^0\right)^2\right] = n_1 \phi_1 \frac{\partial p_1^0}{\partial t}, \tag{4.88}$$

$$\nabla_x \cdot \left(\tilde{K}_2^{\mathrm{IV}} p_2^0 \nabla_x p_2^0\right) + \frac{H}{2}\left[\left(p_1^0\right)^2 - \left(p_2^0\right)^2\right] = n_2 \phi_2 \frac{\partial p^0}{\partial t}, \tag{4.89}$$

in which

$$H = \frac{1}{|\Omega|} \int_{\Gamma} h \, dS, \tag{4.90}$$

and where

$$\tilde{K}_1^{\mathrm{IV}} = \tilde{K}_1^{\mathrm{III}}, \tag{4.91}$$

$$\tilde{K}_2^{\mathrm{IV}} = \tilde{K}_2^{\mathrm{III}}. \tag{4.92}$$

4.5. Model V: $B_l = O(\varepsilon^3)$

The derivation of model V is similar to that of model IV, but with the difference that there is no source term. As a result, the macroscopic behaviour is described by:

$$\nabla_x \cdot \left(\tilde{K}_1^{\mathrm{V}} p_1^0 \nabla_x p_1^0\right) = n_1 \phi_1 \frac{\partial p_1^0}{\partial t}, \tag{4.93}$$

$$\nabla_x \cdot \left(\tilde{K}_2^{\mathrm{V}} p_2^0 \nabla_x p_2^0\right) = n_2 \phi_2 \frac{\partial p^0}{\partial t}, \tag{4.94}$$

in which

$$\tilde{K}_1^{\mathrm{V}} = \tilde{K}_1^{\mathrm{IV}} = \tilde{K}_1^{\mathrm{III}}, \tag{4.95}$$

$$\tilde{K}_2^{\mathrm{V}} = \tilde{K}_2^{\mathrm{IV}} = \tilde{K}_2^{\mathrm{III}}. \tag{4.96}$$

When $B_l = O(\varepsilon^p), p > 3$, the macroscopic behaviour is also described by model V.

5. Conclusions

We have homogenized the problem of compressible fluid flow in a porous medium with interfacial flow barrier. An important conclusion drawn from this study is that the macroscopic description strongly depends upon the relative value of the interfacial conductance with respect to the permeabilities of the constituents. Thus, we have derived five distinct macroscopic models whose domains of validity are related to the value of B (Fig. 2).

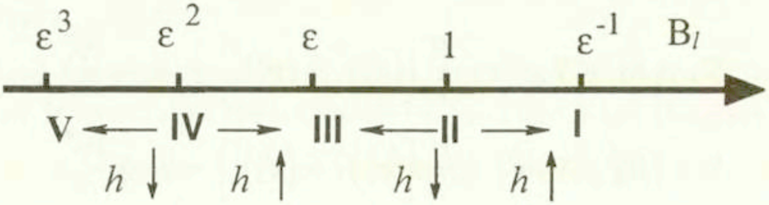

FIG. 2. The five macroscopic models and the corres-ponding orders of magnitude of B_l.

Models I, II and III are written in the form:

$$\nabla \cdot \left(\tilde{K}^{\alpha} p \nabla p\right) = <\phi> \frac{\partial p}{\partial t}, \qquad \alpha = \mathrm{I}, \mathrm{II}, \mathrm{III}. \tag{5.1}$$

They are single-pressure fields models, which means that the pressure field is constant over the REV. As a consequence, these models describe a state of local equilibrium. $\tilde{K}^{\mathrm{I}}$, $\tilde{K}^{\mathrm{II}}$ and $\tilde{K}^{\mathrm{III}}$ are three distinct effective permeabilities. $\tilde{K}^{\mathrm{I}}$ is

the effective permeability in the absence of interfacial barrier ($h = \infty$). $\tilde{K}^{\mathrm{II}}$ is h-dependent. $\tilde{K}^{\mathrm{III}}$ is different from $\tilde{K}^{\mathrm{I}}$ and $\tilde{K}^{\mathrm{II}}$, is independent of h and is such that $\tilde{K}^{\mathrm{III}} = \tilde{K}_1 + \tilde{K}_2$, where $\tilde{K}_1$ and $\tilde{K}_2$ are the effective permeabilities of both constituents.
Models IV and V are two- pressure field models, and as a consequence they describe a state of local non-equilibrium. Model IV is written as:

$$\nabla \cdot \left(\tilde{K}_1 p_1 \nabla p_1\right) = \phi_1 \frac{\partial p_1}{\partial t} - \frac{H}{2}\left((p_1)^2 - ((p_2)^2\right), \tag{5.2}$$

$$\nabla \cdot \left(\tilde{K}_2 p_2 \nabla p_2\right) = \phi_2 \frac{\partial p_2}{\partial t} + \frac{H}{2}\left((p_1)^2 - ((p_2)^2\right), \tag{5.3}$$

where H is the average of the interfacial barrier.
Model V is derived from model IV by taking $H = 0$.

Although these models are strongly non-linear, the effective permeabilities are the same as in the linear seepage problem. The fact that all boundary value problems reduce to linear boundary value problems is a remarkable feature. A few continuous passages between the models are possible (Fig. 2): by either increasing or decreasing h, Models I and III can be derived from Model II and Models III and V can be derived from Model IV.

An important issue to be addressed is the estimation of B_l with respect to ε. Since it is related to the knowledge of the microstructure, the value of B is well defined, whilst ε, as a measure of the separation of scales, depends in particular on the macroscopic characteristic length, which is either the sample size or related to the pressure gradient. As a consequence, if the macroscopic length is modified (e.g. by changing the pressure gradient), then the value of B_l remains the same but its measure in powers of ε may be changed. In the latter case, the macroscopic model itself is also modified. Therefore, the choice of the macroscopic model is conditioned by both the size of the sample and the pressure gradient.

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